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CORRIGENDUM ON: OPTIMALITY CONDITIONS AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH DATA UNCERTAINTY VIA MORDUKHOVICH SUBDIFFERENTIAL (YUGOSLAV JOURNAL OF OPERATIONS RESEARCH, VOL. 31, NO 4, DOI: HTTPS://DOI.ORG/10.2298/YJOR201017013P)

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The author of the article "Optimality Conditions and Duality for Multiobjective Semi-Infinite Programming with Data Uncertainity via Mordukhovich Subdifferential", Thanh-Hung Pham has informed the Editor about necessary corrections of the paper, as follows:

The whole paragraphs, or the parts, starting with Example 13 should be replaced by the text:

"

Example 13. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \frac{x}{4}, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$$

By simple computation, we have

$$\partial^M f(0) = \{\frac{1}{4}, 1\}.$$

It is easy to see that f is $\varepsilon-$pseudo-convex of type II but not $\varepsilon-$pseudo-convex of type I at $x = 0$. We fisrt prove that f is $\varepsilon-$pseudo-convex of type II at $x = 0$. Indeed, take $y = -1, \xi = \frac{1}{4} \in \partial^M f(0) = \{\frac{1}{4}, 1\}$ and $\varepsilon = \frac{1}{4}$. Clearly,

$$f(y) + \sqrt{\varepsilon}|y - x| = -1 + \frac{1}{2} = -\frac{1}{2} \leq 0 = f(x),$$

which implies

$$\langle \xi, y - x \rangle = -\frac{1}{4} \leq 0.$$

We now prove that f is not $\varepsilon-$pseudo-convex of type I at $x = 0$. Indeed, take $y = -1, \xi = \frac{1}{4} \in \partial^M f(0) = \{\frac{1}{4}, 1\}$ and $\varepsilon = \frac{1}{4}$. Clearly,

$$f(y) + \sqrt{\varepsilon}|y - x| = -1 + \frac{1}{2} = -\frac{1}{2} \leq 0 = f(x).$$

However,

$$\langle \xi, y - x \rangle + \sqrt{\varepsilon}|y - x| = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \geq 0.$$

Next, we can derive the following sufficient condition for a quasi $\varepsilon-$solution of (RSIP).

Theorem 14. *Let $\varepsilon \geq 0$ and Ω be convex set. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε. If $f(.)$ is Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$ and $g_t(., \bar{v}_t), t \in T$ is Mordukhovich quasi-convex at $\bar{x}$, then $\bar{x} \in F$ is a quasi $\varepsilon-$solution of (RSIP).*

Proof. Let $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ be satisfied regarding the robust approximate KKT condition with respect to ε. Therefore, there exist $\xi_0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + w + \sqrt{\varepsilon} b = 0. \tag{5}$$

Since $b \in \mathbb{B}, w \in N^M(\bar{x}; \Omega)$ and Ω is convex set, it follows that, for any $x \in F$,

$$\langle w, x - \bar{x} \rangle \leq 0, \langle b, x - \bar{x} \rangle \leq ||x - \bar{x}||.$$

From (5), we have

$$\left\langle \xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0,$$

which means that

$$\langle \xi_0, x - \bar{x} \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq -\left\langle \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle. \tag{6}$$

Moreover, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that $g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t)$ for any $x \in F$ and $t \in T(\lambda)$. By the Mordukhovich quasi-convexity of $g_t(., \bar{v}_t)$ at $\bar{x}$ and $\xi_t \in \partial_x^M g_t(\bar{x}, \bar{v}_t)$, we obtain

$$\langle \xi_t, x - \bar{x} \rangle \leq 0. \tag{7}$$

Combining (6) and (7), we obtain

$$\langle \xi_0, x - \bar{x} \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0.$$

Since $f(.,\bar{u})$ is Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$, it follows from Definition 11 that

$$f(x)+\sqrt{\varepsilon}||x-\bar{x}||\geq f(\bar{x}).$$

Therefore, $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP). This completes the proof. □

Now, we present an example to show the importance of the Mordukhovich $\varepsilon-$pseudo-convexity of type I in Theorem 14 (function $f(.)$ is given in [27] page 87).

