

## PRODUCTION AND PROCESS MANAGEMENT: AN OPTIMAL CONTROL APPROACH

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**Abstract:** Optimal control and efficient management of industrial products are the key for sustainable development in industrial and process engineering. It is well-known that proper maintenance of process performance, ensuring the quality products after a long time operation of the system, is desirable in any industry. Nonlinear dynamical systems may play crucial role to appropriately design the model and obtain optimal control strategy in production and process management. This paper deals with a mathematical model in terms of ordinary differential equations (ODEs) that describe control of production and process arising in industrial engineering. The optimal control technique in the form of maximum principle, used to control the quality products in the operation processes, is applied to analyze the model. It is shown that the introduction of state constraint can be advantageous for obtaining good products during the longer operation process. We investigate the model numerically, using some known nonlinear optimal control solvers, and we present the simulation results to illustrate the significance of introducing state constraint onto the dynamics of the model.

**Keywords:** Optimal Control, State Constraint, Production and Process, Industrial Engineering, Numerical Application.

**MSC:** 49K15, 92D30.

## 1. INTRODUCTION

Optimal control theory has a long history in the literature for diverse applications in engineering science. There may have been some controversies about the birth of the optimal control theory [28], but not about its necessity in practice. It is now widely accepted that optimal control theory has dominant role in applications of many real life problems arising in science, engineering, biology, and biomedicine. According to the argument of Sussmann and Willems [28], the fundamental research in optimal control theory came to light more than three centuries ago with the publication of Johann Bernoulli's solution of the Brachistochrone problem in 1697. However, main theoretical development in this field occurred in the 1950s, more than 65 years ago. The development of optimal control has gained strength by treating multivariable, time varying systems, as well as many nonlinear problems arising in control engineering, biology, and medicine. The Pontryagin Maximum Principle is a milestone in optimal control theory [25]. It extends the classical Euler and Weierstrass conditions from the classical calculus of variations to control settings [29]. The development of Nonsmooth Analysis [17] enhanced a wide scope of research and opened a new horizon in the optimal control theory; we omit the detailed descriptions of the theoretical development of non smooth maximum principle for optimal control problems, referring the readers to ([4, 8, 9, 10, 14, 15] and references therein) for some recent theoretical developments both for state and mixed constrained problems in this area.

Since the last few decades, in parallel to the theoretical development, numerical solutions, as well as applications of optimal control problems have become some of the most challenging and demanding areas of research due to their diverse applications especially in biology and medicine. The crucial issue of the present day research on optimal control theory is to bridge the gap between the theory and the application.

The necessary conditions of optimality (NCO) for optimal control problems are powerful tool in determining the optimal solution and are widely used in developing solvers. Moreover, they can provide qualitative information on the solution and are the basis for the study of regularity of the optimal solution (see for examples, [17, 29]). Necessary condition of optimality for optimal control problems with state constraints has been studied since the very beginning of optimal control theory. It appeared in a very natural way when modeling many real life engineering applications in engineering, life science, biology, and medicine. Since the first application of optimal control in biomedical engineering, around 1980s [23], several control and vaccination strategies for the treatment of infectious diseases in a certain population over a period of time have been successfully modeled as optimal control problems. Among those applications in biomedicine are modeling of infectious diseases and optimal control strategies like HIV/AIDS (see for examples, [1, 2, 6, 7, 19, 21], deadly nipah virus infections [3, 5]; for control of SEIR epidemic disease [10]; modeling the potential impact of global climate change [13], as well as modeling and control of cancer treatments [16, 26]. Other important applications of optimal control theory attracting attention are soft landing and fuel consuming of space vehicles in aerospace engineering [11], and efficient and sustainable managements of forest and ecosystems in life science [12].

This study is concerned with one of such applications of optimal control in production and process management arising in industrial engineering. We study a mathematical

model of the process in the form of optimal control problem in terms of ordinary differential equations (ODE), and present the numerical investigations of the problem (omitting the theoretical details) arising in production and process managements in industrial engineering. We also discuss the necessity of introducing state constraints in the model and show their influence in management for obtaining “good” (quality) products over a certain time of operations.

