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CORRECTION OF THERMOGRAPHIC IMAGES BASED ON THE MINIMIZATION METHOD OF TIKHONOV FUNCTIONAL

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Abstract: The paper considers the method of correction of thermographic images (thermograms) obtained by recording in the infrared range of radiation from the surface of the object under study using a thermal imager. A thermogram with a certain degree of reliability transmits an image of the heat-generating structure inside the body. In this paper, the mathematical correction of images on a thermogram is performed based on an analytical continuation of the stationary temperature distribution as a harmonic function from the surface of the object under study towards the heat sources. The continuation is carried out by solving an ill-posed mixed problem for the Laplace equation in a cylindrical region of rectangular cross-section. To construct a stable solution to the problem, the principle of the minimum of the Tikhonov smoothing functional we used.

Keywords: Termogram, Ill-posed problem, Inverse problem, Cauchy problem for the Laplace equation, Integral equation of the first kind, Tikhonov regularization method.

MSC: 35R25, 35R30.

1. INTRODUCTION

Thermal imaging is one of the most effective method for studying the internal heat-generating structure of an object that is inaccessible to direct research. With the help of a thermal imager that registers thermal electromagnetic radiation from the surface of the object under study in the infrared range, it is possible to obtain a thermogram of the object's surface with an image of the internal heat-generating structure.

In medicine, thermal imaging has become an effective means of early diagnosis [5]. The image on the thermogram, which is a map of the temperature distribution on the surface of the patient's body, makes it possible to assess functional anomalies in the state of its internal organs. At the same time, the image on the thermogram in some cases turns out to be significantly distorted due to the processes of thermal conductivity and heat exchange, surface irregularities.

The paper proposes a method for correcting the image on a thermogram within the framework of a certain mathematical model. As an adjusted thermogram, the image of the temperature distribution function on the plane near the heat sources is considered as more accurately transmitting their structure than the image on the original thermogram. It is proposed to obtain this distribution function as a result of the continuation (similar to the continuation of gravitational fields in geophysics problems [12]) of the temperature distribution from the surface from which the initial thermogram is taken.

The continuation is obtained by solving the inverse problem to some mixed boundary value problem for the Poisson equation. The inverse problem under consideration is incorrectly posed, since significant errors in the solution of the inverse problem may correspond to small errors in the initial data (the initial thermogram, surface data, boundary conditions). To construct its stable approximate solution, the Tikhonov [13] regularization method is used, based on the principle of the minimum of the smoothing functional.

2. STATEMENT OF THE PROBLEM

Let's consider a physical and then a mathematical model, within which we will set the inverse problem.

As a physical model, we consider a homogeneous heat-conducting body in the form of a rectangular cylinder, bounded by the surface S and containing heat sources with a time-independent distribution density function. These sources create a stationary distribution of temperature in the body. We associate the density function of the distribution of heat sources with the object under study. We assume that a given temperature distribution is maintained on the side faces of the cylinder, and on the surface S there is a convective heat exchange with a medium of temperature U_0, described by Newton's law, according to which the density of the heat flow at a point on the surface is directly proportional to the temperature difference inside and outside the body.

Let's move on to the mathematical model. In a rectangular cylinder

$$D^{\infty} = \{(x, y, z) :\, 0 < x < l_x,\, 0 < y < l_y,\, -\infty < z < \infty\} \subset \mathbb{R}^3 \tag{1}$$

we consider a cylindrical domain

$$D(F, \infty) = \{(x, y, z) :\, 0 < x < l_x,\, 0 < y < l_y,\, F(x, y) < z < \infty\}, \tag{2}$$

bounded with a surface

$$S = \{(x, y, z) :\, 0 < x < l_x,\, 0 < y < l_y,\, z = F(x, y) < H\}. \tag{3}$$

Let Γ be the set of side faces of the domain $D(F, \infty)$ of the form (2). In the domain $D(F, \infty)$, we consider the following mixed boundary value problem for the Poisson equation

$$\begin{array}{ll} \Delta u(M) = \rho(M), & M \in D(F, \infty), \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - u)\Big|_S, & \\ u|_\Gamma = f_1, & \\ u \;\text{ bounded when }\; z \to \infty. & \end{array} \tag{4}$$

The problem (4) corresponds to a steady temperature distribution created by heat sources with a distribution density function ρ. On the surface S a third boundary condition is set and corresponds to a convective heat exchange with a medium of temperature U_0 with a constant coefficient h, at the boundary Γ the temperature is set as a function f_1, independent of time.

We assume that the functions ρ, f_1 are such that the solution to the problem (4) exists in $C^2(D(F, \infty)) \bigcap C^1(\overline{D(F, \infty)})$. In particular, the solution to the problem (4) allows us to find $u|_S$. In addition, we assume that the density carrier ρ is located in the domain $z > H$.

Let us now set the inverse problem.

Inverse problem 1. Let the following functions be defined within the frame of the model (4)

$$f = u|_S, \quad f_1 = u|_\Gamma.$$

We need to find a continuous function ρ.

Note that the reconstruction of the density ρ is associated with the same difficulties as the solution of the inverse potential problem [11], for which significant restrictions on uniqueness classes are known. Therefore, to solve the inverse problem, we apply the [12] approach used in geophysics problems. The source of information about the density of ρ will be the function $u|_{z=H}$ on the plane $z = H$, closer to the density carrier ρ than the surface S.

