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EXPECTED UTILITY FOR PROBABILISTIC PROSPECTS AND THE COMMON RATIO PROPERTY

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Abstract: We prove the existence of an expected utility function for preferences over probabilistic prospects satisfying Strict Monotonicity, Indifference, the Common Ratio Property, Substitution and Reducibility of Extreme Prospects. The example in [1] that is inconsistent with the existence of a von Neumann-Morgenstern for preferences over probabilistic prospects, violates the Common Ratio Property. Subsequently, we prove the existence of expected utility functions with piecewise linear Bernoulli utility functions for preferences that are piece-wise linear. For this case a weaker version of the Indifference Assumption that is used in the earlier existence theorems is sufficient. We also state analogous results for probabilistic lotteries. We do not require any compound prospects or mixture spaces to prove any of our results. In the third last section of this paper, we "argue" that the observations related to Allais paradox, do not constitute a violation of expected utility maximization by individuals, but is a likely manifestation of individuals assigning (experiment or menu-dependent?) subjective probabilities to events which disagree with their objective probabilities.

Keywords: Probabilistic prospects, preferences, expected utility, common ratio property, piece-wise linear, Allais' paradox, subjective probability, uncertainty aversion

MSC: 91A30, 91B06, 91B08.

1. INTRODUCTION

Consider an individual with initial monetary wealth $w > 0$ and the maximum amount of money that can be gained by the individual is $M > 0$. Hence the maximum the individual can lose is w and at any point after participating in an uncertain prospect, the individual's wealth is an amount in the closed interval $[0, M + w]$. Thus, the individual can gain any amount $x \in [-w, M]$. Clearly, 0 belongs to the open interval $(-w, M)$.

Uncertain prospects are of two kinds. In one kind, the gains are state dependent allowing for the assessment of (subjective) probabilities of the states. There is another kind of uncertain prospect, where the probabilities of the gains are exogenously specified. In reality, an uncertain prospect of the second kind is also a state dependent uncertain prospect for which the assessment of probabilities has already been completed. A formal and rigorous discussion of the underlying calculus of subjective probability can be found in [2].

Allowing for a slight variation of the framework considered in [1], we define a **binary probabilistic prospect**, as a pair $(x, p) \in ([-w, 0) \cup (0, M]) \times (0,1]$ which denotes that the individual gains a non-zero amount of money x with probability p and with probability $1 - p$ the individual gains zero, (i.e., neither gains nor loses any money) after participating in an uncertain prospect. Negative gains denote losses.

Rubinstein in [1] uses a simplified version of the example used to demonstrate Allais Paradox to show that it violates the Independence Assumption- the latter being a sufficient condition for the existence of a von Neumann-Morgenstern utility function for preferences over probabilistic prospects- and, is inconsistent with the existence of any such utility function. In section 7 we argue that the paradoxical behavior can be explained in terms of an expected utility function for preferences over "state-dependent" prospects if we use subjective probabilities. In any case, the definition of the Independence Assumption that is referred to in [1], explicitly uses the concept of compound lotteries, that is not defined anywhere in the paper.

In this paper, we consider finite arrays of binary probabilistic prospects which we refer to as "probabilistic prospects". Instead of using Independence we use an assumption, that we call "Common Ratio Property". We show that the example in [1] that we referred to earlier violates the common ratio property. The Common Ratio Property originates in Allais [3] under the name "Common Ratio Effect" and is also used by Kahneman and Tversky [4] in the context of the example that is shared by Rubinstein in [1]. It is referred to and briefly explained in [5]. A thorough discussion of the Common Ratio Effect and its implications in choice, pricing as well as on happiness is available in [6]. As mentioned in [5] and [6], almost all non-expected utility theories, including that of Kahneman and Tversky [4], that explain the so-called Allais Paradox (which we dispute in Section 7) satisfy the Common Ratio Property. Our understanding of this property (which is repeated after its formal statement is, the following: Given an array of returns from uncertain prospects, what matters in choosing between them are not the absolute value of risks measured probabilistically, but their relative values. Such an assumption is possible and verifiable, because we define probabilistic prospects to allow for status quo with positive probability.

Our first major result in this note proves that along with the Common Ratio Property, four other assumptions that we refer to as Strong Monotonicity, Indifference, Substitution and Reducibility of Extreme Prospects are together sufficient for the existence of an expected utility function for preferences over probabilistic prospects. All assumptions, except for Substitution, are defined in terms of binary probabilistic prospects. In fact, it is easy to see that the "common ratio property"- which is defined in terms of binary probabilistic prospects - is also a necessary condition for the existence of an expected utility function for preferences over probabilistic prospects.

In a subsequent section we prove the existence of "expected utility functions with piecewise linear Bernoulli utility functions" for "preferences that are piece-wise linear". For this case a weaker version of the Indifference Assumption that is used in the earlier existence theorems is sufficient. Section 13.4 in Eeckhoudt et al. [7] is concerned with such Bernoulli utility functions having two pieces. However, the second and very realistic example (considerably more realistic than the St. Petersburg Paradox) about a hypothetical individual by the name of Sempronius in section 1.1 of [7], shows that although two pieces are sufficient for explaining "loss aversion" more than two pieces are required for explaining preference for a more diversified portfolio of assets over a less diversified one. Before moving on to a discussion of the Allais Paradox, we examine the special and very popular case of probabilistic prospects that allow for gains but no losses. We refer to such probabilistic prospects as probabilistic lotteries. It is easily seen that fewer and a considerably weaker set of assumptions are sufficient for the representation of preferences over probabilistic lotteries by expected utility. We do not require any compound prospects or mixture spaces to prove any of our results presented here.

Section 7 is one of several justifications for our attempt here to provide an easily accessible axiomatic characterization of expected utility maximization in choosing from a finite set of probabilistic prospects, the latter being the backbone of financial decision making, game theory and decision analysis. A small sample of the wide applicability of expected utility maximization can be found in [8]. Further, scientific theories are meant to be robust and context specific-not infallible. There could be genuine reasons-such as lack of information- which prevents exact knowledge of monetary gains and losses arising out of choices. This could be a valid reason for bounded rationality, but not one that justifies an elaborate research agenda on non-expected utility theories of monetary gains and losses. Decision making in the absence of exact information about monetary gains and losses, is the subject matter of a different research endeavor altogether.

It is worth noting that every intertemporal discounted utility maximization problem may be viewed as an expected utility maximization problem, with the discount factor representing the probability of being rewarded in the immediately next period. The most general investigation of intertemporal decision making- including the possibility of "hyperbolic discounting", with a non-empty finite set of available alternatives and in a two-period model, can be found in [9]. In the paper by Cruz-Rambaud and Sánchez Pérez [10] it has been pointed out that the worth of a (binary) probabilistic prospect may depend on when the gains and losses are incurred, i.e. if (x, p, t) denotes a binary prospect in which x is the gain (or loss) that will be realized with probability p, 't' periods later, then then the preference between (x, p, t) and (y, q, s) may be different from the one between (y, q, t) and (x, p, s). In this work we assume that all potential risky gains and losses will be available at the same time in future. Our analysis here does not have any explicit intertemporal dimension. That however does not mean, that the utility functions based on which expected utility comparisons over probabilistic lotteries take place, are not amenable to exponential or hyperbolic discounting. More, generally the utility function itself could be explicitly indexed by time, i.e., the utility function for gains (or losses) that will be available 't' periods later. Intertemporal applications about uncertain future consumptions using non-expected utility can be found in [11].

