Yugoslav Journal of Operations Research
25 (2015), Number 1, 73-91
DOI: 10.2298/YJOR130217035J

OPTIMAL PRICING AND PROMOTIONAL EFFORT CONTROL POLICIES FOR A NEW PRODUCT GROWTH IN SEGMENTED MARKET

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Received: February 2013 / Accepted: October 2013

Abstract: Market segmentation enables the marketers to understand and serve the customers more effectively thereby improving company's competitive position. In this paper, we study the impact of price and promotion efforts on evolution of sales intensity in segmented market to obtain the optimal price and promotion effort policies. Evolution of sales rate for each segment is developed under the assumption that marketer may choose both differentiated as well as mass market promotion effort to influence the uncaptured market potential. An optimal control model is formulated and a solution method using Maximum Principle has been discussed. The model is extended to incorporate budget constraint. Model applicability is illustrated by a numerical example. Since the discrete time data is available, the formulated model is discretized. For solving the discrete model, differential evolution algorithm is used.

Keywords: Market Segmentation, Price and Promotional Effort Policy, Differentiated and Market promotion Effort, Optimal Control Problem, Maximum Principle, Differential Evolution Algorithm.

MSC: 49J15, 49M25, 68Q25, 93B40.

1. INTRODUCTION

Successful introduction and growth of a new product entail creating a sound and efficient marketing strategy for the target market. Such a strategy involves effective planning and decision making with regards to price and promotion that affect product sales, potential profit, and also plays a major role in the survival of a company in the competitive marketplace. Counter to traditional marketing concept which was more about an economic exchange of goods for money, modern marketing focuses on customer satisfaction and delight. Firms today achieve profit maximization but not at the cost of dissatisfied customers. They develop customer oriented marketing strategies based on the needs/desires of the customers. In vast and diversified market scenario, where every customer has an individualistic need and preference, it becomes difficult for firms to satisfy everyone. Firms, therefore, employ a tool of market segmentation and divide the customer groups on the basis of their demand characteristics and traits into distinct segments. Segregating market into segments helps firms to better serve needs of their customers and consequently, to gain higher levels of market share and profitability. Market segmentation divides the customers according to their geographical, demographical, psychographical and/or behavioral characteristics. Market segmentation allows firms to employ buyer oriented marketing, so as to target each of the market segments with the marketing strategies specially developed for the segments, commonly known as differentiated marketing strategy. Typically, marketers also view these segments together as a larger market and develop mass market promotion strategies to cater to the common traits of the customers with a spectrum effect in all segments. In this paper, we study the impact of price and promotional efforts on evolution of sales intensity in segmented market to facilitate determination of optimal price and promotional effort policies. Evolution of sales rate for each segment is determined under joint influence of differentiated and mass market promotion effort. The problem has been formulated as an optimal control problem. Using Maximum Principle [24], optimal price and promotion effort policies have been obtained for the proposed model. The model is extended to incorporate the budget constraint. Further, as the formulated model is continuous in nature and discrete data is available for practical application, discrete counterpart of the model is developed. For solving the discrete model differential evolution algorithm is discussed.

Since past few years, a number of researchers have been working in the area of optimal control models pertaining to advertising expenditure and price in marketing (Thompson and Teng [31]). The simplest diffusion model was due to Bass [1]. Since the landmark work of Bass, the model has been widely used in the diffusion theory. The major limitation of this model is that it does not take into consideration the impact of marketing variables. Many authors have suitably modified Bass model to study the impact of price on new product diffusion (Horsky [9]; Kalish [12,13]; Kamakura and Balasubramanium [14]; Robinson and Lakhani [21]; Sethi and Bass [26]). Also, there are models that incorporate the effect of advertising on diffusion (Dockner and Jørgensen [4]; Horsky and Simon [8]; Simon and Sebastian [28]). Horsky and Simon [8] incorporated the effects of advertising in Bass innovation coefficient. Thompson and Teng [31] incorporated learning curve production cost in their oligopoly price-advertising model. Bass, Krishnan and Jain [2] included both price and advertising in their Generalized Bass Model.

Segmentation serves as a base for many vital marketing decisions. It is an important strategy in modern marketing as it provides an insight into the target pricing and promotion policies. Market segmentation is one of the most widely studied area for academic research in marketing. Quite a few papers have been written in the area of dynamic advertising models that deal with market segmentation (Buratto, Grosset and Viscolani [3]; Grosset and Viscolani [10]; Little and Lodish [15]; Seidmann, Sethi and Derzko [23]). Buratto, Grosset and Viscolani [3] and Grosset and Viscolani [10] discussed the optimal advertising policy for a new product introduction in a segmented market with Narlove-Arrow's [17] linear goodwill dynamics. Little and Lodish [15] analyzed a discrete time stochastic model of multiple media selection in a segmented market. Seidmann, Sethi and Derzko [23] proposed a general sales-advertising model in which the state of the system represented a population distribution over a parameter space. They showed that such models were well posed, and that there existed an optimal control. Further, Jha, Chaudhary and Kapur [11] used the concept of market segmentation in diffusion model for advertising a new product, and studied the optimal advertising effectiveness rate in a segmented market. They discussed the evolution of sales dynamics in the segmented market under two cases. Firstly, assuming that the firm advertises in each segment independently, and further they took the case of a single advertising channel that reaches several segments with a fixed spectrum. Manik, Chaudhary, Singh and Jha [16] formulated an optimal control problem to study the effect of differentiated and mass promotional effort on evolution of sales rate for each segment. They obtained the optimal promotional effort policy for the proposed model. Dynamic behavior of optimal control theory leads to its application in sales-promotion control analysis and provide a powerful tool for understanding the behavior of sales-promotion system where dynamic aspect plays an important role. Numerous papers on the application of optimal control theory in sales-advertising problem exist in the literature [3, 4, 5, 6, 10, 25, 30, 32, 33].