Since the carrier of the function ρ is conditionally located in the domain $z > H$, then the solution to the problem (4) satisfies the Laplace equation in the domain

$$D(F, H) = \{(x, y, z) :\, 0 < x < l_x,\, 0 < y < l_y,\, F(x, y) < z < H\}. \tag{5}$$

The set of side faces of the domain $D(F, H)$ denote by Γ_H. Then, instead of the inverse problem 1, we will solve the following inverse problem

Inverse problem 2. Let following functions be defined within the frame of the model (4)

$$f = u|_S, \quad f_1 = u|_{\Gamma_H}. \tag{6}$$

We want to find in the domain $D(F, H)$ of the form (5) the solution u to the boundary value problem

$$\begin{array}{ll} \Delta u(M) = 0, & M \in D(F, H), \\ u|_S = f, & \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - f)\Big|_S, & \\ u|_{\Gamma_H} = f_1. & \end{array} \tag{7}$$

We assume that the functions f, f_1 in (6), (7) are taken from the set of solutions to the direct problem (4), so the solution to the inverse problem 2 exists in $C^2(D(F, H)) \bigcap C^1(\overline{D(F, H)})$.

Note that in the problem (7) on the surface S of the form (3), Cauchy conditions are set, that is, the boundary values f of the desired function u and the values of its normal derivative are set, so the problem (7) has a unique solution. The boundary $z = H$ of the domain $D(F, H)$ is free and, thus, the problem (7) is similar in properties to the Cauchy problem for the Laplace equation and is unstable with respect to errors in the data, i.e. it is ill-posed.

In the inverse problem 2, the function f corresponds to the original thermogram obtained using a thermal imager. The function $u|_{z=H}$ will be considered as an adjusted thermogram, i.e. as a source of more accurate information about the density ρ.

3. EXACT SOLUTION TO THE PROBLEM

Based on the scheme [7, 3], an explicit representation of the exact solution to the problem (7) is constructed in [8]. We present this solution.

Consider the source function $\varphi(M, P)$ of the Dirichlet problem for the Laplace equation in a cylinder D^∞ of the form (1). This function has the form

$$\varphi(M, P) = \frac{1}{4\pi r_{MP}} + W(M, P), \tag{8}$$

where r_{MP} is the distance between points $M \in D^\infty$ and $P \in D^\infty$, $W(M, P)$ is a harmonic function with respect to M and P satisfying homogeneous boundary conditions of the first kind.

The source function (8) can be obtained by the reflection method as a sum of functions of point sources with a period of $2l_x$ for the variable x and $2l_y$ for the variable y

$$\varphi(M,P)=\frac{1}{4\pi}\sum_{n,m=-\infty}^{\infty}\Big(\frac{1}{r_{1,nm}}-\frac{1}{r_{2,nm}}-\frac{1}{r_{3,nm}}+\frac{1}{r_{4,nm}}\Big),$$

where

$$\begin{aligned} r_{1,nm}&=[(x_M-x_P+2l_xn)^2+(y_M-y_P+2l_ym)^2+(z_M-z_P)^2]^{1/2},\\ r_{2,nm}&=[(x_M+x_P+2l_xn)^2+(y_M-y_P+2l_ym)^2+(z_M-z_P)^2]^{1/2},\\ r_{3,nm}&=[(x_M-x_P+2l_xn)^2+(y_M+y_P+2l_ym)^2+(z_M-z_P)^2]^{1/2},\\ r_{4,nm}&=[(x_M+x_P+2l_xn)^2+(y_M+y_P+2l_ym)^2+(z_M-z_P)^2]^{1/2}, \end{aligned}$$

so $r_{1,00}=r_{MP}$ is the distance between the points $M\in D^\infty$ and $P\in D^\infty$.

We denote

$$\Pi(H)=\{(x,y,z):\ 0<x<l_x,\ 0<y<l_y,\ z=H\}. \tag{9}$$

In the domain $z_M<H$, we introduce the notation

$$\Phi(M)=\int_S\Big[h(U_0-f(P))\varphi(M,P)-f(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P-\\ -\int_{\Gamma_H}\Big[f_1(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P. \tag{10}$$

Then we obtain the solution to the problem (7) in the form [8]:

$$u(M)=v(M)+\Phi(M),\quad M\in D(F,H), \tag{11}$$

where the function Φ is calculated with the known functions f and f_1, and the function v is the solution to the problem

$$\begin{array}{ll} \Delta v(M)=0, & M\in D(-\infty,H),\\ v|_{z=H}=v_H, & \\ v|_{x=0,l_x}=0, & v|_{y=0,l_y}=0,\\ v\to 0 & \text{when}\quad z\to-\infty, \end{array} \tag{12}$$

and can be expressed using the boundary value v_H and the Green function of the problem (12):

$$v(M)=-\int_{\Pi(H)}\frac{\partial G}{\partial n_P}(M,P)v_H(P)dx_Pdy_P,\quad M\in D(-\infty,H), \tag{13}$$

where the kernel of the integral representation can be represented as a decomposition

$$\frac{\partial G}{\partial n_P}(M,P)\Big|_{P\in\Pi(H)}=-\frac{4}{l_xl_y}\sum_{n,m=1}^{\infty}\exp\big\{k_{nm}(-H+z_M)\big\}\times\\ \times\sin\frac{\pi nx_M}{l_x}\sin\frac{\pi my_M}{l_y}\sin\frac{\pi nx_P}{l_x}\sin\frac{\pi my_P}{l_y}, \tag{14}$$

$$k_{nm} = \pi\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right)^{1/2}, \tag{15}$$

over the complete system of functions

$$\left\{\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}\right\}_{n,m=1}^{\infty}. \tag{16}$$

In turn, v_H is the solution of an integral equation of the first kind

$$\int\limits_{\Pi(H)} \frac{\partial G}{\partial n_P}(M,P)v_H(P)dx_P dy_P = \Phi(M), \quad M \in \Pi(a), \tag{17}$$

where $a < \min\limits_{(x,y)} F(x,y)$ and $\Pi(a)$ is the domain of the form (9) for $z = a$.