In several respects, our approach and results are significantly different from the huge literature on the existence of expected utility functions that has been produced till date and is comprehensively surveyed in [12]. In the first place our approach is completely combinatorial, made possible by a simplifying assumption which we refer to as Indifference, rather than assuming "Continuity" which is a topological concept and implies Indifference along with the other assumptions after some routine manipulations. If we agree to refer to the body of mathematics using real or rational numbers that does not depend on anything beyond the ordered field property of the real or rational number system as "basic algebra", then the methodology used here is precisely that-and no more. In particular, nothing that we use requires appealing to the least upper bound property (completeness axiom) of the real number system. This allows our results to be accessible to a much larger audience than is currently the case. In any case, a proof of representability of preferences over alternatives that are uncertain monetary outcomes, by an expected utility function beginning with Continuity, would have to proceed by proving Indifference. Since the objective of this paper is to prove the possibility of the desired representation by an expected utility function using a concept that allows us to ignore mixtures of probabilistic prospects, and not that of providing technical refinements, we choose to simplify the underlying mathematics without compromising on rigor.

Secondly, the conventional framework in which representability by an expected utility function is discussed – such as in Appendix A of Chapter 1 in [5] – is based on lotteries or risky outcomes in the amount of individual wealth. However, as pointed out succinctly in section 13.4 of [7], individuals treat gains and losses differently, manifested in loss aversion. Thus, instead of considering uncertainty of wealth, we consider uncertainty of gains (a loss being a "negative gain"). Regardless of whether or not the possibility of representability of preferences over alternatives representing uncertain gains by expected utility of gains follows from the existing results about representability of preferences over alternatives representing uncertain wealth by expected utility of wealth, we are not aware of any explicit theorem about the former. Further, even if there is any such result that we are unaware of, there certainly aren't any that uses our assumptions, particularly the common ratio property.

Thirdly, the existing theorems concerning representation of preferences over alternatives representing uncertain monetary outcomes by expected utility either use an assumption called Independence, which would require the preferences to be defined on the much larger set of finite mixtures of probabilistic prospects (including degenerate ones), as in [5], or invoke an assumption that permits mixtures of alternatives representing uncertain monetary outcomes, to be identical-and not merely indifferent-to the alternative representing uncertain monetary outcomes, with the latter probabilities being obtained by applying the probability multiplication rule, as in Chapter 8 of [13]. This is known in the literature as "reduction of compound lotteries" (ROCL) axiom. However as noted in thought experiments reported in Ellsberg [14], laboratory experiments reported in [15] and [16], which are all cited in [17], there is a strong correlation between ambiguity aversion and "violation of ROCL", with the intuitive and observed validity of the former implying credibility of the latter. The question that may be asked at this juncture is the following: Since probabilistic prospects imply the non-existence of ambiguity, ROCL should not be problematic in our context and hence what is the problem with invoking ROCL? Our answer to this question, would be along the following lines. The desirability of a property should be intrinsic, and not by default. Consider a "kleptomaniac" whose habit is to steal items sold in supermarkets. Suppose this person relocates to a town or city where there are no supermarkets and goods are sold in grocery stores and similar stores. Clearly, in such a town, the kleptomaniac does not get to reveal his dubious propensities. However, that does not in any way indicate that he has been cured of his moral or psychological disorder. He continues to remain a kleptomaniac regardless of whether he gets the opportunity to give way to his temptations or not. Otherwise, it would be like saying that "a thief ceases to be a thief (or becomes innocent), once arrested by the police and put into prison", which is obviously a very unlikely statement. Hence, not having assumed ROCL explicitly and not including mixtures of probabilistic prospects in our domain for preferences, the only way ROCL could be implicit or for that matter conceivable in our framework is if it was implied by the assumptions, we invoke, which is certainly not the case.

Fourthly, the assumptions and results that we provide in this paper are intended and meant to be applicable only in the context of alternatives representing uncertain monetary gains (and losses). We do not consider the most abstract setting where outcomes could be objects other than money, even if such an extension is possible. With outcomes being other than monetary gains, the field is wide open for theories and decision support systems other than ones using expected utility. That there is ample scope for decision making under uncertainty which deviates from expected utility maximization is documented in [18] and several references therein. We neither intend nor expect the results in this paper to fall in the category of "one size fits all" theories, even if theorizing of this sort is conceptually possible in the social sciences.

The section preceding the concluding section of this paper is a modest tribute to Howard Raiffa, whose seminal work "Decision Analysis: Introductory Lectures on Choices under Uncertainty", is the place where decision analysis started "happening". No amount of axiomatic analysis can be a substitute for the fact that expected utility maximization is applicable and profusely applied for decision making when a choice has to be made between alternatives each of which represent uncertain gains. Raiffa [19] is an authoritative account of such applications that are meaningful, possible and additionally goes on to show how it can be done. Without Raiffa's intellectual legacy, our work and similar others, would be an exercise in futility. Fortunately, with the huge literature on decision analysis based on and inspired by Raiffa [19], that is not the case.

2. THE MATHEMATICAL MODEL

A **probabilistic prospect** is a finite array $L = <(x_1, p_1), \ldots, (x_n, p_n)>$ for some positive integer 'n', where for each $j \in \{1, \ldots, n\}, (x_j, p_j)$ is a "binary probabilistic prospect" and $\sum_{j=1}^{n} p_j$ belongs to the left-open, right-closed interval $(0,1]$. The interpretation of a probabilistic prospect is that at most one of the 'n' binary probabilistic prospects that it comprises of can yield a non-zero gain (loss if the gain is negative) and the probability of getting neither a gain nor a loss, i.e., zero gain, is $1 - \sum_{j=1}^{n} p_j$. In other words, the events associated with non-zero gains/losses and zero gains, are mutually exclusive and exhaustive.

Let $\mathcal{L}$ denote the set of all probabilistic prospects and let $\mathcal{L}^1$ denote the strict subset of all "binary probabilistic prospects".

The following property of probability prospects is implied by the definition of a finitely additive probability measure.

Additivity: *If for* $<(x_1, p_1), \ldots, (x_n, p_n)> \in \mathcal{L}$ *it is the case that for some* $j, k \in \{1, \ldots, n\}$ *with* $j < k$, $x_j = x_k = \alpha$, *then* $<(x_1, p_1), \ldots, (x_n, p_n)> \sim <(y_1, q_1), \ldots, (y_{n-1}, q_{n-1})>$ *where* $(x_h, p_h) = (y_h, q_h)$ *for all* $h \in \{1, \ldots, n\} \backslash \{j\}$ *with* $h < k$, $(y_j, q_j) = (\alpha, p_j + p_k)$ *and* $(y_h, q_h) = (x_{h+1}, p_{h+1})$ *for all* $h \in \{1, \ldots, n\}, h \geq k$.

Additivity is not an assumption, but a property that is satisfied by the set of probabilistic prospects $\mathcal{L}$.

In what follows we require the possibility of **zero gains with probability one** denoted $<(M, 0)>$.

For ease of exposition and when there will be no cause for confusion, we will denote $<(x, p)> \in \mathcal{L}^1 \cup \{<(M, 0)>\}$ by (x, p).

We assume that there is a binary relation $\succcurlyeq$ on $\mathcal{L} \cup \{(M, 0)\}$ such that for all L_1, $L_2 \in \mathcal{L} \cup \{(M, 0)\}$: $L_1 \succcurlyeq L_2$ is interpreted as L_1 is worth at least as much (satisfaction) as L_2. The symmetric part of $\succcurlyeq$ denoted $\sim$ is interpreted as "is worth exactly as much as" and its asymmetric part denoted $\succ$ is interpreted as "is worth more than".