## 2. MATHEMATICAL MODEL

It is well-known that the rate of production of “good” (or quality/non-defective) products in a process management slows down after a long time operation of the systems due to absence of the proper maintenance. However, in some cases, an appropriate introduction of state constraints can help in preventing the decline of “good” items produced over time. Mathematical model can be used to describe this production process in terms of ODEs. This problem is an application of optimal control in management and industrial engineering where the state constraints play an influential role in maintaining the performance of the production process. The dynamic model describing the process is taken as in [22] in terms of the following differential equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2(t)u_1(t) - d(t), \\ \frac{dx_2}{dt} &= -(\alpha + u_2(t))x_2(t) + u_2(t), \\ x_1(t) &\geq 0, \quad a.e. \ t \in [0, T],\end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10} > 0, \quad x_2(0) = x_{20} > 0,$$

and the control constraints

$$0 \leq u_1(t) \leq U_1, \quad 0 \leq u_2(t) \leq U_2, \quad a.e. \ t \in [0, T].$$

In the above dynamic model, one can see that the process consists of two state variables and two control variables. Here, we suppose that the state variable  $x_1(t)$  represents inventory level at time  $t \in [0, T]$  for a fixed final time  $T > 0$ , and  $x_2(t)$  represents the proportion of 'good' units of end items, produced at time  $t$  (also known as process performance). The two control variables,  $u_1(t)$  and  $u_2(t)$  represent scheduled production rate and preventive maintenance rate to reduce the proportion of defective units produced, respectively.  $\alpha(t)$  denotes the obsolescence rate of the process performance in the absence of maintenance and  $d(t)$  is the demand rate. With the above system of differential equations, a production process in industrial engineering is efficiently described. It is worth mentioning that the negative sign in the second dynamic equation shows how the productions of ‘good’ (or non-defective) units of items decrease over time in absence of maintenance. However, an appropriate preventive measure can be applied to the process to slow the rate of declination of process performance. In this

preventive measure, an introduction of lower bound on the number of ‘good’ items produced over time can be imposed as *state constraints* to improve the process performance which may keep the ‘good’ items produced in a minimum level. Thus, we introduce the following state constraint (i.e. a lower bound) on the number of ‘good’ items:

$$x_2(t) \geq l_{\min}, \quad a.e. t \in [0, T] \quad (1)$$

where  $l_{\min}$  is the lower bound of the “good” items produced at time  $t$  and taking values in  $\mathbb{R}$ .

We observe that the constraint  $x_1(t) \geq 0, \quad a.e. t \in [0, T]$  is imposed because all demands must be satisfied, and the state constraint in (1) is crucial in our analysis because it is a lower bound on the number of ‘good’ items produced at time  $t$ .

Our objectives in this optimal control problem are to maximize the total discounted cost as well as the salvage cost with the following objective functional:

$$J(x_1, x_2, u_1, u_2) = \int_0^T (wd - hx_1(t) - \phi(u_1(t)) - cu_2(t))e^{-\rho t} dt + bx_2(T)e^{-\rho T}, \quad (2)$$

where the function  $\phi(u_1(t))$  is called the production cost function and is defined either as the *quadratic function*

$$\phi(u_1(t)) = ru_1^2, \quad r > 0, \quad (3)$$

or the *linear function*

$$\phi(u_1(t)) = qu_1, \quad q > 0, \quad (4)$$

all other parameters and constants (also sometimes regarded as weight parameters balancing the cost), along with their values used in the objective function, are presented in Table 1. Note that in our analysis we will consider and investigate the *quadratic function* only. Observe also that in the objective function, the other control function,  $u_2(t)$ , representing the maintenance cost, appears as a *linear function*. Thus, we are considering the quadratic production cost function and linear maintenance cost functions. Maureret *al.*[22] studied such problem without introducing any state constraint in the model. They investigated state constrained model when both the production cost function and maintenance cost function are linear. In this study, we investigate the state constrained model when the production cost function is quadratic and maintenance cost function is linear. We treat this problem based on numerical analysis, omitting the theoretical details. Before presenting the results of numerical simulations, we sketch the idea of optimality conditions for the optimal solutions of our problem, referring readers to [22, 24, 29] for more detailed studies.