From the equation (17) taking into account the decomposition (14) for $z_M = a$, we obtain the ratio between the Fourier coefficients of the unique solution of the integral equation v_H and the Fourier coefficients of the right part

$$-(\tilde{v_H})_{nm}\exp\left\{-k_{nm}(H-a)\right\} = \tilde{\Phi}_{nm}(a), \tag{18}$$

where $\tilde{\Phi}_{nm}(a)$ — Fourier coefficients of the function $\Phi(M)|_{M\in\Pi(a)}$:

$$\tilde{\Phi}_{nm}(a) = \frac{4}{l_x l_y}\int\limits_{\Pi(a)} \Phi(x,y,a)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy.$$

We note that the formula (18) characterizes the descending of Fourier coefficients of $\tilde{\Phi}_{nm}(a)$ with increasing n and m if the function f and f_1 are such that ensure the existence of solution to the problem (7) and consequently, the function v_H. Expressing the Fourier coefficients $(\tilde{v_H})_{nm}$ from (18) and substituting into (13), we get the function v in the domain $D(-\infty, H)$:

$$v(M) = -\sum_{n,m=1}^{\infty}(\tilde{v_H})_{nm}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y} = -\sum_{n,m=1}^{\infty}\tilde{\Phi}_{nm}(a)\times$$
$$\exp\left\{k_{nm}(z-a)\right\}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}, \quad M(x,y,z) \in D(-\infty, H). \tag{19}$$

Series (19) converges uniformly in $D(-\infty, H-\varepsilon)$ for any arbitrarily small fixed $\varepsilon > 0$, if the solution to the problem (7) exists for the data f and f_1.

The formula (11), where the functions v and Φ are of the form (19) and (10), respectively, gives an explicit expression for the exact solution to the problem (7).

4. APPROXIMATE CALCULATION OF A NORMAL TO AN INACCURATELY DEFINED SURFACE

Since the surface S of the form (3) is given with the equation $z = F(x, y)$, the function f given on S can be considered as a function of the variables x and y on the rectangle Π:

$$\Pi = \{(x, y) :\ 0 < x < l_x,\ 0 < y < l_y\}, \tag{20}$$

then the integral in (10) over the surface S may be reduced to the integral over the variables x_P and y_P. Given that $\dfrac{\partial\varphi}{\partial n} = (\mathbf{n}, \nabla\varphi)$, $\mathbf{n} = \dfrac{\mathbf{n_1}}{n_1}$, $\mathbf{n_1} = (F'_x, F'_y, -1)$, and $d\sigma_P = n_1(x_P, y_P)dx_P dy_P$, let's rewrite (10) in the form

$$\Phi(M) = \int\limits_{\Pi} \Big[h(U_0 - f(x_P, y_P))\varphi(M, P)\big|_{P\in S} n_1(x_P, y_P) - f(x_P, y_P)\times \\ \times (\mathbf{n}_1, \nabla_P\varphi(M, P))\big|_{P\in S}\Big] dx_P dy_P - \int\limits_{\Gamma_H} \Big[f_1(P)\frac{\partial\varphi}{\partial n_P}(M, P)\Big] d\sigma_P. \tag{21}$$

As follows from (21), when forming the right part of the integral equation (17), it is necessary to calculate the vector function of the normal $\mathbf{n}_1$ to the surface S of the form (3), which is the gradient of the function $F(x, y) - z$,

$$\mathbf{n}_1 = grad\,(F(x, y) - z) = \nabla_{xy}F - \mathbf{k}. \tag{22}$$

Let the surface S is given with an error, namely, instead of the exact function F in (3), the function F^μ is known, given on a rectangle Π of the form (9), such that

$$\|F^\mu - F\|_{L_2(\Pi)} \leqslant \mu. \tag{23}$$

For the approximate calculation of the integral (21), it is necessary to calculate the normal to the surface given approximately, which is also an ill-posed problem, since the calculation of the normal $\mathbf{n}_1$ is associated with the calculation of the derivatives of the function F.

To obtain a stable solution to this problem, we use the substitution [9], that is, we consider the problem of calculating the gradient of the function as the problem of calculating values of the unbounded operator [6].

As an approximation to the function $\nabla_{xy}F$, calculated from the known function F^μ, associated with the function F by condition (23), consider the gradient from the extremal of the functional

$$N^\beta\big[W\big] = \Big\|W - F^\mu\Big\|^2_{L_2(\Pi)} + \beta\Big\|\nabla W\Big\|^2_{L_2(\Pi)}, \quad \beta > 0, \tag{24}$$

where Π is the domain of the form (20).

For simplicity of calculating the extremal, we consider such surfaces S, for which

$$F|_{x=0,l_x} = 0, \quad F|_{y=0,l_y} = 0.$$

This condition, in particular, occurs in the case when S can be considered as a perturbation of the plane $z = 0$. Then the extremal of the functional (24) is the solution to the following problem for the Euler equation

$$-\beta\Delta W + W = F^{\mu},$$
$$W|_{x=0,l_x} = 0,\ W|_{y=0,l_y} = 0.$$

The solution of this problem is

$$W^{\mu}_{\beta}(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde{F}^{\mu}_{nm}}{1+\beta k^2_{nm}} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}. \tag{25}$$

It is easy to see that the series (25) converges uniformly on Π.

As an approximate value of the gradient of the function F^{μ}, we'll consider the vector function

$$\begin{aligned}\nabla_{xy} W^{\mu}_{\beta}(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde{F}^{\mu}_{nm}}{1+\beta k^2_{nm}} \times \\ \times \left(\mathbf{i}\frac{\pi n}{l_x}\cos\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y} + \mathbf{j}\frac{\pi m}{l_y}\cos\frac{\pi m y}{l_y}\sin\frac{\pi n x}{l_x}\right).\end{aligned} \tag{26}$$

The series (26) also converges uniformly on Π.