***Assumption concerning** $\succcurlyeq$ **on** $\mathcal{L} \cup \{(M, 0)\}$**:** $\succcurlyeq$ on $\mathcal{L} \cup \{(M, 0)\}$ (i.e., the binary relation "worth paying at least as much as" on $\mathcal{L} \cup \{(M, 0)\}$) is a preference relation i.e., $\succcurlyeq$ is reflexive, complete (or connected or total) and transitive binary relation on $\mathcal{L} \cup \{(M, 0)\}$.*

The symmetric part of $\succcurlyeq$ denoted $\sim$ is interpreted as "is worth exactly as much as" and its asymmetric part denoted $\succ$ is interpreted as "is worth more than".

*An **expected utility function** for $\succcurlyeq$ is a function $W: \mathcal{L} \to \mathbb{R}$ such that there exists a function $u: [-w, M] \to \mathbb{R}$ for which the following are satisfied:*

(i) $u(0) = 0$.

(ii) For all $L = \langle (x_1, p_1), \ldots, (x_n, p_n) \rangle \in \mathcal{L}$, $W(L) = \sum_{j=1}^{n} p_j u(x_j)$.

(iii) For all $L_1, L_2 \in \mathcal{L}$: $L_1 \succcurlyeq L_2$ <u>if and only if</u> $W(L_1) \geq W(L_2)$.

*The function u is called a **Bernoulli utility function** for $\succcurlyeq$.*

3. ASSUMPTIONS ABOUT PREFERENCES ON $\mathcal{L} \cup \{(M, 0)\}$ AND A PRELIMINARY LEMMA

In what follows we invoke the following assumptions for the preference relation $\succcurlyeq$ on $\mathcal{L} \cup \{(M, 0)\}$.

Assumption 1 (Strong Monotonicity): *For all $(x, p), (y, q) \in \mathcal{L}^1$ with $x \geq y$ and $p \geq q$: (i) $[x > y, p \geq q]$ implies $[(x, p) \succ (y, q)]$; (ii) $[x = y > 0,\ p > q]$ implies $[(x, p) \succ (y, q)]$; (iii) $[x = y < 0, p > q]$ implies $[(y, q) \succ (x, p)]$.*

Assumption 1 summarizes the basic and intuitively plausible idea that "gains are preferred to losses".

Assumption 2 (Indifference): *For all $x \in [-w, 0) \cup (0, M]$: (a) if $x > 0$ then there exists $r \in (0,1]$ such that $(x, 1) \sim (M, r)$; (b) if $x < 0$ then there exists $r \in (0,1]$ such that $(x, 1) \sim (-w, r)$.*

Assumption 2 is of a psychological nature. It says that the security associated with a modest sure gain is overall equivalent to the ecstasy experienced with a sudden and unexpectedly high windfall profits. The nagging insecurity associated with a modest sure loss is overall equivalent to unexpected bankruptcy.

After all, some do make a living by taking risks while some prefer a stable income and both have reasons to feel happy and sad.

Assumption 3 (Common Ratio Property): *For all $(x, p), (y, q) \in \mathcal{L}^1$ and $t \in (0,1]$: $[(x, p) \succcurlyeq (y, q)]$ implies $[(x, tp) \succcurlyeq (y, tq)]$.*

Assumption 3 says that given an array of returns from uncertain prospects, what matters in choosing between them are not the absolute value of risks measured probabilistically, but their relative values. Such an assumption is possible and makes sense, because we define probabilistic prospects to allow for status quo with positive probability.

Note 1: *It is easy to see that Assumption 2, i.e., Indifference, is a consequence of Assumption 1 (Strong Monotonicity) and this very weak version of continuity.*

Weak Continuity: *For all* $(x,p)\in\mathcal{L}^1$: *(a)* $(M,1)\succ(x,p)$ *with* $x>0$ *implies that there exists* $\alpha\in[0,1]$ *such that* $(M,\alpha)\sim(x,p)$*; and (b)* $(x,p)\succ(-w,1)$ *with* $x<0$ *implies that there exists* $\alpha\in[0,1]$ *such that* $(-w,\alpha)\sim(x,p)$.

Note 2: *The "common ratio property" in [5] says that preferences are independent of changes in the magnitude of the probabilities so long as monetary gains and the "ratio" of probabilities remain unchanged.*

Note 3: *As a consequence of Assumption 1, it follows that given any* $L\in\mathcal{L}$ *if* $\{x\in[-w,M]\mid L\sim(x,1)\}\neq\phi$*, then it must be a singleton whose unique member* c_L *is called the certainty equivalent of L.*

Lemma 1: *If* $\succcurlyeq$ *satisfies Assumptions 1, 2 and 3, then there exists a strictly increasing function* $v:[-w,M]\to\mathbb{R}$ *satisfying* $v(-w)=-1$ *and* $v(M)=1$*, such that for all* $(x,p),(y,q)\in\mathcal{L}^1$*:* $[(x,p)\succcurlyeq(y,q)]$ *if and only if* $[pv(x)\geq qv(y)]$*.*

Proof*: By Assumptions 1(ii) and 2 for all* $x\in([-w,0)\cup(0,M])$*, there exists a unique real number* $v(x)$ *such that: (a) if* $x>0$*, then* $v(x)>0$, $(x,1)\sim(M,v(x))$ *and* $v(M)=1$*; (b) if* $x<0$*, then* $v(x)<0,(x,1)\sim(-w,-v(x)),v(-w)=-1$*. Let* $v(0)=0$.

For all $p,q\in(0,1]$ with $p>q$, we have by Assumption 1(ii), $(M,p)\succ(M,q)$ and by Assumption 1(iii), $(-w,q)\succ(-w,p)$ for all $p\in(0,1]$.

For all $p,q\in(0,1]$, we have by Assumption 1(i), $(M,p)\succ(-w,q)$.

Clearly v is strictly increasing, since:

(a) If $x>y>0$, then $(x,1)\sim(M,v(x))$ and $(y,1)\sim(M,v(y))$ with $v(x),v(y)>0$ and by Assumption 1(i) $(x,1)\succ(y,1)$, so that $(M,v(x))\succ(M,v(y))$ from which and by Assumption 1(ii), $v(x)>v(y)$.

(b) If $0>x>y>0$, then $(x,1)\sim(-w,-v(x))$ and $(y,1)\sim(-w,-v(y))$ with $v(x),v(y)<0$ and by Assumption 1(i) $(x,1)\succ(y,1)$, so that $(-w,-v(x))\succ(-w,-v(y))$ from which and by Assumption 1(iii), $-v(x)<-v(y)$, i.e. $v(x)>v(y)$.

(c) If $x>0>y$, then by the definition of v, $v(x)>0>v(y)$.

Consider the function $v:[-w,M]\to\mathbb{R}$ defined above.

Thus $v(x)>0$ for all $x\in(0,M],v(x)<0$ for all $x\in[-w,0)$ and $v(0)=0$.

By Assumption 3, for all $p\in(0,1]$ and $x\in(0,M]$: $(x,p)\succcurlyeq(M,pv(x))$ as well as $(M,pv(x))\succcurlyeq(x,p)$; and for all $p\in(0,1]$ and $x\in[-w,0)$: $(x,p)\succcurlyeq(-w,-pv(x))$ as well as $\big(-w,-pv(x)\big)\succcurlyeq(x,p)$.

Thus, for all $p\in(0,1]$ and $x\in(0,M]$: $(x,p)\sim(M,pv(x))$; <u>and</u> for all $p\in(0,1]$ and $x\in[-w,0)$: $(x,p)\sim(-w,-pv(x))$.

Hence, for all $(x,p),(y,q)\in\mathcal{L}^1$:

(a) If $x,y\in(0,M],(x,p)\succcurlyeq(y,q)$ if and only if $(M,pv(x))\succcurlyeq(M,qv(y))$.

(b) If $x\in(0,M]$ and $y\in(0,-w]$ then by Assumption 1(i) $(x,p)\succ(y,q)$ and $pv(x)>0>qv(y)$.

