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HIGHER ORDER FRACTIONAL VARIATIONAL SYMMETRIC DUALITY OVER CONE CONSTRAINTS

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Abstract: The article aims at higher order fractional variational pair of symmetric dual formulations where constraints are defined over cones and explores pertinent duality output applying the idea of higher order η-invexity. Also, we bring into begin a numerical example in order to validate the definition exploited to establish duality results. Moreover, we demonstrate a case study dealing with the static formulation of our considered problem and explore the results carefully.

Keywords: Symmetric duality, variational problem, higher order η-invexity, cone constraints.

MSC: 90C29, 90C30, 90C46.

1. INTRODUCTION

Dorn [1] was the first who introduced symmetric duality. Dantzig [2] extended these results under a more generalized setting for the nonlinear problems under convexity assumptions. Another effort in this direction is taken by Bazarra and Goode [3] who extended these results over arbitrary cones. Computing advantage of second as well as higher order analog of the problems was furnished by Mangasarian [4]. Mond and Weir [5] developed these theorems connecting the relationship between symmetric dual pairs in nonlinear programming problems.

Chandra [6] magnified the results to fractional programming problems. Jayswal and Prasad [7] derived suitable duality results for a Mond-Weir type nondifferentiable second order fractional symmetric dual programs over arbitrary cones while Jayswal *et al.* [8] extended the work to higher order counterpart of the problem and furnished the aforesaid theorems making use of higher order η-invexity. For more details on fractional programming, one can follow [9], [10],[11].

Chen [12] focused on higher-order analog of the problems having more than one objective equipped with support functions and set up the duality criteria. Subsequent extension of these results were done by Khurana [13] by considering a class of Mond-Weir symmetric dual type formulations defined over cones. Kaseem [14] studied multiobjective symmetric first order dual problems under usual convexity assumptions to set up weak, strong, and converse duality criteria. Jayswal *et al.* [15] formulated symmetric variational formulations up to second order consisting of more than one objective under F-convexity. Prasad *et al.* [16] focused on second order fractional variational problems and derived various duality theorems. Yang [17] studied suitable duality results to discuss the symmetric dual first order formulations employing invexity.

Recently, Suneja and Louhan [18] studied symmetric higher order problems and established the duality results assuming the functions as invex. In this connection, Kharbanda and Agarwal [19] introduced nonsmooth multiobjective fractional programming problems involving higher order invex functions. Moreover, a novel approach is developed by Verma *et al.* [20] to handle higher order multiobjective symmetric dual programs under cone invexity. Sharma and Kaur [21] focused on fractional problems under a more generalized setting over cones and discussed suitable results on dual formulations assuming that functions satisfy higher order (Φ, ρ) convexity.

In the present paper, the key idea is to study higher order fractional symmetric variational programs where constraints are defined over cones and set up appropriate duality results. In Section 2, we collect formal definitions along with higher order η-invex function that is needed in the sequel of the paper. A numerical example is also given proper space in order to validate the definitions used in this paper. In Section 3, we have taken steps to cast higher order symmetric fractional variational problems equipped with constraints defined over cones and deduce suitable duality theorems, the static case of our problem, and conclusions in the next two sections.

2. PRELIMINARIES

Definition 1. *[8] A subset C of R^n is known as cone, if it is characterized by*

$$0 \leq \lambda \in \mathbb{R}, \ x \in C \Rightarrow \lambda x \in C.$$

Definition 2. *[15] For any cone C, the polar cone C^* is described mathematically as*

$$C^* = \{y : x^T y \leqq 0, \ x \text{ being a member of } C\}.$$

Definition 3. *The functional $\int_{\gamma_1}^{\gamma_2} f(t,s,\dot{s})\,dt$ is termed as higher order η-invex at a point $\sigma \in \aleph \subset \mathbb{R}^n$ corresponding to $h : I \times \aleph \times \aleph \times \mathbb{R}^n \mapsto \mathbb{R}$, if there exists a function $\eta : I \times \aleph \times \aleph \mapsto \mathbb{R}^n$ such that*

$$\int_{\gamma_1}^{\gamma_2} f(t,s,\dot{s})\,dt - \int_{\gamma_1}^{\gamma_2} f(t,\sigma,\dot{\sigma})\,dt - \int_{\gamma_1}^{\gamma_2} h(t,\sigma,\dot{\sigma},p)dt + \int_{\gamma_1}^{\gamma_2} p^t \nabla_p h(t,\sigma,\dot{\sigma},p)dt$$

$$\geqq \int_{\gamma_1}^{\gamma_2} [\eta(t,s,\sigma)^T \{\nabla_s f(t,\sigma,\dot{\sigma}) + D\nabla_{\dot{s}} f(t,\sigma,\dot{\sigma}) + \nabla_p h(t,\sigma,\dot{\sigma},p)\}]dt,$$

$\forall (s,p) \in \aleph \times \mathbb{R}^n$.

Next, we layout an appropriate example to display that higher order η-invex function exist that is not second order η-invex.

Example 4. *Let $\pi = (\pi_1,\pi_2)$, $\omega = (\omega_1,\omega_2) \in \aleph \subset \mathbb{R}^2_+$ and $a = (a_1,a_2) \in \mathbb{R}^2$. Define $f : I \times \aleph \times \aleph \mapsto \mathbb{R}$ by $f(t,\pi,\dot{\pi}) = 1 + \sin\pi_1 + \sin\pi_2$; $\eta : I \times \aleph \times \aleph \mapsto \mathbb{R}^2$ by $\eta(t,\pi,\omega) = (\cos\pi_1 + \sin\omega_1 + 2,\ \cos\pi_2 + \sin\omega_2 + 2)$ and $h : I \times \aleph \times \aleph \times \mathbb{R}^2 \mapsto \mathbb{R}$ by $h(t,\omega,\dot{\omega},p) = -p_1\cos\omega_1 - p_2\cos\omega_2$. We take $\omega = (0,0)$; $p = (1,1)$ and $I = [0,1]$.*

$$\int_0^1 f(t,\pi,\dot{\pi})\,dt - \int_0^1 f(t,\omega,\dot{\omega})\,dt - \int_0^1 h(t,\omega,\dot{\omega},p)dt + \int_0^1 p^t \nabla_p h(t,\omega,\dot{\omega},p)dt$$

$$- \int_0^1 [\eta(t,\pi,\omega)^T \{\nabla_\pi f(t,\omega,\dot{\omega}) + D\nabla_{\dot{\pi}} f(t,\omega,\dot{\omega}) + \nabla_p h(t,\omega,\dot{\omega},p)\}]dt$$

$$= \int_0^1 (\sin\pi_1 + \sin\pi_2 - \sin\omega_1 - \sin\omega_2)dt - \int_0^1 (-p_1\cos\omega_1 - p_2\cos\omega_2)$$

