Yugoslav Journal of Operations Research
32 (2022), Number 3, 357–376
DOI: https://doi.org/10.2298/YJOR210617006C

EXISTENCE AND STABILITY OF SOLUTIONS OF A GLOBAL SETVALUED OPTIMIZATION PROBLEM

Binayak S. CHOUDHURY
Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
binayak12@yahoo.co.in

Nikhilesh METIYA
Department of Mathematics, Sovarani Memorial College, Jagatballavpur, Howrah-711408, West Bengal, India
metiya.nikhilesh@gmail.com

Sunirmal KUNDU
Department of Mathematics, Government General Degree College, Salboni, Paschim Mednipur 721516, West Bengal, India
sunirmalkundu2009@rediffmail.com

Pranati MAITY
Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
pranati.math@gmail.com

Received: June 2021 / Accepted: March 2022

Abstract: In this paper we consider a best proximity point problem whose purpose is to determine the minimum distance between two sets. It is a global optimization problem by its very nature which is solved by converting it into a problem of finding an optimal approximate solution of a fixed point inclusion for a coupled setvalued mapping. Two solutions are obtained simultaneously through an iteration. We introduce certain definitions which are used in our theorems. We investigate the data dependence property of the proximity point sets and establish a weak stability result for the proximity point sets. There are some illustrative examples. The broad area of the present study is setvalued optimization.

Keywords: Metric Space, Coupled Best Proximity Point, Proximal α-dominating Mapping, Generalized Coupled Proximal Contraction, Data Dependence Property, Stability.

MSC: 47H10, 54H10, 54H25.

1. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

The problem which is considered in this paper is that of determining the minimum distance between two given sets. Such problems for different types of objects arise in various branches of mathematics for which different approaches are available in the literatures. An instance of it is the study of geodesics in geometry which is obtained by optimizing the path length between two points. Our candidates here are two sets, more specifically two closed subsets of a metric space. The approach to the problem is analytic. It is solved here by constructing a coupled setvalued function and then by globally solving the corresponding coupled setvalued fixed point problem for its optimal approximate solution although an exact solution does not in general exist. The approach which we adopt towards the problem has been discussed in a good number of papers appearing in recent literatures. It is known under the formal name of proximity point problem. The corresponding theorems in this study being optimality results are different from best approximation theorems like those in [21, 30]. One early result in this domain is the work of Eldered et al. [20] in the year 2006 in which a nonself singlevalued contraction was utilized of the above purpose. Coupled mappings have been utilized in these problems in works like [14, 24, 28, 34]. The special feature of these utilizations is that the optimal value is realized simultaneously through two different pairs of points. Setvalued mappings have also been considered in this category of problems in works like [2, 10, 22, 33]. In this paper we combine the above two approaches, that is, we consider setvalued coupled mapping for the above purpose. Some other important references from this line of research are [1, 4, 5, 8, 12, 19, 23, 27, 29, 35]. Most of the above mentioned works are in metric spaces. Some works have also appeared in generalized and extended metric structures. In our study, we introduce the coupled proximal α-dominating mapping which is a extended version of admissibility condition which is quite extensively used in fixed point theory now-a-days. It was introduced in the work of Samet et al. [32] and was further elaborated through works like [3, 16, 17].

We establish a data dependence result corresponding to our problem. In the present context, data dependence problem is to estimate the distance between the proximity point sets of the two mappings. Our problem of data dependence is with coupled mappings and their coupled proximity point sets. Several research papers on data dependence of fixed point sets have been published in recent literatures which we mention a few in references [7, 16, 25, 31]. Such problems for coupled fixed point sets have already appeared in work of Chifu et al [6].

Also we establish a weak stability theorem for the set of solutions of the problem which, in the present case is the set of proximity points of the corresponding coupled mappings. Convergence of fixed point sets of a sequence of mappings, known as the stability of fixed points, has also been widely studied in various settings [9, 16, 18]. Here we define the weak stability of proximity point sets of a sequence of multivalued mappings and establish the weak stability of proximity point sets of that sequence of mappings.

In the following we describe the technical background of the work.

Let $(X,\ d)$ be a metric space. Then $X \times X$ is also a metric space under the metric ρ defined by $\rho((x,y),\ (u,v)) = \max\ \{d(x,u),\ d(y,v)\}$, for all (x,y) and $(u,v) \in X \times X$.

Let $N(X)$ denote the collection of all nonempty subsets of X, $B(X)$ denote the collection of all nonempty and bounded subsets of X, $CL(X)$ denote the collection of all nonempty closed subsets of X, $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X and $K(X)$ denote the collection of all nonempty compact subsets of X. We use following notations and definitions :

$$D(x,B) = \inf\ \{d(x,y) : y \in B\},\ \text{where } x \in X \text{ and } B \in CB(X),$$

$$D(A,B) = \inf\ \{d(a,b) : a \in A,\ b \in B\},\ \text{where } A,B \in CB(X),$$

$$H(A,B) = \max\ \{\sup_{x\in A} D(x,B),\ \sup_{y\in B} D(y,A)\},\ \text{where } A,B \in CB(X).$$

H is known as the Hausdorff metric on $CB(X)$ [26]. Further, if (X,d) is complete then $(CB(X),H)$ is also complete and so is $(X \times X,\rho)$. Let H_ρ be the Hausdorff metric induced by ρ.

For any two subsets A,B of a metric space $(X,\ d)$, we write the distance between A and B as

$$d(A,\ B) = \inf\{d(x,\ y) : x \in A,\ y \in B\}.$$

We use the two sets described below.

$$A_0 = \{a \in A : d(a,\ b) = d(A,\ B),\ \text{for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,\ b) = d(A,\ B),\ \text{for some } a \in A\}.$$

Lemma 1 ([15]). Let $(X,\ d)$ be a metric space and $B \in K(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

Best proximity point and coupled best proximity point results are related to the problem of finding minimum distances which is by itself a classical problem considered in many areas of mathematics. In our case the objects are subsets of metric spaces. Here the minimum distance between pairs of subsets is realized by utilizing best proximity points or coupled best proximity points of nonself mappings. Let A and B be two nonintersecting subsets of a metric space $(X,\ d)$. An element $x \in A$ is said to be a best proximity point of the mapping $S\colon X \to X$ with respect to the pair $(A,\ B)$ if $d(x,\ Sx) = d(A,\ B)$, $Sx \in B$. An element $(x,\ y) \in A\times B$ is called a coupled best proximity point of the mapping $F\colon X\times X \to X$ with respect to the pair $(A,\ B)$ if $F(x,\ y) \in B$, $F(y,\ x) \in A$, $d(x,\ F(x,\ y)) = d(A,\ B)$ and $d(y,\ F(y,\ x)) = d(A,\ B)$.

Definition 2 ([11]). Let $(X,\ d)$ be a metric space and $(A,\ B)$ be a pair of nonempty subsets of X. Let $F : X \to CL(X)$ be a multivalued mapping. Then a point $x \in A$ is said to be a best proximity point of the mapping F with respect to the pair $(A,\ B)$ if $Fx \subset B$, $D(x,\ Fx) = d(A,\ B)$.

Definition 3 ([13]). Let $(X,\ d)$ be a metric space and $(A,\ B)$ be a pair of nonempty subsets of X. Let $F : X \times X \to CL(X)$ be a multivalued mapping. Then a point $(x,\ y) \in A \times B$ is said to be a coupled best proximity point of the mapping F with respect to the pair $(A,\ B)$ if $F(x,\ y) \subset B$, $F(y,\ x) \subset A$, $D(x,\ F(x,\ y)) = d(A,\ B)$ and $D(y,\ F(y,\ x)) = d(A,\ B)$.

We denote the collection all coupled best proximity points of F by $P(F)$.