(c) If $x,y\in(0,-w],(x,p)\succcurlyeq(y,q)$ if and only if $(-w,-pv(x))\succcurlyeq(-w,-qv(y))$.

Thus, for all $(x,p),(y,q)\in\mathcal{L}^1$:

(a1) By Assumption 1(iii), if $x,y\in(0,M]$, $(x,p)\succcurlyeq(y,q)$ if and only if $pv(x)\geq qv(y)$.

(b1) By Assumption 1 (i), if $x\in(0,M]$ and $y\in(0,-w],(x,p)\succ(y,q)$ and $pv(x)>0>qv(y)$.

(c1) By Assumption 1 (iv) If $x,y\in(0,-w],(x,p)\succcurlyeq(y,q)$ if and only if $-pv(x)\leq-qv(y)$.

Clearly, $(x,p) \succcurlyeq (y,q)$ if and only if $pv(x) \geq qv(y)$. Q.E.D.

Note 4: *The version of the paradox in [1] violates the requirement (that follows from Assumption 3) which says that: if $(x,1) \succ (y,\frac{4}{5})$, then multiplying the probabilities by $\frac{1}{4}$, we should get $(x,\frac{1}{4}) \succcurlyeq (y,\frac{1}{5})$.*

Further, note that for the main existence result in this section, Assumption 3 holds on $\mathcal{L}$ as well, i.e., for all $<(x_1,p_1),\dots,(x_n,p_n)>$, $<(y_1,q_1),\dots,(y_m,q_m)> \in \mathcal{L}$: if $<(x_1,p_1),\dots,(x_n,p_n)> \succcurlyeq <(y_1,q_1),\dots,(y_m,q_m)>$ and $t\in(0,1]$, then $<(x_1,tp_1),\dots,(x_n,tp_n)> \succcurlyeq <(y_1,tq_1),\dots,(y_m,tq_m)>$. However, we will not be requiring Assumption 3 in its full generality; its use restricted to $\mathcal{L}^1$, as has already been done in Lemma 1, is enough for our purpose.

We will require the following additional assumptions for the main result in this section.

Assumption 4 (Substitution): *If $<(x_1,p_1),\dots,(x_n,p_n)>$, $<(y_1,q_1),\dots,(y_n,q_n)> \in \mathcal{L}$, $<(z,r)> \in \mathcal{L}^1$, $<(z,r)> \sim <(x_j,p_j)>$ for some $j\in\{1,\dots,n\}$ and $r \leq p_j$, then $<(x_1,p_1),\dots,(x_n,p_n)> \sim <(y_1,q_1),\dots,(y_n,q_n)>$ where $(x_k,p_k)=(y_k,q_k)$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ and $(y_j,q_j) = (z,r)$.*

Since preferences are defined over prospects – binary or otherwise – in order to avoid misunderstanding, in the statement of Substitution we have purposely written: "… $<(z,r)> \in \mathcal{L}^1$, $<(z,r)> \sim <(x_j,p_j)> \cdots$", even though we have agreed to identify and denote binary prospects such $<(z,r)>$ by (z,r). Substitution does not require a probabilistic prospect to be a component of another probabilistic prospect. It just requires extracting (z,r) from $<(z,r)>$ and replacing (x_j,p_j) by (z,r).

Note that instead of $r \leq p_j$ we could have used the less restrictive criteria $\sum_{k=1}^{n} p_k - p_j + r\in(0,1]$ in Assumption 4 and continue our analysis without hindrance.

Let us agree to refer to probabilistic prospects of the form $<(M,p),(-w,q)>$ as **extreme prospects**.

Assumption 5 (Reducibility of Extreme Prospects): *(i) For all $p\in(0,1]$: $(M,p) \succ (M,0)$ and $(M,0) \succ (-w,p)$. (ii) There exists $\alpha,\beta > 0$, satisfying the following condition: For all $p,q\in(0,1]$:*

(a) $<(M,p),(-w,q)> \sim (M,p-\frac{\beta}{\alpha}q)$ if $\alpha p - \beta q \geq 0$;

(b) $<(M,p),(-w,q)> \sim (-w,q-\frac{\alpha}{\beta}p)$ if $\alpha p - \beta q < 0$.

4. THE MAIN RESULT

We now present the main result of this paper.

Proposition 1: *If $\succcurlyeq$ on $\mathcal{L} \cup \{(M,0)\}$ satisfies Assumptions 1 to 5, then there exists an expected utility function for $\succcurlyeq$.*

Proof: *By Lemma 1 we know that there exists a function v: $[-w,M]\to[-1,1]$ satisfying the following properties:*

(1) $v(M)=1, v(0)=0, v(-w)=-1, v(x)>0$ for all $x\in[M,0)$ and $v(x)<0$ for all $x\in[-w,0)$.
(2) For all $(x,p)\in \mathcal{L}^1$: (i) $x>0$ implies $(x,p)\sim(M,pv(x))$; (ii) $x<0$ implies $(x,p)\sim(-w,-pv(x))$.
(3) For all $(x,p),(y,q)\in \mathcal{L}^1$: $(x,p) \succcurlyeq (y,q)$ if and only if $pv(x) \geq qv(y)$.

Let $\alpha, \beta > 0$ be the given real numbers in Assumption 5. Let $u: [-w, M] \to \mathbb{R}$ be defined as $u(x) = \alpha v(x)$ for all $x \in [0, M]$ and $u(x) = \beta v(x)$ for all $x \in [-w, 0)$.

Thus:

(1*) $u(M) = \alpha, u(0) = 0, u(-w) = -\beta, u(x) > 0$ for all $x \in [M, 0)$ and $u(x) < 0$ for all $x \in [-w, 0)$.

(2*) For all $(x,p),(y,q) \in \mathcal{L}^1$: $(x,p) \succcurlyeq (y,q)$ if and only if $pu(x) \geq qu(y)$.

Let $< (x_1, p_1), \ldots, (x_n, p_n) > \in \mathcal{L}$.

Note that it is not possible for both $\sum_{\{j|x_j \geq 0\}} p_j v(x_j)$ as well as $\sum_{\{j|x_j < 0\}} p_j v(x_j)$ to be equal to zero.

It is easy to see that $\sum_{j=1}^{n} p_j u(x_j) = \alpha \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \beta \sum_{\{j|x_j < 0\}} p_j v(x_j)$.

By repeated application of Assumption 4 and (2) above we get that $< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (y_1, p_1 v(x_1)), \ldots, (y_n, p_n v(x_n)) >$ where for all $y_j = M$ if $x_j > 0$ and $y_j = -w$ if $x_j < 0$.

Thus:

If $\sum_{\{j|x_j < 0\}} p_j v(x_j) = 0$, then $< (x_1, p_1), \ldots, (x_n, p_n) > \sim (M, \sum_{\{j|x_j \geq 0\}} p_j v(x_j)) = (M, \sum_{j=1}^{n} p_j v(x_j))$.

If $\sum_{\{j|x_j \geq 0\}} p_j v(x_j) = 0$, then $< (x_1, p_1), \ldots, (x_n, p_n) > \sim (-w, -\sum_{\{j|x_j < 0\}} p_j v(x_j)) = (-w, -\sum_{j=1}^{n} p_j v(x_j))$.

If $\sum_{\{j|x_j \geq 0\}} p_j v(x_j) \neq 0$ and $\sum_{\{j|x_j < 0\}} p_j v(x_j) \neq 0$, then

$< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (M, \sum_{\{j|x_j \geq 0\}} p_j v(x_j)), (-w, -\sum_{\{j|x_j < 0\}} p_j v(x_j)) >$.