While price, differentiated and mass market promotion play a central role in determining the acceptability, growth and profitability of the product, to the best of our knowledge, existing literature doesn't incorporate all the three parameters simultaneously in the optimal control model. In this paper we analyze the effect of price along with promotion (differentiated and mass market) policies on the evolution of sales of a product marketed in segmented market to obtain optimal price and promotion policies for a segment specific new product with an aim to maximize the profit. The formulated problem is solved using Maximum Principle [24]. The control model is extended to include the budgetary constraint. The proposed model is a continuous time model, but in practical application often discrete time data are available. So, the equivalent discrete formulation of the proposed model is developed. The discrete model can't be solved by using maximum principle applicable to continuous time models. For solving the discrete model, differential evolution (DE) algorithm is discussed as it is NP-hard in nature and mathematical programming procedures can't be used to solve such problems. DE algorithm is a useful tool for solving complex and intricate optimization problems otherwise difficult to be solved by the traditional methods. It is a powerful tool for global optimization, easy to implement, simple to use, fast and reliable. There is no particular structural requirement on the model before using DE.

The article is organized as follows. Section 2 presents the diffusion model and optimal control formulation, where the segmented sales rate is developed; the assumption is that the firm promotes its product by using differentiated promotion in each segment to target the segment potential, and the mass promotion campaign that influences all segments with a fixed spectrum effect. Solution methodology of the problem is also discussed in this section. Particular cases of the problem have been presented in section 2.1. Differential evolution algorithm for solving discretized problem is presented in section 3. Numerical example has been discussed in section 4. Conclusions and the scope for a future research are given in section 5.

2. MODEL FORMULATION

2.1. *Notations*

M	:	number of segments in the market (>1)
$\overline{N}_i$	:	expected number of potential customers in i^{th} segment, i=1,2,…,M
$N_i(t)$	:	number of adopters of the product in i^{th} segment by time t, i=1,2,…,M
$x_i(t)$	:	promotional effort rate for i^{th} segment at time t, i=1,2,…,M
$x(t)$	:	mass market promotional effort rate at time t
α_i	:	segment specific spectrum rate $i=1,\ldots,M;\ \alpha_i>0,\ \forall\, i=1,\ldots,M;\ \sum_i \alpha_i = 1$
$b_i(t)$	:	adoption rate per additional adoption in i^{th} segment, i=1,2,…,M
p_i/q_i	:	coefficient of external/internal influence in segment i, i=1,2,…,M
$u_i(x_i(t))$	:	differentiated market promotional effort cost
$v(x(t))$	:	mass market promotional effort cost
ρ	:	discounted profit
$P_i(t)$	:	sales price for i^{th} segment which depends upon time, i=1,2,…,M
D_i	:	price coefficients for i^{th} segment, i=1,2,…,M
$C_i(N_i(t))$	:	total production cost of i^{th} segment, i=1,2,…,M

We assume that segments are disjoint from each other and the value $\sum_{i=1}^{M} \overline{N}_i$ represents the total number of potential customers of the product. Sales rate, assumed to be a function of price, differentiated and mass market promotion effort, and remaining market potential evolution of sales intensity are described by the following differential equation

$$\frac{dN_i(t)}{dt} = b_i(t)\left(x_i(t) + \alpha_i x(t)\right) \left(\overline{N}_i - N_i(t)\right)e^{-D_i P_i(t)} \quad i = 1,2,\ldots,M \tag{1}$$

where, $\left(\overline{N}_i - N_i(t)\right)$, i=1,2,…,M is unsaturated portion of the market in i^{th} segment by time t, and $b_i(t)$, i=1,2,…,M is the adoption rate per additional adoption. Parameter α_i represents the rate with which mass promotion influence a segment i, i=1,2,…,M. Price effects are represented by the expression $e^{-D_i P_i(t)}$.

$b_i(t)$ can be represented either as a function of time or a function of the number of previous adopters. Since the latter approach is used most widely, it is applied here, too. Therefore, we assume that the adoption rate per additional adoption is $b_i(t) = \left(p_i + q_i \frac{N_i(t)}{\overline{N}_i}\right)$ [1], and consequently, sales intensity takes the following form

$$\frac{dN_i(t)}{dt} = \left(p_i + q_i \frac{N_i(t)}{\overline{N}_i}\right)\left(x_i(t) + \alpha_i x(t)\right) \left(\overline{N}_i - N_i(t)\right)e^{-D_i P_i(t)} \quad i = 1,2,\ldots,M \tag{2}$$

Under the initial condition

$$N_i(0) = N_{i0},\ i = 1,2,\ldots,M \tag{3}$$

The firm aims at maximizing the total present value of profit over the planning horizon in segmented market. Thus, the optimal control problem to determine optimal price, differentiated market and mass market promotional effort rates $P_i(t)$, $x_i(t)$, $x(t)$ for the new product is given by

$$\text{Max } J = \int_0^T e^{-\rho t}\left(\sum_{i=1}^{M}\left[\begin{matrix}\left(P_i(t) - C_i\left(N_i(t)\right)\right)N_i'(t) \\ -u_i\left(x_i(t)\right)\end{matrix}\right] - v\left(x(t)\right)\right)dt \tag{4}$$

subject to system equations (2) and (3), where $C_i(N_i(t))$ is production cost that is continuous and differentiable with the assumption that $C_i'(.) > 0$, and $P_i(t) - C_i\left(N_i(t)\right) > 0$ for all segments.