Definition 4 ([35]). Let A and B be two nonempty subsets of a metric, space $(X,\ d)$ with $A_0 \neq \emptyset,\ B_0 \neq \emptyset$. We say that $(A,\ B)$ satisfies the P-property if for $x_1,\ x_2 \in A_0$ and $y_1,\ y_2 \in B_0$,

$$d(x_1,\ y_1) = d(x_2,\ y_2) = d(A,\ B) \Rightarrow d(x_1,\ x_2) = d(y_1,\ y_2).$$

We propose the following definitions of stability and weak stability of coupled best proximity points.

Definition 5. Let $(X,\ d)$ be a metric space and $(A,\ B)$ be a pair of nonempty subsets of X. Let $\{F_n : X \times X \to CL(X)\}$ be a sequence of multivalued mappings and $F : X \times X \to CL(X)$ such that $F = \lim_{n\to\infty} F_n$. Suppose that $\{P(F_n)\}$ is the sequence of coupled best proximity point sets of the sequence $\{F_n\}$ and $P(F)$ is the coupled best proximity points set of F and $P(F_n),\ P(F) \in CB(X)$. Then we say that

(i) the best proximity point sets of $\{F_n\}$ are stable if $\lim_{n\to\infty} H_\rho(P(F_n), P(F)) = 0$;

(ii) the best proximity point sets of $\{F_n\}$ are weakly stable if there exists $R > 0$ such that $\limsup_{n\to\infty} H_\rho(P(F_n),\ P(F)) \leq R$.

The above definition is a modification of the similar concepts which can be found in [9, 16].

Definition 6 ([32]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. Then X is said to have α-regular property if for every sequence $\{x_n\}$ in X, $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n and $x_n \to x$ as $n \to \infty$ imply $\alpha(x_n,\ x) \geq 1$, for all n.

We now introduce the following definitions of coupled proximal α-dominating mapping and generalized coupled proximal contraction in case of single and multivalued mappings.

Definition 7. Let A and B be two nonempty subsets of a metric space $(X,\ d)$ and $\alpha : X \times X \to [0,\ \infty)$. A mapping $F : X \times X \to X$ is said to be a coupled proximal α-dominating with respect to the pair $(A,\ B)$ if for $(x,\ y),\ (u,\ v) \in A \times B$,

$$d(u,\ F(x,y)) = d(v,\ F(y,x)) = d(A,B) \Rightarrow \alpha(x,\ u) \geq 1 \text{ and } \alpha(y,v) \geq 1.$$

Definition 8. Let A and B be two nonempty subsets of a metric space $(X,\ d)$ and $\alpha : X \times X \to [0,\ \infty)$. A multivalued mapping $F : X \times X \to CL(X)$ is said to be a coupled proximal α-dominating with respect to the pair $(A,\ B)$ if for $(x, y), (u, v) \in A \times B$,

$$D(u,\ F(x,y)) = D(v,\ F(y,x)) = d(A,B) \Rightarrow \alpha(x,u) \geq 1 \text{ and } \alpha(y,v) \geq 1.$$

Let Ψ be the collection of all functions $\psi : [0,\ \infty)^6 \to [0,\ \infty)$ such that

(i) ψ is continuous and nondecreasing in each coordinate;

(ii) $\sum_{n=1}^{\infty} \theta^n(t) < \infty$, where $\theta(t) = \psi(t,\ t,\ t,\ t,\ t,\ t)$;

(iii) $\psi(0,\ 0,\ 0,\ 0,\ 0,\ 0) = 0$.

Definition 9. Let A and B be two nonempty subsets of a metric space $(X,\ d)$. Let $\alpha : X \times X \to [0,\ \infty)$ and $\psi \in \Psi$. A mapping $F : X \times X \to X$ is called generalized coupled proximal contraction with respect to the pair $(A,\ B)$ if for $(x, y), (u, v) \in (A \times B) \cup (B \times A)$ with $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$,

$$d(F(x,\ y),\ F(u,\ v)) \leq N(x,\ y,\ u,\ v), \tag{1}$$

where

$$\begin{aligned} N(x,y,u,v) = \psi\Big(&d(x,\ u),\ d(y,\ v),\ d(x,\ F(x,\ y)) - d(A,\ B),\\ &d(u,F(x,y)) - d(A,B),\ d(y,F(y,x)) - d(A,B),\\ &d(v,F(y,x)) - d(A,B)\Big). \end{aligned}$$

Definition 10. Let A and B be two nonempty subsets of a metric space $(X,\ d)$. Let $\alpha : X \times X \to [0,\ \infty)$ and $\psi \in \Psi$. A multivalued mapping $F : X \times X \to CL(X)$ is called generalized coupled proximal contraction with respect to the pair $(A,\ B)$ if for $(x, y), (u, v) \in (A \times B) \cup (B \times A)$ with $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$,

$$H(F(x,\ y),\ F(u,\ v)) \leq M(x,\ y,\ u,\ v), \tag{2}$$

where

$$\begin{aligned} M(x,y,u,v) = \psi\Big(&d(x,\ u),\ d(y,\ v),\ D(x,\ F(x,\ y)) - d(A,\ B),\\ &D(u,F(x,y)) - d(A,B),\ D(y,F(y,x)) - d(A,B),\\ &D(v,F(y,x)) - d(A,B)\Big). \end{aligned}$$

The above definitions are modifications of our concepts introduced in [16].

2. MAIN RESULT

Theorem 11. Let (X, d) be a complete metric space having α-regular property, where $\alpha : X \times X \to [0, \infty)$, and (A, B) be a pair of nonempty closed subsets of X having P-property with $A_0, B_0 \neq \emptyset$. Let $F : X \times X \to K(X)$ be a multivalued mapping such that

(i) $F(x, y) \subseteq B_0$ and $F(y, x) \subseteq A_0$, for all $(x, y) \in A \times B$;

(ii) F is coupled proximal α-dominating with respect to the pair (A, B);

(iii) there exists $\psi \in \Psi$ for which F is a generalized coupled proximal contraction with respect to the pair (A, B).

Then the coupled best proximity point set of F, that is, $P(F)$ is nonempty.

Proof. Let $(x_0, y_0) \in A \times B$ with $u_0 \in F(x_0, y_0)$ and $v_0 \in F(y_0, x_0)$. By the assumption (i) and the definitions of A_0, B_0, there exist $x_1 \in A_0$ and $y_1 \in B_0$ such that

$$d(x_1, u_0) = d(A, B) \text{ and } d(y_1, v_0) = d(A, B).$$

Now,

$$d(A, B) \leq D(x_1, F(x_0, y_0)) \leq d(x_1, u_0) = d(A, B)$$

and

$$d(A, B) \leq D(y_1, F(y_0, x_0)) \leq d(y_1, v_0) = d(A, B).$$

Then we have

$$D(x_1, F(x_0, y_0)) = d(A, B) \quad \text{and} \quad D(y_1, F(y_0, x_0)) = d(A, B).$$

By the assumption (ii), we have

$$\alpha(x_0, x_1) \geq 1 \quad \text{and} \quad \alpha(y_0, y_1) \geq 1.$$

By Lemma 1, there exist $u_1 \in F(x_1, y_1)$ and $v_1 \in F(y_1, x_1)$ such that

$$d(u_0, u_1) = D(u_0, F(x_1, y_1)) \quad \text{and} \quad d(v_0, v_1) = D(v_0, F(y_1, x_1)).$$