By Assumptions 5 (ii), if $\sum_{\{j|x_j \geq 0\}} p_j v(x_j) \neq 0$ and $\sum_{\{j|x_j < 0\}} p_j v(x_j) \neq 0$, then:

(i) $< (x_1, p_1), \ldots, (x_n, p_n) > \sim <(M, \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \frac{\beta}{\alpha} \sum_{\{j|x_j < 0\}} p_j v(x_j)>$ if $\alpha \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \beta \sum_{\{j|x_j < 0\}} p_j v(x_j) \geq 0$;

(ii) $< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (-w, \sum_{\{j|x_j < 0\}} p_j v(x_j) - \frac{\alpha}{\beta} \sum_{\{j|x_j \geq 0\}} p_j v(x_j)) >$ if $\alpha \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \beta \sum_{\{j|x_j < 0\}} p_j v(x_j) < 0$.

The above can be written as follows.

If $\sum_{\{j|x_j \geq 0\}} p_j v(x_j) \neq 0$ and $\sum_{\{j|x_j < 0\}} p_j v(x_j) \neq 0$, then:

(i) $< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (M, \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \frac{\beta}{\alpha} \sum_{\{j|x_j < 0\}} p_j v(x_j) >$ if $\sum_{j=1}^{n} p_j u(x_j) \geq 0$;

(ii) $< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (-w, \sum_{\{j|x_j < 0\}} p_j v(x_j) - \frac{\alpha}{\beta} \sum_{\{j|x_j \geq 0\}} p_j v(x_j))>$ if $\sum_{j=1}^{n} p_j u(x_j) < 0$

Recall that for $< (x_1, p_1), \ldots, (x_n, p_n) > \in \mathcal{L}$, $\sum_{j=1}^{n} p_j u(x_j) = \alpha \sum_{\{j|x_j \geq 0\}} p_j v(x_j) - \beta \sum_{\{j|x_j < 0\}} p_j v(x_j)$.

Thus, given $< (x_1, p_1), \ldots, (x_n, p_n) >, < (y_1, q_1), \ldots, (y_m, q_m) > \in \mathcal{L}$:

(a) It follows from Lemma 1, that: if $\sum_{j=1}^{n} p_j v(x_j) \neq 0$ and $\sum_{j=1}^{m} q_j v(y_j) \neq 0$ then: $< (x_1, p_1), \ldots, (x_n, p_n) > \succcurlyeq < (y_1, q_1), \ldots, (y_m, q_m) >$ if and only if $\sum_{j=1}^{n} p_j u(x_j) \geq \sum_{j=1}^{m} q_j u(y_j)$.

(b) It follows from Lemma 1 and Assumption 5 (i), that if $\sum_{j=1}^{n} p_j v(x_j) \neq 0$ and $\sum_{j=1}^{m} q_j v(y_j) = 0$ then: (i) $< (x_1, p_1), \ldots, (x_n, p_n) > \succ < (y_1, q_1), \ldots, (y_m, q_m) >$ <u>if</u> $\sum_{j=1}^{n} p_j u(x_j) > 0 = \sum_{j=1}^{m} q_j u(y_j)$; and (ii) $< (y_1, q_1), \ldots, (y_m, q_m) > \succ < (x_1, p_1), \ldots, (x_n, p_n) >$ <u>if</u> $\sum_{j=1}^{n} p_j u(x_j) < 0 = \sum_{j=1}^{m} q_j u(y_j)$.

(c) It follows from reflexivity and transitivity of $\succcurlyeq$ (i), that if $\sum_{j=1}^{n} p_j v(x_j) = 0 = \sum_{j=1}^{m} q_j v(y_j)$, then: $< (x_1, p_1), \ldots, (x_n, p_n) > \sim (M, 0) \sim (M, 0) \sim < (y_1, q_1), \ldots, (y_m, q_m) >$ and hence $< (x_1, p_1), \ldots, (x_n, p_n) > \sim < (y_1, q_1), \ldots, (y_m, q_m) >$.

For $L \in \mathcal{L}$, let $W(L) = \sum_{j=1}^{n} p_j u(x_j)$ if $L = < (x_1, p_1), \ldots, (x_n, p_n) >$.

Thus for $< (x_1, p_1), \ldots, (x_n, p_n) >, < (y_1, q_1), \ldots, (y_m, q_m) > \in \mathcal{L}$,

$< (x_1, p_1), \ldots, (x_n, p_n) > \succcurlyeq < (y_1, q_1), \ldots, (y_m, q_m) >$

if and only if $W(< (x_1, p_1), \ldots, (x_n, p_n) >) \geq W(< (y_1, q_1), \ldots, (y_m, q_m) >) = 0$. Q.E.D.

5. PIECEWISE LINEAR PREFERENCES

Of particular interest in decision theory (see section 13.4 of [7]) are preference relations on $\mathcal{L} \cup \{(M,0)\}$ satisfying the following "piece-wise linearity" property.

A preference relation $\succcurlyeq$ on $\mathcal{L} \cup \{(M, 0)\}$ is said to be **piece-wise linear** if for some positive integer 'n' with $n \geq 2$, there exists a non-empty finite set of real numbers $\{x_j | j = 0, 1, \ldots, n\}$ including '0' and satisfying $-w = x_0 < x_1 < \cdots < x_n = M$, such that for all $p \in (0,1)$ and $j \in \{0, \ldots, n-1\}$: (a) If $x_j \neq 0 \neq x_{j+1}$ then $(px_j + (1-p)x_{j+1}, 1) \sim < (x_j, p), (x_{j+1}, 1-p) >$; (b) If $x_j = 0 \neq x_{j+1}$ then $(px_j + (1-p)x_{j+1}, 1) \sim (x_{j+1}, 1-p)$; and (c) If $x_j \neq 0 = x_{j+1}$ then $(px_j + (1-p)x_{j+1}, 1) \sim (x_j, p)$.

In this case $\succcurlyeq$ is said to be a **piece-wise linear preference relation** on $\mathcal{L} \cup \{(M, 0)\}$ **generated by** $\{x_j | j = 0, 1, \ldots, n\}$.

In what follows, we denote the set $\{x_j | j = 0, 1, \ldots, n\}$ by X.

A piece-wise linear preference relation $\succcurlyeq$ on $\mathcal{L} \cup \{(M, 0)\}$ generated by X is said to satisfy:

Assumption 2' (Indifference): *For all $x \in X$: (a) if $x > 0$ then there exists $r \in (0,1]$ such that $(x, 1) \sim (M, r)$; (b) if $x < 0$ then there exists $r \in (0,1]$ such that $(x, 1) \sim (-w, r)$.*

Assumption 2' is considerably weaker than Assumption 2.

The proof of the following lemma is almost exactly the same as the proof of Lemma 1.

Lemma 2: *If $\succcurlyeq$ is a piece-wise linear preference relation on $\mathcal{L} \cup \{(M,0)\}$ generated by X that satisfies Assumptions 1, 2' and 3, then there exists a set of 'n' distinct real numbers $\{u(x_j) | j = 0, 1, \ldots, n\}$ satisfying $u(x_{j+1}) > u(x_j)$ for $j \in \{0, \ldots, n-1\}$ with $u(-w) = -1, u(0) = 0$ and $u(M) = 1$, such that for all $(x, p) \in (([-w, 0) \cup (0, M]) \cap X) \times (0,1]$: (a) $(x, p) \sim (M, pu(x))$ if $x > 0$; (b) $(x, p) \sim (-w, -pu(x))$ if $x < 0$; and (c) for all $(y, q) \in (([-w, 0) \cup (0, M]) \cap X) \times (0,1], (x, p) \succcurlyeq ((y, q)$ if and only if $pu(x) \geq qu(y)$.*

An immediate corollary of Lemma 2 and Substitution (Assumption 4) is the following, whose proof includes very easy arguments analogous to the ones used in Proposition 1.

