OPTIMAL REPLENISHMENT POLICY FOR ITEMS WITH THREE-PARAMETER WEIBULL DETERIORATION, STOCK-LEVEL DEPENDENT DEMAND AND PARTIAL BACKLOGGING

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Abstract: Our focus in this paper is on determining an optimal replenishment policy for items with three-parameter Weibull distribution deterioration and inventory-level dependent demand. We developed and analyzed the model under the assumption of partial backlogging. The three-parameter Weibull hazard rate captures the impact of already deteriorated items that are received into the inventory, as well as those items that may start deteriorating in future. The inventory-level demand reflects a real market demand for product whose sales is enhanced by stock on display. We study a case base examples to gain some quantitative insight into the proposed model and we perform sensitivity analysis to draw some managerial implications.

Keywords: Inventory Model, Stock-Dependent Demand, Differential Equation, Optimal Policy, Hazard function.

MSC: 90B05, 34K06, 91B74.

1. INTRODUCTION

We consider the problem of finding a replenishment policy for inventory system of deteriorating items with stock dependent demand. The main target

will be to determine how much quantity of an item should be stocked and when so that customers' orders can be reasonably and promptly met from stock until next replenishment. These decisions are largely influenced by the fact that the on-hand inventory depletes due to the combined effects of deterioration and demand.

Most practical inventory items deteriorate over time and the Weibull hazard rate offers excellent approximation of the actual instantaneous rate of decay of inventory items. Cooray[5] suggested that when modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shape. Many researchers of inventory models have long identified the importance of modelling decay of inventory systems with the use of Weibull hazard function. Covert and Philip [6] developed an inventory model for items with two-parameter Weibull deterioration and constant demand rate. Philip [14] reconsidered the inventory system of [6] and replaced the distribution of the time to deterioration with three-parameter Weibull distribution. Chakrabarty et al [2] further developed the work of [14] by taking linear demand rate to capture the rate of withdrawal of items from inventory. Samanta and Bhowmick[16] proposed an inventory model with two parameter Weibull distribution decay rate and allowed shortages in the inventory. They studied two cases; where the inventory starts with shortages and the other case where the system starts without shortages.

It is evident from the foregoing that demand structure occupies central position in inventory modelling. Different demand patterns have been considered in literature-constant, time-dependent, stock-dependent, price-dependent demand pattern, etc. Thus, researchers have usually used varying demand patterns to reflect sales in different phases of product life cycle. These demand structures tend to approximate the actual rate of withdrawal of items from the inventory. Another factor of an inventory system is that demand, when an item is out of stock, can be either lost completely, fully backordered or partially backordered. In practice, partial backlogging occurs more often. An excellent review of research efforts with different demand patterns is given by [12].

In real life, it is often observed that large amount of stock on display has motivational effect on customers' demand for many products. Levin et al [11] suggested that large piles of consumer goods displayed in a supermarket would attract customers to buy more items. This phenomenon has impelled researchers to investigate the effects of stock-dependent demand structures on inventory models. In particular, Baker and Urban [1] examined a power form stock-dependent demand on inventory model with no deterioration and assumed zero ending inventory. They derived the expressions for the optimal cycle length and replenishment quantity of the system. Detta and Pal [7] developed an inventory model for items with ramp type power function of inventory level and assumed that the replenishment ends with zero inventory. They also assumed that the withdrawal rate of an item from inventory is dependent on the stock displayed until a certain level of inventory is attained, after which demand rate becomes asymptotically constant. Giri and Chaudhuri[9] proposed an economic order quantity model for

items with a power form inventory-level dependent demand rate and with constant rate of decay. They derived optimal decision variables of the system under nonlinear holding cost and zero ending inventory. Detta and Paul[8] presented an inventory system for items with stock-dependent, price-sensitive demand rate and derived the expression for the optimal order quantity and optimal cycle length. Wu et al [20] developed a replenishment policy for non-instantaneous decaying items with stock-demand rate under partial backlogging. It is assumed that constant fraction of the on-hand inventory deteriorates per unit time after a fixed time, t_d. Their work is focused on the necessary and sufficient conditions of existence and uniqueness of the optimal replenishment solution for the considered inventory system. Chang et al[4] considered the problem of determining the optimal selling price and optimal order quantity for inventory system of deteriorating items. They assumed that the demand rate depends not only on the on-display stock level but also on the selling price per unit, as well as the limited shelf space.