As $u_1 \in F(x_1, y_1) \subseteq B_0$ and $v_1 \in F(y_1, x_1) \subseteq A_0$, by the definitions of A_0, B_0, there exist $x_2 \in A_0$ and $y_2 \in B_0$ such that

$$d(x_2, u_1) = d(A, B) \quad \text{and} \quad d(y_2, v_1) = d(A, B).$$

Now,

$$d(A, B) \leq D(x_2, F(x_1, y_1)) \leq d(x_2, u_1) = d(A, B)$$

and

$$d(A, B) \leq D(y_2, F(y_1, x_1)) \leq d(y_2, v_1) = d(A, B).$$

Then we have

$$D(x_2,\ F(x_1,\ y_1)) = d(A,\ B) \text{ and } D(y_2,\ F(y_1,\ x_1)) = d(A,\ B).$$

By the assumption (ii), we have

$$\alpha(x_1,\ x_2) \geq 1 \ \text{ and } \ \alpha(y_1,\ y_2) \geq 1.$$

By Lemma 1, there exist $u_2 \in F(x_2,\ y_2)$ and $v_2 \in F(y_2,\ x_2)$ such that

$$d(u_1,\ u_2) = D(u_1,\ F(x_2,\ y_2)) \ \text{ and } \ d(v_1,\ v_2) = D(v_1,\ F(y_2,\ x_2)).$$

From the above discussion we have $x_1, x_2, v_0, v_1 \in A_0$ and $y_1, y_2,\ u_0, u_1 \in B_0$ with $d(x_1, u_0) = d(A, B)$; $d(x_2, u_1) = d(A, B)$ and $d(y_1, v_0) = d(A, B)$; $d(y_2, v_1) = d(A, B)$. Using the P-property, we have

$$d(x_1,\ x_2) = d(u_0,\ u_1) \ \text{ and } \ d(y_1,\ y_2) = d(v_0,\ v_1).$$

As $u_2 \in F(x_2,\ y_2) \subseteq B_0$ and $v_2 \in F(y_2,\ x_2) \subseteq A_0$, by the definitions of $A_0,\ B_0$, there exist $x_3 \in A_0$ and $y_3 \ \in B_0$ such that

$$d(x_3,\ u_2) = d(A,\ B) \text{ and } \ d(y_3,\ v_2) = d(A,\ B).$$

Now,

$$d(A,\ B) \leq D(x_3,\ F(x_2,\ y_2)) \leq d(x_3,\ u_2) = d(A,\ B)$$

and

$$d(A,\ B) \leq D(y_3,\ F(y_2,\ x_2)) \leq d(y_3,\ v_2) = d(A,\ B).$$

Then we have

$$D(x_3,\ F(x_2,\ y_2)) = d(A,\ B) \text{ and } \ D(y_3,\ F(y_2,\ x_2)) = d(A,\ B).$$

By the assumption (ii), we have

$$\alpha(x_2,\ x_3) \geq 1 \ \text{ and } \ \alpha(y_2,\ y_3) \geq 1.$$

Again $x_2,\ x_3,\ v_1,\ v_2 \in A_0$ and $y_2,\ y_3,\ u_1,\ u_2 \in B_0$ with $d(x_2,\ u_1) = d(A,\ B)$; $d(x_3,\ u_2) = d(A,\ B)$ and $d(y_2,\ v_1) = d(A, B)$; $d(y_3, v_2) = d(A, B)$. Using the P-property, we have

$$d(x_2,\ x_3) = d(u_1,\ u_2) \text{ and } d(y_2,\ y_3) = d(v_1,\ v_2).$$

Continuing in this way, we obtain four sequences $\{x_n\},\ \{v_n\}$ in A_0 and $\{y_n\},\ \{u_n\}$ in B_0 such that for all $n \geq 0$,

$$u_n \in F(x_n,\ y_n) \subseteq B_0 \ \text{ and } \ v_n \in F(y_n,\ x_n) \subseteq A_0, \tag{3}$$

$$d(x_{n+1},\ u_n) = d(A,\ B) \ \text{ and } \ d(y_{n+1},\ v_n) = d(A,\ B), \tag{4}$$

$$\alpha(x_n,\ x_{n+1}) \geq 1 \ \text{ and } \ \alpha(y_n,\ y_{n+1}) \geq 1, \tag{5}$$

$$d(x_{n+1},\ x_{n+2}) = d(u_n,\ u_{n+1}) \ \text{ and } \ d(y_{n+1},\ y_{n+2}) = d(v_n,\ v_{n+1}) \tag{6}$$

and

$$d(u_n, u_{n+1}) = D(u_n, F(x_{n+1}, y_{n+1})) \text{ and } d(v_n, v_{n+1}) = D(v_n, F(y_{n+1}, x_{n+1})). \tag{7}$$

Let

$$r_n = \max\ \{d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1})\}, \quad \text{for all n} \geq 0. \tag{8}$$

Since $(x_n,\ y_n),\ (x_{n+1},\ y_{n+1}) \in A \times B$ with $\alpha(x_n,\ x_{n+1}) \geq 1$ and $\alpha(y_n,\ y_{n+1}) \geq 1$, using the assumption (iii), (3) - (8) and a property of ψ, we have

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(u_n, u_{n+1}) = D(u_n,\ F(x_{n+1}, y_{n+1})) \\
&\leq H(F(x_n, y_n),\ F(x_{n+1}, y_{n+1})) \leq M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) \\
&\leq \psi\Big(d(x_n, x_{n+1}),\ d(y_n, y_{n+1}),\ D(x_n, F(x_n, y_n)) - d(A, B), \\
&\qquad D(x_{n+1}, F(x_n, y_n)) - d(A, B), D(y_n, F(y_n, x_n)) - d(A, B), \\
&\qquad\qquad D(y_{n+1}, F(y_n, x_n)) - d(A, B)\Big) \\
&\leq \psi\Big(d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1}),\ d(x_n,\ u_n) - d(A,\ B), \\
&\qquad d(x_{n+1},\ u_n) - d(A,\ B),\ d(y_n,\ v_n) - d(A,\ B), \\
&\qquad\qquad d(y_{n+1},\ v_n) - d(A,\ B)\Big) \\
&\leq \psi\Big(d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(x_n, x_{n+1}) + d(x_{n+1}, u_n) - d(A, B), \\
&\qquad d(A, B) - d(A, B),\ d(y_n, y_{n+1}) + d(y_{n+1}, v_n) - d(A, B), \\
&\qquad\qquad d(A, B) - d(A, B)\Big) \\
&= \psi\Big(d(x_n, x_{n+1}),\ d(y_n, y_{n+1}),\ d(x_n, x_{n+1}) + d(A, B) - d(A, B), \\
&\qquad\qquad 0,\ d(y_n, y_{n+1}) + d(A, B) - d(A, B),\ 0\Big) \\
&= \psi(d(x_n, x_{n+1}),\ d(y_n, y_{n+1}),\ d(x_n, x_{n+1}),\ 0,\ d(y_n, y_{n+1}),\ 0) \\
&\leq \psi(r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n) = \theta(r_n).
\end{aligned} \tag{9}$$

