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MODIFIED CONVERSE DUALITY FOR MULTIOBJECTIVE HIGHER ORDER WOLFE TYPE DUAL PROGRAM WITH CONE CONSTRAINTS

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Abstract: In this paper, we obtain a converse duality theorem for higher order Wolfe type multiobjective programming with cone constraints under appropriate assumptions. This fills some gaps in the work of Kim et al. [Kim, D.S., Kang, H.S., Lee, Y.J., Seo, Y.Y., Higher order duality in multiobjective programming with cone constraints, Optimization, **59**(1), 29–43, (2010)].

Keywords: Multiobjective programming, converse duality theorem, higher order duality, cones, efficient solutions.

MSC: 90C29, 90C26, 49N15, 90C46.

1. INTRODUCTION

Kim et al. [1] studied the duality relations for the following higher order multiobjective dual problems:

Primal Problem (MCP)

$$\begin{aligned}&\text{Minimize} f(x)\\ &\text{s.t.} - g(x) \in C_2^*,\ x \in C_1.\end{aligned}$$

Mond-Weir type dual problem (MMCD)

$$\begin{aligned}\text{Maximize}\quad & f(u) + (\lambda^T h(u,p))e - p^T(\nabla_p(\lambda^T h(u,p)))e\\ \text{s.t.}\quad & \nabla_p(\lambda^T h(u,p)) = \nabla_p(y^T k(u,p)),\\ & g(u) + k(u,p) - p^T \nabla_p k(u,p) \in C_2^*,\\ & y \in C_2,\ \lambda > 0,\ \lambda^T e = 1.\end{aligned}$$

Wolfe type dual problem (MWCD)

$$\begin{aligned}\text{Maximize}\quad & f(u) - y^T g(u)e + (\lambda^T h(u,p) - y^T k(u,p))e\\ & -p^T(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p)))e\\ \text{s.t.}\quad & \nabla_p(\lambda^T h(u,p)) = \nabla_p(y^T k(u,p)),\\ & y \in C_2,\ \lambda \geqq 0,\end{aligned}$$

where

(1) $f = (f_1, f_2, \ldots, f_l) : R^n \to R^l$ and $g = (g_1, g_2, \ldots, g_m) : R^n \to R^m$ are differentiable functions,
(2) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interiors respectively,
(3) C_1^* and C_2^* are polar cones of C_1 and C_2 respectively,
(4) $e = (1, 1, \ldots, 1)^T$ is vector in R^l,
(5) $h : R^n \times R^n \to R^l$ and $k : R^n \times R^n \to R^m$ are differentiable functions; $\nabla_p(h_j(u,p))$ denotes the $n \times 1$ gradient of h_j with respect to p and $\nabla_p(y^T k(u,p))$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p. For other notations and definitions, we refer to Kim et al. [1].

2. DISCUSSION

In Mond-Weir type dual model (MMCD) studied in Kim et al. [1], the constraint $\lambda^T e = 1$ appears in dual problem, while in Wolfe type dual model (MWCD), this constraint ($\lambda^T e = 1$) has not been taken. As pointed out in Gulati and Ahmad [2], this constraint plays an important role in the proof of weak duality theorem for Wolfe type models, but is not needed to prove the weak duality theorem for Mond-Weir type models. Hence this constraint should be included in Wolfe type dual model in Kim et al. [1].

Recently, Yang et al. [3] have pointed out that the proofs of converse duality theorem for Mond-Weir and Wolfe type dual models in Kim et al. [1] were erroneous and then they gave the rectified proof of converse duality theorem for higher order Mond-Weir type dual model (MMCD) under mild assumptions. As pointed out by Yang et al. [3], the higher order Wolfe type converse duality theorem in

Kim et al. [1] is also erroneous and it still has not been obtained and remains an open problem.