Example 15. *Let $x\in\mathbb{R}, t\in T=[0,1], \Omega=[0,+\infty)$ and $v_t\in\mathcal{V}_t=[2-t,2+t]$ for any $t\in T$. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathcal{V}_t\to\mathbb{R}$ be defined by*

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0,\end{cases}$$

and

$$g_t(x,v_t)=tx^2-2v_tx.$$

Then, $F=[0,2]$ and $N^M(\bar{x};\Omega)=N^M(\bar{x};[0,+\infty))=(-\infty,0]$. Let us consider $\bar{x}=0, \bar{\lambda}_t=0$ and $\bar{v}_t=2-t$. Note that $f(.)$ is locally Lipschitz at $\bar{x}$ and $g_t(.,\bar{v}_t)$ is convex at $\bar{x}$. We have,

$$\partial^M f(\bar{x})=[-1,1](\text{ see [27] page 87}) \text{ and } \partial^M_x g_t(\bar{x},\bar{v}_t)=\{2(t-2)\}.$$

We prove that $f(.)$ is not Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$. Indeed, take $\bar{y}=\frac{2}{3\pi}, \xi=0\in\partial^M f(\bar{x})=[-1,1]$ and $0\leq\sqrt{\varepsilon}\leq\frac{2}{3\pi}$. Clearly,

$$\langle\xi,\bar{y}-\bar{x}\rangle+\sqrt{\varepsilon}|\bar{y}-\bar{x}|=\sqrt{\varepsilon}|\bar{y}-\bar{x}|\geq 0.$$

However,

$$f(\bar{y})+\sqrt{\varepsilon}|\bar{y}-\bar{x}|=-\frac{4}{9\pi^2}+\sqrt{\varepsilon}.\frac{2}{3\pi}\leq 0=f(\bar{x}).$$

Now, take an arbitrarily $0\leq\sqrt{\varepsilon}\leq\frac{2}{3\pi}$. Then, $(\bar{x},\bar{\lambda}_t,\bar{v}_t)\in F\times\mathbb{R}^{(T)}_+\times\mathcal{V}_t$ satisfies the robust approximate KKT conditions with respect to ε. Indeed, let us select $\sqrt{\varepsilon}=\frac{1}{9}, \bar{x}=0, \bar{\lambda}_t=0, \bar{v}_t=2-t$ and $\mathbb{B}=[-1,1]$. Then,

$$0\in\left(-\infty,\frac{4}{3}\right]=\partial^M f(\bar{x})+\sum_{t\in T}\bar{\lambda}_t\partial^M_x g_t(\bar{x},\bar{v}_t)+N^M(\bar{x};\mathbb{R})+\sqrt{\varepsilon}\mathbb{B},$$

and $\bar{\lambda}_t g(\bar{x},\bar{v}_t)=0$.

However, $\bar{x}=0$ is not a quasi $\varepsilon-$solution of (RSIP). In order to see this, let us take $x=\frac{2}{3\pi}\in F$ and $\sqrt{\varepsilon}=\frac{1}{9}$. Then,

$$f(x)+\sqrt{\varepsilon}|x-\bar{x}|=-\frac{4}{9\pi^2}+\frac{2}{27\pi}<0=f(\bar{x}).$$

In the special case when $\mathcal{V}_t$ is a singleton, we can obtain the following result.

Corollary 16. *Consider problem (SIP). Let $\varepsilon\geq 0$ and Ω be convex set. Assume that $(\bar{x},\bar{\lambda}_t)\in F\times\mathbb{R}^{(T)}_+$ satisfies approximate KKT condition with respect to ε. If f is Mordukhovich $\varepsilon-$pseudo-convex of type I at $\bar{x}$ and $g_t, t\in T$ is Mordukhovich quasi-convex at $\bar{x}$, then $\bar{x}\in F$ is a quasi $\varepsilon-$solution of (SIP).*

In the following theorem, we give another sufficient optimality condition for robust $\varepsilon-$quasi-minimum of (RSIP).