$$+ (p_1,p_2)\begin{bmatrix} -\cos\omega_1 \\ -\cos\omega_2 \end{bmatrix} dt$$

$$= \int_0^1 (\sin\pi_1 + \sin\pi_2 - \sin\omega_1 - \sin\omega_2)dt - \int_0^1 (-2p_1\cos\omega_1 - 2p_2\cos\omega_2)\,dt$$

$$= \int_0^1 \sin\pi_1 + \sin\pi_2 + 4\,dt \geq 0,$$

which display that f is higher order invex. Moreover, f is not second order invex as explained below:

$$\int_0^1 \Bigg(f(t,\pi,\dot{\pi}) - f(t,\omega,\dot{\omega}) + \frac{1}{2} p^T \nabla_{\pi\pi} f(t,\omega,\dot{\omega})p$$

$$- \eta(t,\pi,\omega)^T (\nabla_\pi f(t,\omega,\dot{\omega}) + \nabla_{\pi\pi} f(t,\omega,\dot{\omega})p)\Bigg)dt$$

$$= \int_0^1 \Bigg((\sin \pi_1 + \sin \pi_2 - \sin \omega_1 - \sin \omega_2) + \frac{1}{2}(p_1, p_2) \begin{bmatrix} -\sin \omega_1 & 0 \\ 0 & -\sin \omega_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$- (\cos \pi_1 + \sin \omega_1 + 2, \ \ \cos \pi_2 + \sin \omega_2 + 2) \begin{bmatrix} \cos \omega_1 \\ \cos \omega_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -\sin \omega_1 & 0 \\ 0 & -\sin \omega_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \Bigg) dt$$

$$= \int_0^1 \Bigg((\sin \pi_1 + \sin \pi_2 - \sin \omega_1 - \sin \omega_2) + \frac{1}{2}[{p_1}^2 \sin \omega_1 + {p_2}^2 \sin \omega_2]$$

$$- (\cos \pi_1 + \sin \pi_1 + 2, \ \ \cos \pi_2 + \sin \omega_2 + 2) \begin{bmatrix} \cos \omega_1 - p_1 \sin \omega_1 \\ \cos \omega_2 - p_2 \sin \omega_2 \end{bmatrix} \Bigg) dt$$

$$= \int_0^1 (\sin \pi_1 + \sin \pi_2 - \cos \pi_1 - \cos \pi_2 - 4) dt$$

$$\leq 0, \ \ \forall \ \pi \in \aleph.$$

In next couple of sections, C_1 and C_2 represent cones with non-void interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Further, we also assume that C_1 and C_2 are closed and convex cones. We take $\aleph \subset \mathbb{R}^n$ and $\tilde{\aleph} \subset \mathbb{R}^m$ in such a way that $C_1 \times C_2 \subset \aleph \times \tilde{\aleph}$. Also, $\eta_1 : I \times \aleph \times \aleph \mapsto \mathbb{R}^n$ and $\eta_2 : I \times \tilde{\aleph} \times \tilde{\aleph} \mapsto \mathbb{R}^m$.

3. PRIMAL - DUAL FORMULATION

In this paper, we investigate the following primal-dual doublet defined over cone constraints:

Primal (FVSP)

$$\text{Minimize} \quad \left(\frac{\int_{\gamma_1}^{\gamma_2} (f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)) dt}{\int_{\gamma_1}^{\gamma_2} (g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)) dt} \right)$$

subject to

$$\varrho(\gamma_1) = 0 = \varrho(\gamma_2), \quad \dot{\varrho}(\gamma_1) = 0 = \dot{\varrho}(\gamma_2),$$

$$\delta(\gamma_1) = 0 = \delta(\gamma_2), \quad \dot{\delta}(\gamma_1) = 0 = \dot{\delta}(\gamma_2),$$

$$\nabla_\delta f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) - D \nabla_{\dot{\delta}} f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)$$

$$-\left(\frac{\int_{\gamma_1}^{\gamma_2}(f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)-p^T\nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)\,dt}{\int_{\gamma_1}^{\gamma_2}(g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)-p^T\nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)\,dt}\right)$$
$$(\nabla_\delta g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})-D\nabla_{\dot{\delta}}g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+\nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p))\in C_2^*,$$

$$\delta^T[\nabla_\delta f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})-D\nabla_{\dot{\delta}}f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+\nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)$$
$$-\left(\frac{\int_{\gamma_1}^{\gamma_2}(f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)-p^T\nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)\,dt}{\int_{\gamma_1}^{\gamma_2}(g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)-p^T\nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)\,dt}\right)$$
$$(\nabla_\delta g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})-D\nabla_{\dot{\delta}}g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+\nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p))]\geqq 0,$$

$$\varrho(t)\in C_1,\quad t\in I.$$

Dual (FVSD)

$$\text{Maximize}\quad \left(\frac{\int_{\gamma_1}^{\gamma_2}(f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))\,dt}{\int_{\gamma_1}^{\gamma_2}(g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))\,dt}\right)$$

subject to

$$\sigma(\gamma_1)=0=\sigma(\gamma_2),\quad \dot{\sigma}(\gamma_1)=0=\dot{\sigma}(\gamma_2),$$
$$\vartheta(\gamma_1)=0=\vartheta(\gamma_2),\quad \dot{\vartheta}(\gamma_1)=0=\dot{\vartheta}(\gamma_2),$$

$$-[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})-D\nabla_{\dot{\varrho}}f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-\left(\frac{\int_{\gamma_1}^{\gamma_2}(f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)\,dt}{\int_{\gamma_1}^{\gamma_2}(g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)\,dt}\right)$$
$$(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})-D\nabla_{\dot{\varrho}}g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))]\in C_1^*,$$

$$\sigma^T[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})-D\nabla_{\dot{\varrho}}f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-\left(\frac{\int_{\gamma_1}^{\gamma_2}(f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)\,dt}{\int_{\gamma_1}^{\gamma_2}(g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)-q^T\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)\,dt}\right)$$
$$(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})-D\nabla_{\dot{\varrho}}g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})+\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))]\leqq 0,$$

$$\vartheta(t)\in C_2,\quad t\in I,$$

where
(i) $f : I \times \aleph \times \aleph \times \tilde{\aleph} \times \tilde{\aleph} \to \mathbb{R}_+$ and $g : I \times \aleph \times \aleph \times \tilde{\aleph} \times \tilde{\aleph} \to \mathbb{R}_+ \setminus \{0\}$,
(ii) $H,\ G : I \times \aleph \times \aleph \times \tilde{\aleph} \times \tilde{\aleph} \times \mathbb{R}^m \mapsto \mathbb{R}$ are differentiable functions,
(iii) $X,\ Y : I \times \aleph \times \aleph \times \tilde{\aleph} \times \tilde{\aleph} \times \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable functions,
(iv) p and q are vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