### 3. CHARACTERIZATION OF OPTIMAL CONTROL

The above mentioned dynamic model along with the state constraint can be rewritten in the following optimal control problem:

$$(P) \begin{cases} \text{Maximize}_{(u_1, u_2) \in U} l(x(T)) + \int_0^T L(t, x(t), u(t)) \\ \text{subject to} \\ \dot{x}(t) = f(t, x(t), u(t)), \\ g(t, x(t)) \leq 0, \quad \forall t \in [0, T], \\ (u_1(t), u_2(t)) \in U \text{ a.e. } t \in [0, T], \\ x_1(t) \geq 0, \text{ a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where  $x(t) = (x_1(t), x_2(t))$ ,

$$\begin{aligned} f(x) &= (x_2(t)u_1(t) - d(t), -(\alpha + u_2(t))x_2(t) + u_2(t)), \\ g(t, x(t)) &= -x_2(t) + l_{\min}, \\ l(x(T)) &= bx_2(T)e^{-\rho T}, \quad L(t, x(t), u(t)) = (wd - hx_1(t) - 2u_1^2(t) - cu_2(t))e^{-\rho t} \end{aligned}$$

and the control functions representing the percentage are measurable functions defined in the set

$$U = \{(u_1, u_2) : 0 \leq u_1(t) \leq U_1, \quad 0 \leq u_2(t) \leq U_2, \text{ a.e. } t \in [0, T]\}.$$

Here problem (P) is a well-known optimal control problem with state constraint which coincides with *standard* optimal control problem in absence of state constraint. However, the optimal solutions of such problem can be characterized by the Pontryagin Maximum Principle (PMP), the pioneer works of Pontryagin *et al.* [25], which satisfies the necessary conditions of optimality for optimal control problems having the novelty of being sufficient conditions for the normal linear convex problems. The optimality systems are characterized by the two dynamic equations associated with two adjoint equations in turns of multipliers. For the optimality systems, we define the pseudo Hamiltonian in normal form (i.e. for  $\lambda = 1$ ):

$$H(x_1, x_2, u_1, u_2, p, 1) = \langle p, f(t, x, u) \rangle - L(t, x, u) \tag{5}$$

Suppose that  $(x^*, u^*)$  is the optimal solution of the problem (P). Then the maximum principle in [29] asserts the existence of an absolutely continuous function  $p$  and a scalar  $\lambda$  such that

$$(i) \|p\|_{\infty} + 1 = 0,$$

$$(ii) (-\dot{p}(t), 0) = p(t) \cdot f_{x,u}(t, x^*(t), u^*(t)) - L_{x,u}(t, x^*(t), u^*(t)),$$

$$(iii) \forall (u_1, u_2) \in U,$$

$$p(t) \cdot f_u(t, x^*(t), u(t)) - L_u(t, x^*(t), u(t)) \leq p(t) \cdot f_u(t, x^*(t), u^*(t)) - L_u(t, x^*(t), u^*(t)) \text{ a.e.},$$

Together with the transversality conditions  $p(T) = (v, b)$ , where  $v \in \mathbb{R}$ .

Now consider that  $p(t) = (p_{x_1}, p_{x_2})$ . We deduce from (iii) an explicit characterization of the optimal control for the production cost given in terms of the multipliers  $p$ :

$$u_1^*(t) = \frac{p_{x_1}(t)x_2(t)}{2r}.$$

Observe a special feature of  $(P)$  that the dynamics is linear, both in the state and control variables separately, and the cost is convex with respect to the production cost (control) function. For such problem, we can a priori get an explicit optimal control. However, although we refrain from writing here the necessary conditions in the form of a maximum principle, it is well known that the presence of explicit state constraints introduces an additional multiplier which, a priori, is a non-negative Radon measure (see, for example, [29]). This fact itself complicates analytical analysis and prevents determination of a closed form for the optimal control. Given the special features of  $(P)$ , we could hope to get some additional information taking into account some literature, like [20, 27], on the regularity of the optimal control. We also refer readers to [24] for the study of both necessary and sufficient conditions of optimality for such state constrained problems.

#### 4. NUMERICAL RESULTS AND DISCUSSIONS

We solve the optimality systems by numerical simulations to obtain the optimal performance of the process for our model in different scenarios: without state constraint and with state constraint. To do these simulations, we use the Imperial College London Optimal control Software – ICLOCS—version 0.1b [18]. ICLOCS is an optimal control interface, implemented in MATLAB, for solving the optimal control problems with general path and boundary constraints, and free or fixed final time. ICLOCS uses the IPOPT—Interior Point OPTimizer—solver which is an open-source software package for large-scale nonlinear optimization [30]. Considering a fixed time  $T = 1$ , a time-grid with 1000 nodes was created, i.e., for  $t \in [0, 1]$  we get  $\nabla t = 0.001$ . According to [22], the values for all parameters and constants we use in this model are presented in Table 1.