Also,

$$\begin{aligned}
d(y_{n+1}, y_{n+2}) &= d(v_n, v_{n+1}) = D(v_n,\ F(y_{n+1}, x_{n+1})) \\
&\leq H(F(y_n, x_n),\ F(y_{n+1}, x_{n+1})) \leq M(y_n,\ x_n,\ y_{n+1},\ x_{n+1}) \\
&\leq \psi\Big(d(y_n, y_{n+1}),\ d(x_n, x_{n+1}),\ D(y_n,\ F(y_n, x_n)) - d(A, B), \\
&\quad D(y_{n+1}, F(y_n, x_n)) - d(A, B),\ D(x_n, F(x_n, y_n)) - d(A, B), \\
&\quad D(x_{n+1},\ F(x_n, y_n)) - d(A, B)\Big) \\
&\leq \psi\Big(d(y_n, y_{n+1}), d(x_n, x_{n+1}), d(y_n, v_n) - d(A, B), \\
&\quad d(y_{n+1}, v_n) - d(A, B),\ d(x_n,\ u_n) - d(A,\ B), \\
&\quad d(x_{n+1},\ u_n) - d(A,\ B)\Big) \\
&\leq \psi\Big(d(y_n, y_{n+1}), d(x_n, x_{n+1}), d(y_n, y_{n+1}) + d(y_{n+1}, v_n) - d(A, B), \\
&\quad d(A, B) - d(A, B),\ d(x_n, x_{n+1}) + d(x_{n+1}, u_n) - d(A, B), \\
&\quad d(A, B) - d(A, B)\Big) \\
&\leq \psi\Big(d(y_n, y_{n+1}),\ d(x_n, x_{n+1}),\ d(y_n, y_{n+1}) + d(A, B) - d(A, B), \\
&\quad 0,\ d(x_n, x_{n+1}) + d(A, B) - d(A, B),\ 0\Big) \\
&= \psi(d(y_n,\ y_{n+1}),\ d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1}),\ 0,\ d(x_n,\ x_{n+1}),\ 0) \\
&\leq \psi(r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n) = \theta(r_n). \qquad (10)
\end{aligned}$$

Combining (9) and (10), we have

$$r_{n+1} = \max\ \{d(x_{n+1},\ x_{n+2}),\ d(y_{n+1},\ y_{n+2})\} \leq \theta(r_n). \qquad (11)$$

By repeated application of (11), we have

$$r_n \leq \theta(r_{n-1}) \leq \theta^2(r_{n-2}) \leq ... \leq \theta^n(r_0). \qquad (12)$$

Using (11), (12) and a property of ψ, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} \theta^n(r_0) < \infty$$

and

$$\sum_{n=1}^{\infty} d(y_n, y_{n+1}) \leq \sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} \theta^n(r_0) < \infty,$$

which imply that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A and B respectively. Since $d(x_{n+1}, x_{n+2}) = d(u_n,\ u_{n+1})$ and $d(y_{n+1},\ y_{n+2}) = d(v_n,\ v_{n+1})$, we have

$$\sum_{n=1}^{\infty} d(u_n,\ u_{n+1}) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} d(v_n,\ v_{n+1}) < \infty,$$

which imply that both $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in B and A respectively.

As A and B are closed in X, there exist $x,\ v \in A$ and $y,\ u \in B$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty} v_n = v. \tag{13}$$

Taking limit as $n \to \infty$ in (4) and using (13), we have

$$d(x,\ u) = d(A,\ B) \text{ and } d(y,\ v) = d(A,\ B). \tag{14}$$

By (5), (13) and the regularity assumption of the space, we have $\alpha(x_n, x) \geq 1$ and $\alpha(y_n, y) \geq 1$, for all n. Using the assumption (iii) and the properties of ψ, we have

$$\begin{aligned} D(u_n,\ F(x,\ y)) &\leq H(F(x_n,\ y_n),\ F(x,\ y)) \leq M(x_n,\ y_n,\ x,\ y) \\ &\leq \psi\Big(d(x_n,\ x),\ d(y_n,\ y),\ D(x_n,\ F(x_n,\ y_n)) - d(A,\ B), \\ &\qquad D(x, F(x_n, y_n)) - d(A, B), D(y_n, F(y_n, x_n)) - d(A, B), \\ &\qquad\quad D(y, F(y_n, x_n)) - d(A, B)\Big) \\ &\leq \psi\Big(d(x_n, x), d(y_n, y), d(x_n, u_n) - d(A, B),\ d(x, u_n) - d(A, B), \\ &\qquad\quad d(y_n, v_n) - d(A, B),\ d(y, v_n) - d(A, B)\Big). \end{aligned} \tag{15}$$

Taking limit as $n \to \infty$ in (15) and using (13), (14) and the properties of ψ, we have

$$\begin{aligned} D(u,\ F(x,\ y)) &\leq \psi\Big(0,\ 0,\ d(A,\ B) - d(A,\ B),\ d(A,\ B) - d(A,\ B), \\ &\qquad d(A,\ B) - d(A,\ B),\ d(A,\ B) - d(A,\ B)\Big) \\ &= \psi(0,\ 0,\ 0,\ 0,\ 0,\ 0) = 0, \end{aligned}$$

which implies that $D(u, F(x, y)) = 0$. Since $F(x, y)$ is compact, it is closed. Now, $D(u, F(x, y)) = 0$ implies that $u \in \overline{F(x, y)} = F(x, y)$, where $\overline{F(x, y)}$ is the closure of $F(x, y)$. Using (14), we have

$$d(A,\ B) \leq D(x,\ F(x,\ y)) \leq d(x,\ u) = d(A, B),$$

which implies that

$$D(x,\ F(x,\ y)) = d(A, B). \tag{16}$$

By (5), (13) and the regularity assumption of the space, we have $\alpha(y_n, y) \geq 1$ and

$\alpha(x_n, x) \geq 1$. Using the assumption (iii) and the properties of ψ, we have

$$\begin{aligned} D(v_n,\ F(y,\ x)) &\leq H(F(y_n,\ x_n),\ F(y,\ x)) \leq M(y_n,\ x_n,\ y,\ x) \\ &\leq \psi\Big(d(y_n,\ y),\ d(x_n,\ x),\ D(y_n,\ F(y_n,\ x_n)) - d(A,\ B), \\ &\qquad D(y, F(y_n, x_n)) - d(A, B),\ D(x_n, F(x_n, y_n)) - d(A, B), \\ &\qquad D(x,\ F(x_n,\ y_n)) - d(A, B)\Big) \\ &\leq \psi\Big(d(y_n, y),\ d(x_n, x),\ d(y_n, v_n) - d(A, B),\ d(y, v_n) - d(A, B), \\ &\qquad d(x_n, u_n) - d(A, B),\ d(x, u_n) - d(A, B)\Big). \end{aligned} \tag{17}$$

Taking limit as $n \to \infty$ in (17) and using (13), (14) and the properties of ψ, we have

$$\begin{aligned} D(v,\ F(y,\ x)) &\leq \psi\Big(0,\ 0,\ d(A,\ B) - d(A,\ B),\ d(A,\ B) - d(A,\ B), \\ &\qquad d(A, B) - d(A, B),\ d(A, B) - d(A, B)\Big) \\ &= \psi(0,\ 0,\ 0,\ 0,\ 0,\ 0) = 0, \end{aligned}$$

which implies that $D(v, F(y, x)) = 0$. Since $F(y, x)$ is compact, it is closed. Now, $D(v, F(y, x)) = 0$ implies that $v \in \overline{F(y, x)} = F(y, x)$, where $\overline{F(y, x)}$ is the closure of $F(y, x)$. Using (14), we get

$$d(A,\ B) \leq D(y,\ F(y,\ x)) \leq d(y,\ v) = d(A,\ B),$$

which implies that

$$D(y,\ F(y,\ x)) = d(A,\ B). \tag{18}$$

From (16) and (18), we have (x, y) is a coupled best proximity point of F. Hence the coupled best proximity point set of F, that is, $P(F)$ is nonempty. □

Example 12. Let $X = \mathbb{R}^2$ ($\mathbb{R}$ denotes the set of real numbers) and d be a metric on X defined as $d(x,\ y) = \max\ \{|\ x_1 - x_2\ |, |\ y_1 - y_2\ |\}$, for $x = (x_1,\ y_1),\ y = (x_2,\ y_2) \in X$. Let

$$A = \{(t,\ 0) : 0 \leq t \leq 1\} \cup \{(0,\ t) : -\infty \leq t \leq -5\}$$

and

$$B = \{(t,\ \frac{1}{32}) : 0 \leq t \leq 1\} \cup \{(0,\ t) : 5 \leq t \leq \infty\}.$$