In order to make the problem suitably defined, we consider the numerator is nonnegative in the feasible region while on the other hand denominator is constrained to be positive. First of all, we convert our problem into the parametric problem by introducing l and m defined as below:

$$l = \frac{\int_{\gamma_1}^{\gamma_2} (f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p))\, dt}{\int_{\gamma_1}^{\gamma_2} (g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p))\, dt},$$

$$m = \frac{\int_{\gamma_1}^{\gamma_2} (f(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}) + X(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}, q) - q^T \nabla_q X(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}, q))\, dt}{\int_{\gamma_1}^{\gamma_2} (g(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}) + Y(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}, q) - q^T \nabla_q Y(t, \sigma, \dot{\sigma}, \vartheta, \dot{\vartheta}, q))\, dt}.$$

Equivalent form of the above problems are:

Primal (FVSP$'$)

Minimize $\quad l$

subject to

$$\varrho(\gamma_1) = 0 = \varrho(\gamma_2), \quad \dot{\varrho}(\gamma_1) = 0 = \dot{\varrho}(\gamma_2),$$
$$\delta(\gamma_1) = 0 = \delta(\gamma_2), \quad \dot{\delta}(\gamma_1) = 0 = \dot{\delta}(\gamma_2),$$

$$\int_{\gamma_1}^{\gamma_2} (f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)\, dt)$$
$$- l \int_{\gamma_1}^{\gamma_2} (g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p) - p^T \nabla_p G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)\, dt) = 0, \quad (1)$$

$$\nabla_\delta f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) - D \nabla_{\dot{\delta}} f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)$$
$$- l(\nabla_\delta g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) - D \nabla_{\dot{\delta}} g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + \nabla_p G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)) \in C_2^*, \quad (2)$$

$$s^T [\nabla_\delta f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) - D \nabla_{\dot{\delta}} f(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + \nabla_p H(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p)$$
$$- l(\nabla_\delta g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) - D \nabla_{\dot{\delta}} g(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}) + \nabla_p G(t, \varrho, \dot{\varrho}, \delta, \dot{\delta}, p))] \geqq 0, \quad (3)$$

$$\varrho(t) \in C_1, \quad t \in I.$$

Dual (FVSD$'$)

Maximize $\quad m$

subject to

$$\sigma(\gamma_1) = 0 = \sigma(\gamma_2), \quad \dot{\sigma}(\gamma_1) = 0 = \dot{\sigma}(\gamma_2),$$
$$\vartheta(\gamma_1) = 0 = \vartheta(\gamma_2), \quad \dot{\vartheta}(\gamma_1) = 0 = \dot{\vartheta}(\gamma_2),$$

$$\int_{\gamma_1}^{\gamma_2} (f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q) - q^T\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))\,dt$$
$$-m\int_{\gamma_1}^{\gamma_2} (g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q) - q^T\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))\,dt = 0, \quad (4)$$

$$-[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-m(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))] \in C_1^*, \quad (5)$$

$$\sigma^T[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-m(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))] \leqq 0, \quad (6)$$

$$\vartheta(t) \in C_2, \quad t \in I.$$

4. DUALITY THEOREMS

The present section discusses the well-suited duality results under higher order invexity along with suitably chosen conditions. The following duality results have been discussed for (FVSP$'$) and (FVSD$'$) but apply evenly to (FVSP) and (FVSD) to the same extent.

Theorem 5. *(Weak duality). Assume that (ϱ,δ,l,p) and (σ,ϑ,m,q) denote feasible solutions to (FVSP$'$) and (FVSD$'$), respectively. Moreover, the following conditions are also imposed*

(*a*) $\int_{\gamma_1}^{\gamma_2} (f(t,.,.,\vartheta(t),\dot{\vartheta}(t))\,dt$ *is higher order invex at* $\sigma(t)$ *w.r.t.*η_1 *and* $X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$ *and* $\int_{\gamma_1}^{\gamma_2} -g(t,.,.,\vartheta(t),\dot{\vartheta}(t))\,dt$ *is higher order invex at* $\sigma(t)$ *w.r.t.* η_1 *and* $-Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$,

(*b*) $-\int_{\gamma_1}^{\gamma_2} f(t,\varrho(t),\dot{\varrho}(t),.,.)\,dt$ *is higher order invex at* $\delta(t)$ *w.r.t.* η_2 *and* $-H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)$ *and* $\int_{\gamma_1}^{\gamma_2} g(t,\varrho(t),\dot{\varrho}(t),.,.)\,dt$ *is higher order invex at* $\delta(t)$ *w.r.t.* η_2 *and* $G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)$,

(*c*) $(\eta_1(t,\varrho(t),\sigma(t)) + \sigma)^T \in C_1, \forall\ \varrho(t) \in C_1,\ t \in I$, *(d)* $(\eta_2(t,\vartheta(t),\delta(t))+\delta)^T \in C_2, \forall\ \vartheta(t) \in C_2,\ t \in I.$

Then $l \geqq m$.

Proof. Since (ϱ,δ,l,p) and (σ,ϑ,m,q) are feasible solutions to formulations (FVSP$^{'}$) and (FVSD$^{'}$), respectively. Therefore, with the help of (c) and condition (5), we obtain

$$-(\eta_1(t,\varrho,\sigma)+\sigma)^T[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-m(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))] \leqq 0,$$

which on account of (6) gives

$$(\eta_1(t,\varrho,\sigma))^T[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-m(\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - D\nabla_{\dot{\varrho}} g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q))] \geqq 0. \tag{7}$$

Using constraint (2) in (d), we obtain

$$(\eta_2(t,\vartheta,\delta)+\delta)^T[\nabla_\delta f(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) - D\nabla_{\dot{\delta}} f(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) + \nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)$$
$$-\,l(\nabla_\delta g(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) - D\nabla_{\dot{\delta}} g(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) + \nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p))] \leqq 0,$$

which on account of (3) settles down to

$$(\eta_2(t,\vartheta,\delta))^T[\nabla_\delta f(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) - D\nabla_{\dot{\delta}} f(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) + \nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)$$
$$-\,l(\nabla_\delta g(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) - D\nabla_{\dot{\delta}} g(t,\varrho,\dot{\varrho},\delta,\dot{\delta}) + \nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p))] \leqq 0. \tag{8}$$

From the assumption (a), we obtain

$$\int_{\gamma_1}^{\gamma_2} f(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta}) - f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})dt \geqq (\eta_1(t,\varrho,\sigma))^T[\nabla_\varrho f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})$$
$$+\,D\nabla_{\dot{\varrho}} f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) + \nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)] + X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$-\,q^T\nabla_q X(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q) \tag{9}$$

and

$$\int_{\gamma_1}^{\gamma_2} -g(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta}) + g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})dt \geqq (\eta_1(t,\varrho,\sigma))^T[-\nabla_\varrho g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta})$$
$$-\,D\nabla_{\dot{\varrho}} g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - \nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)] - Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q)$$
$$+\,q^T\nabla_q Y(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta},q). \tag{10}$$