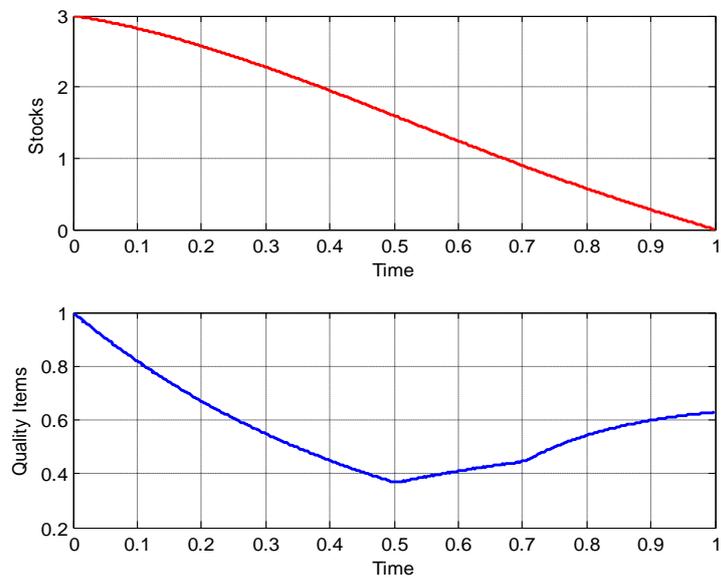
**Table 1:** Definitions and values of parameters and constants:

Parameters and constants	Definitions	Values
$d$	Demand rate	4
$\alpha$	Obsolescence rate of process performance in absence of maintenance	2
$x_{10}$	Initial value of inventory level	3
$x_{20}$	Initial value of process performance	1
$U_1$	Upper limit of scheduled production rate	3
$U_2$	Upper limit of preventive maintenance rate	4
$\rho$	Discount rate (positive constant)	0.1
$w$	Positive constant	8
$h$	Positive constant	1
$c$	Positive constant	2.5
$b$	Positive constant	10
$r$	Positive constant	2
$q$	Positive constant	4
$l_{\min}$	Lower bound on number of 'good' items	0.5

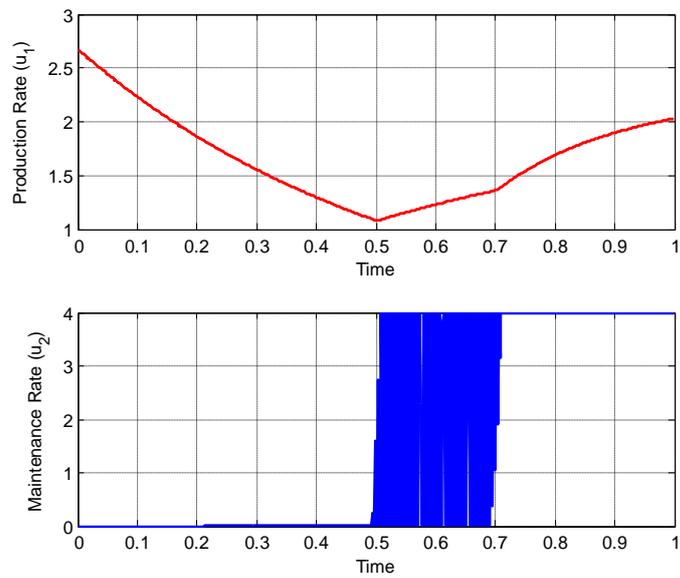
(Source: [22])

Since we used a direct method, and consequently, an iterative approach, we imposed an acceptable convergence tolerance at each step of  $\varepsilon_{rel} = 10^{-9}$  (see [18] for more details).

We first solve the optimality systems for our problem ( $P$ ) when the state constraint is not imposed. We run the program written in MATLAB code using the aforementioned solver 'ICLOCS' and the optimal solution found. The simulation results for the optimal states and controls are presented in Figures 1 and 2, respectively. Figure 1 shows how the state trajectories for stocks and productions of 'good' items slow down in absence of maintenance during the operations of the process over time. Figure 2 represents the behaviors of the control functions where the production cost control is singular, while the maintenance cost control is bang-bang with some strange behaviors from 0.5 to 0.7 in the time interval.