Then $A_0 = \{(t,\ 0) : 0 \leq t \leq 1\}$ and $B_0 = \{(t,\ \frac{1}{32}) : 0 \leq t \leq 1\}$.
Let $F : X \times X \to K(X)$, $\psi : [0,\ \infty)^6 \to [0,\ \infty)$ and $\alpha : X \times X \to [0,\ \infty)$ be

respectively defined as follows:

$$F(a,\ b) = \begin{cases} [0,\ \frac{x+u}{128}] \times \{\frac{1}{32}\}, & \text{if} \quad a = (x,\ 0) \in A_0 \text{ and } \ b = (u,\ \frac{1}{32}) \in B_0, \\ [0,\ \frac{x+u}{128}] \times \{0\}, & \text{if} \quad a = (x,\ \frac{1}{32}) \in B_0 \text{ and } \ b = (u,\ 0) \in A_0, \\ \{(0,\ \frac{1}{32})\}, & \text{if} \quad (a,\ b) \in (A \times B) - (A_0 \times B_0), \\ \{(0,\ 0)\}, & \text{if} \quad (a,\ b) \in (B \times A) - (B_0 \times A_0), \\ \{(\frac{1}{128},\ \frac{1}{32})\}, & \text{otherwise,} \end{cases}$$

$$\psi(t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6) = \frac{5}{16}\ (t_1 + t_2 + t_3)$$

and

$$\alpha(x,\ y) = \begin{cases} 1, & \text{if} \quad (x,\ y) \in (A_0 \times A_0) \cup (B_0 \times B_0), \\ 0, & \text{otherwise.} \end{cases}$$

Here all of the conditions of Theorem 11 are satisfied and $((0,0),\ (0,\frac{1}{32}))$ is a coupled best proximity point of F.

The following theorem can be obtained from Theorem 11 if one treats $T : X \times X \to X$ as a multivalued mapping, that is, $T(x,y)$ is a singleton set for every $(x,y) \in X \times X$.

Theorem 13. Let (X,d) be a complete metric space having α-regular property, where $\alpha : X \times X \to [0,\ \infty)$, and (A,B) be a pair of nonempty closed subsets of X having P-property with $A_0,\ B_0 \neq \emptyset$. Let $T : X \times X \to X$ be a mapping such that

(i) $T(x,y) \in B_0$ and $T(y,x) \in A_0$, for all $(x,y) \in A \times B$;

(ii) T is coupled proximal α-dominating with respect to the pair (A,B);

(iii) there exists $\psi \in \Psi$ for which T is a generalized coupled proximal contraction with respect to the pair (A,B).

Then the coupled best proximity point set of T, that is, $P(T)$ is nonempty.

Proof. Define a multivalued mapping $S : X \times X \to K(X)$ as $S(x,y) = \{T(x,y)\}$ for $(x,y) \in X \times X$.

Let $(a,b) \in A \times B$. By assumption (i) of the theorem, $T(a,b) \in B_0$ and $T(b,a) \in A_0$, which imply that $S(a,b) \subseteq B_0$ and $S(b,a) \subseteq A_0$, for $(a,b) \in A \times B$. Since $(a,b) \in A \times B$ is arbitrary, it follows that $S(x,y) \subseteq B_0$ and $S(y,x) \subseteq A_0$, for all $(x,y) \in A \times B$. So, S satisfies the assumption (i) of Theorem 11.

Let $(x,y),(u,v) \in A \times B$ and $D(u,S(x,y)) = D(v,S(y,x)) = d(A,B)$. Then $d(u,\ T(x,y)) = d(v,\ T(y,x)) = d(A,B)$. As T is coupled proximal α-dominating with respect to the pair (A,B), we have $\alpha(x,u) \geq 1$ and $\alpha(y,v) \geq 1$, which imply that S is coupled proximal α-dominating with respect to the pair (A,B). So, the assumption (ii) of the theorem reduces to the assumption (ii) of Theorem 11 for the mapping S.

Let $(x,\ y),\ (u,\ v) \in A \times B \cup B \times A$ with $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$. As T is a generalized coupled proximal contraction with respect to the pair $(A,\ B)$, we have

$$\begin{aligned} H(S(x,y),\ S(u,v)) &= d(T(x,y),\ T(u,v)) \leq N(x,y,u,v) \\ &\leq \psi\Big(d(x,u),\ d(y,v),\ d(x,T(x,y)) - d(A,B),\ d(u,T(x,y)) - d(A,B), \\ &\qquad d(y,T(y,x)) - d(A,B),\ d(v,T(y,x)) - d(A,B)\Big), \\ &= \psi\Big(d(x,u),\ d(y,v), D(x,\ S(x,y)) - d(A,B),\ D(u,S(x,y)) - d(A,B), \\ &\qquad D(y,S(x,y)) - d(A,B),\ D(v,S(x,y)) - d(A,B)\Big) \\ &= M(x,\ y,\ u,\ v), \end{aligned}$$

which implies that S is a generalized coupled proximal contraction with respect to the pair (A,B), that is, S satisfies the assumption (iii) of Theorem 11.

Therefore, S satisfies all the assumptions of Theorem 11. Hence by application of Theorem 11, S has a coupled proximity point, that is there exists $(x,y) \in A \times B$ such that $D(x,S(x,y)) = D(y,S(y,x)) = d(A,B)$, which implies that $d(x,T(x,y)) = d(y,T(y,x)) = d(A,B)$. Therefore, (x,y) is a coupled best proximity point of T, that is, $P(T)$ is nonempty. □

Now we present a few special cases illustrating the applicability of Theorem 11 and Theorem 13.

Remark 14. Taking different suitable functions α and ψ in Theorem 11 and Theorem 13, we have different corollaries. For examples, we respectively mention some of the corollaries by taking $\alpha(x,\ y) = 1$, for all $(x,\ y) \in X \times X$ and choosing ψ as

(i) $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = k \max\{t_1,\ t_2\}$;

(ii) $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = k \max\{t_3,\ t_5\}$;

(iii) $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = k \max\{t_4,\ t_6\}$;

(iv) $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = k \max\{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6\}$;

where $0 \leq k < 1$.

Corollary 15. Let $(X,\ d)$ be a complete metric space and $(A,\ B)$ be a pair of nonempty closed subsets of X having P-property with $A_0,\ B_0 \neq \emptyset$. Let $F : X \times X \to K(X)$ be a multivalued mapping such that $F(x,\ y) \subseteq B_0$ and $F(y,\ x) \subseteq A_0$, for all $(x,\ y) \in A \times B$. Then the coupled best proximity point set of F, that is, $P(F)$ is nonempty if for $(x,y),\ (u,v) \in X \times X$ one of the following inequalities holds:

(i) $H(F(x,y),F(u,v)) \leq k \max\{d(x,u),\ d(y,v)\}$;

(ii) $H(F(x,y),F(u,v)) \leq k \max\{D(x,F(x,y))-d(A,B), D(y,F(y,x))-d(A,B)\}$;

(iii) $H(F(x,y),F(u,v)) \leq k\max\{D(u,F(x,y))-d(A,B),D(v,F(y,x))-d(A,B)\}$;
(iv) $H(F(x,y),F(u,v)) \leq k\ \max\{d(x,\ u),\ d(y,\ v),D(x,\ F(x,y))-d(A,B),$
$D(u,F(x,y))-d(A,B),D(y,F(y,x))-d(A,B),D(v,F(y,x))-d(A,B)\}$;
where $k \in [0,\ 1)$.