Multiply equation (10) by m and resultant is added to (9) so as to get

$$\int_{\gamma_1}^{\gamma_2}[f(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta}) - f(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}) - m(g(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta}) - g(t,\sigma,\dot{\sigma},\vartheta,\dot{\vartheta}))]dt$$

$$\geqq \eta_1(t,\varrho,\sigma)^T[\nabla_\varrho f(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta) + D\nabla_{\dot\varrho} f(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta)$$
$$+ \nabla_q X(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q) + m(-\nabla_\varrho g(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta) - D\nabla_{\dot\varrho} g(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta)$$
$$- \nabla_q Y(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q))] + [X(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q) - q^T\nabla_q X(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q)$$
$$+ m(-Y(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q) + q^T\nabla_q Y(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q))], \tag{11}$$

which by equation (4) reduces to

$$\int_{\gamma_1}^{\gamma_2}[f(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta) - m(g(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta)]dt \geqq \eta_1(t,\varrho,\sigma)^T[\nabla_\varrho f(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta)$$
$$+ D\nabla_{\dot\varrho} f(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta) + \nabla_q X(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q) - m(\nabla_\varrho g(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta)$$
$$+ D\nabla_{\dot\varrho} g(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta) + \nabla_q Y(t,\sigma,\dot\sigma,\vartheta,\dot\vartheta,q))]. \tag{12}$$

The above inequality together with equation (7) gives

$$\int_{\gamma_1}^{\gamma_2}(f(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta) - mg(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta))dt \geqq 0. \tag{13}$$

In the same way, by condition (b), we obtain

$$\int_{\gamma_1}^{\gamma_2} -f(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta) + f(t,\varrho,\dot\varrho,\delta,\dot\delta)dt \geqq (\eta_2(t,\vartheta,\delta))^T[-\nabla_\delta f(t,\varrho,\dot\varrho,\delta,\dot\delta)$$
$$- D\nabla_{\dot\delta} f(t,\varrho,\dot\varrho,\delta,\dot\delta) - \nabla_p H(t,\varrho,\dot\varrho,\delta,\dot\delta,p)] - [H(t,\varrho,\dot\varrho,\delta,\dot\delta,p)$$
$$+ p^T\nabla_p H(t,\varrho,\dot\varrho,\delta,\dot\delta,p)] \tag{14}$$

and

$$\int_{\gamma_1}^{\gamma_2}[g(t,\varrho,\dot\varrho,\delta,\dot\delta) - g(t,\varrho,\dot\varrho,\delta,\dot\delta)]dt \geqq (\eta_2(t,\vartheta,\delta))^T[\nabla_\delta g(t,\varrho,\dot\varrho,\delta,\dot\delta)$$
$$+ D\nabla_{\dot\delta} g(t,\varrho,\dot\varrho,\delta,\dot\delta) + \nabla_p G(t,\varrho,\dot\varrho,\delta,\dot\delta,p)] + [G(t,\varrho,\dot\varrho,\delta,\dot\delta,p)$$
$$- p^T\nabla_p G(t,\varrho,\dot\varrho,\delta,\dot\delta,p)]. \tag{15}$$

Multiplying (14) by l and adding the resultant to (15), we have

$$\int_{\gamma_1}^{\gamma_2}[-f(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta) + f(t,\varrho,\dot\varrho,\delta,\dot\delta) - l(g(t,\varrho,\dot\varrho,\vartheta,\dot\vartheta) - g(t,\varrho,\dot\varrho,\delta,\dot\delta))]dt$$
$$\geqq (\eta_2(t,\vartheta,\delta))^T[-\nabla_\delta f(t,\varrho,\dot\varrho,\delta,\dot\delta) - D\nabla_{\dot\delta} f(t,\varrho,\dot\varrho,\delta,\dot\delta) - \nabla_p H(t,\varrho,\dot\varrho,\delta,\dot\delta,p)$$
$$+l[(\nabla_\delta g(t,\varrho,\dot\varrho,\delta,\dot\delta) + D\nabla_{\dot\delta} g(t,\varrho,\dot\varrho,\delta,\dot\delta) + \nabla_p G(t,\varrho,\dot\varrho,\delta,\dot\delta,p))]dt$$
$$-H(t,\varrho,\dot\varrho,\delta,\dot\delta,p) + p^T\nabla_p H(t,\varrho,\dot\varrho,\delta,\dot\delta,p) + l(G(t,\varrho,\dot\varrho,\delta,\dot\delta,p)$$
$$-p^T\nabla_p G(t,\varrho,\dot\varrho,\delta,\dot\delta,p)), \tag{16}$$

which by equation (1) reduces to

$$\int_{\gamma_1}^{\gamma_2}[-f(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta})+l(g(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta})]dt \geqq (\eta_2(t,\vartheta,\delta))^T[-\nabla_\delta f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})$$

$$-D\nabla_{\dot{\delta}} f(t,\varrho,\dot{\varrho},\delta,\dot{\delta})-\nabla_p H(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p)+l(\nabla_\delta g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})$$

$$+D\nabla_{\dot{\delta}} g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})+\nabla_p G(t,\varrho,\dot{\varrho},\delta,\dot{\delta},p))].$$

On account of the above inequality and equation (8), we can write

$$\int_{\gamma_1}^{\gamma_2}[-f(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta})+l(g(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta})]dt \geqq 0. \tag{17}$$

On adding equations (13) and (17), we get

$$\int_{\gamma_1}^{\gamma_2}(l-m)g(t,\varrho,\dot{\varrho},\vartheta,\dot{\vartheta})dt \geqq 0.$$

Since $\int_{\gamma_1}^{\gamma_2} g(t,\varrho,\dot{\varrho},\delta,\dot{\delta})dt > 0$, we get

$$l \geqq m.$$

Hence the theorem is configured. □

Theorem 6. *(Strong Duality). Let $(\bar{\varrho},\bar{\delta},\bar{l},\bar{p})$ be a solution of $(FVSP')$ which is also assumed to be local optimal. Under the suitable conditions*

(i) $\nabla_\varrho H(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_q X(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$,
$\nabla_\varrho G(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_q Y(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$
$\nabla_{\varrho'} H(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q'} X(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$,
$\nabla_{\varrho'} G(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q'} Y(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$
$\nabla_{\varrho''} H(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q''} X(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$,
$\nabla_{\varrho''} G(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q''} Y(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0),\ldots,$
$\nabla_{\varrho^{(2n)}} H(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q^{(2n)}} X(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$,
$\nabla_{\varrho^{(2n)}} G(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)=\nabla_{q^{(2n)}} Y(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\sigma},\dot{\bar{\sigma}},0)$.