Along the same lines, Chang [3] developed a finite planning inventory model for non-instantaneous deteriorating items with linear stock-dependent consumption rate under inflation. It was assumed that the decay rate is constant and that unfill demand is partially backlogged according to exponential function. Recently, Roa and Muleneh[15] proposed a production inventory model for three parameter Weibull decaying items with stock dependent production rate. They assumed that the rate of withdrawal of an item from inventory is a function of the selling price and time and they derived the optimal quantities under the simplifying modelling assumption that shortages are completely backlogged. Quite recently, Sarker and Sarker [19] presented an improved inventory model for items whose time to deterioration follows Rayleigh distribution with the scale parameter, $\sigma^2 = 1/\theta$ and in which the demand rate is a function of stock. They derived model under partial backlogging and obtained the optimal quantities at the point where various system costs are minimized. Khedlekar and Shukla[10] studied a dynamic pricing model with a logarithmic price sensitive demand structure that evolves over time. Sanni and Chukwu [18] presented an inventory system with quadratic demand rate and three parameter Weibull decay function. The system allows erratic demand patterns and backordering. Mishra[13] studied an inventory with two-parameter weibull decay rate where the demand rate is trapezoidal-type function of time and the production rate largely depends on the demand rate. Their models are developed and analysed under two inventory modelling conditions:(1) model without shortages and (2) model with shortages that are fully backlogged.

The present paper addresses the key issues in stock . None of the references cited above study inventory system with three-parameter Weibull decay rate and linear stock -dependent rate of withdrawal of items under partial backlogging. Here, we attempt to formulate and solve a real-life inventory system for deteriorating items in which the demand rate is taken to be stock dependent demand during the time interval $[0, t_1]$. Deterioration of items begins after a certain time from the moment of their arrival in stock. The model is derived under condition of partial

backlogging. The research focus is to establish the analytic frame of the assumed inventory system, provide an optimal replenishment policy for the model and perform sensitivity study to gain useful insight into how the optimal inventory policy for the proposed model respond to changes of the model parameters. Thus, this work brings together, for the first time, the realistic inventory features; three-parameter Weibull hazard rate of deterioration, linear stock-dependent demand rate and partial backlogging rate.

2. MATHEMATICAL MODEL and ANALYSIS

The mathematical model in this work is developed on the basis of the following assumptions and notation.

Notation

C_1: inventory holding cost per unit per unit time.
C_2: shortage cost per unit per unit time.
C_3: ordering cost per order.
C_4: unit cost.
C_5: unit cost of lost sales.
$I(t)$: inventory level at any time, t.
$D(t)$: demand rate at any time $t \geq 0$.
T: cycle length.
Q: size of the initial inventory.
S: quantity not affected by deterioration.
δ: backlogging parameter.
$\theta(t) = \alpha\beta(t - \gamma)^{\beta-1}$: instantaneous rate function for a three-parameter Weibull distribution; where $\alpha > 0$ is the scale parameter, β is the shape parameter, $\beta > 0$ and γ is the location parameter, $\gamma > 0$. Also, t is time to deterioration.
t_1: time during which inventory level is zero.
T^*: optimal value of T.
Q^*: optimal value of Q.
t_1^*: optimal value of t_1.
I^* : optimal order quantity.

Assumptions

(i) The inventory system under consideration deals with a single item.
(ii) The time horizon is infinite and a typical planning schedule of cycle of length T is considered.

(iii) The demand rate is a function of stock , i.e. $D(t) = \begin{cases} a + bI(t), & I(t) > 0 \\ a, & I(t) \leq 0 \end{cases}$

where $a > 0$ stands for the initial demand rate and $b > 0$ is a coefficient of stock influence on sales.

(iv) Shortages in the inventory are allowed and partially backlogged so that at inventory level zero some arrived demand is permitted. We assume that backlog-

ging rate depends on the length of the waiting time for the next replenishment. The proportion of customers who would like to accept backorder at time, t decreases with the waiting time, $(T - t)$ before the next replenishment. Thus, we define the backlogging rate as $B(t) = \frac{1}{1+\delta(T-t)}$.