Corollary 16. Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X having P-property with $A_0,B_0 \neq \emptyset$. Let $T : X \times X \to X$ be a mapping such that $T(x,\ y) \in B_0$ and $T(y,\ x) \in A_0$, for all $(x,\ y) \in A \times B$. Then the coupled best proximity point set of T, that is, $P(T)$ is nonempty if for $(x,y),\ (u,v) \in X \times X$ one of the inequalities holds:
(i) $d(T(x,y),T(u,v)) \leq k\ \max\{d(x,u),\ d(y,v)\}$;
(ii) $d(T(x,y),T(u,v)) \leq k\ \max\{d(x,T(x,y))-d(A,B),\ d(y,T(y,x))-d(A,B)\}$;
(iii) $d(T(x,y),T(u,v)) \leq k\ \max\{d(u,T(x,y))-d(A,B),d(v,T(y,x))-d(A,B)\}$;
(iv) $d(T(x,y),T(u,v)) \leq k\ \max\{d(x,\ u),\ d(y,\ v),\ d(x,\ T(x,y))-d(A,B)$
$d(u,T(x,y))-d(A,B),d(y,T(y,x))-d(A,B),\ d(v,T(y,x))-d(A,B)\}$,
where $k \in [0,\ 1)$.

3. DATA DEPENDENCE RESULT

In this section, we obtain a data dependence result for the proximity point sets of multivalued coupled mappings consider in the previous section. Result is utilized to obtain to a weak stability theorem in the following section.

Theorem 17. Let (X,d) be a complete metric space having α-regular property, where $\alpha : X \times X \to [0,\infty)$, and (A,B) be a pair of nonempty closed subsets of X having P-property with $A_0,B_0 \neq \emptyset$. Let $F_j : X\times X \to K(X)$ $(j=1,2)$ be two multivalued mappings such that F_2 satisfies the conditions (i), (ii) and (iii) of Theorem 11. Then $P(F_2) \neq \emptyset$. Moreover, let $P(F_1) \neq \emptyset$ and there exists $M > 0$ such that $H(F_1(x,y),F_2(x,y)) \leq M$, for all $(x,y) \in X \times X$. Then for every $(x^*,y^*) \in P(F_1)$ there exists $(x,y) \in P(F_2)$ such that $\max\{d(x^*,\ x),\ d(y^*,y)\} \leq \Phi(M+2d(A,B))$. Moreover, if $P(F_2) \in CB(X \times X)$, then $\sup_{z\in P(F_1)} D_\rho(z,\ P(F_2)) \leq \Phi(M + 2d(A,B))$, where $\Phi(t) = \sum_{n=1}^{\infty}\theta^n(t)$.

Proof. By Theorem 11, we have that $P(F_2) \neq \emptyset$. Suppose that $P(F_1) \neq \emptyset$. Let $(x^*,\ y^*) \in P(F_1)$. Take $(x^*,\ y^*) = (x_0,\ y_0)$. Then

$$D(x_0,F_1(x_0,y_0)) = d(A,B) \text{ and } D(y_0,F_1(y_0,x_0)) = d(A,B). \tag{19}$$

By Lemma 1 and (19), there exist $p_0 \in F_1(x_0,y_0)$ and $q_0 \in F_1(y_0,x_0)$ such that

$$\left.\begin{array}{c} d(x_0,\ p_0) = D(x_0,\ F_1(x_0,y_0)) = d(A,\ B) \\ and \\ d(y_0,\ q_0) = D(y_0,\ F_1(y_0,x_0)) = d(A,\ B). \end{array}\right\} \tag{20}$$

Also by Lemma 1, there exist $u_0 \in F_2(x_0,\ y_0)$ and $v_0 \in F_2(y_0,\ x_0)$ such that

$$d(p_0,\ u_0) = D(p_0,\ F_2(x_0,\ y_0)) \text{ and } d(q_0,\ v_0) = D(q_0,\ F_2(y_0,\ x_0)). \tag{21}$$

From (20) and (21), we have

$$\begin{aligned} d(x_0,\ u_0) &\le d(x_0,\ p_0) + d(p_0,\ u_0) = d(A,\ B) + D(p_0,\ F_2(x_0,\ y_0)) \\ &\le d(A,\ B) + H(F_1(x_0,\ y_0),\ F_2(x_0,\ y_0)) \le d(A,\ B) + M. \end{aligned} \tag{22}$$

Again from (20) and (21), we have

$$\begin{aligned} d(y_0,\ v_0) &\le d(y_0,\ q_0) + d(q_0,\ v_0) = d(A,\ B) + D(q_0,\ F_2(y_0,\ x_0)) \\ &\le d(A,\ B) + H(F_1(y_0,\ x_0),\ F_2(y_0,\ x_0)) \le d(A,\ B) + M. \end{aligned} \tag{23}$$

Arguing similarly as in the proof of Theorem 11, we construct four sequences $\{x_n\},\ \{v_n\}$ in A_0 and $\{y_n\},\ \{u_n\}$ in B_0 such that for all $n \ge 0$,

$$u_n \in F_2(x_n,\ y_n) \ \text{ and } \ v_n \in F_2(y_n,\ x_n), \tag{24}$$

$$\left.\begin{array}{c} d(u_n,\ u_{n+1}) = D(u_n,\ F_2(x_{n+1},\ y_{n+1})) \\ and \\ d(v_n,\ v_{n+1}) = D(v_n,\ F_2(y_{n+1},\ x_{n+1})) \end{array}\right\} \tag{25}$$

and also (4)- (6), (8), (11) and (12) are satisfied. Arguing similarly as in the proof of Theorem 11, we prove $\{x_n\},\ \{v_n\}$ and $\{y_n\},\ \{u_n\}$ are Cauchy sequences in A_0 and in B_0 respectively and we have (13), that is,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} u_n = u \ \text{ and } \ \lim_{n\to\infty} v_n = v.$$

Similarly as in the proof of Theorem 11, we also have $(x,\ y)$ is a coupled best proximity point of F_2. Now using (11), (12) and the properties of ψ, we have

$$\begin{aligned} d(x_0,\ x) &\le \sum_{k=0}^{n} d(x_k,\ x_{k+1}) + d(x_{n+1},\ x) \le \sum_{k=0}^{n} r_k + d(x_{n+1},\ x) \\ &\le \sum_{k=0}^{n} \theta^k(r_0) + d(x_{n+1},\ x). \end{aligned} \tag{26}$$

Taking limit as $n \to \infty$ in (26), using (22), (23) and (4) and property of ψ, we have

$$\begin{aligned} d(x_0,\ x) &\le \sum_{k=0}^{\infty} \theta^k(r_0) = \Phi(r_0) = \Phi(\max\{d(x_0,\ x_1),\ d(y_0,\ y_1)\}) \\ &\le \Phi(\max\{d(x_0,\ u_0) + d(u_0,\ x_1),\ d(y_0,\ v_0) + d(v_0,\ y_1)\}) \\ &\le \Phi(\max\{M + d(A,B) + d(A,B),\ M + d(A,B) + d(A,B)\}) \\ &\le \Phi(M + 2d(A,\ B)). \end{aligned} \tag{27}$$

Similarly, we have

$$d(y_0,\ y) \le \Phi(M + 2d(A,\ B)). \tag{28}$$

Combining (27) and (28), we have

$$\max\{d(x_0,\ x),\ d(y_0,\ y)\} \le \Phi(M + 2d(A,\ B)).$$

Therefore, for $(x_0,\ y_0) \in P(F_1)$, there exists $(x,\ y) \in P(F_2)$ such that

$$\max\{d(x_0,\ x),\ d(y_0,\ y)\} \le \Phi(M + 2d(A,\ B)),$$

which implies that

$$\max\{d(x^*,\ x),\ d(y^*,\ y)\} \le \Phi(M + 2d(A,\ B)).$$

Let $P(F_2) \in B(X \times X)$. Then

$$\rho((x^*,\ y^*),\ (x,\ y)) = \max\{d(x^*,\ x),\ d(y^*,\ y)\} \le \Phi(M + 2d(A,\ B)).$$

For $(x^*, y^*) \in P(F_1)$ there exists $(x, y) \in P(F_2)$ such that $\rho((x^*, y^*),\ (x, y)) \le \Phi(M + 2d(A, B))$. So, $D_\rho((x^*,\ y^*),\ P(F_2)) \le \Phi(M + 2\ d(A, B))$. Since $(x^*,\ y^*) \in P(F_1)$ is arbitrary, it follows that $\sup_{z \in P(F_1)} D_\rho(z,\ P(F_2)) \le \Phi(M + 2\ d(A,\ B))$. Hence the proof of the theorem is completed. □

4. STABILITY ANALYSIS

In this section, we perform stability analysis of coupled proximity point sets of a sequence of multivalued coupled mappings. For this purpose we first prove the following lemma.