Let $F^{-}-$ be an odd-periodic continuation of the function F with a period of $2l_x$ for the variable x and with a period of $2l_y$ for the variable y, i.e.

$$\begin{aligned}&F^{-}(x,y) = F(x,y), \quad (x,y)\in\Pi,\\ &F^{-}(-x,y) = -F(x,y), \quad (x,y)\in\Pi,\\ &F^{-}(x,-y) = -F(x,y), \quad (x,y)\in\Pi,\\ &F^{-}(-x,-y) = F(x,y), \quad (x,y)\in\Pi,\\ &F^{-}(x+2l_x n, y+2l_y m) = F^{-}(x,y), \quad (x,y)\in\mathbb{R}^2, \quad n,m=\pm1,\pm2,....\end{aligned}$$

Theorem 1. [9] *Let* $F^{-}\in C^2(\mathbb{R}^2)$, $\beta=\beta(\mu)>0$, $\beta(\mu)\to 0$ *and* $\mu/\sqrt{\beta(\mu)}\to 0$ *when* $\mu\to 0$. *Then*

$$\left\|\nabla_{xy}W_{\beta(\mu)} - \nabla_{xy}F\right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}\left\|\Delta F\right\|_{L_2(\Pi)} \to 0 \quad \textit{when} \quad \mu\to 0.$$

Based on the theorem, we can use the formula (26) to approximate the normal to the surface using the formula (22):

$$\mathbf{n}^{\mu}_{1,\beta} = \nabla_{xy}W^{\mu}_{\beta} - \mathbf{k}. \tag{27}$$

With a known estimate

$$\left\|\Delta F\right\|_{L_2(\Pi)} \leqslant M,$$

it follows from the statement of the theorem

$$\left\|\mathbf{n}^{\mu}_{1,\beta} - \mathbf{n}_1\right\|_{L_2(\Pi)} = \left\|\nabla_{xy}W^{\mu}_{\beta} - \nabla_{xy}F\right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}M.$$

The maximum for theβ expression on the right is achieved when

$$\beta(\mu) = \frac{\mu}{M}$$

and, thus denoting in accordance with (27)

$$\mathbf{n}_1^\mu = \mathbf{n}_{1,\beta(\mu)}^\mu = \nabla_{xy} W_{\beta(\mu)}^\mu - \mathbf{k}, \tag{28}$$

we'll obtain:

$$\left\| \mathbf{n}_1^\mu - \mathbf{n}_1 \right\|_{L_2(\Pi)} \leqslant \sqrt{M\mu} \xrightarrow[\mu \to 0]{} 0. \tag{29}$$

It is also not difficult to obtain an estimate

$$\left\| W_{\beta(\mu)} - F \right\|_{L_2(\Pi)} \leqslant 2\mu. \tag{30}$$

The surface defined by the equation $z = W_{\beta(\mu)}^\mu(x,y)$, we denote as

$$S^\mu = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = W_{\beta(\mu)}^\mu(x,y)\}. \tag{31}$$

5. CONSTRUCTION OF AN APPROXIMATE SOLUTION

Let the functions f, and f_1 in the problem (7) be given with an error, that is, instead of f, and f_1, the functions f^δ, and f_1^δ are given, such that

$$\|f^\delta - f\|_{L_2(\Pi)} \leqslant \delta, \quad \|f_1^\delta - f_1\|_{L_2(\Gamma_H)} \leqslant \delta. \tag{32}$$

In this case, we assume that the surface S of the form (3) is given approximately with the condition (23).

We'll assume that we also know that

$$a_1 < F(x,y) < a_2, \quad (x,y) \in \Pi. \tag{33}$$

Let's construct an approximate solution to the problem (7).

Using the results of the previous paragraph, the right part of the integral equation (17) of the form (21) will be calculated approximately in this case on a rectangle

$$\Pi(a) = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = a\}, \qquad a < \min_{(x,y)} W_{\beta(\mu)}^\mu(x,y), \quad a < a_1 \tag{34}$$

in accordance with the formula (21) as a function

$$\Phi^{\delta,\mu}(M) = \int\limits_{\Pi} \Big[h(U_0 - f^\delta(x_P,y_P))\varphi(M,P)\big|_{P\in S^\mu} n_1^\mu(x_P,y_P) - f^\delta(x_P,y_P)\times \\ \times (\mathbf{n}_1^\mu, \nabla_P\varphi(M,P))\big|_{P\in S^\mu}\Big] dx_P dy_P - \int\limits_{\Gamma_H} \Big[f_1^\delta(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \tag{35}$$

where the surface S^μ has the form (31), the approximate normal $\mathbf{n}_1^\mu$ is calculated by the formula (28).

Let's estimate the error in calculating the function $\Phi^{\delta,\mu}$ of the form (35) with respect to the function Φ of the form (21) – the right side of the integral equation (17), i.e. we estimate the difference

$$\left|\Phi^{\delta,\mu}(M)-\Phi(M)\right| \leqslant \left|\Phi^{\delta,\mu}(M)-\Phi^{\delta,\mu,1}(M)\right| + \left|\Phi^{\delta,\mu,1}(M)-\Phi^{\delta}(M)\right| + \\ + \left|\Phi^{\delta}(M)-\Phi(M)\right|, \quad M\in\Pi(a). \quad (36)$$

In this estimation is introduced the function $\Phi^{\delta,\mu,1}$ of the form (35), where formally the approximate normal $\mathbf{n}_1^\mu$ is replaced by the exact normal $\mathbf{n}_1$ (Note that $\mathbf{n}_1(x_P,y_P)|_{P\in S^\mu} = \mathbf{n}_1(x_P,y_P)|_{P\in S}$):

$$\Phi^{\delta,\mu,1}(M) = \int\limits_{\Pi}\Big[h(U_0-f^\delta(x_P,y_P))\varphi(M,P)\big|_{P\in S^\mu} n_1(x_P,y_P) - f^\delta(x_P,y_P)\times \\ \times(\mathbf{n}_1,\nabla_P\varphi(M,P))\big|_{P\in S^\mu}\Big]dx_P dy_P - \int\limits_{\Gamma_H}\Big[f_1^\delta(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P,$$

and is also introduced the function Φ^δ of the form (35), which is calculated on an exactly specified surface

$$\Phi^{\delta}(M) = \int\limits_{\Pi}\Big[h(U_0-f^\delta(x_P,y_P))\varphi(M,P)\big|_{P\in S} n_1(x_P,y_P) - f^\delta(x_P,y_P)\times \\ \times(\mathbf{n}_1,\nabla_P\varphi(M,P))\big|_{P\in S}\Big]dx_P dy_P - \int\limits_{\Gamma_H}\Big[f_1^\delta(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P. \quad (37)$$