(v) Replenishment is instantaneous and lead time is zero.

(vi) Deteriorated unit is not repaired or replaced during a given cycle.

(vii) The holding cost, ordering cost, shortage cost and unit cost remain constant over time.

(viii) The characteristic rate of decay, α, lies in the interval, $0 < \alpha << 1$; this seems reasonable because of advancement in modern storage facility.

(ix) The distribution of the time to deterioration of the items follows three-parameter Weibull distribution, i.e $f(t) = \alpha\beta(t - \gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}, t > 0$. The instantaneous rate function is $\theta(t) = \alpha\beta(t - \gamma)^{\beta-1}$.

We consider an inventory system that starts with Q units of item and in which the rate of withdrawal of items from the system is affected by the amount of stock on display. During the time interval, $[0, \gamma]$, inventory depletes due to demand only and in the interval, $[\gamma, t_1]$, the inventory depletes due mainly to demand and partly to deterioration. At t_1, the inventory level is zero and all the demand hereafter (i.e. $T - t_1$) is partially backlogged. The total number of backordered items is replaced by the next replenishment. The deterioration time of items follows three-parameter Weibull distribution. The graphical representation of the described inventory system is given in Figure 1.
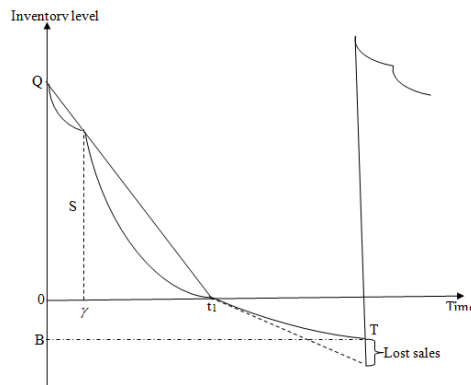

Figure 1: Simplified inventory system for deterioration items

The time evolution of the inventory system is described by the differential equations:

$$\frac{dI(t)}{dt} = -\big(a + bI(t)\big), \ \ 0 \le t \le \gamma \tag{1}$$

$$\frac{dI(t)}{dt} + \alpha\beta(t - \gamma)^{\beta-1}I(t) = -\big(a + bI(t)\big), \ \ \gamma \le t \le t_1 \tag{2}$$

$$\frac{dI(t)}{dt} = -\frac{a}{1 + \delta(T - t)}, \ \ t_1 \le t \le T \tag{3}$$

with boundary conditions $I(0) = Q$, $I(\gamma) = S$, $I(t_1) = 0$ and $I(T) = 0$. Note that $I(T) = 0$ is a terminal condition: at the end of each cycle, the negative amount (i.e. backordered of $-B$) is brought to zero by the positive amount B when an order is received.

The solution of the differential equations with conditions $I(0) = Q$ and $I(t_1) = 0$ is;

$$I(t) = \begin{cases} \left[-\frac{a}{b}\big(\exp\{bt\} - 1\big) + Q\right]\exp\{-bt\}, \ \ 0 \le t \le \gamma \\[2mm] \left[-a \int_t^{t_1} \exp\{(bh + \alpha(h - \gamma)^\beta)\} \, dh\right]\exp\{-\big(bt + \alpha(t - \gamma)^\beta\big)\}, \ \ \gamma \le t \le t_1 \\[2mm] \frac{a}{\delta}\left[\ln[1 + \delta(T - t)] - \ln[1 + \delta(T - t_1)]\right], \ \ t_1 \le t \le T. \end{cases} \tag{4}$$

Using $I(\gamma) = S$ in eqn(4) for the $0 \le t \le \gamma$ interval case and for the equation in the interval $\gamma \le t \le t_1$, we obtain;

$$S \ = \ \left(-\frac{a}{b}\big(\exp\{b\gamma\} - 1\big) + Q\right)\exp\{-b\gamma\} \tag{5}$$

$$S \ = \ \left(-a \int_\gamma^{t_1} \exp\{(bh + \alpha(h - \gamma)^\beta)\} \, dh\right)\exp\{-b\gamma\}. \tag{6}$$

Hence, the order size function becomes

$$Q = a\left[\frac{1}{b}\big(\exp\{b\gamma\} - 1\big) - \int_\gamma^{t_1} \exp\{(bh + \alpha(h - \gamma)^\beta)\} \, dh\right]. \tag{7}$$

The above result in (7) sufficiently yields the lot size for the inventory system but a convenient approximation may be needed to characterise the inventory process.