Theorem 17. *Let $\varepsilon \geq 0$ and Ω be convex set. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε. If $f(.)$ is Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$ and $g_t(., \bar{v}_t), t \in T$ is Mordukhovich $\varepsilon-$quasi-convex at $\bar{x}$, then $\bar{x} \in F$ is a quasi $\varepsilon-$solution of (RSIP).*

Proof. Similarly to the proof of Theorem 14, there exist $\xi_0 \in \partial^M f(\bar{x})$, $\xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\langle \xi_0, x - \bar{x} \rangle \geq -\sqrt{\varepsilon}||x - \bar{x}|| - \left\langle \sum_{t \in T} \bar{\lambda}_t \xi_t, x - \bar{x} \right\rangle. \tag{8}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that $g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t)$ for any $x \in F$ and $t \in T(\lambda)$. By the Mordukhovich $\varepsilon-$quasi-convexity of $g_t(., \bar{v}_t)$ at $\bar{x}$ and $\xi_t \in \partial_x^M g_t(\bar{x}, \bar{v}_t)$, we obtain

$$\langle \xi_t, x - \bar{x} \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \leq 0. \tag{9}$$

Combining (8) and (9), we obtain

$$\langle \xi_0, x - \bar{x} \rangle \geq 0.$$

Since $f(., \bar{u})$ is Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$, it follow from Definition 11 that

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}).$$

Therefore, $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP). This completes the proof. □

Now, we present an example to show the importance of the Mordukhovich $\varepsilon-$pseudo-convexity of type II in Theorem 17.

Example 18. *Let $f, g_t, t \in T, \Omega$ and $\mathcal{V}_t$ be defined as in Example 15. Then, $F = [0, 2]$ and $N^M(\bar{x}; \Omega) = N^M(\bar{x}; [0, +\infty)) = (-\infty, 0]$. Let us consider $\bar{x} = 0, \bar{\lambda}_t = 0$, and $\bar{v}_t = 2 - t$. Note that $f(.)$ is locally Lipschitz at $\bar{x}$ and $g_t(., \bar{v}_t)$ is convex at $\bar{x}$. We have,*

$$\partial^M f(\bar{x}) = [-1, 1] \text{ and } \partial_x^M g_t(\bar{x}, \bar{v}_t) = \{2(t - 2)\}.$$

We prove that $f(., \bar{u})$ is not Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$. Indeed, take $\bar{y} = \frac{2}{3\pi}, \xi = 0 \in \partial^M f(\bar{x}) = [-1, 1]$ and $0 \leq \sqrt{\varepsilon} \leq \frac{2}{3\pi}$. Clearly,

$$\langle \xi, \bar{y} - \bar{x} \rangle = 0 \geq 0.$$

However,

$$f(\bar{y}) + \sqrt{\varepsilon}|\bar{y} - \bar{x}| = -\frac{4}{9\pi^2} + \sqrt{\varepsilon}.\frac{2}{3\pi} \leq 0 = f(\bar{x}).$$

Now, take an arbitrarily $0 \leq \sqrt{\varepsilon} \leq \frac{2}{3\pi}$. From Example 15, $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT conditions with respect to ε. By virtue of Example 15, $\bar{x} = 0$ is not a quasi $\varepsilon-$solution of (RSIP).

In the special case when $\mathcal{V}_t$ is a singleton, we can obtain the following result.