Let's estimate the first difference on the right side of the inequality (36):

$$\left|\Phi^{\delta,\mu}(M)-\Phi^{\delta,\mu,1}(M)\right|_{M\in\Pi(a)} = \\ = \Big|\int\limits_{\Pi}\Big[h(U_0-f^\delta(x_P,y_P))\varphi(M,P)\big|_{P\in S^\mu}(n_1^\mu(x_P,y_P)-n_1(x_P,y_P)) - \\ - f^\delta(x_P,y_P)(\mathbf{n}_1^\mu-\mathbf{n}_1,\nabla_P\varphi(M,P))\big|_{P\in S^\mu}\Big]dx_P dy_P\Big| \leqslant \\ \leqslant \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}}\left|\varphi(M,P)\right|\int\limits_{\Pi}\left|h(U_0-f^\delta(x,y))\right|\cdot|n_1^\mu(x,y)-n_1(x,y)|dxdy + \\ + \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}}\left|\nabla_P\varphi(M,P)\right|\int\limits_{\Pi}\left|f^\delta(x,y)\right|\cdot|\mathbf{n}_1^\mu(x,y)-\mathbf{n}_1(x,y)|dxdy \leqslant \\ \leqslant C_1\int\limits_{\Pi}\left|h(U_0-f^\delta(x,y))\right|\cdot|\mathbf{n}_1^\mu-\mathbf{n}_1|dxdy + C_2\int\limits_{\Pi}\left|f^\delta(x,y)\right|\cdot|\mathbf{n}_1^\mu-\mathbf{n}_1|dxdy.$$

Using the Cauchy-Bunyakovsky inequality, the estimate (29) and the estimate $\|f^\delta\| \leqslant \|f\| + \delta$, for the first difference on the right in the inequality (36), we obtain

$$\left|\Phi^{\delta,\mu}(M) - \Phi^{\delta,\mu,1}(M)\right| \leqslant C_1 h\|U_0 - f^\delta\| \cdot \|\mathbf{n}_1^\mu - \mathbf{n}_1\| + C_2\|f^\delta\| \cdot \|\mathbf{n}_1^\mu - \mathbf{n}_1\| \leqslant \leqslant (C_1 h(\|U_0 - f\| + \delta) + C_2(\|f\| + \delta))\sqrt{M\mu} \leqslant C_3\sqrt{\mu}, \quad M \in \Pi(a). \quad (38)$$

Let's estimate the second difference on the right side of the inequality (36)

$$\left|\Phi^{\delta,\mu,1}(M) - \Phi^{\delta}(M)\right|_{M\in\Pi(a)} = \\ = \Big|\int\limits_{\Pi}\Big[h(U_0 - f^\delta(x_P,y_P))n_1(x_P,y_P)(\varphi(M,P)\big|_{P\in S^\mu} - \varphi(M,P)\big|_{P\in S}) - \\ - f^\delta(x_P,y_P)(\mathbf{n}_1, \nabla_P\varphi(M,P)\big|_{P\in S^\mu} - \nabla_P\varphi(M,P)\big|_{P\in S})\Big]dx_P dy_P\Big|.$$

Using the Lagrange formula we obtain

$$\left|\Phi^{\delta,\mu,1}(M) - \Phi^{\delta}(M)\right|_{M\in\Pi(a)} = \\ = \Big|\int\limits_{\Pi}\Big[h(U_0 - f^\delta(x_P,y_P))n_1(x_P,y_P)\Big(\frac{\partial}{\partial z_P}\varphi(M,P^*)(z_P|_{P\in S^\mu} - z_P|_{P\in S})\Big) - \\ - f^\delta(x_P,y_P)\Big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P^{**})(z_P|_{P\in S^\mu} - z_P|_{P\in S})\Big)\Big]dx_P dy_P\Big|.$$

Since according to (31) $z_P|_{P\in S^\mu} = W^\mu_{\beta(\mu)}(x_P,y_P)$ and $z_P|_{P\in S} = F(x_P,y_P)$, hence we obtain

$$\left|\Phi^{\delta,\mu,1}(M) - \Phi^{\delta}(M)\right|_{M\in\Pi(a)} = \Big|\int\limits_{\Pi}\Big[h(U_0 - f^\delta(x_P,y_P))n_1(x_P,y_P)\times \\ \times\frac{\partial}{\partial z_P}\varphi(M,P^*)\Big(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P)\Big) - f^\delta(x_P,y_P)\times \\ \times\Big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P^{**})(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big)\Big]dx_P dy_P\Big|. \quad (39)$$

We introduce the notation using (33)

$$\begin{aligned} z_1(x_P,y_P) &= \min\{W^\mu_{\beta(\mu)}(x_P,y_P), a_1\}, \\ z_2(x_P,y_P) &= \max\{W^\mu_{\beta(\mu)}(x_P,y_P), a_2\}. \end{aligned} \quad (40)$$