Tight Approximation

Lemma 2.1. *for* $0 < \alpha \le 1$ *and* $|b(y - x)| \le 1$,

$$\int_x^y \exp\{(bh + \alpha(h - \gamma)^\beta)\} \, dh \approx \left[(y-x) + \frac{b}{2}(y^2 - x^2) + \frac{\alpha\big((y - \gamma)^{\beta+1} - (x - \gamma)^{\beta+1}\big)}{\beta + 1}\right]. \tag{8}$$

The Lemma is obtained by taking the first order series approximation of the exponential term on the integrand and thus integrate to arrive at the required result.

Evoking Lemma 2.1 in eqn(4) for the $\gamma \leq t \leq t_1$ interval case, we get the simplifying result;

$$I(t) \approx a\left[(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{\alpha\left((t_1 - \gamma)^{\beta+1} - (t - \gamma)^{\beta+1}\right)}{\beta + 1}\right]\exp\{-\left(bt + \alpha(t - \gamma)^{\beta}\right)\}.$$

Putting the condition $I(\gamma) = S$ in eqn(4) for the $0 \leq t \leq \gamma$ interval case and for the equation in the interval $\gamma \leq t \leq t_1$, we get;

$$S = \left[-\frac{a}{b}\left(\exp\{b\gamma\} - 1\right) + Q\right]\exp\{-b\gamma\} \tag{9}$$

$$S = a\left[(t_1 - \gamma) + \frac{b}{2}(t_1^2 - \gamma^2) + \frac{\alpha(t_1 - \gamma)^{\beta+1}}{\beta + 1}\right]\exp\{-b\gamma\} \tag{10}$$

Eliminate S by equating eqn(9) and eqn(10), we obtained the initial inventory level for the system,

$$Q \approx a\left[(t_1 + \frac{b}{2}t_1^2) - \gamma(\gamma + 1) + \frac{\alpha(t_1 - \gamma)^{\beta+1}}{\beta + 1} + \frac{1}{b}\left(\exp\{b\gamma\} - 1\right)\right]. \tag{11}$$

Applying the condition $I(T) = 0$ in eqn(4) for the equation in the interval $t_1 \leq t \leq T$, we obtained the maximum amount of demand backlogged per cycle as;

$$B = -I(T) = \frac{a}{\delta}\ln[1 + \delta(T - t_1)]. \tag{12}$$

Remark: The bound $|bT| \leq 1$ as implied in Lemma 2.1 limits the cycle length and initial stock level displayed in inventory without leaving a negative impression on customers. Excessively Large amount of deteriorating goods stock for a very long time may create negative impression on customers.
The total demand during period $[0, t_1]$ is

$$D = \int_0^{t_1} D(t)\,\mathrm{d}t = aT - b\int_0^{t_1} I(t)\,\mathrm{d}t = (a + \frac{b}{2}Q)t_1 \tag{13}$$

Note that we have used the approximation $\int_0^{t_1} I(t)\,\mathrm{d}t = \frac{1}{2}Qt_1$ in eqn(13) (see [2] for similar treatment).
The deterioration cost per cycle is

$$DC = C_4\left[Q - \int_{\gamma}^{t_1}(a + bI(t_1))\,\mathrm{d}t\right] = C_4\left[Q - a(t_1 - \gamma)\right]. \tag{14}$$

The total inventory holding cost is

$$HC = C_1 \Big[\int_0^{t_1} I(t)\, dt \Big] \approx \frac{1}{2} C_1 Q t_1. \tag{15}$$

We approximate the inventory depletion curve with a straight line to obtain the holding cost per cycle (see Figure 1). This approximation is adopted for analytical convenience. A similar approach is used by [14],[2] and [17]. Alternatively, the inventory holding cost can be approximated using

$$HC = C_1 \Big[\int_0^{\gamma} I(t)\, dt + \int_{\gamma}^{t_1} I(t)\, dt \Big] \tag{16}$$