Corollary 19. *Consider problem (SIP). Let $\varepsilon \geq 0$ and Ω be convex set. Assume that $(\bar{x}, \bar{\lambda}_t) \in F \times \mathbb{R}_+^{(T)}$ satisfies approximate KKT condition with respect to ε. If f is Mordukhovich $\varepsilon-$pseudo-convex of type II at $\bar{x}$ and $g_t, t \in T$ is Mordukhovich $\varepsilon-$quasi-convex at $\bar{x}$, then $\bar{x} \in F$ is an $\varepsilon-$quasi-minimum of (SIP).*

Motivated by the definition of generalized convexity due to [8, 9] and [20], we introduce a new concept of generalized convexity as follows:

Definition 20. *Let $g_T := (g_t)_{t \in T}, \varepsilon \geq 0$.*

(i) We say that (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, if for any $x \in F, \xi_0 \in \partial^M f(\bar{x})$ and $\xi_t \in \partial_x^M g_t(\bar{x}, v_t), v_t \in \mathcal{V}_t, t \in T$, there exists $w \in \mathbb{R}^n$ such that

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}),$$

$$g_t(x, v_t) \leq g_t(\bar{x}, v_t) \Rightarrow \langle \xi_t, w \rangle \leq 0, \forall t \in T,$$

and

$$\langle b, w \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

(ii) We say that (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, if for any $x \in F, \xi_0 \in \partial^M f(\bar{x})$ and $\xi_t \in \partial_x^M g_t(\bar{x}, v_t), v_t \in \mathcal{V}_t, t \in T$, there exists $w \in \mathbb{R}^n$ such that

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}||x - \bar{x}|| > f(\bar{x}),$$

$$g_t(x, v_t) \leq g_t(\bar{x}, v_t) \Rightarrow \langle \xi_t, w \rangle \leq 0, \forall t \in T,$$

and

$$\langle b, w \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

Now, let us provide an example illustrating our Definition 20 (i).

Example 21. *Let $x \in \mathbb{R}, t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [-t-1, -t]$ for any $t \in T, \mathbb{B} = [-1, 1]$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by*

$$f(x) = |x| + x^3 \text{ and } g_t(x, v_t) = v_t x^2.$$

Then $F = \mathbb{R}$. Let us consider $\bar{x} = 0$, we have $\partial^M f(\bar{x}) = [-1, 1]$ and $\partial_x^M g(\bar{x}, v_t) = \{0\}$. Let us consider $x = -1 \in F = \mathbb{R}, \xi_0 = 0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, v_t), 0 \leq \varepsilon \leq 1$, by taking $w = x = -1$, it follows that $w \in \mathbb{R}$,

$$\langle \xi_0, w \rangle + \sqrt{\varepsilon}|x - \bar{x}| = \sqrt{\varepsilon} \geq 0 \Rightarrow f(x) + \sqrt{\varepsilon}|x - \bar{x}| = \sqrt{\varepsilon} \geq 0 = f(\bar{x}),$$

$$g_t(x, v_t) = v_t \leq g_t(\bar{x}, v_t) = 0 \Rightarrow \langle \xi_t, w \rangle = 0 \leq 0, t \in T,$$

and

$$\langle b, w \rangle = -b \leq ||x - \bar{x}|| = 1, \forall b \in [-1, 1].$$

This shows that (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x} \in F$.

Next, we give sufficient conditions for a feasible point of problem (RSIP) to be a quasi $\varepsilon-$solution and a quasi weakly $\varepsilon-$solution.

Theorem 22. *Let $\varepsilon \geq 0$. Assume that $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT conditions with respect to ε.*

(i) If (f, g_T) is Mordukhovich $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $\bar{x}$ is a quasi weakly $\varepsilon-$solution of (RSIP).

(ii) If (f, g_T) is Mordukhovich strictly $\varepsilon-$quasi generalized convex on F at $\bar{x}$, then $\bar{x}$ is a quasi $\varepsilon-$solution of (RSIP).