Now from (39) using (40) we obtain

$$\left|\Phi^{\delta,\mu,1}(M) - \Phi^{\delta}(M)\right| = \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}}\left|\frac{\partial}{\partial z_P}\varphi(M,P)n_1\right|\int\limits_{\Pi}\left|h(U_0 - f^\delta(x,y))\right|\times$$

$$\times \Big| W^{\mu}_{\beta(\mu)}(x,y) - F(x,y) \Big| dxdy + \max_{\substack{M\in\Pi(a)\\ P: z_1<z_P<z_2}} \Big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P \varphi(M,P)\Big)\times$$
$$\times \int\limits_{\Pi} \Big| f^{\delta}(x,y) \Big| \cdot \Big| W^{\mu}_{\beta(\mu)}(x,y) - F(x,y) \Big| dxdy, \quad M\in\Pi(a).$$

Applying the Cauchy-Bunyakovsky inequality, assuming that $\delta < \delta_0$, and using the estimate (30), we obtain

$$\Big| \Phi^{\delta,\mu,1}(M) - \Phi^{\delta}(M) \Big| \leqslant C_4 h \|U_0 - f^{\delta}\| \cdot \|W^{\mu}_{\beta(\mu)} - F\| + C_5 \|f^{\delta}\| \cdot \|W^{\mu}_{\beta(\mu)} - F\| \leqslant$$
$$\leqslant \Big(C_4 h(\|U_0 - f\| + \delta) + C_5(\|f\| + \delta)\Big)\|W^{\mu}_{\beta(\mu)} - F\| \leqslant C_6\mu,\ M\in\Pi(a). \tag{41}$$

Let's estimate the third difference on the right side of the inequality (36) using (37) and (21)

$$\Big| \Phi^{\delta}(M) - \Phi(M) \Big|_{M\in\Pi(a)} \leqslant$$
$$\leqslant \int\limits_{\Pi} \Big| h(f^{\delta}(x_P,y_P) - f(x_P,y_P))\varphi(M,P)\big|_{P\in S} n_1(x_P,y_P) \Big| dx_P dy_P +$$
$$+ \int\limits_{\Pi} \Big| (f^{\delta}(x_P,y_P) - f(x_P,y_P))(\mathbf{n}_1, \nabla_P\varphi(M,P))\big|_{P\in S} \Big| dx_P dy_P +$$
$$+ \int\limits_{\Gamma_H} \Big| (f_1^{\delta}(P) - f_1(P))\frac{\partial\varphi}{\partial n_P}(M,P) \Big| d\sigma_P.$$

Using the Cauchy-Bunyakovsky inequality, as well as (32), we obtain from here

$$|\Phi^{\delta}(M) - \Phi(M)|_{M\in\Pi(a)} \leqslant$$
$$\leqslant h \max_{M\in\Pi(a)} \Big(\int\limits_{\Pi} \varphi^2(M,P)\big|_{P\in S} dx_P dy_P\Big)^{1/2} \|f^{\delta} - f\|_{L_2(\Pi)} +$$
$$+ \max_{M\in\Pi(a)} \Big(\int\limits_{\Pi} \big[(\mathbf{n}_1, \nabla_P\varphi(M,P))\big|_{P\in S}\big]^2 dx_P dy_P\Big)^{1/2} \|f^{\delta} - f\|_{L_2(\Pi)} +$$
$$+ \max_{M\in\Pi(a)} \Big(\int\limits_{\Gamma_H} \big[\frac{\partial\varphi}{\partial n_P}(M,P)\big]^2 d\sigma_P\Big)^{1/2} \|f_1^{\delta} - f_1\|_{L_2(\Gamma_H)} \leqslant C_7\delta. \tag{42}$$

Collecting the estimates (38), (41), (42) and assuming that $\mu < \mu_0$, from (36) we obtain when $M\in\Pi(a)$

$$\big| \Phi^{\delta,\mu}(M) - \Phi(M) \big| \leqslant C_3\sqrt{\mu} + C_6\mu + C_7\delta \leqslant \overline{C}_1\sqrt{\mu} + \overline{C}_2\delta = \gamma(\mu,\delta) \xrightarrow[\substack{\mu\to 0\\ \delta\to 0}]{} 0, \tag{43}$$

where $\overline{C}_1, \overline{C}_2$ are constants. From (43) we obtain an estimate in L_2 of the error of the approximate right part of the integral equation (17)

$$\left\|\Phi^{\delta,\mu} - \Phi\right\|_{L_2(\Pi(a))} \leqslant \frac{2}{\sqrt{l_x l_y}}\gamma(\mu,\delta) \xrightarrow[\substack{\mu\to 0\\ \delta\to 0}]{} 0. \tag{44}$$

Equation (17) as an integral equation of the first kind is an ill-posed problem. As an approximate solution to the equation (17) with the right side (35) under the condition (44) we will consider the extremal of the Tikhonov functional [13] with a zero-order stabilizer

$$M^{\alpha}[w] = \Big\| \int\limits_{\Pi(H)} \frac{\partial G}{\partial n} w d\sigma - \Phi^{\delta,\mu} \Big\|^2_{L_2(\Pi(a))} + \alpha\|w\|^2_{L_2(\Pi(H))}, \quad \alpha > 0, \tag{45}$$

where $\Pi(a)$ and $\Pi(H)$ are domains of the form (34) and (9).

In case of additional constraints on the solution, the optimization problem can be solved using the results [1, 2].

The extremal of the functional (45) can be obtained as a solution of the Euler equation, that in the Fourier coefficients $\tilde{w}_{nm}$ of the desired function w has the form

$$\exp\left\{-2k_{nm}(H-a)\right\}\tilde{w}_{nm} + \alpha\tilde{w}_{nm} = -\exp\left\{-k_{nm}(H-a)\right\}\tilde{\Phi}^{\delta,\mu}_{nm}(a),$$

where

$$\tilde{\Phi}^{\delta,\mu}_{nm}(a) = \frac{4}{l_x l_y} \int\limits_{\Pi(a)} \Phi^{\delta,\mu}(x,y,a) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dxdy \tag{46}$$

— Fourier coefficients of the function $\Phi^{\delta,\mu}(M)|_{M\in\Pi(a)}$.