See Appendix A for approximate solution of (16). We use (15) because it reduces computational time and it gives about the same result as (16).
The shortage cost per cycle is

$$SC = C_2 \Big[\int_{t_1}^T -I(t)\, dt \Big] = \frac{dC_2}{\delta} \Big[(T - t_1) - \frac{1}{\delta} \ln[1 + \delta(T - t_1)] \Big]. \tag{17}$$

The opportunity cost due to lost sales per cycle is

$$PC = C_5 \Big[\int_{t_1}^T a[1 - \frac{1}{1 + \delta(T - t)}]\, dt \Big] = aC_5 \Big[(T - t_1) - \frac{1}{\delta} \ln[1 + \delta(T - t_1)] \Big]. \tag{18}$$

The total inventory system cost per cycle length cost consists of inventory holding cost, deteriorations cost per cycle length, ordering cost per cycle, shortage cost per cycle length and opportunity cost due to lost sales per unit time. That is;

$$\varphi(T) = \frac{1}{T} \Big[HC + DC + OC + SC + PC \Big]. \tag{19}$$

Substituting the various cost components into eqn(19), we get

$$C(T, t_1) = \frac{1}{T} \Big[\Big(C_4 + \frac{1}{2} C_1 t_1 \Big) Q + C_3 - aC_4(t_1 - \gamma) + \frac{a(C_2 + \delta C_5)}{\delta} \Big((T - t_1) - \frac{1}{\delta} \ln[1 + \delta(T - t_1)] \Big) \Big]. \tag{20}$$

To get the optimal quantities T^* and t_1^*, the inventory problem can be formulated as a non-linear optimisation problem as follows:

min$\{C(T, t_1)\}$ subject to $t_1 > 0, T > t_1.$

The above optimisation problem can be implemented via Mathematical software to seek the exact global minimum of the inventory cost function in eqn(20).
Having obtained T^* and t_1, we substitute t_1 into eqn(7) to get the optimal initial inventory level for the considered system.
The optimal order quantity for the system is:

$$I^* = Q^* + \frac{a}{\delta} \ln[1 + \delta(T^* - t_1^*)]. \tag{21}$$

Thus, the inventory policy is as follows:
Order I^* units for every T^* time units and use B^* units to settle backorder with a corresponding inventory cost of $C(T^*, t_1^*)$.

3. NUMERICAL EXAMPLES

Example 3.1. *SanFruit, a major distributor of guava, is interested in using an optimal inventory policy. Over the last several weeks, management has estimated the demand trend as*

$$D(t) = \begin{cases} 25 + 0.3I(t), & I(t) > 0 \\ 25, & I(t) \leq 0 \end{cases}$$

Other inventory data are:
δ = 2.5, C_1 = \$0.32 per kg of guava/week, C_2 = \$1.75 per kg of guava/week, C_3 = \$14/order, C_4 = \$6.84 per kg and C_5 = \$9.88 per kg. The time-to-deterioration of guava is known to follow Weibull distribution with decay parameters α = 0.02, β = 12 and γ = 0.6.
Suggest optimal inventory policy for SanFruit.
The optimal solutions, got by the proposed model, are found to be:
T^* = 1.5450 weeks, t_1^* = 1.3620 weeks, S^* = 14.2045 kg of guava, Q^* = 33.4406 kg of guava, B^* = 3.7674 kg of guava, I^* = 37.2080 kg of guava and $C(T^*, t_1^*)$ = \$83.0275. The solutions satisfy Lemma 2.1 since $0 < \alpha = 0.02 < 1$ and $|b(t_1^* - \gamma)| < |bT^*| = 0.4635 < 1$.