Corollary of Lemma 2: *If $\succcurlyeq$ is a piece-wise linear preference relation on $\mathcal{L} \cup \{(M, 0)\}$ generated by X that satisfies Assumptions 1, 2', 3, 4 and 5 then there exists a strictly increasing function $u: [-w, M] \to \mathbb{R}$ with $u(-w) = -1, u(0) = 0$ and $u(M) = 1$, such that for all $j \in \{0, \ldots, n-1\}, p \in [0,1], x = px_j + (1-p)x_{j+1}, x \neq 0$: $u(x) = pu(x_j) + (1-p)u(x_{j+1})$ and*

(a) If $0 < x_j < x_{j+1}$ and $p \in (0,1)$, then $(x, 1) \sim (M, pu(x_j) + (1-p)u(x_{j+1})) = (M, u(x))$;
(b) If $0 > x_j > x_{j+1}$ and $p \in (0,1)$, then $(x, 1) \sim (-w, -pu(x_j) - (1-p)u(x_{j+1})) = (-w, -u(x))$;
(c) If $0 > x_{j+1}$ and $px_j = 0$ then $(x, 1) = ((1-p)x_{j+1}, 1) \sim (x_{j+1}, 1-p) \sim (-w, -(1-p)u(x_{j+1})) = (-w, -u(x))$;

(d) If $0 > x_j$, and $(1-p)x_{j+1} = 0$ then $(x,1) = (px_j, 1) \sim (x_j, p) \sim (-w, -pu(x_j)) = (-w, -u(x))$;
(e) If $x_{j+1} > 0$ and $px_j = 0$, then $(x,1) = ((1-p)x_{j+1}, 1) \sim (x_{j+1}, 1-p) \sim (M, (1-p)u(x_{j+1})) = (M, u(x))$; and
(f) If $x_j > 0$ and $(1-p)x_{j+1} = 0$, then $(x,1) \sim (px_j, 1) \sim (x_j, p) \sim (M, pu(x_j)) = (M, u(x))$.

However, with the same assumptions, we can prove the following stronger result.

Lemma 3: *If $\succcurlyeq$ is a piece-wise linear preference relation on $\mathcal{L} \cup \{(M,0)\}$ generated by X that satisfies Assumptions 1, 2', 3, 4 and 5 then there exists a strictly increasing function $u: [-w, M] \to \mathbb{R}$ with $u(-w) = -1$, $u(0) = 0$ and $u(M) = 1$ satisfying the following properties:*

(i) For all $j \in \{0, \ldots, n-1\}$, $p \in [0,1]$, $x = px_j + (1-p)x_{j+1}$, $x \neq 0$: $u(x) = pu(x_j) + (1-p)u(x_{j+1})$.
(ii) For all $p \in (0,1]$ and $x \in (0, M]$: $(x,p) \sim (M, pu(x))$; and for all $p \in (0,1]$ and $x \in [-w, 0)$: $(x,p) \sim (-w, -pu(x))$.
(iii) For all $(x,p), (y,q) \in \mathcal{L}^1$: $(x,p) \succcurlyeq (y,q)$ if and only if $pu(x) \geq qu(y)$.

Equipped with Lemma 3, we are in a position to prove the following proposition.

Proposition 2: *If $\succcurlyeq$ is a piece-wise linear preference relation on $\mathcal{L} \cup \{(M,0)\}$ generated by X that satisfies Assumptions 1, 2', 3, 4 and 5 then there exists an expected utility function for the preference relation, and the Bernoulli utility function u for any such expected utility function is piece-wise linear, i.e., for all $j \in \{0, \ldots, n\}$ and $p \in [0,1]$, if $x = px_j + (1-p)x_{j+1}$, then $u(x) = pu(x_j) + (1-p)u(x_{j+1})$.*

6. THE SPECIAL CASE OF PROBABILISTIC LOTTERIES

In this section we study the possibility of representing preference relations over an important special case of probabilistic prospects that do not allow for the possibility of losses. We call such prospects, probabilistic lotteries.

A **binary probabilistic lottery** is a pair $(x,p) \in (0, M]) \times (0,1]$ which denotes that the individual gains a positive amount of money x with probability p and with probability $1-p$ the individual gains zero, (i.e., neither gains nor loses any money) after participating in an uncertain prospect.

A **probabilistic lottery** is a finite array $L = < (x_1, p_1), \ldots, (x_n, p_n) >$ for some positive integer 'n', where for each $j \in \{1, \ldots, n\}$, (x_j, p_j) is a "binary probabilistic lottery" and $\sum_{j=1}^{n} p_j$ belongs to the left-open, right-closed interval $(0,1]$.

Let $\mathcal{L}^+$ denote the set of all probabilistic lotteries and identifying probabilistic lotteries of the form $< (x,p) >$ with its only binary probabilistic lottery in the array, let $\mathcal{L}^{1+}$ denote the strict subset of all "binary probabilistic prospects".

We assume that there is a binary relation $\succcurlyeq$ on $\mathcal{L}^+$ whose symmetric part of is denoted by $\sim$ and whose asymmetric part is denoted by $\succ$.

An **expected utility function** for $\succcurlyeq$ is a function $V: \mathcal{L}^+ \to \mathbb{R}$ such there exists a function $u: [0, M] \to \mathbb{R}$ for which the following are satisfied:

(i) $u(0) = 0$.
(ii) For all $L = < (x_1, p_1), \ldots, (x_n, p_n) > \in \mathcal{L}^+$, $V(L) = \sum_{j=1}^{n} p_j u(x_j)$.
(iii) For all $L_1, L_2 \in \mathcal{L}^+$: $L_1 \succcurlyeq L_2$ if and only if $(W(L_1) \geq W(L_2)$.

As before the function u is called a **Bernoulli utility function** for $\succcurlyeq$.

Assumption 6 (Restricted Strong Monotonicity): *For all* $(x,p),(y,q) \in \mathcal{L}^{1+}$: $x \geq y, p \geq q$ *with at least one strict inequality implies* $(x,p) \succ (y,q)$.

Assumption 7 (Restricted Indifference): *For all* $x \in (0,M]$, *there exists* $r \in (0,1]$ *such that* $(x,1) \sim (M,r)$.

Assumption 8 (Common Ratio Property): *For all* $(x,p),(y,q) \in \mathcal{L}^{1+}$ *and* $t \in (0,1]$: $(x,p) \succcurlyeq (y,q)$ *implies* $(x,tp) \succcurlyeq (y,tq)$.

The following lemma and its proof are similar to that of lemma 1 and its proof.

Lemma 4: *If* $\succcurlyeq$ *satisfies Assumptions 6, 7 and 8, then there exists a strictly increasing function* $v\colon [0,M] \to [0,1]$ *satisfying* $v(0) = 0$ *and* $v(M) = 1$ *such that for all* $(x,p),(y,q) \in \mathcal{L}^{1+}$: $(x,p) \succcurlyeq (y,q)$ *if and only if* $pv(x) \geq qv(y)$.

We now invoke the following additional assumption.

Assumption 9 (Restricted Substitution): *If* $< (x_1,p_1),\ldots,(x_n,p_n) >$, $< (y_1,q_1),\ldots,(y_n,q_n) > \in \mathcal{L}^{+}$, $< (z,r) > \in \mathcal{L}^{1+}$, $< (z,r) > \sim <(x_j,p_j)>$ *for some* $j \in \{1,\ldots,n\}$ *and* $r \leq p_j$, *then* $< (x_1,p_1),\ldots,(x_n,p_n) > \sim < (y_1,q_1),\ldots,(y_m,q_m) >$ *where* $(x_k, p_k) = (y_k, q_k)$ *for all* $k \in \{1,\ldots,n\} \setminus \{j\}$ *and* $(y_j,q_j) = (z,r)$.