Proof. Since $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ satisfies the robust approximate KKT condition with respect to ε, there exists $\xi_0 \in \partial^M f(\bar{x}), \xi_t \in \partial_x^M g(\bar{x}, \bar{v}_t), \forall t \in T$ with $w \in N^M(\bar{x}; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + w + \sqrt{\varepsilon} b = 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

or, equivalent

$$\xi_0 + \sum_{t \in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b = -w. \tag{10}$$

We first prove (i). Suppose on contrary that $\bar{x}$ is not a quasi weakly $\varepsilon-$solution of (RSIP). It then follows that there exists $x \in F$ satisfying

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \leq f(\bar{x}). \tag{11}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t), \text{ for any } x \in F \text{ and } t \in T(\lambda). \tag{12}$$

By the Mordukhovich $\varepsilon-$quasi generalized convexity of (f, g_T) on $\mathcal{F}$ at $\bar{x}$ and (11), (12), there exists $d \in \mathbb{R}^n$ such that $(x \neq \bar{x})$

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d \rangle \leq 0, t \in T,$$

and

$$\langle b, d \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}. \tag{13}$$

Therefore, we have

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0.$$

On the other hand, by (13), one has

$$\left\langle \xi_0 + \sum_{t\in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b, d \right\rangle < 0,$$

which contradicts (10).

We now prove (ii). Suppose on contrary that $\bar{x}$ is not a quasi $\varepsilon-$solution of (RSIP). It then follows that there exists $x \in F$ satisfying

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| < f(\bar{x}). \tag{14}$$

On the other hand, if $t \in T(\lambda)$, then $g_t(\bar{x}, \bar{v}_t) = 0$. Note that for any $x \in F$, then $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. It follows that

$$g_t(x, \bar{v}_t) \leq g_t(\bar{x}, \bar{v}_t), \text{ for any } x \in F \text{ and } t \in T(\lambda). \tag{15}$$

By the Mordukhovich strictly $\varepsilon-$quasi generalized convexity of (f, g_T) on F at $\bar{x}$ and (14), (15), there exists $d \in \mathbb{R}^n$ such that

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d \rangle \leq 0, t \in T,$$

and

$$\langle b, d \rangle \leq ||x - \bar{x}||, \forall b \in \mathbb{B}. \tag{16}$$

Therefore, we have

$$\langle \xi_0, d \rangle + \sum_{t\in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0.$$

On the other hand, by (16), one has

$$\left\langle \xi_0 + \sum_{t\in T} \bar{\lambda}_t \xi_t + \sqrt{\varepsilon} b, d \right\rangle < 0,$$

which contradicts (10). This completes the proof. □

4. MOND-WEIR TYPE DUALITY IN ROBUST APPROXIMATE OPTIMIZATION PROBLEM

In this section, we investigate some results for $\varepsilon-$Mond-Weir type robust duality for robust optimization problems under Mordukhovich $\varepsilon-$quasi generalized convexity assumptions.

Now, we consider the Mond–Weir type dual problem (RUD) of (RSIP) as given by

$$
\text{(RUD)} \quad \begin{cases} \max & f(y) \\ \text{s.t.} & 0 \in \partial^M f(y) + \sum_{t\in T} \lambda_t \partial_x^M g_t(y, v_t) + N^M(y;\Omega) + \sqrt{\varepsilon}\mathbb{B}, \\ & \lambda_t g_t(y, v_t) \geq 0, \\ & y \in \Omega, \lambda_t \in \mathbb{R}_+^{(T)}, \varepsilon \geq 0, v_t \in \mathcal{V}_t, t \in T. \end{cases}
$$

The feasible set of (RUD) is defined by

$$
F_{\text{RUD}} = \{(y, \lambda_t, v_t) \in \Omega \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t \mid 0 \in \partial^M f(y) + \sum_{t\in T} \lambda_t \partial_x^M g_t(y, v_t) + N^M(y;\Omega)
$$

$$
+\sqrt{\varepsilon}\mathbb{B}, \lambda_t g_t(y, v_t) \geq 0.\}.
$$

Now, we give the following definition of a robust approximate quasi-solution for (RUD).