Example 3.2. *Consider data taken from Example 1 of Wu et al[20] :*
C_1 = \$0.5, C_2 = \$2.5, C_3 = \$250, C_4 = \$1.5, C_5 = \$2, a = 600, b = 0.1, δ = 2, α = 0.08 and γ = 1/12. In addition, let the rate of change of decay per unit time, β = 1.
Solving the above inventory problem, by using proposed model, we have the following results:
T^* = 1.2170 unit time, t_1^* = 1.0379 unit time, S^* = 617.615 units, Q^* = 672.992 units, B^* = 91.818 units, I^* = 764.81 units and $C(T^*, t_1^*)$ = \$514.132. The solutions satisfy Lemma 2.1 since $0 < \alpha = 0.08 < 1$ and $|b(t_1^* - \gamma)| < |bT^*| = 0.1217 < 1$. Further,

the holding cost over the period is $HC = \frac{1}{2}C_1 Q t_1 = 143.496$ or $HC = \frac{C_1}{b^2}\left[(Qb + a)\left(1 - \right.\right.$

$\left.\left. \exp\{-b\gamma\}\right) - ab\gamma + \frac{b^2}{2}(t_1 - \gamma)S\right] = 143.208$.

To show the convex structure of the cost function, a 2D plot of the inventory system cost curve is given in Figure 2. We plot only the portion of the curve that encloses the minimum point.
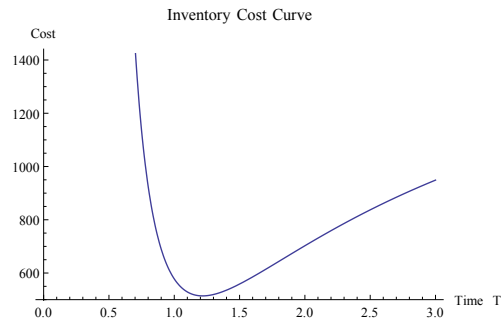
Figure 2: Inventory System Cost Curve for Example 3.2

This result compares well with Wu et al [20]. It is found that the optimal order quantity increases by 2.12%, the optimal cycle length increases with about 4.13%, and there is roughly 23.82% increase in the total inventory cost per unit time. The significant percent increase in the inventory total cost per cycle is brought about by β (i.e. a rough measure of the slope of deterioration over time) in the proposed model. Thus, ignoring the effect of β in inventory model development would lead to under-assessment of the total inventory cost as evident in [20].

Example 3.3. *BreadSmile Bakery is a local distributor of breads. Its inventory manager is reassessing the inventory policy for one popular brand; Whole White Honey Bread(WWHB). The manager suspects that sales are greatly enhanced by the quantity of bread on display and has guessed the daily withdrawal rate of WWHB as:*

$$D(t) = \begin{cases} 3000 + bI(t), & I(t) > 0 \\ 3000, & I(t) \le 0 \end{cases}$$

with $\delta = 0.25$ and she/he has also observed that stock reaches zero level at every $t_1 = 2/3$ day.
Other relevant inventory data are:
$C_1 = \$0.002$ /bread/day, $C_2 = 4.25$, $C_3 = \$100$/order, $C_4 = \$3.22$ /bread and $C_5 = \$3.89$. The time-to-deterioration of bread is known to follow Weibull distribution with decay parameters $\alpha = 0.35$, $\beta = 4$ and $\gamma = 0.045$.
What would the optimal inventory policy be?
The optimal solutions, got by using the proposed model, are found to be:
$b^* = 2.3310 \times 10^{-8}$, $T^* = 0.7207$ day, $S = 1878.42$ breads, $B = 160.951$ breads, $Q^* = 2013.42$ breads, and $C(T^*) = \$835.208$.

Next, we perform sensitivity analysis to determine how the optimal quantities respond to changes in the model parameters using Example 3.1. We generate Table 1 that shows what the optimal quantities would be if the model parameters were increased or decreased by the following percentages:-50%, -25%, 25%, 50%.