After establishing weak and strong duality theorem for above mentioned Wolfe type higher order multiobjective dual problem (MWCD), Kim et al. [1] proved the following converse duality theorem:

Theorem 1. *Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ *be an efficient solution of (MWCD). Assume that*

(i) $h(\bar{x}, 0) = 0,\ k(\bar{x}, 0) = 0,\ \nabla_p h(\bar{x}, 0) = \nabla f(\bar{x}),\ \nabla_p k(\bar{x}, 0) = \nabla g(\bar{x})$;

(ii) *the matrix* $\nabla_p[\nabla(\bar{\lambda}^T f(\bar{x})) + \nabla(\bar{\lambda}^T h(\bar{x}, \bar{p})) - \nabla(\bar{y}^T g(\bar{x})) - \nabla(\bar{y}^T k(\bar{x}, \bar{p}))]$ *is positive or negative definite;*

(iii) *the vectors* $\{\nabla_p^2 \bar{\lambda}_i h_i(\bar{x}, \bar{p})\}_{i=1,2,\ldots,l}$ *and* $\{\nabla_p^2 \bar{y}_j k_j(\bar{x}, \bar{p})\}_{j=1,2,\ldots,m}$ *are linearly independent.*

If the conditions of Theorem 3.1 *in [1] are satisfied, then* $\bar{x}$ *is an efficient solution to (MCP).*

Remark 2. *To obtain erroneous reason of the converse duality theorem of Wolfe type dual model, one can refer to Remark* 2.2 *in Yang et al. [3].*

3. CONCLUSIONS/RESULTS

1 The above discussion clearly shows that the Wolfe type dual model (MWCD) in Kim et al. [1] is erroneous.

2 The rectified proof of converse duality theorem for Wolfe type dual model (MWCD) has not been obtained yet. In this paper, we provide a modified version of converse duality theorem for Wolfe type dual model (MWCD).

Now we present higher order Wolfe type dual model (WCD) along with the constraint $\lambda^T e = 1$. Also we establish a converse duality theorem which is a modification of Theorem 1. The assumption (ii) and (iii) in Theorem 1 has been replaced by assumption (ii) and (iii) in Theorem 3 given below.

Wolfe type dual problem (WCD)

$$\begin{aligned} \text{Maximize} \quad & f(u) - y^T g(u)e + (\lambda^T h(u,p) - y^T k(u,p))e \\ & -p^T(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p)))e \\ \text{s.t.} \quad & \nabla_p(\lambda^T h(u,p)) = \nabla_p(y^T k(u,p)), \\ & y \in C_2,\ \lambda > 0, \lambda^T e = 1. \end{aligned}$$

4. MODIFIED CONVERSE DUALITY THEOREM

Theorem 3. *Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ *be an efficient solution of (WCD). Assume that*

(i) $h(\bar{x}, 0) = 0,\ k(\bar{x}, 0) = 0,\ \nabla_p h(\bar{x}, 0) = \nabla f(\bar{x}),\ \nabla_p k(\bar{x}, 0) = \nabla g(\bar{x})$,

(ii) $\alpha^T \nabla f(\bar{x}) - (\alpha^T e)\nabla \bar{y}^T g(\bar{x}) + (\alpha^T e)(\nabla \bar{\lambda}^T h(\bar{x}, \bar{p}) - \nabla \bar{y}^T k(\bar{x}, \bar{p})) = 0$ *implies that* $\bar{p} = 0$ *for some* $\alpha \in R^l$ $(\alpha \geq 0)$,

(iii) $\nabla_{pp}(\bar{\lambda}^T h(\bar{x},\bar{p}) - \bar{y}^T k(\bar{x},\bar{p}))$ *is nonsingular.*

If the conditions of Theorem 3.1 *in* [1] *are satisfied, then* $\bar{x}$ *is an efficient solution to (MCP).*