The following proposition is similar to proposition 1 and its proof is much easier than the proof of proposition 1

Proposition 3: *If* $\succcurlyeq$ *on* $\mathcal{L}^{+}$ *satisfies Assumptions 6 to 9, then there exists an expected utility function for* $\succcurlyeq$.

The analogous version of Assumption 5, i.e., Reducibility of Extreme Prospects, is not required to prove the existence of an expected utility function for the preference relation $\succcurlyeq$ on $\mathcal{L}^{+}$.

As in the previous section, say that a preference relation $\succcurlyeq$ on $\mathcal{L}^{+}$ is **piece-wise linear** if or some positive integer 'n' with $n \geq 2$, there exists a non-empty finite set of real numbers $\{x_j \mid j = 0,1,\ldots,n\}$ including '0' and satisfying $0 = x_0 < x_1 <\ldots < x_n = M$, such that for all $p \in (0,1)$:

(a) if $j \in \{1,\ldots,n-1\}$ then $(px_j + (1-p)x_{j+1},1) \sim <(x_j,p), (x_{j+1},1-p)>$;

(b) $(px_0 + (1-p)\, x_1, 1) \sim (x_1, 1-p)$.

In this case $\succcurlyeq$ is said to be a piece-wise linear preference relation on $\mathcal{L}^{+}$ generated by $\{x_j \mid j = 0,1,\ldots,n\}$.

A piece-wise linear preference relation $\succcurlyeq$ on $\mathcal{L}^{+}$ generated by $\{x_j \mid j = 0,1,\ldots,n\}$ is said to satisfy:

Assumption 7' (Restricted Indifference): *For all* $j \in \{1,\ldots,n\}$, *there exists* $r_j \in (0,1]$ *such that* $(x_j,1) \sim (M,r_j)$.

As before Assumption 7' is considerably weaker than Assumption 7.

Following a path similar to the one traversed in the previous section, we can state and prove the following proposition.

Proposition 4: *If* $\succcurlyeq$ *is a piece-wise linear preference relation on* $\mathcal{L}^{+}$ *generated by* $\{x_j \mid j = 0,1,\ldots,n\}$ *that satisfies Assumptions 6, 7', 8 and 9 then there exists an expected utility function for the preference relation, and the Bernoulli utility function u for any such expected utility function is piece-wise linear, i.e., for all* $j \in \{0,\ldots,n\}$ *and* $p \in [0,1]$, *if* $x = px_j + (1-p)x_{j+1}$, *then* $u(x) = pu(x_j) + (1-p)u(x_{j+1})$.

7. ALLAIS' PARADOX REPRESENTS DISAGREEMENT OVER PROBABILITIES

In this section we discuss an issue that became a compelling thought for me while reading [1]. Our understanding and interpretation of the issue reinforces the reasons for continuing with conventional framework, (i.e., that of expected utility maximization) in which problem of choices from among a finite set of probabilistic lotteries – particularly the famous Allais paradox in [3] – is formulated.

The simplified version of the Allais paradox that Rubinstein considers and which goes as follows:

Experiment 1:

Choose between $L_1 = (\$4000, 0.8)$ and $L_2 = (\$3000, 1)$.

Experiment 2:

Choose between $L_3 = (\$4000, 0.2)$ and $L_4 = (\$3000, 0.25)$.

If (Rubinstein's version of) Allais Paradox is to be believed then in the first Experiment most individuals chose L_2 and in the second Experiment most individuals chose L_3.

On the basis of this observation, we are told that if the individuals were expected utility maximizers with the Bernoulli utility function for monetary prizes being u and satisfying $u(0) = 0$, then $u(3000) > 0.8u(4000)$ for Experiment 1 should imply $0.25\,u(3000) > 0.2u(4000)$ implying that contrary to observation L_4 should be chosen and not L_3.

Hence, we are expected to conclude that individuals may not be behaving as though they are expected utility maximizers.

It has been known for a long time-at least since Friedman and Savage [20] – that decision theory is concerned with as if behavior of individuals which may not necessarily reflect conscious behavior. The way Rubinstein in [1] defines (probabilistic) lotteries and the way the Experiments above are framed, imposes on the individuals some exogenously given probabilities, with which they are expected to do the expected utility calculations-consciously or unconsciously – and make choices. Hence, it is not unrealistic to conclude from the above observations that if individuals are faced with binary probabilistic lotteries, then the individuals may not be maximizing expected utility using the exogenously given probabilities, and no more than that. There is no guarantee, that choosers don't impute (antagonistic?) motives to those offering choices and use other probabilities for EU maximization. It does not mean that individuals may use probabilities different from the ones that are exogenously given for the purpose of expected utility maximization; nor and more importantly does it mean that individuals "do not" maximize expected utility if a money prize is associated with an uncertain event and the individual is allowed to conjecture his/her own-possibly experiment or "menu" dependent-subjective probability of the event, i.e., the assigned subjective probabilities may depend on the set of probabilistic prospects that the individual has to choose from. In fact, most probabilistic lotteries in the real-world are such state dependent lotteries and not probabilistic lotteries.

What would be the implication of Allais' paradox if instead of choosing from probabilistic lotteries individuals were required to choose from state dependent lotteries?

Let X be the set of all possible outcomes of an experiment any non-empty subset of which is called an event and denoted by E.

A "simple state dependent lottery" (SSDL) is a pair (x, E) where x is a non-negative real number denoting the money prize an individual receives if the event E occurs and nothing if E does not occur.

Consider an urn containing 55 black balls, 25 red balls and 20 white balls all of which are evenly distributed in the urn in such a way that if the experiment of "picking a ball randomly from the urn, noting its color and then returning the ball to the urn" was repeated a large number of times then relative frequency with which each ball-regardless of its color-would be chosen, is approximately 1%.

The above information is shared with all individuals who participate in the following two experiments.

Experiment 3:

Choose between L_1 = (\$4000, E) and L_2 = (\$3000, X), where E is the event that the color of a ball randomly chosen from the urn is either black or red but not white.

Experiment 4:

Choose between L_3 = (\$4000, E_1) and L_4 = (\$3000, E_2), where E_1 is the event that the color of a ball randomly chosen from the urn is white and E_2 is the event that the color of a ball randomly chosen from the urn is red.

In what follows we will – in keeping with existing tradition introduced in Section 2 of this paper – refer to the concept of approximate relative frequency of the occurrence of an event in a very large number of trials of the experiment in which the event could occur simply as objective probability.

What anyone who participates in the two experiments of "picking a ball randomly from the urn, noting its color and then returning the ball to the urn" knows are the following:

(1) the objective probability of E is 80%.

(2) the objective probability of E_1 is 20%.

(3) the objective probability of E_2 is 25%.

(4) each participant in the experiment knows that he/she will get exactly one attempt at each experiment and definitely not a large number of attempts in any.

Given this information which is shared with all the participants, we need to weigh it against the other piece of information which is almost universally known but rarely invoked by any decision theory: many individuals consider themselves unfortunate (unlucky), compared to a small number of individuals who consider themselves fortunate (lucky).

From the point of view of the chooser, E = Lucky and E^c = Unlucky.

Consider Experiment 3:

Choose between L_1 = (\$4000, E) and L_2 = (\$3000, X), where objective probability of E is 80%.