Table 1: Sensitivity analysis of the inventory model parameters

Parameter	% Change	% Change in			
		T^*	Q^*	B^*	$C(T^*, t_1^*)$
C_1	+50	-3.34	- 6.17	+3.34	+2.74
	+25	-1.73	-3.19	+1.70	+1.39
	-25	+1.84	+3.41	-1.75	-1.45
	-50	+3.79	+7.04	-3.56	-2.95
C_2	+50	-0.34	+0.28	-3.61	+0.21
	+25	-0.17	+1.43	-1.84	+0.11
	-25	+0.18	-0.15	+1.91	-0.11
	-50	+0.37	-0.30	+3.89	-0.23
C_3	+50	+5.15	+6.91	+6.54	+5.32
	+25	+2.63	+3.53	+3.29	+2.69
	-25	-2.73	-3.66	-3.34	-2.77
	-50	-5.56	-7.45	-6.73	-5.61
C_4	+50	+4.65	-4.91	+49.17	+36.96
	+25	+2.06	-2.46	+23.93	+18.84
	-25	Infeasible	Infeasible	Infeasible	Infeasible
	-50	Infeasible	Infeasible	Infeasible	Infeasible
C_5	+50	Infeasible	Infeasible	Infeasible	Infeasible
	+25	-1.88	+1.61	-20.93	+1.22
	-25	+3.73	-2.82	+35.94	-2.13
	-50	+14.21	-8.89	+111.63	-6.69
a	+50	-3.66	+42.63	+43.28	+44.44
	+25	-2.18	+21.35	+21.66	+22.24
	-25	+3.48	-21.49	-21.72	-22.32
	-50	+1.86	-43.41	-43.53	-44.80
b	+50	-9.08	-11.45	+27.53	+21.54
	+25	-5.41	-6.44	+13.94	+11.18
	-25	+7.32	+7.95	-14.46	-12.23
	-50	+13.87	+12.81	-29.47	-25.66
δ	+50	-2.14	+2.21	-30.09	+1.68
	+25	Infeasible	Infeasible	Infeasible	Infeasible
	-25	+1.91	-1.98	+27.51	-1.4968
	-50	+5.19	-5.39	+76.20	-4.07
α	+50	-0.06	-0.10	0.00	0.00
	+25	-0.03	-0.05	0.00	0.00
	-25	+0.03	+0.05	0.00	0.00
	-50	+0.06	+0.10	0.00	0.00
β	+50	+0.10	+0.17	0.00	0.00
	+25	+0.07	+0.11	0.00	0.00
	-25	-0.16	-0.24	+0.01	+0.01
	-50	-0.47	-0.75	+0.06	+0.05
γ	+50	-10.43	-41.54	-14.22	-12.03
	+25	-2.48	-15.74	-3.93	-3.26
	-25	-2.94	+5.62	-2.23	-1.85
	-50	-11.29	+1.36	-8.55	-9.28

Note that "Infeasible" in Table 1 indicates solution with imaginary root. Interpreting solutions with imaginary roots is unrealistic in the sense of cost analysis. The following insights are gained from results in Table1,

- As the ordering cost, C_3, increases, the optimal cycle length, T^*, the optimal order quantity, Q^*, optimal back-order quantity , B^*, and the optimal total inventory cost, $C(T^*)$, are significantly increased and vice-versa. The managerial implication is as follows: SanFruit needs to order more quantity if the ordering cost is high so as to reduce the order frequency and cost.

- T^* and Q^* decrease while $C(T^*)$ increases with increase in the unit purchase cost, C_4. Also T^* and Q^* increase while $C(T^*)$ deceases with decrease in the unit purchase cost, C_4, which thus, obey the law of low cost high volume.

- T^* and Q^* decrease with increase in the holding cost, C_1 while the optimal inventory cost, $C(T^*)$ increases slightly with increase in C_1. Further, T^* and Q^* increase with decrease in the holding cost, C_1 while the optimal inventory cost, $C(T^*)$ decreases moderately with decrease in C_1. The economic interpretation is as follows: when holding cost decreases, Management of SanFruit needs to order more quantity and stock more quantity for a longer time so as take advantage of low inventory holding cost and implicitly reducing the order frequency and cost. When C_1 increases, it is reasonable that management shorten the cycle length and the optimal order quantity as an adjustment strategy to keep inventory cost as low as possible.

- As the rate of withdrawal of item a increases, Q^* and $C(T^*)$ increase (larger quantity) but the optimal cycle length, T^*, decreases (more frequent order) and vice versa. An obvious implication of this is that when the demand rate is decreasing, SanFruit should order less quantity and broaden the cycle length an intervention measure to keep inventory cost resealable.

- Results indicate that higher value of the stock-dependent parameter, b leads to higher value for the optimal inventory cost while it will result in lower values for the optimal order quantity and optimal cycle length. The economic interpretation is as follows: As b increases, management of SanFruit store needs to shorten the cycle length to restrict b to a reasonable degree so as to prevent excessive display of item for unduely long duration which might create negative impression.