Definition 23. *Let* $\varepsilon \geq 0$.

(i) We say that $(\bar{y}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$ *is a quasi* $\varepsilon-$*solution of (RUD) if for any* $(y, \lambda_t, v_t) \in F_{\text{RUD}}$,

$$
f(\bar{y}) + \sqrt{\varepsilon}||y - \bar{y}|| \geq f(y).
$$

(ii) We say that $(\bar{y}, \bar{\lambda}_t, \bar{v}_t) \in F_{\text{RUD}}$ *is a quasi weakly* $\varepsilon-$*solution of (RUD) if for any* $(y, \lambda_t, v_t) \in F_{\text{RUD}}$,

$$
f(\bar{y}) + \sqrt{\varepsilon}||y - \bar{y}|| > f(y).
$$

Now, we establish the following approximate weak duality theorem, which holds between (RSIP) and (RUD).

Theorem 24. *Let* $\varepsilon \geq 0$ *and* $x \in F$. *Suppose that* $(\bar{x}, \bar{\lambda}_t, \bar{v}_T) \in F_{\text{RUD}}$.

(i) If (f, g_T) *is Mordukhovich* $\varepsilon-$*quasi generalized convex on* F *at* $\bar{x}$, *then*

$$
f(x) > f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.
$$

(ii) If (f, g_T) *is Mordukhovich strictly* $\varepsilon-$*quasi generalized convex on* F *at* $\bar{x}$, *then*

$$
f(x) \geq f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.
$$

Proof. Since $(\bar{x}, \bar{\lambda}_t, \bar{v}_t) \in F_{RUD}$, we have $\bar{x} \in \Omega, \bar{v}_t \in \mathcal{V}_t, \bar{\lambda}_t \geq 0, t \in T$ and

$$
0 \in \partial^M f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^M g_t(\bar{x}, \bar{v}_t) + N^M(\bar{x};\Omega) + \sqrt{\varepsilon}B, \tag{17}
$$

From (17), there exist $\xi_0 \in \partial^M f(x), \xi_t \in \partial_x^M g(x, v_t), \forall t \in T$ with $w \in N^M(x; \Omega)$ and $b \in \mathbb{B}$, such that

$$\xi_0 + \sum_{t \in T} \lambda_t \xi + \sqrt{\varepsilon} b = -w. \tag{18}$$

We first prove (i). Let $x \in F$. Suppose on contrary that

$$f(x) \le f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||. \tag{19}$$

Note that for any $x \in F$, $g_t(x, \bar{v}_t) \le 0$ for any $t \in T$ and $\bar{\lambda}_t \ge 0, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \ge 0, \bar{v}_t \in \mathcal{V}_t, t \in T$. It follows that

$$g_t(x, \bar{v}_t) \le 0 \le g_t(\bar{x}, \bar{v}_t). \tag{20}$$

By the Mordukhovich $\varepsilon-$quasi generalized convexity of (f, g_T) on F at $\bar{x}$ and (19), (20), there exists $d \in \mathbb{R}^n$ such that $(x \neq \bar{x})$

$$\langle \xi_0, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0,$$

$$\langle \xi_t, d \rangle \le 0, t \in T,$$

$$\langle b, d \rangle \le ||x - \bar{x}||, \forall b \in \mathbb{B}.$$

Therefore, we have

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0. \tag{21}$$

On the other hand, by (18), one has

$$\langle \xi_0, d \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, d \rangle + \sqrt{\varepsilon}||x - \bar{x}|| = -\langle w, d \rangle \ge 0,$$

which contradicts (21). Thus,

$$f(x) > f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||.$$

We now prove (ii). Let $x \in F$. Suppose on contrary that

$$f(x) < f(\bar{x}) - \sqrt{\varepsilon}||x - \bar{x}||. \tag{22}$$

".

The Author appologizes for the inconveniences he has made to the readers and the Editors.