The optimal control model formulated above consists of $2M$+1 control variables ($P_i(t)$, $x_i(t)$, $x(t)$) and M state variables ($N_i(t)$). Using the Maximum Principle [24], Hamiltonian can be defined as

$$H=\left(\sum_{i=1}^{M}\left[\left(P_i(t)-C_i(N_i(t))+\lambda_i(t)\right)\begin{pmatrix}\left(p_i+q_i\dfrac{N_i(t)}{\bar{N}_i}\right)\\ \left(x_i(t)+\alpha_i x(t)\right)\left(\bar{N}_i-N_i(t)\right)e^{-D_iP_i(t)}\end{pmatrix}-u_i\left(x_i(t)\right)\right]-v\left(x(t)\right)\right) \tag{5}$$

The Hamiltonian represents the overall profit of the various policy decisions where both the immediate and the future effects are taken into account. Assuming the existence of an optimal control solution (the maximum principle provides the necessary optimality conditions), there exists a piecewise continuously differentiable function $\lambda_i(t)$ for all $t\in[0,T]$, where $\lambda_i(t)$ is known as an adjiont variable, and the value of $\lambda_i(t)$ at time t describes future effect on profits upon making a small change in $N_i(t)$.

From the optimality conditions [27], we have

$$\begin{aligned}&\frac{\partial H^*}{\partial x(t)}=0;\ \frac{\partial H^*}{\partial x_i(t)}=0;\ \frac{\partial H^*}{\partial P_i(t)}=0,\\ &\frac{d\lambda_i(t)}{dt}=\rho\lambda_i(t)-\frac{\partial H^*}{\partial N_i(t)},\ \lambda_i(T)=0\end{aligned} \tag{6}$$

The Hamiltonian H of each of the segments is strictly concave in $P_i(t)$, $x_i(t)$ and $x(t)$; according to the Mangasarian Sufficiency Theorem [24,27], there exist unique values of price control $P_i^*(t)$ and promotional effort control $x_i^*(t)$ and $x^*(t)$ for each segment, respectively. From equation (5) and (6), we get

$$P_i^*(t)=\frac{1}{D_i}+C_i(N_i(t))-\lambda_i(t),\ i=1,2,...,M \tag{7}$$

$$x_i^*(t)=\phi_i\begin{pmatrix}\left(P_i(t)-C_i(N_i(t))+\lambda_i(t)\right)\left(p_i+q_i\dfrac{N_i(t)}{\bar{N}_i}\right)\\ \left(\bar{N}_i-N_i(t)\right)e^{-D_iP_i(t)}\end{pmatrix},\ i=1,2,...,M \tag{8}$$

$$x^*(t)=\varphi\left(\sum_{i=1}^{M}\begin{bmatrix}\left(P_i(t)-C_i(N_i(t))+\lambda_i(t)\right)\alpha_i\left(p_i+q_i\dfrac{N_i(t)}{\bar{N}_i}\right)\\ \left(\bar{N}_i-N_i(t)\right)e^{-D_iP_i(t)}\end{bmatrix}\right) \tag{9}$$

where, $\varphi_i(.)$ and $\varphi(.)$ are the inverse functions of u_i and v, respectively. Optimal price policy suggests that price which maximizes immediate profits for a firm is the price that equates marginal revenue with marginal cost. The consideration of factor such as discounting alters the nature of the price. The optimal control promotional policy shows that when market is almost saturated, then differentiated market promotional expenditure rate and mass market promotional expenditure rate, respectively, should be zero (i.e. there is no need of promotion in the market).

For optimal control policy, the optimal sales trajectory using optimal values of price ($P_i^*(t)$), differentiated market promotional effort ($x_i^*(t)$) and mass market promotional effort ($x^*(t)$) rates are given by

$$N_i^*(t) = \frac{\bar{N}_i\left(\left(\frac{\left(p_i + q_i\frac{N_i(0)}{\bar{N}_i}\right)}{\bar{N}_i - N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left((x_i(t)+\alpha_i x(t))e^{-D_iP_i(t)}\right)dt\right) - p_i\right)}{\frac{q_i}{\bar{N}_i} + \left(\frac{\left(p_i + q_i\frac{N_i(0)}{\bar{N}_i}\right)}{\bar{N}_i - N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left((x_i(t)+\alpha_i x(t))e^{-D_iP_i(t)}\right)dt\right)} \quad \forall i \tag{10}$$

If $N_i(0)=0$, then we get the following result

$$N_i^*(t) = \bar{N}_i\left(\frac{1-\exp\left(-(p_i+q_i)\int_0^t\left((x_i(t)+\alpha_i x(t))e^{-D_iP_i(t)}dt\right)\right)}{1+\frac{q_i}{p_i}\exp\left(-(p_i+q_i)\int_0^t\left((x_i(t)+\alpha_i x(t))e^{-D_iP_i(t)}dt\right)\right)}\right) \quad \forall i \tag{11}$$

and adjoint trajectory is given as

$$\frac{d\lambda_i(t)}{dt} = \rho\lambda_i(t) - \left\{\left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\left(\partial\dot{N}_i(t)\big/\partial N_i(t)\right) - \dot{N}_i(t)\left(\partial C_i(N_i(t))\big/\partial N_i(t)\right)\right\} \tag{12}$$

with transversality condition $\lambda_i(T)=0$.

Integrating (12), we have the future benefit of having one more unit of sale

$$\lambda_i(t) = e^{-\rho t}\int_t^T e^{-\rho s}\left(\left\{\begin{array}{l}\left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\left(\partial\dot{N}_i(t)\big/\partial N_i(t)\right)\\ -\dot{N}_i(t)\left(\partial C_i(N_i(t)\big/\partial N_i(t)\right)\end{array}\right\}\right)ds \tag{13}$$