- The optimal solutions are highly sensitive to changes in the backlogging parameter, δ. Our computations indicate that some changes in δ could result in infeasible solution. The implication is thus; management of SanFruit should exercise great care in estimating the backlogging parameter, δ, as wrong estimation of δ could lead to unattainable solution.

- The optimal solutions are highly insensitive to change in characteristic rate of decay, α.

- The optimal solutions are slightly sensitive to changes in the rate of decay per unit time, β. Our computations indicate that the total inventory cost increases with decrease in β. The implication is thus; management of SanFruit should exercise great care in estimating the rate of change of deterioration of item, β, as wrong estimation of β could lead to under/over assessment of the total inventory cost per unit time.

- As the time at which inventory starts deteriorating, γ, increases, the optimal cycle length and the optimal order quantity slightly increase while the optimal total inventory cost decreases and vice versa. The economic interpretation is as follows: when γ has higher value, management needs to order some quantity more and slightly extend the cycle length so as to take advantage of reduced deterioration cost.

To sum up, it is observed in Table 1 that the optimal values, T^*, Q^*, and B^* are insensitive to change in the inventory cost components while the solutions are highly sensitive to change in δ, and γ. Furthermore, the solutions, Q^* and $C(T^*, t_1^*)$ are slightly sensitive to changes in α.

4. CONCLUSION

We have proposed an inventory model for items with stock dependent demand rate, in which the deterioration time of items follows three parameter Weibull distribution. The model is developed and analysed under the assumption that shortages are allowed and are partially backlogged. The paper discusses the optimal replenishment policy regarding the quantity of items and the time to stock them in order to minimise inventory cost per cycle. The current model differs from similar ones in considering: (1) the usage of three-parameter Weibull to model deterioration time, (2) linear-stock dependent rate of withdrawal and (3) allowing partial backlogging. These simultaneous considerations make the model more flexible, mimicking a realistic situation. The model considered here is suitable for items whose inventory level is decoratively displayed to attract customers and thus to increase sales. The three-parameter Weibull instantaneous rate incorporated into the model captures the impact of the already deteriorated items that are received into the inventory as well as those items that may start deteriorating in future. We study a case based examples to gain some quantitative insight into the proposed model and we conduct sensitivity analysis to draw some managerial implications.

The results from the sensitive analysis indicate the possibility of strategically adjusting the model parameters to take advantage of further inventory cost reduction. For example, management of SanFruit can explore the fact that a reduction of holding cost in the interval $-50 \leq C_2 \leq -25$ may bring about substantial reduction in the total inventory cost with slight change in the optimal replenishment quantity. In addition, 50% reduction in the characteristic rate of deterioration α would bring about 2.34% reduction in the total inventory cost. Thus, the proposed

model yield optimal replenishment policy at point where different system costs are minimized, and the sensitivity study captures the implications of drifting away from the present optimal solutions.

In terms of future work, the order size can be assumed to contain imperfect quality items and a stochastic consideration could be developed. Furthermore, trade credit financing structure could be incorporated into the model to encourage sales if the retail is facing declining demand.

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Appendix A. Derivation of the approximate total holding cost

$$
\begin{aligned}
HC &= C_1 \Big[\int_0^\gamma I(t)\, \mathrm{d}t + \int_\gamma^{t_1} I(t)\, \mathrm{d}t \Big] \\
&= C_1 \Big[(\frac{Q}{b} + \frac{a}{b^2})(1 - \exp\{-b\gamma\}) - \frac{a\gamma}{b} + \int_\gamma^{t_1} I(t)\, \mathrm{d}t \Big].
\end{aligned}
$$

For analytical convenience, we approximate the inventory depletion curve in the interval $[\gamma, t_1]$ with a straight line. That is

$$
\int_\gamma^{t_1} I(t)\, \mathrm{d}t \approx \frac{1}{2}(t_1 - \gamma)S.
$$

Thus, the total average holding cost for the system is approximately

$$
HC = \frac{C_1}{b^2}\Big[(Qb + a)\big(1 - \exp\{-b\gamma\}\big) - ab\gamma + \frac{b^2}{2}(t_1 - \gamma)S \Big].
$$

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