(ii) the specified Hessian matrix $\nabla_{pp} H(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\delta},\dot{\bar{\delta}})-\bar{l}\nabla_{pp} G(t,\bar{\varrho},\dot{\bar{\varrho}},\bar{\delta},\dot{\bar{\delta}})$ *is positive or negative definite,*

(iii) $(\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\neq 0$,

(iv) for chosen $\bar{p}\in\mathbb{R}^m$,

$$\bar{p}^T[(\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)]=0$$

indicate that $\bar{p}=0$ *and*

(v)

$$\Big[D\Big((\nabla_{\varrho'} f+\nabla_{q'} X)-\bar{l}(\nabla_{\varrho'} g+\nabla_{q'} Y)\Big)+D^2\Big((\nabla_{\varrho''} f+\nabla_{q''} X)-\bar{l}(\nabla_{\varrho''} g+\nabla_{q''} Y)\Big)$$
$$-D^3\Big((\nabla_{\varrho'''} f+\nabla_{q'''} X)-\bar{l}(\nabla_{\varrho'''} g+\nabla_{q'''} Y)\Big)+\dots$$
$$+D^{2n}\Big((\nabla_{\varrho^{(2n)}} f+\nabla_{q^{(2n)}} X)-\bar{l}(\nabla_{\varrho^{(2n)}} g+\nabla_{q^{(2n)}} Y)\Big)\Big]=0.$$

$(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{q}=0)$ *becomes feasible to (FVSD′) and the two objectives yield equal output. If, in addition to above, the conditions mentioned in Theorem 4.1 are also contented for every solutions feasible to considered problems (FVSP′) and (FVSD′), then* $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{p}=0)$ *and* $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{q}=0)$ *become absolute optima to considered formulations (FVSP′) and (FVSD′), respectively.*

Proof. Since $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{p})$ represents optimal solution to (FVSP′), there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \in C_2$ and $\xi \in \mathbb{R}$ satisfying the following Fritz John optimality conditions at the point $(\bar{\varrho}(t), \bar{\delta}(t), \bar{l}, \bar{p}(t))$:

$$\beta\Big[\Big((\nabla_\varrho f+\nabla_\varrho H-\bar{p}^T\nabla_{p\varrho} H)-\bar{l}(\nabla_\varrho g+\nabla_\varrho G-\bar{p}^T\nabla_{p\varrho} G)\Big)$$
$$-D\Big((\nabla_{\varrho'} f+\nabla_{\varrho'} H-\bar{p}^T\nabla_{p\varrho'} H)-\bar{l}(\nabla_{\varrho'} g+\nabla_{\varrho'} G-\bar{p}^T\nabla_{p\varrho'} G)\Big)$$
$$+D^2\Big((\nabla_{\varrho''} f+\nabla_{\varrho''} H-\bar{p}^T\nabla_{p\varrho''} H)-\bar{l}(\nabla_{\varrho''} g+\nabla_{\varrho''} G-\bar{p}^T\nabla_{p\varrho''} G)\Big)$$
$$-D^3\Big((\nabla_{\varrho'''} f+\nabla_{\varrho'''} H-\bar{p}^T\nabla_{p\varrho'''} H)-\bar{l}(\nabla_{\varrho'''} g+\nabla_{\varrho'''} G-\bar{p}^T\nabla_{p\varrho'''} G)\Big)+\dots$$
$$+D^{2n}\Big((\nabla_{\varrho^{(2n)}} f+\nabla_{\varrho^{(2n)}} H-\bar{p}^T\nabla_{p\varrho^{(2n)}} H)-\bar{l}(\nabla_{\varrho^{(2n)}} g+\nabla_{\varrho^{(2n)}} G$$
$$-\bar{p}^T\nabla_{p\varrho^{(2n)}} G)\Big)\Big]+(\gamma-\xi\bar{\delta})^T\Big[\Big((\nabla_{\delta\varrho} f-D\nabla_{\delta'\varrho} f+\nabla_{p\varrho} H)-\bar{l}(\nabla_{\delta\varrho} g$$
$$-D\nabla_{\delta'\varrho} g+\nabla_{p\varrho} G)\Big)-D\Big((\nabla_{\delta\varrho'} f-D\nabla_{\delta'\varrho'} f+\nabla_{p\varrho'} H)-\bar{l}(\nabla_{\delta\varrho'} g$$
$$-D\nabla_{\delta'\varrho'} g+\nabla_{p\varrho'} G)\Big)+D^2\Big((\nabla_{\delta\varrho''} f-D\nabla_{\delta'\varrho''} f+\nabla_{p\varrho''} H)-\bar{l}(\nabla_{\delta\varrho''} g$$
$$-D\nabla_{\delta'\varrho''} g+\nabla_{p\varrho''} G)\Big)-D^3\Big((\nabla_{\delta\varrho'''} f-D\nabla_{\delta'\varrho'''} f+\nabla_{p\varrho'''} H)-\bar{l}(\nabla_{\delta\varrho'''} g$$
$$-D\nabla_{\delta'\varrho'''} g+\nabla_{p\varrho'''} G)\Big)+\dots+D^{2n}\Big((\nabla_{\delta\varrho^{2n}} f-D\nabla_{\delta'\varrho^{2n}} f+\nabla_{p\varrho^{2n}} H)$$
$$-\bar{l}(\nabla_{\delta\varrho^{2n}} g-D\nabla_{\delta'\varrho^{2n}} g+\nabla_{p\varrho^{2n}} G)\Big)\Big](\varrho(t)-\bar{\varrho}(t))\geqq 0,\ t\in I, \forall\ \varrho\in C_1, \quad (18)$$

$$\beta\Big[\Big((\nabla_\delta f+\nabla_\delta H-\bar{p}^T\nabla_{p\delta}H)-\bar{l}(\nabla_\delta g+\nabla_\delta G-\bar{p}^T\nabla_{p\delta}G)\Big)$$
$$-D\Big((\nabla_{\delta'} f+\nabla_{\delta'} H-\bar{p}^T\nabla_{p\delta'}H)-\bar{l}(\nabla_{\delta'} g+\nabla_{\delta'} G-\bar{p}^T\nabla_{p\delta'}G)\Big)$$
$$+D^2\Big((\nabla_{\delta''} f+\nabla_{\delta''} H-\bar{p}^T\nabla_{p\delta''}H)-\bar{l}(\nabla_{\delta''} g+\nabla_{\delta''} G-\bar{p}^T\nabla_{p\delta''}G)\Big)$$
$$-D^3\Big((\nabla_{\delta'''} f+\nabla_{\delta'''} H-\bar{p}^T\nabla_{p\delta'''}H)-\bar{l}(\nabla_{\delta'''} g+\nabla_{\delta'''} G-\bar{p}^T\nabla_{ps'''}G)\Big)+\dots$$
$$+D^{2n}\Big((\nabla_{\delta^{2n}} f+\nabla_{\delta^{2n}} H-\bar{p}^T\nabla_{p\delta^{2n}}H)-\bar{l}(\nabla_{\delta^{2n}} g+\nabla_{\delta^{2n}} G-\bar{p}^T\nabla_{p\delta^{2n}}G)\Big)$$
$$+(\gamma-\xi\bar{\delta})^T\Big[\Big((\nabla_{\delta\delta} f-D\nabla_{\delta'\delta} f+\nabla_{p\delta}H)-\bar{l}(\nabla_{\delta\delta} g-D\nabla_{\delta'\delta} g+\nabla_{p\delta}G)\Big)$$
$$-D\Big((\nabla_{\delta\delta'} f-D\nabla_{\delta'\delta'} f+\nabla_{p\delta'}H)-\bar{l}(\nabla_{\delta\delta'} g-D\nabla_{\delta'\delta'} g+\nabla_{p\delta'}G)\Big)$$
$$+D^2\Big((\nabla_{\delta\delta''} f-D\nabla_{\delta'\delta''} f+\nabla_{p\delta''}H)-\bar{l}(\nabla_{\delta\delta''} g-D\nabla_{\delta'\delta''} g+\nabla_{p\delta''}G)\Big)$$
$$-D^3\Big((\nabla_{\delta\delta'''} f-D\nabla_{\delta'\delta'''} f+\nabla_{p\delta'''}H)-\bar{l}(\nabla_{\delta\delta'''} g-D\nabla_{\delta'\delta'''} g+\nabla_{p\delta'''}G)\Big)+\dots$$
$$+D^{2n}\Big((\nabla_{\delta\delta^{2n}} f-D\nabla_{\delta'\delta^{2n}} f+\nabla_{p\delta^{2n}}H)-\bar{l}(\nabla_{\delta\delta^{2n}} g-D\nabla_{\delta'\delta^{2n}} g+\nabla_{p\delta^{2n}}G)\Big)$$
$$-\xi\Big((\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\Big)=0,\ t\in I, \forall\delta\in\mathbb{R}^n. \quad (19)$$