Lemma 18. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) and $\alpha : X \times X \to [0,\ \infty)$. Assume that a sequence of mappings $\{F_n : X \times X \to K(X) : n \in \mathbb{N}\}$ converges to a mapping $F : X \times X \to K(X)$. Suppose there exists $\psi \in \Psi$ such that each $F_n,\ (n \in \mathbb{N})$ is a generalized coupled proximal contraction with respect to the pair $(A,\ B)$. Then F is a generalized coupled proximal contraction with respect to the pair $(A,\ B)$.

Proof. Let $(x, y),\ (u, v) \in (A \times B) \cup (B \times A)$ with $\alpha(x, u) \ge 1$ and $\alpha(y, v) \ge 1$. By the hypothesis of the lemma, each F_n $(n \in \mathbb{N})$ is a generalized coupled proximal contraction with respect to the pair (A, B), that is,

$$\begin{aligned} H(F_n(x,\ y),\ F_n(u,\ v)) &\le M(x,\ y,\ u,\ v) \\ &= \psi\Big(d(x,\ u),\ d(y,\ v),\ D(x,\ F_n(x,\ y)) - d(A,\ B), \\ &\qquad D(u, F_n(x, y)) - d(A, B),\ D(y, F_n(y, x)) - d(A, B), \\ &\qquad\quad D(v, F_n(y, x)) - d(A, B)\Big). \end{aligned}$$

As F_n converges to F, taking limit as $n \to \infty$ in above inequality and using the continuity of ψ, we have

$$\begin{aligned}H(F(x,\ y),\ F(u,\ v)) &\le M(x,\ y,\ u,\ v)\\ &= \psi\Big(d(x,\ u),\ d(y,\ v),\ D(x,\ F(x,\ y)) - d(A,\ B),\\ &\qquad D(u,\ F(x,\ y)) - d(A,\ B),\ D(y,\ F(y,\ x)) - d(A,\ B),\\ &\qquad D(v,\ F(y,\ x)) - d(A,\ B)\Big),\end{aligned}$$

which shows that F is a generalized coupled proximal contraction with respect to the pair $(A,\ B)$. $\square$

Theorem 19. Let $(X,\ d)$ be a complete metric space having α-regular property, where $\alpha : X \times X \to [0,\ \infty)$, and (A, B) be a pair of nonempty closed subsets of X having P-property with $A_0,\ B_0 \neq \emptyset$. Assume that a sequence of mappings $\{F_n : X \times X \to K(X) : n \in \mathbb{N}\}$ converges uniformly to a mapping $F : X \times X \to K(X)$ and each F_n $(n \in \mathbb{N})$ satisfies the conditions (i), (ii) and (iii) of Theorem 11 and F satisfies the conditions (i) and (ii) of Theorem 11. Then $P(F_n) \neq \emptyset$, for all $n \in \mathbb{N}$ and $P(F) \neq \emptyset$. Also let (a) Φ is continuous, where $\Phi(t) = \sum_{n=1}^{\infty} \theta^n(t)$; (b) each $P(F_n)$ $(n \in \mathbb{N})$ and $P(F)$ are closed and bounded subsets of X and (c) $M_n = \sup_{(x,\ y) \in X \times X} H(F(x,\ y),\ F_n(x,\ y))$ exists for each $n \in \mathbb{N}$. Then the coupled best proximity point sets of the sequence of mappings $\{F_n\}$ are weakly stable.

Proof. By the hypothesis of the theorem and Lemma 18, F is a generalized coupled proximal contraction with respect to the pair (A, B). By Theorem 11, $P(F_n) \neq \emptyset$, for all $n \in \mathbb{N}$ and $P(F) \neq \emptyset$. By Theorem 17, we have for each $n \in \mathbb{N}$,

$$\sup_{z \in P(F_n)} D_\rho(z,\ P(F)) \le \Phi(M_n + 2\ d(A,\ B))$$

and

$$\sup_{z \in P(F)} D_\rho(z,\ P(F_n)) \le \Phi(M_n + 2\ d(A,\ B)),$$

where $M_n = \sup_{(x,\ y) \in X \times X} H(F(x,\ y),\ F_n(x,\ y))$.

Combining these, we have

$$H_\rho(P(F),\ P(F_n)) \le \Phi(M_n + 2d(A,\ B)), \quad \text{for all} \ \ n \in \mathbb{N}. \tag{29}$$

As F_n converges to F uniformly, we have

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup_{(x,\ y) \in X \times X} H(F(x,\ y),\ F_n(x,\ y)) = 0. \tag{30}$$

Taking limit supremum on both sides of (29) and using (30) and the continuity of Φ, we have

$$\limsup_{n\to\infty} H_\rho(P(F),\ P(F_n)) \le \Phi(2d(A,\ B)) = R, \ \text{where} \ R = \Phi(2d(A,\ B)).$$

Hence the coupled best proximity point sets of the sequence of mappings $\{F_n\}$ are weakly stable. □

Remark 20. *It may be noted from the above that in case F is a singlevalued coupled mapping, that is, F is from $X \times X$ to X, then the above weak stability result reduces to a stability result.*

Example 21. We take the metric space $(X,\ d)$, the sets $A,\ B$ and the mappings ψ and α, as taken in Example 12. Let $F_n,\ F : X \times X \to K(X)$ be defined as follows:

$$F_n(a,\ b) = \begin{cases} [0,\ \frac{x+u}{128+n}] \times \{\frac{1}{32}\}, & \text{if } \ a = (x,\ 0) \in A_0 \text{ and } \ b = (u,\ \frac{1}{32}) \in B_0, \\ [0,\ \frac{x+u}{128+n}] \times \{0\}, & \text{if } \ a = (x,\ \frac{1}{32}) \in B_0 \text{ and } \ b = (u,\ 0) \in A_0; \\ \{(0,\ \frac{1}{32})\}, & \text{if } \ (a,\ b) \in (A \times B) - (A_0 \times B_0), \\ \{(0,\ 0)\}, & \text{if } \ (a,\ b) \in (B \times A) - (B_0 \times A_0), \\ \{(\frac{1}{128},\ \frac{1}{32})\}, & \text{otherwise} \end{cases}$$

and

$$F(a,\ b) = \begin{cases} \{(0,\ \frac{1}{32})\}, & \text{if } \ a = (x,\ 0) \in A_0 \text{ and } \ b = (u,\ \frac{1}{32}) \in B_0, \\ \{(0,\ 0)\}, & \text{if } \ a = (x,\ \frac{1}{32}) \in B_0 \text{ and } \ b = (u,\ 0) \in A_0, \\ \{(\frac{1}{128},\ \frac{1}{32})\}, & \text{otherwise.} \end{cases}$$

Here all the conditions of Theorem 19 are satisfied and $P(F_n) = \{(0,0),\ (0, \frac{1}{32})\}$, for every $n \in \mathbb{N}$ and $P(F) = \{(0,0),\ (0, \frac{1}{32})\}$. Hence by an application of theorem 4.1, the coupled best proximity point sets of the sequence of mappings $\{F_n\}$ are here weakly stable, precisely stable.