2.1. *Particular Cases of General Formulation*

2.1.1. Differentiated market promotional effort and mass market promotional effort costs are linear functions

Let us assume that differentiated market promotional effort and mass market promotional effort costs take the following linear forms – $u_i(x_i(t)) = k_i\, x_i(t)$, $v(x(t)) = a\, x(t)$ and $0 \le x_i(t) \le \overline{A}_i$, $0 \le x(t) \le \overline{A}$, where $\overline{A}_i$, $\overline{A}$ are positive constants which are maximum acceptable promotional effort rates ($\overline{A}_i, \overline{A}$ are determined by the promotion budget etc.), k_i is cost per unit of promotion effort per unit time towards i^{th} segment, and a is cost per unit of promotional effort per unit time towards mass market. Now, Hamiltonian can be defined as

$$H = \left(\sum_{i=1}^{M} \left[\left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right) \begin{pmatrix} \left(p_i + q_i \dfrac{N_i(t)}{\overline{N}_i(t)} \right) \\ \left(x_i(t) + \alpha_i x(t)\right)\left(\overline{N}_i - N_i(t)\right) e^{-D_i P_i(t)} \end{pmatrix} - k_i x_i(t) \right] - ax(t) \right) \tag{14}$$

Optimal price policy does not depend directly on $x_i(t)$ and $x(t)$ therefore, for the particular case, it will be the same as in case of general scenario.

$$P_i^*(t) = \frac{1}{D_i} + C_i(N_i(t)) - \lambda_i(t), \; i = 1, 2, \ldots, M \tag{15}$$

Since Hamiltonian is linear in $x_i(t)$ and $x(t)$, optimal differentiated market promotional effort and mass market promotional effort as obtained by the maximum principle are given by

$$x_i^*(t) = \begin{cases} 0 & if \quad W_i \le 0 \\ \overline{A}_i & if \quad W_i > 0 \end{cases} \tag{16}$$

$$x^*(t) = \begin{cases} 0 & if \quad B \le 0 \\ \overline{A} & if \quad B > 0 \end{cases} \tag{17}$$

where, $W_i = -k_i + \left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\left(p_i + q_i \dfrac{N_i(t)}{\overline{N}_i} \right)\left(\overline{N}_i - N_i(t)\right) e^{-D_i P_i(t)}$

and $B = -a + \sum_{i=1}^{M} \left\{ \alpha_i \left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\left(p_i + q_i \dfrac{N_i(t)}{\overline{N}_i} \right)\left(\overline{N}_i - N_i(t)\right) e^{-D_i P_i(t)} \right\}$

W_i and B are promotional effort switching functions. In the optimal control theory terminology, this type of control is called “Bang-Bang” control. However, interior control is possible on an arc along $x_i(t)$ and $x(t)$. Such an arc is known as “Singular arc” [24,27]. There are four sets of optimal control values of differentiated market promotional effort ($x_i(t)$) and mass market promotional effort ($x(t)$) rate: 1) $x_i^*(t)=0,\ x^*(t)=0$; 2) $x_i^*(t)=0, x^*(t)=\overline{A}$; 3) $x_i^*(t)=\overline{A_i}, x^*(t)=0$; 4) $x_i^*(t)=\overline{A_i}, x^*(t)=\overline{A}$.

The optimal sales trajectory and adjoint trajectory, respectively using optimal values of price ($P_i^*(t)$), differentiated market promotional effort ($x_i^*(t)$) and mass market promotional effort ($x^*(t)$) rate are given by

$$N_i^*(t)=\frac{\overline{N}_i\left(\left(\frac{\left(p_i+q_i\frac{N_i(0)}{\overline{N}_i}\right)}{\overline{N}_i-N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left(\left(\overline{A_i}+\alpha_i\overline{A}\right)e^{-D_iP_i(t)}\right)dt\right)-p_i\right)}{\frac{q_i}{\overline{N}_i}+\left(\frac{\left(p_i+q_i\frac{N_i(0)}{\overline{N}_i}\right)}{\overline{N}_i-N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left(\left(\overline{A_i}+\alpha_i\overline{A}\right)e^{-D_iP_i(t)}\right)dt\right)}\quad\forall i \tag{18}$$

If $N_i(0)=0$, then we get the following result

$$N_i^*(t)=\overline{N}_i\left(\frac{1-\exp\left(-(p_i+q_i)\int_0^t\left(\left(\overline{A_i}+\alpha_i\overline{A}\right)e^{-D_iP_i(t)}dt\right)\right)}{1+\frac{q_i}{p_i}\exp\left(-(p_i+q_i)\int_0^t\left(\left(\overline{A_i}+\alpha_i\overline{A}\right)e^{-D_iP_i(t)}dt\right)\right)}\right)\quad\forall i \tag{19}$$

which is similar to Bass model [1] sales trajectory, and the adjiont variable is given by

$$\lambda_i(t)=e^{-\rho t}\int_t^T e^{-\rho s}\left(\begin{Bmatrix}\left(P_i(t)-C_i(N_i(t))+\lambda_i(t)\right)\left(\partial\dot{N}_i(t)\big/\partial N_i(t)\right)\\-\dot{N}_i(t)\left(\partial C_i(N_i(t))\big/\partial N_i(t)\right)\end{Bmatrix}\right)ds \tag{20}$$

2.1.2. Differentiated market promotional effort and mass market promotional effort costs are quadratic functions

Promotional efforts towards differentiated market and mass market are costly. Let us assume that differentiated market promotional effort and mass market promotional effort costs take the following quadratic forms – $u_i(x_i(t))=k_{1i}\,x_i(t)+\frac{k_{2i}}{2}x_i^2(t)$ and $v(x(t))=a_1\,x(t)+\frac{a_2}{2}x^2(t)$, where $a_1\geq0$; $k_{1i}\geq0$ and $a_2>0$; $k_{2i}>0$ are positive constants. The constants k_{1i} and a_1 are fixed cost per unit of promotional effort per unit time towards i^{th} segment and towards mass market, respectively. And the value of k_{2i} and a_2 represent the magnitude of promotional effort rate per unit time towards i^{th} segment and towards mass market, respectively. This assumption is common in literature [30], where promotion cost is quadratic. Now, Hamiltonian can be defined as

$$H = \left(\sum_{i=1}^{M} \left[\begin{array}{l} \left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right) \\ \left(\left(p_i + q_i \frac{N_i(t)}{\bar{N}_i} \right) \left(x_i(t) + \alpha_i x(t)\right) \left(\bar{N}_i - N_i(t)\right) e^{-D_i P_i(t)} \right) \\ -\left(k_{1i}\, x_i(t) + \frac{k_{2i}}{2} x_i^2(t) \right) \end{array} \right] \\ -\left(a_1\, x(t) + \frac{a_2}{2} x^2(t) \right) \right) \tag{21}$$