Solving the equation with respect to Fourier coefficients of the extremal and substituting the extremal $w^{\delta,\mu}_{\alpha}$ instead of v_H in (13), we find an approximation of $v^{\delta,\mu}_{\alpha}$ to the function v in the domain $D(-\infty, H)$:

$$v^{\delta,\mu}_{\alpha}(M) = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}^{\delta,\mu}_{nm}(a)\exp\{k_{nm}(z_M - a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}. \tag{47}$$

We note that members of the series (47) differs from members of the series (19) by the regularizing factor $(1+\alpha\exp\{2k_{nm}(H-a)\})^{-1}$, that ensures the convergence of series.

In accordance with (11), we obtain an approximate solution to the problem (7) in the form

$$u^{\delta,\mu}_{\alpha}(M) = v^{\delta,\mu}_{\alpha}(M) + \Phi^{\delta,\mu}(M), \quad M \in D(W,H), \tag{48}$$

where $v^{\delta,\mu}_{\alpha}$ and $\Phi^{\delta,\mu}$ are functions of the form (47) and (37).

Theorem 2. *Let the solution to the problem (7) exist. Then for any* $\alpha = \alpha(\gamma) > 0$ *such that* $\alpha(\gamma) \to 0$, $\gamma/\sqrt{\alpha(\gamma)} \to 0$ *when* $\gamma \to 0$, *the function* $u^{\delta,\mu}_{\alpha(\gamma)}$ *of the form (48), where according to (43)* $\gamma = \gamma(\mu, \delta) = \overline{C}_1\sqrt{\mu} + \overline{C}_2\delta$, *converges to the exact solution to the problem (7) uniformly when* $\delta \to 0$, $\mu \to 0$ *on any compact set* $K \subset D(F, H)$.

Proof. On the compact set K, in accordance with (48) and (11), we estimate the difference

$$|u^{\delta,\mu}_{\alpha(\gamma)} - u| \leqslant |v^{\delta,\mu}_{\alpha(\gamma)} - v| + |\Phi^{\delta,\mu} - \Phi|. \tag{49}$$

Obviously, there exists $\varepsilon > 0$, such that $K \subset D(-\infty, H - \varepsilon)$. For the difference $v^{\delta,\mu}_{\alpha(\gamma)} - v$ in the domain $D(-\infty, H - \varepsilon)$ we get

$$|v^{\delta,\mu}_{\alpha} - v| \leqslant |v^{\delta,\mu}_{\alpha} - v_\alpha| + |v_\alpha - v|, \tag{50}$$

where v_α is a function of the form (47) for exact f and f_1:

$$v_\alpha(M) = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}(a)\exp\{k_{nm}(z_M - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}.$$

Let's estimate the difference $v^{\delta,\mu}_{\alpha} - v_\alpha$ in (50) for $z_M < H - \varepsilon$ using (43)

$$\begin{aligned}
&|v^{\delta,\mu}_{\alpha}(M) - v_\alpha(M)| \leqslant \\
&\leqslant \Big|\sum_{n,m=1}^{\infty} \frac{\exp\{k_{nm}(z_M - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}}\Big| \cdot 4 \max_{P\in\Pi(a)} \big|\Phi^{\delta,\mu}(P) - \Phi(P)\big| \leqslant \\
&\leqslant 4\gamma \sum_{n,m=1}^{\infty} \frac{\exp\{k_{nm}(H - \varepsilon - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}} \leqslant \\
&\qquad\qquad \leqslant 4\gamma \max_x \Big[\frac{e^x}{1 + \alpha e^{2x}}\Big] \sum_{n,m=1}^{\infty} \exp\{-k_{nm}\varepsilon\} \leqslant C_8 \frac{\gamma}{\sqrt{\alpha}}.
\end{aligned} \tag{51}$$

Let's estimate the difference $v_\alpha - v$ in (50) for $z_M < H - \varepsilon$:

$$|v_\alpha - v| \leqslant \sum_{n,m=1}^{\infty} \frac{\alpha\exp\{2k_{nm}(H - a)\}\exp\{k_{nm}(H - \varepsilon - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}} \big|\tilde{\Phi}_{nm}(a)\big|.$$

Using (18) and applying the Cauchy-Bunyakovsky inequality, we obtain

$$\begin{aligned}
&|v_\alpha - v| = \sum_{n,m=1}^{\infty} \frac{\alpha\exp\{2k_{nm}(H - a)\}\exp\{-k_{nm}\varepsilon\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}} \big|(\tilde{v_H})_{nm}\big| \leqslant \\
&\leqslant \Big[\sum_{n,m=1}^{\infty} \Big(\frac{\alpha\exp\{2k_{nm}(H - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}}\Big)^2 \exp\{-2k_{nm}\varepsilon\}\Big]^{1/2} \cdot \frac{2}{\sqrt{l_x l_y}} \|v_H\|_{L_2}.
\end{aligned}$$

Since the series depending on the parameter α is majorized by a convergent numerical series with the coefficients $\exp\{-2\varepsilon k_{nm}\}$, then a limit transition with respect to α is possible and, thus,

$$|v_\alpha - v| \to 0 \quad \text{when} \quad \alpha \to 0. \tag{52}$$

From (50), (51) and (52) and conditions of the theorem it follows that

$$|v^{\delta,\mu}_{\alpha(\gamma)} - v| \to 0 \quad \text{when} \quad \mu \to 0,\, \mu \to 0. \tag{53}$$

The second difference on the right side of (49) is evaluated similarly (43) when $M \in K$

$$\big|\, \Phi^{\delta,\mu}(M) - \Phi(M) \,\big|_{M\in K} \leqslant \overline{C}_3\sqrt{\mu} + \overline{C}_4\delta.$$

Hence, as well as from (49) and (53), the statement of the theorem follows.