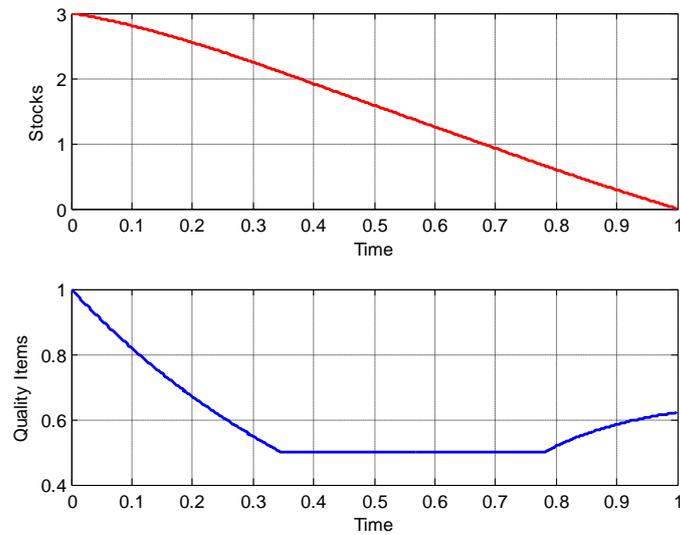


**Figure 1:** Optimal processes of stocks and quality products (i.e. 'good' items) without state constraint.

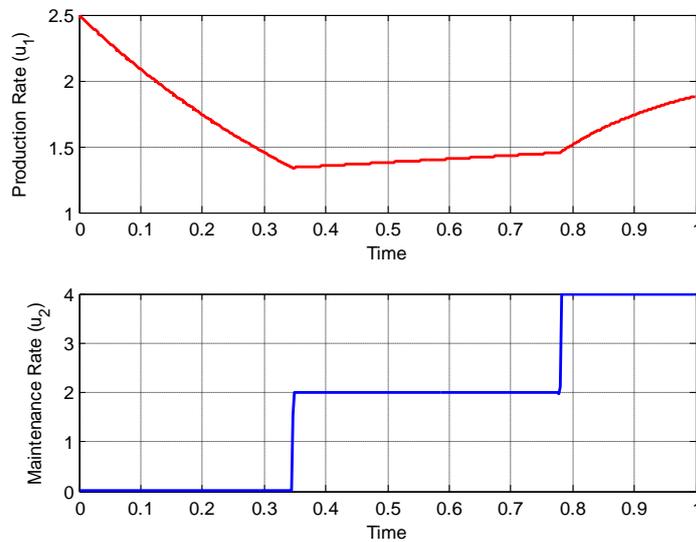


**Figure 2:** Optimal control representing the production rate and maintenance rate without state constraint.

Now we run the program for the optimality systems of problem ( $P$ ) in presence of state constraint. Here we again find the optimal solution satisfying the necessary conditions of optimality. Our results show that the rate of declination of the number of 'good' items produced over time can be halted from being slowed down at a certain minimum level. The simulation results of the optimal trajectories for state and control have been illustrated in Figures 3 and 4, respectively. From Figure 3, we see that the number of quality products can be halted from becoming slow down at  $l_{\min} = 0.5$  due to imposing state constraint, and thus, increases again over a fixed interval of time. The behaviors of control functions presented in Figure 4 are quite better than those in Figure 2.



**Figure 3:** Optimal processes of stocks and quality products (i.e. 'good' items) with state constraint.



**Figure 4:** Optimal control representing the production rate and maintenance rate with state constraint.

Our simulation results show that state constrained model can provide more information as well as better performance for maintenance a production process in industry if appropriate design of controller for better approximation can be made. As we omit here the detailed analytical investigations, further extensive study requires more accurate control strategy. Moreover, due to the presence of state constraint, the involvement measure requires validation of sufficient conditions of optimality, as well as the regularity of minimizers. Our future work will focus on these directions.

#### 4. CONCLUSIONS

Optimal control theory has become the distinct area of extensive research in dynamic optimization because it is necessary needed in diverse applications. It has been used as an essential tool for the optimal managements of all kinds of recourses, which are the key ingredients of sustainable development of a country. This paper mainly focuses on one of such applications in some real life problems, emphasizing management of production and maintenance in industrial engineering. A mathematical model, describing the production process in an industry over time, in terms of ordinary differential equations has been studied, and a numerical solution for the optimality systems has been presented using Pontryagin maximum principles. In many real situations, it is sometimes natural to impose constraints on the state variables to obtain desired optimal outputs from dynamic control problems. An important feature of this paper is that the model discussed introduced state constraint, showing that such constraint can be advantageous for the maintenance of process performance, ensuring the quality (or 'good' items) products after a long time operation of the system. As this result is based only on a numerical treatment, further analytical investigation is needed due to the presence of state

constraint. Our future research on the validation of sufficient optimality conditions and the regularity of minimizers will be focused on both detailed theoretical and numerical approaches.

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