Optimal price policy does not depend directly on $x_i(t)$ and $x(t)$ therefore, for the quadratic case, it will be the same as in case of general scenario.

$$P_i^*(t) = \frac{1}{D_i} + C_i(N_i(t)) - \lambda_i(t), \ i = 1, 2, \ldots, M \tag{22}$$

From the optimality necessary conditions (6), the optimal differentiated market promotional effort and mass market promotional effort are given by

$$x_i^*(t) = \frac{1}{k_{2i}} \left(\left[\begin{array}{l} \left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\left(p_i + q_i \frac{N_i(t)}{\bar{N}_i} \right) \\ \left(\bar{N}_i - N_i(t)\right) e^{-D_i P_i(t)} \end{array} \right] - k_{1i} \right), \ i = 1, 2, \ldots, M \tag{23}$$

$$x^*(t) = \frac{1}{a_2} \left(\sum_{i=1}^{M} \left[\begin{array}{l} \left(P_i(t) - C_i(N_i(t)) + \lambda_i(t)\right)\alpha_i \left(p_i + q_i \frac{N_i(t)}{\bar{N}_i} \right) \\ \left(\bar{N}_i - N_i(t)\right) e^{-D_i P_i(t)} \end{array} \right] - a_1 \right) \tag{24}$$

The optimal sales trajectory and adjoint trajectory, respectively, using optimal values of price ($P_i^*(t)$), differentiated market promotional effort ($x_i^*(t)$) and mass market promotional effort ($x^*(t)$) rate are given by

$$N_i^*(t) = \frac{\bar{N}_i\left(\left(\frac{\left(p_i + q_i\frac{N_i(0)}{\bar{N}_i}\right)}{\bar{N}_i - N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left(\left(x_i^*(t)+\alpha_i x^*(t)\right)e^{-D_iP_i^*(t)}\right)dt\right)\right) - p_i}{\frac{q_i}{\bar{N}_i} + \left(\frac{\left(p_i + q_i\frac{N_i(0)}{\bar{N}_i}\right)}{\bar{N}_i - N_i(0)}\right)\exp\left((p_i+q_i)\int_0^t\left(\left(x_i^*(t)+\alpha_i x^*(t)\right)e^{-D_iP_i^*(t)}\right)dt\right)} \quad \forall i \qquad (25)$$

If $N_i(0)=0$, then we get the following result

$$N_i^*(t) = \bar{N}_i\left(\frac{1-\exp\left(-(p_i+q_i)\int_0^t\left(\left(x_i^*(t)+\alpha_i x^*(t)\right)e^{-D_iP_i^*(t)}\right)dt\right)}{1+\frac{q_i}{p_i}\exp\left(-(p_i+q_i)\int_0^t\left(\left(x_i^*(t)+\alpha_i x^*(t)\right)e^{-D_iP_i^*(t)}\right)dt\right)}\right) \quad \forall i \qquad (26)$$

which is similar to Bass model [1] sales trajectory, and the adjiont variable is given by

$$\lambda_i(t) = e^{-\rho t}\int_t^T e^{-\rho s}\left(\left\{\begin{matrix}\left(P_i(t)-C_i(N_i(t))+\lambda_i(t)\right)\left(\partial\dot{N}_i(t)\big/\partial N_i(t)\right) \\ -\dot{N}_i(t)\left(\partial C_i(N_i(t))\big/\partial N_i(t)\right)\end{matrix}\right\}\right)ds \qquad (27)$$

Now, to illustrate the applicability of the formulated model through a numerical example, the discounted continuous optimal problem (4) is transformed into an equivalent discrete problem [22], which can be solved by using DE. The discrete optimal control problem can be written as follows

$$\text{Max } J=\sum_{r=1}^{T}\left(\left[\left(\sum_{i=1}^{M}\left(P_i(r)-C_i(N_i(r))\right)\left(N_i(r+1)-N_i(r)\right)-u_i\left(x_i(r)\right)\right)-v\left(x(r)\right)\right]\left(\frac{1}{\left(1+\rho\right)^{r-1}}\right)\right) \qquad (28)$$

s.to

$$N_i(r+1)=N_i(r)+\left(\left(p_i+q_i\frac{N_i(r)}{\bar{N}_i(r)}\right)\left(x_i(r)+\alpha_i x(r)\right)\left(\bar{N}_i-N_i(r)\right)e^{-D_iP_i(r+1)}\right) \quad i=1,2,\ldots,M \qquad (29)$$

Usually, firms employ promotional efforts to increase sales of their products by transforming potential customers from the state of unawareness to that of action. Despite the fact that promotion is essential to increase sales of the firm's product, firms cannot go on promoting their products indefinitely due to scarcity of promotional resources and short product life cycles. Also as time progresses, consumer adoption pattern changes. Hence, to make a more realistic problem, it becomes imperative to introduce a budget constraint in the above written optimal control problem. The budgetary problem can be written as

Max $$J = \int_0^T e^{-\rho t}\left(\sum_{i=1}^{M}\begin{bmatrix}\left(P_i(t)-C_i(N_i(t))\right)\\ N_i'(t)-k_i x_i(t)\end{bmatrix}-ax(t)\right)dt \tag{30}$$

s.to

$$\frac{dN_i(t)}{dt}=\left(p_i+q_i\frac{N_i(t)}{\bar{N}_i(t)}\right)\left(x_i(t)+\alpha_i x(t)\right)\left(\overline{N}_i-N_i(t)\right)e^{-D_iP_i(t)} \quad i=1,2,...,M \tag{31}$$

$$\left(\sum_{i=1}^{M}\int_0^T k_i x_i(t)dt+\int_0^T ax(t)\,dt\right)\le Z_0 \tag{32}$$

$$N_i(0) = N_{i0},\ i = 1,2,...,M \tag{33}$$

where Z_0 is the total budget for differentiated market promotion and mass market promotion. The equivalent discrete optimal control of the budgetary problem can be written as follows