6. NUMERICAL SOLUTION TO THE OPTIMIZATION PROBLEM

The effectiveness of the proposed method for solving the problem (7) is shown in the following model example.

In the problem (4), let the surface S be the plane $\Pi(0)$, $f_1 = U_0 = 24$, $\quad h = 0.5, l_x = 30, l_y = 30, H = 1.4$. The function ρ corresponds to four point sources at points in the plane $\Pi(H)$: $(x_1, y_1) = (8, 8)$, $(x_2, y_2) = (10, 8)$, $(x_3, y_3) = (10, 10)$, $(x_4, y_4) = (6, 10)$. The boundary value of the solution of the model problem (4) in this case has the form

$$f(x,y) = U_0 + \sum_{n,m=1}^{\infty} \sum_{i=1}^{4} q_i \frac{e^{-k_{nm}H}}{k_{nm}+h} \sin\frac{\pi n x_i}{l_x} \sin\frac{\pi m y_i}{l_y} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{54}$$

where k_{nm} is calculated using the formula (15) and $q_i = 100., i = 1, 2, 3, 4$.

To set the inverse problem (7), we consider that the function f, calculated by the formula (54), a known function. Also $f_1 = U_0 = 24$, $\quad h = 0.5$, $l_x = 30$, $l_y = 30$, $H = 1.4$ are known.

To solve the inverse problem (7), we use the formulas (48), (47), (46), (37). In the formula (37) we use the representation for the fundamental solution

$$\varphi(M,P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_M - z_P|}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}, \tag{55}$$

when $z_M = a$, $z_P = 0$. The Fourier coefficients in the formula (46) are calculated without calculating the function Φ, similarly to [10]. When using the formula (46), integration is performed under the sign of the integral in (37) and under the sign

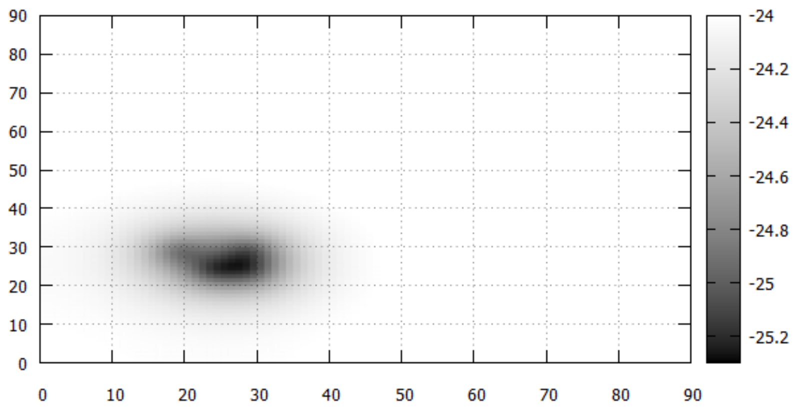

Figure 1: Initial thermogram on the surface.

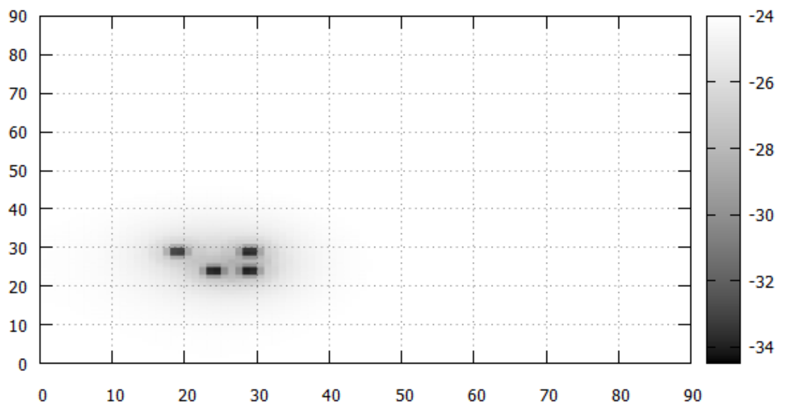

Figure 2: The corrected thermogram obtained as an approximate solution of the inverse problem.

of the sum in (55). Taking into account the orthogonality of the system of functions (16), the calculation formulas for Fourier coefficients Φ_{nm} are significantly simplified.

To obtain a numerical result, the problems (4), (7) are discretized. A uniform grid of 91x91 points is introduced on the rectangles $\Pi(a), a = -0.5$ and $\Pi(H)$. The Hamming algorithm [4, p.83] is used for summing discrete Fourier series.

The calculation results are shown in Fig.1 and Fig.2. Fig.1 shows the initial data of the inverse problem – the function f calculated from the discrete analog of the formula (54). The relative magnitude of the added error is 0.28%. Four sources are perceived as a single whole. Fig.2 shows the result of restoring the $u|_{z=H}$ function using the formulas (48), (47), (46), (37). Four sources are clearly visible. Regularization parameter $\alpha = 10^{-8}$. With the regularization parameter $\alpha = 0$, the solution is destroyed.

7. CONCLUSION

The inverse problem (7) and its stable solution can be used for mathematical processing of thermographic images (thermograms), in particular, in medicine [5], in order to correct the image on the thermogram. The thermogram obtained with the help of a thermal imager reproduces with a certain degree of reliability the image of the structure of heat sources located inside the body. Image correction on the thermogram can be obtained based on the solution to the problem (7). In this case, the function f will be associated with the original thermogram, and the function $u|_{z=H}$ will be considered the result of mathematical processing of the thermogram. Since the function $u|_{z=H}$ represents the temperature distribution on a plane closer to the studied heat sources than the original surface S, we can expect a more accurate reproduction of the image of sources on the calculated thermogram $u|_{z=H}$. The results of calculations carried out on the model example show the effectiveness of the proposed method and algorithm based on the formulas (48), (47), (46), (37), and can be applied for processing thermographic images.

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