$$\Big[\alpha-\beta(g+G-p^T\nabla_p G)+(\gamma-\xi\bar{\delta}(t))^T(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\Big]=0,\ t\in I, \quad (20)$$

$$(\gamma-\xi\bar{\delta}-\beta\bar{p})^T(\nabla_{pp}H-\bar{l}\nabla_{pp}G)=0,\ t\in I, \quad (21)$$

$$\gamma^T\Big((\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\Big)=0,\ t\in I, \quad (22)$$

$$-\xi\bar{\delta}^T\Big((\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\Big)=0,\ t\in I, \quad (23)$$

$$(\alpha,\beta,\gamma,\xi)\neq 0,\ \alpha>0,\ \gamma\in C_2,\ \xi\geq 0. \quad (24)$$

Hypothesis (ii), and Equation (21) yield

$$(\gamma-\xi\bar{\delta}-\beta\bar{p})=0. \quad (25)$$

We infer that $\beta\neq 0$. In case, if $\beta=0$, then Equation (25) turns out to be

$$\gamma=\xi\bar{\delta} \quad (26)$$

and Equation (19) gives

$$\xi\Big((\nabla_\delta f-D\nabla_{\delta'} f+\nabla_p H)-\bar{l}(\nabla_\delta g-D\nabla_{\delta'} g+\nabla_p G)\Big)=0, \quad (27)$$

which by hypothesis (iii) yields $\xi = 0$ and from equation (26), we get $\gamma = 0$ and hence from equation (20), we have $\alpha = 0$. Thus, we get$(\alpha, \beta, \gamma, \xi) \neq 0,\ t \in I$ contradicting equation (24). Hence $\beta > 0$. Once we subtract equation (23) from equation (22), we have

$$(\gamma - \xi\bar{\delta})^T\Big((\nabla_\delta f - D\nabla_{\delta'} f + \nabla_p H) - \bar{l}(\nabla_\delta g - D\nabla_\delta g + \nabla_p G)\Big) = 0.$$

Equation (25) and the fact that $\beta \neq 0$ give rise to

$$\bar{p}^T\Big((\nabla_\delta f - D\nabla_{\delta'} f + \nabla_p H) - \bar{l}(\nabla_\delta g - D\nabla_{\delta'} g + \nabla_p G)\Big) = 0.$$

By hypothesis (iv), we have $\bar{p} = 0$. Therefore, we obtain from equation (26) that $\gamma = \xi\bar{\delta}$. Hence, $\gamma \in C_2$. As $\gamma = \xi\bar{\delta}$, from equation (18), we obtain

$$\begin{aligned}&\beta\Big[\Big((\nabla_\varrho f+\nabla_\varrho H)-\bar{l}(\nabla_\varrho g+\nabla_\varrho G)\Big)-D\Big((\nabla_{\varrho'} f+\nabla_{\varrho'} H)-\bar{l}(\nabla_{\varrho'} g+\nabla_{\varrho'} G)\Big)\\&\quad+D^2\Big((\nabla_{\varrho''} f+\nabla_{\varrho''} H)-\bar{l}(\nabla_{\varrho''} g+\nabla_{\varrho''} G)\Big)-D^3\Big((\nabla_{\varrho'''} f+\nabla_{\varrho'''} H)\\&\quad-\bar{l}(\nabla_{\varrho'''} g+\nabla_{\varrho'''} G)\Big)+\cdots+D^{2n}\Big((\nabla_{\varrho^{(2n)}} f+\nabla_{\varrho^{(2n)}} H)-\bar{l}(\nabla_{\varrho^{(2n)}} g+\nabla_{\varrho^{(2n)}} G)\Big)\Big]^T\\&\qquad(\varrho(t)-\bar{\varrho}(t)) \geq 0, \forall\ \varrho \in C_1. \end{aligned} \tag{28}$$

From assumption (i) for $\bar{p} = 0$ the above inequality turns out to be

$$\begin{aligned}&\Big[\Big((\nabla_\varrho f+\nabla_q X)-\bar{l}(\nabla_\varrho g+\nabla_q Y)\Big)-D\Big((\nabla_{\varrho'} f+\nabla_{q'} X)-\bar{l}(\nabla_{\varrho'} g+\nabla_{q'} Y)\Big)\\&\quad+D^2\Big((\nabla_{\varrho''} f+\nabla_{q''} X)-\bar{l}(\nabla_{\varrho''} g+\nabla_{q''} Y)\Big)-D^3\Big((\nabla_{\varrho'''} f+\nabla_{q'''} X)\\&\quad-\bar{l}(\nabla_{\varrho'''} g+\nabla_{q'''} Y)\Big)+\cdots+D^{2n}\Big((\nabla_{\varrho^{(2n)}} f+\nabla_{q^{(2n)}} X)-\bar{l}(\nabla_{\varrho^{(2n)}} g+\nabla_{q^{(2n)}} Y)\Big)\Big]^T\\&\quad(\varrho(t)-\bar{\varrho}(t)) \geq 0. \end{aligned} \tag{29}$$