Max $$J=\sum_{r=1}^{T}\left(\left[\left(\sum_{i=1}^{M}\left(P_i(r)-C_i(N_i(r))\right)(N_i(r+1)-N_i(r))-k_ix_i(r)\right)-ax(r)\right]\left(\frac{1}{(1+\rho)^{r-1}}\right)\right) \tag{34}$$

s.to

$$N_i(r+1)=N_i(r)+\begin{pmatrix}\left(p_i+q_i\dfrac{N_i(r)}{\bar{N}_i(r)}\right)\left(x_i(r)+\alpha_i x(r)\right)\\ \left(\overline{N}_i-N_i(r)\right)e^{-D_iP_i(r+1)}\end{pmatrix} \quad i=1,2,...,M \tag{35}$$

$$\sum_{i=1}^{M}\left(\sum_{r=0}^{T}\left(k_ix_i(r)+ax(r)\right)\right)\le Z_0 \tag{36}$$

The discrete model formulated above cannot be solved by using maximum principle. Mathematical programming methods can be applied to solve the discrete model, but the proposed model is NP-hard in nature, differential evolution algorithm is discussed to solve the discrete formulation. Subsequent section presents procedure for applying DE algorithm.

3. DIFFERENTIAL EVOLUTION ALGORITHM

Differential evolution is an evolutionary algorithm introduced by Price and Storn [18]. DE is simple, easy to implement, efficient, fast and reliable [7,18-20,29]. Like any other evolutionary algorithm, DE also works with some randomly generated initial population, which is then improved by using selection, mutation, and crossover operations. Numerous methods exist to determine a stopping criterion for DE, but usually, a desired accuracy between the maximum and minimum value of fitness function (objective function) provides an appropriate stopping condition. The fitness function under consideration is

$$J=\sum_{r=1}^{T}\left(\left[\left(\sum_{i=1}^{M}\left(P_i(r)-C_i(r)\right)(N_i(r+1)-N_i(r))-k_i x_i\left(r\right)\right)-ax\left(r\right)\right]\left(\frac{1}{\left(1+\rho\right)^{r-1}}\right)\right)$$

The elementary DE algorithm is described in detail as follows

Start

Step 1: Randomly initialize all the solution vectors in a population

Step 2: Generate a new population by repeating the following steps until the stopping criterion is reached

- **[Selection]** Select the random individuals for reproduction
- **[Reproduction]** Create new individuals from selected ones by mutation and crossover
- **[Evolution]** Compute the fitness values of the individuals
- **[Advanced Population]** Select the new generation from target (initial individuals and trial (crossover) individuals

End steps.

3.1. *Initialization*

The optimal control problem at hand has $x_i(t)$, i=1,2,…,M; $x(t)$ and $P_i(t)$, i=1,2,…,M as the control parameters. Now, in order to optimize a function of 2M+1 (say D) number of control parameters, a population of size NP is selected, where NP parameter vectors have the following form

$$X_{j,G}=(P_{i1,j,G}, P_{i2,j,G},..., P_{il,j,G}, x_{il+1,j,G}, x_{il+2,j,G},..., x_{im,j,G}, x_{m+1,j,G}, x_{m+2,j,G},..., x_{D,j,G})$$

here, D is dimension, j is an individual index, and G represents the number of generations.

To begin with, all the solution vectors in a population are randomly generated between the lower and upper bounds l={l_1,l_2,…,l_D}and u={u_1,u_2,…,u_D}using the equations

$$P_{ik,j,0} = l_k + rand_{j,k}[0,1]\times(u_k - l_k)$$
$$x_{ik,j,0} = l_k + rand_{j,k}[0,1]\times(u_k - l_k)$$
$$x_{k,j,0} = l_k + rand_{j,k}[0,1]\times(u_k - l_k)$$

where, j is an individual index, k is component index, and $rand_{j,k}[0,1]$ is an uniformly distributed random number lying between 0 and 1. This randomly generated population of vectors $X_{j,0}=(P_{i1,j,0}, P_{i2,j,0},..., P_{il,j,0}, x_{il+1,j,0}, x_{il+2,j,0},..., x_{im,j,0}, x_{m+1,j,0}, x_{m+2,j,0},..., x_{D,j,0})$ is known as target vectors.

3.2. *Mutation*

Mutation expands the search space and ensures that the algorithm converges towards near optimal solution. In DE, mutation takes place with 100% intensity. For the parameter vector $X_{j,G}$, three vectors $X_{r_1,G}, X_{r_2,G}, X_{r_3,G}$ are randomly selected such that the indices j, r_1, r_2, r_3 are distinct. The j^{th} mutant vector, $V_{j,G}$, is then generated based on the three chosen individuals as follows

$$V_{j,G} = X_{r_1,G} + F \times (X_{r_2,G} - X_{r_3,G})$$

where, $r_1,r_2,r_3 \in \{1,2,..., NP\}$ are randomly selected, such that $r_1 \neq r_2 \neq r_3 \neq j$, $F \in (0,1.2]$ and the scaled difference between two randomly chosen vectors, $F \times (X_{r_2,G} - X_{r_3,G})$, defines magnitude and direction of mutation.