Given that in Experiment 3, if L_1 is chosen, then the occurrence of event E with an objective probability of 80%, yields a prize of \$4000/–, whereas if L_2 is chosen, then one gets \$3000/– for sure, is it very strange to expect that for a large number of such individuals the "subjective probability of E", in the sense of Ramsey [21] and de Finetti [22], is slightly less than 3/4 given that the "objective probability of E" is 4/5? In other words, is it very strange to expect that for a large number of such individuals the "subjective odds against E" are slightly greater than 1/3 given that the "odds against E based on projected relative frequencies" is 1/4?

The implication of the above is that for an individual who measures satisfaction in money units and is also "pessimistic", the latter being a very likely observation, if $\frac{P_4(E_2)-P_4(E_1)}{P_4(E_1)} < \frac{1}{3} < \frac{1-P_3(E)}{P_3(E)}$ = subjective odds against E (where for $i = 1, 2$: $P_i(event)$ is the subjective probability the individual assigns to the event in Experiment i), then maximization of "expected monetary value"- which is expected utility maximization for risk neutral individuals-is consistent with the observations related to Allais' paradox.

Note that any $P_4(E_1), P_4(E_2) \in (0,1)$ which is a constant multiple of the relative frequencies projected for E_1 and E_2 satisfy the property that the percentage difference between $P_4(E_2)$and $P_4(E_1)$is less than the percentage difference between the money prizes associated with E_2 and E_1 respectively. In such a situation, the only additional assumption we require most individuals to satisfy in order for their behavior to be consistent with "maximization of expected monetary value" is that given a choice between receiving \$3000 for sure and a lottery that gives a money prize of \$4000 with some amount of uncertainty, the individuals would assign a subjective probability to the uncertain event E that is less than 3/4, which is (not significantly) less than the projected relative frequency, i.e. 4/5.

What we intend to say above is that the expected utility maximization problem that is "implicitly" solved by the decision maker depends on subjective probabilities which reflect the decision maker's "attitude"-as we mentioned earlier-towards risk, or more generally the SSDL's under consideration, leading to the possibility of their dependence on the SSDL's that the decision maker has to choose from. Further this attitude for a large number of individuals, if not most, is such, that when faced with a choice between two SSDL's, with one rewarding a sure money prize and the other a slightly higher money prize that is conditional on the occurrence of an uncertain event, many-if not most- individuals assign subjective odds against the uncertain event greater than those based on objective probabilities. Such an attitude may be called uncertainty aversion. It is aptly expressed by the proverb: "A bird in the hand is worth two in the bush".

Clearly, uncertainty aversion- the way we define it- requires the possibility of assigning menu-dependent subjective probabilities by the decision maker (chooser).

Note that given a choice between SSDL's all of which yield prizes conditional on uncertain events, it is quite possible for the assigned subjective probabilities to be equal to the corresponding projected relative frequencies for expected monetary value maximization to be consistent with claims similar to the ones made for the Allais paradox. The observations related to the Allais paradox does not constitute a violation of expected utility maximization; it only says that subjective probabilities used by expected utility maximizers may disagree with objective probabilities, due to uncertainty aversion.

8. AN EXAMPLE OF EXPECTED UTILITY THEORY IN ACTION

In this section we provide the basic problem discussed in section 2 of Chapter 1 of [19], which is the "almost universally" accepted template for motivating decision analysis under uncertainty based on expected utility maximization, when the decision maker is required to choose an alternative from a set of alternatives each representing uncertain monetary gains and losses.

As in section 1.2 of [19], "please imagine a collection of 1000 urns, each of which has one of the labels", θ_1 or θ_2 pasted on its front. Each urn contains only red balls and black balls, the particular composition of which depends on the label. The composition of each type is specified in section 1.2 of [19]. The "gentleman" who owns the urns selects one of them at "random", i.e., by a procedure such that if it was repeated a large number of times the relative frequency with which an urn of each type would be chosen is approximately 50%. Needless to say, the problem could be made more interesting by letting the relative frequencies depend on the type of the urn. Alternatively, the decision maker, who is different from the "experimenter" (who we referred to as the "gentleman") based on information available to him assigns a probability of 1/2 to an urn of each type being chosen. Once the urn is chosen, it is placed on a table and its label-known by the experimenter- is quickly removed, so that the decision maker does not know the type of the urn that is placed on the table.

The decision maker knows that there are two possible "states of nature": E_1= the chosen urn is of type θ_1 and E_2 = the chosen urn is of type θ_2.

Subsequently the decision maker is given a choice between (a) participating in a procedure which requires him to choose one of two "state-dependent dependent prospects" the first yielding a gain of \$40 if the state of nature is E_1 and losing \$20 if the state of nature is E_2, the second yielding a gain of \$100 if the state of nature is E_1 and losing \$50 if the state of nature is E_2; (b) not participating in the procedure described in (a).

The decision maker's first problem is to assess the probability of E_i denoted $P(E_i)$ for $i = 1{,}2$, based on the information available to him about the total number of urns and the "long run" relative frequencies of an urn of each type being chosen. Once that is done, the choice problem faced by the decision maker reduces to the following.

Choose between:

(a) Given probabilistic prospects
$< (40, P(E_1)), (-20, P(E_1)) >$ and $< 50, P(E_1)), (100, P(E_2)) >$
choose one, where the number represent gains in dollars.

(b) Don't choose (a).

Assuming that the decision maker is an expected utility maximizer with a Bernoulli utility function u, the decision maker will choose (a) if and only if $max\{u(40)P(E_1) + u(-20)P(E_2), u(-50)P(E_1) + u(100)P(E_2)\} \geq u(0) = 0$; otherwise he will choose (b).

If $max\{u(40)P(E_1) + u(-20)P(E_2), u(-50)P(E_1) + u(100)P(E_2)\} \geq 0$, he will choose $< (40, P(E_1)), (-20, P(E_2)) >$ if and only if $u(40)P(E_1) + u(-20)P(E_2) \geq u(-50)P(E_1) + u(100)P(E_2)$; otherwise he will choose $< (-50, P(E_1)), (100, P(E_2)) >$.

9. CONCLUSION

Finally, we wish to add clarifications for some very common conceptual misunderstanding that often leads to much confusion.

(a) It is unfortunate that the concept of "loss aversion" has been mis-understood by many as conflicting with "expected utility maximization", when in reality its real purpose is to assert that utility functions display risk-aversion and not risk neutrality at initial wealth, the latter being the characteristic of expected monetary value maximisers. However, this may create some difficulty for results in mathematical finance, for instance the arbitrage theorem, but by itself does not conflict with expected utility maximization.

(b) The piecewise linear utility functions with just two linear segments used to explain loss aversion may be far too simplistic and fails to explain the "diversification" paradox of Bernoulli.

(c) On page 298, Friedman and Savage in [20], explicitly states the "as if" argument to explain individual behaviour, implying that the validity of theories in the physical sciences cannot be questioned on the basis of what one believes about the cognitive ability of what physical sciences are concerned with.

Even if one were to disagree with Friedman and Savage or question the need for such theories in operations research or the decision sciences, the least that one would have to concede is the normative appeal of expected utility theory and hence its use for professionally and technically qualified consultants, provided they are well informed about the gains and losses associated with each alternative. It is with them in mind that this paper has been written and the mathematical prerequisite for understanding it being nothing more than basic algebra and certainly not a graduate course in real analysis. If expected utility theory fails to describe human behaviour due to its computational complexity or framing effects, it can certainly be used for the purposes of decision aiding or as a decision support system, as it is already done in industry these days. In fact, with the increasing use of professional decision aids in markets and industry any argument against expected utility maximization based on the "cognitive limitations" of decision makers is becoming less credible day by day. The potential constraint on the applicability of expected utility maximization, particularly in business and management, has very little to do with its plausibility, desirability or computational complexity. Such constraints are usually due to lack of exact information about the state-dependent payoffs associated with each probabilistic prospect, but that is the subject matter of a related, but different line of research.

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