Suppose $\varrho(t) \in C_1$, then $\varrho(t) + \bar{\varrho(t)} \in C_1$, so equation (29) implies

$$\begin{aligned}&\Big[\Big((\nabla_\varrho f+\nabla_q X)-\bar{l}(\nabla_\varrho g+\nabla_q Y)\Big)-D\Big((\nabla_{\varrho'} f+\nabla_{q'} X)-\bar{l}(\nabla_{\varrho'} g+\nabla_{q'} Y)\Big)\\&\quad+D^2\Big((\nabla_{\varrho''} f+\nabla_{q''} X)-\bar{l}(\nabla_{\varrho''} g+\nabla_{q''} Y)\Big)-D^3\Big((\nabla_{\varrho'''} f+\nabla_{q'''} X)\\&\quad-\bar{l}(\nabla_{\varrho'''} g+\nabla_{q'''} Y)\Big)+\cdots+D^{2n}\Big((\nabla_{\varrho^{(2n)}} f+\nabla_{q^{(2n)}} X)-\bar{l}(\nabla_{\varrho^{(2n)}} g+\nabla_{q^{(2n)}} Y)\Big)\Big]^T\\&\quad\varrho(t) \geq 0, \forall\ \varrho \in C_1. \end{aligned}$$

By a property of polar cone, we have

$$-\Big[\Big((\nabla_\varrho f+\nabla_q X)-\bar{l}(\nabla_\varrho g+\nabla_q Y)\Big)-D\Big((\nabla_{\varrho'} f+\nabla_{q'} X)-\bar{l}(\nabla_{\varrho'} g+\nabla_{q'} Y)\Big)$$

$$+D^2\Big((\nabla_{\varrho''} f + \nabla_{q''} X) - \bar{l}(\nabla_{\varrho''} g + \nabla_{q''} Y)\Big) - D^3\Big((\nabla_{\varrho'''} f + \nabla_{q'''} X) - \bar{l}(\nabla_{\varrho'''} g$$
$$+\nabla_{q'''} Y)\Big) + \cdots + D^{2n}\Big((\nabla_{\varrho^{(2n)}} f + \nabla_{q^{(2n)}} X) - \bar{l}(\nabla_{\varrho^{(2n)}} g + \nabla_{q^{(2n)}} Y)\Big)\Big] \in {C_1}^*.$$

Let $\varrho(t) = 0$ and $\varrho(t) = 2\bar{\varrho}(t)$ in equation (29), we have

$$\bar{\varrho}(t)^T\Big[\Big((\nabla_{\varrho} f + \nabla_q X) - \bar{l}(\nabla_{\varrho} g + \nabla_q Y)\Big) - D\Big((\nabla_{\varrho'} f + \nabla_{q'} X) - \bar{l}(\nabla_{\varrho'} g + \nabla_{q'} Y)\Big)$$
$$+D^2\Big((\nabla_{\varrho''} f + \nabla_{q''} X) - \bar{l}(\nabla_{\varrho''} g + \nabla_{q''} Y)\Big) - D^3\Big((\nabla_{\varrho'''} f + \nabla_{q'''} X) - \bar{l}(\nabla_{\varrho'''} g$$
$$+\nabla_{q'''} Y)\Big) + \cdots + D^{2n}\Big((\nabla_{\varrho^{(2n)}} f + \nabla_{q^{(2n)}} X) - \bar{l}(\nabla_{\varrho^{(2n)}} g + \nabla_{q^{(2n)}} Y)\Big)\Big] = 0. \quad (30)$$

Now, using assumption (v) in above equation, we get

$$\bar{\varrho}(t)^T\Big[\Big((\nabla_{\varrho} f + \nabla_q X) - \bar{l}(\nabla_{\varrho} g + \nabla_q Y)\Big)\Big] = 0. \quad (31)$$

Thus, it becomes clear that $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{p} = 0)$ is a feasible solution to (FVSD′) and both objectives yields equal values. Also, due to Theorem 4.1, $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{p} = 0)$ and $(\bar{\varrho}, \bar{\delta}, \bar{l}, \bar{q} = 0)$ represents globally optimal solution to (FVSP′) and (FVSD′), respectively. □

Theorem 7. *(Converse Duality). Let $(\bar{\sigma}, \bar{\vartheta}, \bar{m}, \bar{q})$ be a local optimal solution of $(FVSP^{'})$. Under the suitable conditions*

(i) $\nabla_{\varrho} H(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_p X(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$,
$\nabla_{\varrho} G(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_p Y(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$
$\nabla_{\varrho'} H(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p'} X(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$,
$\nabla_{\varrho'} G(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p'} Y(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$
$\nabla_{\varrho''} H(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p''} X(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$,
$\nabla_{\varrho''} G(t, \sigma, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p''} Y(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)\ldots$
$\nabla_{\varrho^{(2n)}} H(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p^{(2n)}} X(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$,
$\nabla_{\varrho^{(2n)}} G(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0) = \nabla_{p^{(2n)}} Z(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}, 0)$.

(ii) *the specified Hessian matrix $\nabla_{qq} X(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}}) - \bar{m}\nabla_{qq} Y(t, \bar{\sigma}, \dot{\bar{\sigma}}, \bar{\vartheta}, \dot{\bar{\vartheta}})$ is positive or negative definite,*

(iii) $(\nabla_{\varrho} f - D\nabla_{\varrho'} f + \nabla_q X) - \bar{m}(\nabla_{\varrho} g - D\nabla_{\varrho'} g + \nabla_q Y) \neq 0$,

(iv) *for chosen $\bar{q} \in \mathbb{R}^n$,*

$$\bar{q}^T[(\nabla_{\varrho} f - D\nabla_{\varrho'} f + \nabla_q X) - \bar{m}(\nabla_{\varrho} g - D\nabla_{\varrho'} g + \nabla_q Y)] = 0$$

implies $\bar{q} = 0$ and

(*v*) *suppose*

$$D\Big((\nabla_{\delta'}f+\nabla_{p'}X)-\bar{m}(\nabla_{\delta'}g+\nabla_{p'}Y)\Big)+D^2\Big((\nabla_{\delta''}f+\nabla_{p''}X)-\bar{m}(\nabla_{\delta''}g+\nabla_{p''}Y)\Big)$$
$$-D^3\Big((\nabla_{\delta'''}f+\nabla_{p'''}X)-\bar{m}(\nabla_{\delta'''}g+\nabla_{p'''}Y)\Big)+\dots$$
$$+D^{2n}\Big((\nabla_{\delta^{(2n)}}f+\nabla_{p^{(2n)}}X)-\bar{m}(\nabla_{\delta^{(2n)}}g+\nabla_{p^{(2n)}}Y)\Big)=0.$$

$(\bar{\sigma},\bar{\vartheta},\bar{m},\bar{p}=0)$ *becomes solution feasible to* (FVSD′) *and both objectives yield equal output. If, in addition to the above, conditions mentioned in Theorem 4.1 are contented every feasible solutions to considered problems* (FVSP′) *and* (FVSD′)*, then* $(\bar{\sigma},\bar{\vartheta},\bar{m},\bar{q}=0)$ *and* $(\bar{\sigma},\bar{\vartheta},\bar{m},\bar{p}=0)$ *becomes absolute optima to the considered* (FVSP′) *and* (FVSD′)*, respectively.*