3.3. *Crossover*

The mutant vector $V_{j,G}=(v_{1,j,G},v_{2,j,G},...,v_{l,j,G},v_{l+1,j,G},v_{l+2,j,G}, ...,v_{m,j,G},v_{m+1,j,G},v_{m+2,j,G}, ...,v_{D,i,G})$ and the current population member, $X_{j,G}=(P_{i1,j,G}, P_{i2,j,G},..., P_{il,j,G}, x_{il+1,j,G}, x_{i,l+2,j,G},..., x_{im,j,G}, x_{m+1,j,G}, x_{m+2,j,G},..., x_{D,j,G})$ then undergo crossover, that finally generates the population of candidates known as "*trial*" vectors, $U_{j,G}=(u_{1,j,G},u_{2,j,G},...,u_{l,j,G},u_{l+1,j,G},u_{l+2,j,G}, ...,u_{m,j,G},u_{m+1,j,G},u_{m+2,j,G}, ...,u_{D,i,G})$, as follows

$$u_{k,j,G} = \begin{cases} v_{k,j,G} & if\ rand_{j,k}[0,1] \le C_r \forall\ k = k_{rand} \\ x_{k,j,G} & \text{otherwise} \end{cases}$$

where, $C_r \in [0,1]$ is a crossover probability, $k_{rand} \in \{1,2,...,D\}$ is a random parameter's index, chosen once for each j.

3.4. *Selection*

To select population for the next generation, individuals in the trial vector are compared with the individuals in a current population. If the trial vector has equal or better objective value, then it replaces the current population in the next generation. That is,

$$X_{j,G+1} = \begin{cases} U_{j,G} & if\ J(U_{j,G}) \ge J(X_{j,G}) \\ X_{j,G} & \text{otherwise} \end{cases}$$

where, $J(.)$ is the objective function value. Therefore, the average objective value of the population will never worsen, making DE an elitist method. Mutation, recombination, and selection continue until stopping criterion is reached.

3.5. *Constraint Handling in Differential Evolution*

$$\text{Let } g_i = N_i(r+1) - N_i(r) + \left(p_i + q_i \frac{N_i(r)}{\bar{N}_i}\right)\left(x_i(r) + \alpha_i x(r)\right)\left(\overline{N}_i - N_i(r)\right)e^{-D_i P_i(r+1)} \quad i = 1, 2, ..., M$$

$$\text{and } g_{M+1} = \sum_{i=1}^{M}\left(\sum_{r=0}^{T}\left(k_i x_i(r) + a x(r)\right)\right) \le Z_0$$

There exist different constraints handing techniques in DE, but the most common approach adopted to deal with constrained search spaces is the use of Pareto ranking method. In this method, rank of the sum of the constraints violation is calculated at target and trial vectors, i.e.

$$\sum_{i=1}^{M}\left(\begin{array}{l} N_i(r+1) - N_i(r) + \\ \left(p_i + q_i \frac{N_i(r)}{\bar{N}_i}\right)\left((x_{il+1,j,G}(r), x_{il+2,j,G}(r), ..., x_{im,j,G}(r)) + (\alpha_{m+1}x_{m+1,j,G}(r), \alpha_{m+2}x_{m+2,j,G}(r), ..., \alpha_D x_{D,j,G}(r))\right) \\ \left(\overline{N}_i - N_i(r)\right)e^{-D_i(P_{i1,j,G}(r+1), P_{i2,j,G}(r+1), ..., P_{il,j,G}(r+1))} \end{array}\right) +$$

$$\sum_{i=1}^{M}\left(\sum_{r=0}^{T}\left(k_i(x_{il+1,j,G}(r), x_{il+2,j,G}(r), ..., x_{im,j,G}(r)) + a(x_{m+1,j,G}(r), x_{m+2,j,G}(r), ..., x_{D,j,G}(r))\right)\right) \le Z_0$$

is compared with

$$\sum_{i=1}^{M}\left(\begin{array}{l} N_i(r+1) - N_i(r) + \\ \left(p_i + q_i \frac{N_i(r)}{\bar{N}_i}\right)\left((u_{1,j,G}(r), u_{2,j,G}(r), ..., u_{l,j,G}(r)) + (\alpha_{l+1}u_{l+1,j,G}(r), \alpha_{l+2}u_{l+2,j,G}(r), ..., \alpha_m u_{m,j,G}(r))\right) \\ \left(\overline{N}_i - N_i(r)\right)e^{-D_i(u_{m+1,j,G}(r+1), u_{m+2,j,G}(r+1), ..., u_{D,j,G}(r+1))} \end{array}\right) +$$

$$\sum_{i=1}^{M}\left(\sum_{r=0}^{T}k_i(u_{1,j,G}(r), u_{2,j,G}(r), ..., u_{l,j,G}(r)) + a(u_{l+1,j,G}(r), u_{l+2,j,G}(r), .., u_{m,j,G}(r))\right) \le Z_0$$

then, the selection is made based on the following three rules:

1) Between two feasible solutions, the one with the best value of the objective function is preferred;

2) If one solution is feasible and the other one is infeasible, the one which leads to feasible solution is preferred;

3) If both solutions are infeasible, the one with the lowest sum of constraint violation is preferred.

3.6. *Stopping Criterion*

DE algorithm stops when either

1) Maximum number of generations are reached,

or

2) Desired accuracy is achieved i.e., $\left|J_{\max} - J_{\min}\right| \le \varepsilon$.

In the first stopping criterion, maximum number of generations may exhaust without reaching a near optimal solution. So, we usually chose the second criterion which ensures reaching a near optimal solution.

4. NUMERICAL ILLUSTRATION

In countries like India, where every region has its own distinct (regional) language, many firms adopt differentiated market strategy to promote their products in various regional languages, such as Marathi, Malayalam, Punjabi, Gujarati, Bengali to name a few. Also, they adopt mass market strategy to promote their products in national languages (Hindi and English) to persuade the large customer base. The mass market promotion influences various segments (regions) with a fixed spectrum. Hence, in such a situation, it is essential to allocate at least 30-40% of the total promotional budget to mass market promotion and the remaining for differentiated market.