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ is an efficient solution of (WCD), by the generalized Fritz John Theorem in [4], there exist $\alpha \in R^l,\ \beta \in R^n$, $\mu \in R^l$ and $\gamma \in R$ such that

$$-\alpha^T\nabla f(\bar{x}) + (\alpha^T e)\nabla\bar{y}^T g(\bar{x}) - (\alpha^T e)(\nabla\bar{\lambda}^T h(\bar{x},\bar{p}) - \nabla\bar{y}^T k(\bar{x},\bar{p})) + ((\alpha^T e)\bar{p}+\beta)^T[\nabla_{xp}\bar{\lambda}^T h(\bar{x},\bar{p}) - \nabla_{xp}\bar{y}^T k(\bar{x},\bar{p})] = 0, \tag{1}$$

$$((\alpha^T e)(g(\bar{x}) + k(\bar{x},\bar{p})) - ((\alpha^T e)\bar{p}+\beta)^T(\nabla_p k(\bar{x},\bar{p})))(y-\bar{y}) \geqq 0,\quad \forall y \in C_2, \tag{2}$$

$$(\alpha^T e)h(\bar{x},\bar{p}) - ((\alpha^T e)\bar{p}+\beta)^T(\nabla_p h(\bar{x},\bar{p})) + \mu - \gamma e = 0, \tag{3}$$

$$((\alpha^T e)\bar{p}+\beta)^T(\nabla_{pp}(\bar{\lambda}^T h(\bar{x},\bar{p}) - \bar{y}^T k(\bar{x},\bar{p}))) = 0, \tag{4}$$

$$\nabla_p(\bar{\lambda}^T h(\bar{x},\bar{p})) - \nabla_p(\bar{y}^T k(\bar{x},\ \bar{p})) = 0, \tag{5}$$

$$\mu^T\bar{\lambda} = 0, \tag{6}$$

$$\gamma[\bar{\lambda}^T e - 1] = 0, \tag{7}$$

$$(\alpha,\mu) \geqq 0, \tag{8}$$

$$(\alpha,\beta,\mu,\gamma) \neq 0. \tag{9}$$

Now hypothesis (iii) and (4) imply that

$$(\alpha^T e)\bar{p} + \beta = 0. \tag{10}$$

We claim that $\alpha \neq 0$. If $\alpha = 0$, from (10), $\beta = 0$. Also (6) and $\bar{\lambda} > 0$ give $\mu = 0$. Further the equation (3) gives $\gamma = 0$. Consequently $(\alpha,\beta,\mu,\gamma) = 0$, contradicting (9).

$$\text{Hence, } \alpha \neq 0 \text{ and } \alpha^T e > 0. \tag{11}$$

Using (1) and (10), we get

$$\alpha^T\nabla f(\bar{x}) - (\alpha^T e)\nabla\bar{y}^T g(\bar{x}) + (\alpha^T e)(\nabla\bar{\lambda}^T h(\bar{x},\bar{p}) - \nabla\bar{y}^T k(\bar{x},\bar{p})) = 0. \tag{12}$$

Now (12) and hypothesis (ii) implies that $\bar{p} = 0$.
Now equation (10) and $\bar{p} = 0$ gives $\beta = 0$.
By using $\alpha^T e > 0$, (2), (10) and $\bar{p} = 0$, we have
$[g(\bar{x}) + k(\bar{x},0)]^T(y-\bar{y}) \geqq 0, \forall\ y \in C_2$.
Then, by hypothesis (i), we obtain

$$g(\bar{x})^T(y-\bar{y}) \geqq 0, \forall\ y \in C_2. \tag{13}$$

Since $y \in C_2$, therefore $y + \bar{y} \in C_2$ and (13) implies

$$y^T g(\bar{x}) \geqq 0, \forall\ y \in C_2.$$

that is $-g(\bar{x}) \in C_2^*$.

Now, letting $y = 0$ and $y = 2\bar{y}$ in (13), we get

$$\bar{y}^T g(\bar{x}) = 0.$$

In view of that $\bar{p} = 0$ and hypothesis (i), we find that the corresponding value of (MCP) and (WCD) are equal. If the conditions of Theorem 3.1 in [1] are satisfied, then $\bar{x}$ is an efficient solution of (MCP). □

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