5. STATIC FORMULATION

If the time dependency of the problems (FVSP) and (FVSD) are waived off, then our problems transform into the following form:

Primal Problem (SFVSP)

$$\text{Minimize}\qquad \frac{(f(\varrho,\delta)+H(\varrho,\delta,p)-p^T\nabla_pH(\varrho,\delta,p))}{(g(\varrho,\delta)+G(\varrho,\delta,p)-p^T\nabla_pG(\varrho,\delta,p))}$$

subject to

$$\nabla_\delta f(\varrho,\delta)-D\nabla_{\dot{\delta}}f(\varrho,\delta)+\nabla_pH(\varrho,\delta,p)-\left(\frac{f(\varrho,\delta)+H(\varrho,\delta,p)-p^T\nabla_pH(\varrho,\delta,p)}{g(\varrho,\delta)+G(\varrho,\delta,p)-p^T\nabla_pG(\varrho,\delta,p)}\right)$$
$$(\nabla_\delta g(\varrho,\delta)-D\nabla_{\dot{\delta}}g(\varrho,\delta)+\nabla_pG(\varrho,\delta,p))\in C_2^*,$$

$$\delta^T[\nabla_\delta f(\varrho,\delta)-D\nabla_{\dot{\delta}}f(\varrho,\delta)+\nabla_pH(\varrho,\delta,p)-\left(\frac{f(\varrho,\delta)+H(\varrho,\delta,p)-p^T\nabla_pH(\varrho,\delta,p)}{g(\varrho,\delta)+G(\varrho,\delta,p)-p^T\nabla_pG(\varrho,\delta,p)}\right)$$
$$(\nabla_\delta g(\varrho,\delta)-D\nabla_{\dot{\delta}}g(\varrho,\delta)+\nabla_pG(\varrho,\delta,p))]\geqq 0,$$
$$\varrho(t)\in C_1.$$

Dual Problem (SFVSD)

$$\text{Maximize}\qquad \frac{(f(\sigma,\vartheta)+Y(\sigma,\vartheta,q)-q^T\nabla_qY(\sigma,\vartheta,q))}{(g(\sigma,\vartheta)+Z(\sigma,\vartheta,q)-q^T\nabla_qZ(\sigma,\vartheta,q))}$$

subject to

$$-[\nabla_\varrho f(\sigma,\vartheta)-D\nabla_{\dot{\varrho}}f(\sigma,\vartheta)+\nabla_qX(\sigma,\vartheta,q)-\left(\frac{f(\sigma,\vartheta)+Y(\sigma,\vartheta,q)-q^T\nabla_qX(\sigma,\vartheta,q)}{g(\sigma,\vartheta)+Z(\sigma,\vartheta,q)-q^T\nabla_qY(\sigma,\vartheta,q)}\right)$$

$$(\nabla_{\varrho} g(\sigma,\vartheta) - D\nabla_{\dot{\varrho}} g(\sigma,\vartheta) + \nabla_q Y(\sigma,\vartheta,q))] \in C_1^*,$$

$$\sigma^T[\nabla_{\varrho} f(\sigma,\vartheta) - D\nabla_{\dot{\varrho}} f(\sigma,\vartheta) + \nabla_q X(\sigma,\vartheta,q) - \left(\frac{f(\sigma,\vartheta) + Y(\sigma,\vartheta,q) - q^T\nabla_q X(\sigma,\vartheta,q)}{g(\sigma,\vartheta) + Z(\sigma,\vartheta,q) - q^T\nabla_q Y(\sigma,\vartheta,q)}\right)$$
$$(\nabla_{\varrho} g(\sigma,\vartheta) - D\nabla_{\dot{\varrho}} g(\sigma,\vartheta) + \nabla_q Y(\sigma,\vartheta,q))] \leqq 0,$$

$$\vartheta(t) \in C_2.$$

Equivalent formulations in the parametric form can be constructed as follows:

Primal Problem (SFVSP$'$)

Minimize $\quad l$

subject to

$$(f(\varrho,\delta) + H(\varrho,\delta,p) - p^T\nabla_p H(\varrho,\delta,p)) - l(g(\varrho,\delta) + G(\varrho,\delta,p)$$
$$- p^T\nabla_p G(\varrho,\delta,p)) = 0,$$

$$\nabla_{\delta} f(\varrho,\delta) - D\nabla_{\dot{\delta}} f(\varrho,\delta) + \nabla_p H(\varrho,\delta,p) - l(\nabla_{\delta} g(\varrho,\delta) - D\nabla_{\dot{\delta}} g(\varrho,\delta)$$
$$+ \nabla_p G(\varrho,\delta,p)) \in C_2^*,$$

$$\delta^T[\nabla_{\delta} f(\varrho,\delta) - D\nabla_{\dot{\delta}} f(\varrho,\delta) + \nabla_p H(\varrho,\delta,p) - l(\nabla_{\delta} g(\varrho,\delta) - D\nabla_{\dot{\delta}} g(\varrho,\delta)$$
$$+ \nabla_p G(\varrho,\delta,p))] \geqq 0,$$

$$\varrho(t) \in C_1.$$

Dual Problem (SFVSD$'$)

Maximize $\quad m$

subject to

$$(f(\sigma,\vartheta) + X(\sigma,\vartheta,q) - q^T\nabla_q X(\sigma,\vartheta,q)) - m(g(\sigma,\vartheta) + Y(\sigma,\vartheta,q)$$
$$- q^T\nabla_q Y(\sigma,\vartheta,q)) = 0,$$

$$-[\nabla_{\varrho} f(\sigma,\vartheta) - D\nabla_{\dot{\varrho}} f(\sigma,\vartheta) + \nabla_q X(\sigma,\vartheta,q) - m(\nabla_{\varrho} g(\sigma,\vartheta) - D\nabla_{\dot{\varrho}} g(\sigma,\vartheta)$$

$$+ \nabla_q Y(\sigma, \vartheta, q))] \in C_1^*,$$

$$\sigma^T[\nabla_\varrho f(\sigma, \vartheta) - D\nabla_{\dot{\varrho}} f(\sigma, \vartheta) + \nabla_q X(\sigma, \vartheta, q) - m(\nabla_\varrho g(\sigma, \vartheta) - D\nabla_{\dot{\varrho}} f(\sigma, \vartheta)$$

$$+ \nabla_q Y(\sigma, \vartheta, q))] \leqq 0,$$

$$\vartheta(t) \in C_2.$$

We can easily establish the weak and strong duality results. One can refer for details Jayswal and Prasad [8].

6. CONCLUSION

Our investigation in this article established higher order η-invexity as a tool under which we can derive weak and strong duality for the parametrized higher order variational symmetric duals considered over more general settings of cone constraints. Well-suited duality results for higher order variational symmetric duals can be easily interpreted with the results established in this paper. The future work can be grounded by extending the present work to corresponding multiobjective variants and also to the problems when objective functions subsumed support functions making it nondifferentiable.

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