The optimal control model formulated in this paper incorporates the impact of price, mass and differentiated promotional effort to obtain the optimal pricing and promotion planning. The discrete optimal control problem developed in this paper is solved by using DE algorithm. Parameters of DE are given in Table 1. A desired accuracy of .001 between maximum and minimum values of fitness function is taken as the terminating criteria of the algorithm. Total promotional budget is assumed to be Rs.1,50,000, 30-40% of which is allocated for mass market promotion, and the rest for segment specific promotion (i.e. differentiated market promotion). We further assume that the time horizon is divided into 10 equal time periods. The number of market segments are assumed to be six (i.e. M=6). Value of parameters a and ρ are taken to be 0.2 and 0.095, respectively, and the values of the rest of parameters are given in Table 2.

Table 1: Parameters of Differential Evolution

Parameter	Value	Parameter	Value
Population Size	200	Scaling Factor (F)	.7
Selection Method	Roulette Wheel	Crossover Probability (C_r)	.9

Table 2: Parameters

Segment	$\overline{N}_i$	C_i	k_i	D_i	p_i	q_i	α_i
S1	52000	9850	0.0016	0.00003	0.0000521	0.000626	0.1513
S2	46520	12360	0.0019	0.00004	0.0000493	0.000526	0.2138
S3	40000	10845	0.0022	0.000028	0.0000610	0.000631	0.1268
S4	29100	13055	0.0017	0.000035	0.0000551	0.00055	0.2204
S5	35000	11841	0.0021	0.000041	0.0000541	0.00055	0.1465
S6	25000	10108	0.0018	0.000033	0.0000571	0.000568	0.1412
Total	227620						1

When the model is solved using this data set and budget, DE gives a compromised solution by increasing the promotional budget from Rs.1,50,000 to Rs.5,00,000; this clearly indicates the need for firms to more aggressively promote their product to effectively tap the enhanced uncaptured market, which may be attributed primarily to the impact of price alongside promotion. Best possible allocations of promotional effort resources for each segment are given in Table 3; price is tabulated in Table 4; the corresponding sales with these resources, price and the percentage of adoption (sales) for each segment out of total potential market are tabulated in Table 5.

Table 3: Segment-Wise Promotional Effort Allocation

Segment	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	Total
S1	7105	5802	3208	8016	6585	6452	5543	1907	6719	7885	59223
S2	3204	5028	6841	6716	5804	8143	7238	5411	5807	6845	61038
S3	4518	5417	14922	5423	5814	2818	4129	6851	4386	6454	60733
S4	1905	1918	4115	5802	3986	7754	2812	6195	2948	6717	44153
S5	7111	4509	7108	7761	4635	8011	8019	2942	7755	4247	62099
S6	5805	2556	2684	7759	6845	5417	2819	4634	7238	8406	54164
MPA*	13823	10678	11419	11234	12715	53414	12156	10494	11605	11048	158587

*Mass Promotional Allocation

Table 4: Price (in Rs.)

Segment	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
S1	10100	10050	10030	10085	10136	10097	10106	10200	10112	10131
S2	12505	12619	12833	12732	12625	12843	12736	12622	12525	12633
S3	11100	11207	11388	11107	11010	11087	10997	11118	11299	11082
S4	13290	13240	13307	13420	13306	13493	13477	13423	13198	13327
S5	12016	12096	12216	11959	11997	12123	12023	12184	12059	11994
S6	10373	10262	10349	10388	10281	10470	10550	10364	10284	10293

Table 5: Sales and Percentage of Adoption for Each Segment from Total Potential Market

Segment	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	Total	% of Captured Market Size
S1	3900	3380	2860	4264	3692	4212	3276	2860	3744	4212	36400	70.00
S2	2559	2745	3396	3349	3024	4466	3536	2884	3024	3396	32378	69.60
S3	2200	2480	6720	2480	2600	2200	2200	2920	2200	2800	28800	72.00
S4	1601	1601	1601	1892	1601	5820	1601	1979	1601	2095	21389	73.50
S5	2625	1925	2625	2800	1960	3395	2870	1925	2800	1925	24850	71.00
S6	1625	1375	1375	2000	1825	3750	1375	1400	1900	2125	18750	75.00
Total											162566	71.42

5. CONCLUSION AND SCOPE FOR FUTURE RESEARCH

Market segmentation is an evolving field that has attracted interests of many researchers. The purpose behind segregating the market into segments is to better serve the diversified customers with varying needs, and to have a competitive edge over the competitors. In this paper, we have studied the effect of price along with differentiated market and mass market promotion effort on the evolution of sales rate in the segmented market, where mass market promotion influences each segment with a fixed spectrum. We have formulated an optimal control model using an innovation diffusion model, and then the problem has been extended by adding a budgetary constraint. Maximum principle has been used to obtain the solution of the proposed problem. Using the optimal control techniques, the main objective here was to determine optimal price and promotional effort policies. Two particular cases of the proposed optimal control problem have also been discussed – first, with linear differentiated and mass market promotional effort costs and second, with quadratic differentiated and mass market promotional effort costs. After discretizing the problem with linear costs, a numerical example has been solved by using differential evolution algorithm to illustrate the applicability of the approach. The findings highlight that as per the optimal price policy, price that maximizes immediate profits for a firm is the price that equates marginal revenue with marginal cost. Further, the consideration of factor such as discounting alters the nature of the price. Optimal control promotional effort policy emphasizes that when market is almost saturated, promotional effort diminishes. Also the numerical example clearly indicates that when price impacts the uncaptured market potential alongside promotion, firms need to promote their product more aggressively. The developed optimal control model can be further extended in several ways. For instance, factors such as price, quality and cost can be incorporated along with differentiated and mass market promotional effort expenditure. Further, this monopolistic model can be extended to competitive duopolistic or oligopolistic markets. Also, the model can be extended to obtain optimal control policies for two and/or more generations' product in the market.

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