5. CONCLUSIONS

This paper is a contribution to the vast field of setvalued optimization in which an analytic method is applied. There are further scopes of obtaining results of similar nature by employing other types of coupled setvalued contractions. The geometric property, that is the P-property we have used may possibly be weakened in the contexts of other appropriate coupled mappings. In the study of stability of the solution sets the relation between stability and weak stability may be an interesting study. These considerations may provide motivations for future works.

Acknowledgement: The learned reviewers' suggestions are gratefully acknowledged.

REFERENCES

[1] Abkar, A., and Gabeleh, M., "Global optimal solutions of noncyclic mappings in metric spaces", *Journal of Optimization Theory and Applications*, 153 (2012) 298–305.

[2] Abkar, A., and Gabeleh, M., "The existence of best proximity points for multivalued non-self-mappings", *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 107 (2013) 319–325.

[3] Alizadeh, S., Moradlou, F., and Salimi, P., "Some fixed point results for $(\alpha - \beta)$ - $(\psi - \varphi)$ - contractive mappings", *Filomat*, 28 (3) (2014) 635–647.

[4] Basha, S. S., and Veeramani, P., "Best proximity pair theorems for multi functions with open fibres", *Journal of Approximation Theory*, 103 (2000) 119–129.

[5] Caballero, J., Harjani, J., and Sadarangani, K., "A best proximity point theorem for Geraghty-contractions", *Fixed Point Theory and Applications*, 231 (1) (2012) .

[6] Chifu, C., and Petruşel, G., "Coupled fixed point results for (φ, G)-contractions of type (b) in b-metric spaces endowed with a graph", *Journal of Nonlinear Sciences and Applications*, 10 (2017) 671–683.

[7] Chiş-Novac, A., Precup, R., and Rus, A. I., "Data dependence of fixed points for non-self generalized contractions", *Fixed Point Theory*, 10 (1) (2009) 73–87.

[8] Choudhury, S. B., Maity, P., and Konar, P., "A global optimality result using nonself mappings", *OPSEARCH*, 51 (2) (2014) 312–320.

[9] Choudhury, S., Metiya, B., Som, N., T., and Bandyopadhyay, C., "Multivalued fixed point results and stability of fixed point sets in metric spaces", *Facta Universitatis (NIŠ) Series: Mathematics and Informatics*, 30 (2015) 501–512.

[10] Choudhury, S., Maity, B. P., and Metiya, N., "Best proximity point theorems with cyclic mappings in setvalued analysis", *Indian Journal of Mathematics*, 57 (2015) 79–102.

[11] Choudhury, S., Maity, B. P., and Metiya, N., "Best proximity point results in setvalued analysis", *Nonlinear Analysis: Modelling and Control*, 21 (3) (2016) 293–305.

[12] Choudhury, S. B., and Maity, P., "Best proximity point results in generalized metric spaces", *Vietnam Journal of Mathematics*, 44 (2016) 339–349.

[13] Choudhury, S., Maity, B., Metiya, P. N., and Postolache, M., "Approximating distance between sets by multivalued coupling with application to uniformly convex Banach spaces", *Journal of Inequalities and Applications*, 2018 (2018) 130.

[14] Choudhury, S., Maity, B., Metiya, P. N., and Postolache, M., "Simultaneous determination of distance between sets by multivalued kannan type coupling", *International Journal of Analysis and Applications*, 16 (5) (2018) 712–732.

[15] Choudhury, S., Metiya, B., Bandyopadhyay, N. C., and Maity, P., "Fixed points of multivalued mappings satisfying hybrid rational Pata-type inequalities", *The Journal of Analysis*, 27 (2019) 813–828.

[16] Choudhury, S., Metiya, B. N., and Kundu, S., "Existence, data dependence and stability of coupled fixed point sets of some multivalued operators", *Chaos, Solitons and Fractals*, 133 (2020) 109678.

[17] Choudhury, S., Metiya, B., Kundu, N. S., and Maity, P., "Multivalued coupled fixed point results involving Mizoguchi-Takahashi function in metric spaces", *Pan-American Mathematical Journal* 1 (31) (2021) 29–44.

[18] Choudhury, S. B., Gnana Bhaskar, T., Metiya, N., and Kundu, S., "Existence and stability of coupled fixed point sets for multivalued mappings", *Fixed point Theory*, 22 (2) (2021) 571–586.

[19] Dimri, C. R., and Semwal, P., "Best proximity results for multivalued mappings", *International Journal of Mathematical Analysis*, 7 (2013) 1355–1362.

[20] Eldred, A. A., and Veeramani, P., "Existence and convergence of best proximity points", *Journal of Mathematical Analysis and Applications*, 323 (2006) 1001–1006.

[21] Fan, K., "Extensions of two fixed point theorems of F.E. Browder", *Mathematische Zeitschrift*, 122 (1969) 234–240.

[22] Gabeleh, M., "Best proximity points: global minimization of multivalued non-self mappings", *Optimization Letters* 8 (2014) 1101–1112.

[23] Gupta, A., and Rohilla, M., "On coupled best proximity points and Ulam-Hyers stability", *Journal of Fixed Point Theory and Applications*, 22 (28) (2020) 1–13.

[24] Kumam, P., Pragadeeswarar, V., Marudai, M., and Sitthithakerngkiet, K., "Coupled best proximity points in ordered metric spaces", *Fixed Point Theory and Applications*, 2014 (1) (2014) 107.

[25] Mot, G., and Petrusel, A., "Fixed point theory for a new type of contractive multivalued operators", *Nonlinear Analysis*, 70 (9) (2009) 3371–3377.

[26] Nadler, B. Jr. S., "Multivalued contraction mappings", *Pacific Journal of Mathematics* 30 (1969) 475–488.

[27] Pant, R., Shukla, R., and Rakočević, V., "Approximating best proximity points for Reich type non-self nonexpansive mappings", *RACSAM*, 114 (4) (2020) 197.

[28] Pragadeeswarar, V., Marudai, M., Kumam, P., and Sitthithakerngkiet, K., "The existence and uniqueness of coupled best proximity point for proximally coupled contraction in a complete ordered metric space", *Abstract and Applied Analysis*, 2014 (2014), 274062 .

[29] Raj, S. V., "A best proximity point theorem for weakly contractive non-self-mappings", *Nonlinear Analysis*, 74 (2011) 4804–4808.

[30] Reich, S., "Approximate selections, best approximations, fixed points, and invariant sets", *Journal of Mathematical Analysis and Applications*, 62 (1) (1978) 104–113.

[31] Rus, A., Petruşel, I. A., and Sîntămărian, A., "Data dependence of the fixed points set of multivalued weakly picard operators", *Studia Univ. "Babeş - Bolyai"*, Mathematica XLVI (2) (2001) 111–121.

[32] Samet, B., Vetro, C., and Vetro, P., "Fixed point theorems for $\alpha - \psi$-contractive type mappings", *Nonlinear Analysis* 75 (2012) 2154–2165.

[33] Shahzad, N., Alghamdi, M., and Alnemer, G., "On best proximity points of multivalued nonexpansive mappings", *Dynamic Systems and Applications*, 30 (3) (2021) 411–424.

[34] Shatanawi, W., and Pitea, A., "Best proximity point and best proximity coupled point in a complete metric space with (P)-property", *Filomat*, 29 (2015) 63–74.

[35] Zhang, J., Su, Y., and Cheng, Q., "A note on 'A best proximity point theorem for Geraghty-contractions", *Fixed Point Theory and Applications*, 2013 (1) (2013) 99.