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METRIC ON THE SPACE OF SYSTEMS BEHAVIOR FUNCTIONS REPRESENTED BY FUZZY MEASURES

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Abstract: G. Klir proposed to describe the behavior of complex systems using behavior functions (BFs) - invariant constraints on the set of system states. BFs are one of the most productive tools for studying the functioning of systems. To study systems, it is necessary to have a metric for measuring of the difference between two BFs. To describe BFs modern researchers do not use distributions other than probability or possibility. But these distributions can be considered as special cases of Sugeno fuzzy measures, the use of which greatly expands the possibilities in the study of systems. However, metrics to measure the difference between fuzzy measures have not been developed. Therefore, in this article, the authors proposed a new metric and an algorithm for its calculation for the case when BFs are described by Sugeno fuzzy measures. This metric is based on the Cartesian product of fuzzy measures and the use of our proposed concentration function. The metric makes it possible to compare the behavior of systems in the case of describing BFs by Sugeno fuzzy measures with different modalities, as well as to ensure the priority of taking into account the set of the most significant states of the system.

Keywords: System, behavior function, metric, metric space, fuzzy measure, fuzzy integral, distance function, norm.

MSC: 00A71, 03E72, 28E10

1. INTRODUCTION

In accordance with the approach proposed by G. Klir [1], the behavior of the system can be uniquely described by BF, which is a mapping of the form $f(c): C \to [0,1]$. Unlike differential or difference equations, BF defines an invariant constraint on the set of system states C with respect to some parameter (for example, time, space, or group). Therefore, BF allows the researcher to present and easily analyze all admissible states of the system in a holistic way. If time is chosen as a parameter, then the dynamics of the output variables of the system can be easily defined as the composition of the input variables and BF.

Examples of solving system problems can be found in many studies, for example, in [2, 3]. To solve these problems, it is often necessary to estimate the distance between BFs. For example, it is necessary to determine the degree of stability of behavior for a system that has been affected. Or another example – from a variety of systems, it is necessary to choose a system that best performs a given target behavior. In the latter case, the problem can be formulated as the problem of choosing the best purposeful system (we will consider the solution of this problem below):

$$f^*(c_i) = arg \min_{f(c_i)\in\mathcal{F}} \left(d\left(f(c_i), f^{goal}(c_i)\right)\right), \tag{1}$$

where:
$\mathcal{F} = \{f_j(c_i), j = \overline{1, N_{\mathcal{F}}}\}$ is the set of BFs of possible systems;
$N_{\mathcal{F}} = Card(\mathcal{F})$ is size of the set $\mathcal{F}$;
$f^{goal}(c_i)$ is the target BF;
$d(\cdot,\cdot)$ is the distance function between BFs.

BF $f(c)$ formalizes the distribution of confidence in the appearance of the state $c \in C$. The set of states is n – dimensional unit hypercube $C = [0,1]^n$. Significant states for the system are states that are observed with maximum confidence on a specified parametric set. The state of the system $c_i \in C, i = \overline{1, n}, n = Card(C)$ can be described by the values of the set X of sampling variables of the system. Each variable $x_m \in X, m = \overline{1, N_X}$ (where N_X is the number of sampling variables) is defined on its own set of values V_m. Then the state of the system $c_i \in C$ is a vector in the space defined on the Cartesian product $V_1 \times V_2 \times \cdots \times V_m$.

BF can be defined using various mathematical constructions, the properties of which directly affect the productivity of BF as a system analysis tool. In particular, G. Klir [1] used distributions of the probability measure and the measure of possibility. The same measures have been used in other studies, for example [4-6]. However, as shown by D. Pospelov [7], both the probability distribution and the possibility distribution are partial cases of fuzzy measures, in particular, Sugeno's fuzzy measure [8].

The Sugeno fuzzy measure $f(\cdot)$ is a non-additive function of the set $f(\cdot): 2^C \to [0,1]$, where 2^C is the set of all subsets of the universal set C. The following rule is used to define the Sugeno fuzzy measure. Let $A, B \subseteq X, A \cap B = \emptyset$. Then $f(A \cup B) = f(A) + f(B) + \lambda \cdot f(A) \cdot f(B)$, where λ is the normalization parameter of the fuzzy measure $f(\cdot)$. If $\lambda = 0$ the fuzzy measure is a probability measure. If $\lambda = -1$, the fuzzy measure is a possibility measure. The parameter λ reflects modality, that is, the attitude of the content of the judgment (assessments represented by this measure) to reality. In other words, modality is a qualification of judgments, according to which they are distinguished as affirming or denying the possibility, impossibility, randomness or necessity of their content.

Thus, Sugeno fuzzy measure is a generalization of probability and possibility measures. Therefore, its use for BFs representation expands the capabilities for describing the functioning of complex systems. In particular, determining the distance between two fuzzy measures makes it possible to improve the solution of system problems through the use of estimates with different modalities. Although G. Klir mentioned the possibility of using fuzzy measures, he did not develop this thesis. In addition, we did not find such references in the known literature. To compare BFs, researchers most often suggest using the class of Minkowski distances [9] or the information distance between BFs [1]. However, as we have shown below, in the case of arbitrary fuzzy measures, the application of these distances can lead to incorrect results. This became the main motivation for our study, the purpose of which is to develop a distance function $d(\cdot,\cdot)$ between two Sugeno fuzzy measures.

This distance must meet several general requirements. First, based on the practice of solving system problems, it is desirable that the distance $d(\cdot,\cdot)$ satisfy the properties of the metrics. Secondly, for two systems similar in behavior, the distance should have a minimum value: $d(f_1, f_2) \to 0$. Here, as similar in behavior, we will consider systems for which the sets of their significant states coincide as much as possible. That is, if $A \subseteq C$ and $B \subseteq C$ are sets of significant states for two systems, respectively, then the condition of approximate equality $A \cong B$ must be satisfied. Significant states of the system as a whole should not depend on the modality of fuzzy measures with which BFs are described. Therefore, for the case of fuzzy measures, the condition $\forall \lambda_{f_1}, \lambda_{f_2},\ d(f_1, f_2) \to 0$ must be satisfied, where $\lambda_{f_1}, \lambda_{f_2}$ are the normalization parameters of Sugeno fuzzy measures. Thirdly, if the systems are fundamentally different in their behavior, then the distance should have a maximum value, and the sets of significant states should not intersect $A \cap B = \emptyset$.

Thus, the task of our study is to develop the distance between two Sugeno fuzzy measures that are used to describe the behavior of complex systems. We also need to develop an algorithm for calculating this distance and prove that the distance satisfies the general requirements discussed above, as well as demonstrate the productivity of the distance in solving system problems in practice.

To solve this problem, we proposed a limited metric and an algorithm for its calculation. The metric allows you to compare two Sugeno fuzzy measures, taking into account the most significant states of the system. We have shown that the metric satisfies the general requirements discussed above. We also considered an example of solving a system problem, demonstrating the performance of this metric and the possibility of obtaining erroneous solutions using the well-known Minkowski metric.

2. LITERATURE REVIEW AND ANALYSIS OF KNOWN APPROACHES

To ensure the constructiveness of the review of the literature and the analysis of known approaches, let us clarify the general requirements for the metrics of the BFs space. Then we will correlate known approaches with these requirements.

Requirements for the distance function $d(\cdot,\cdot)$.

So, the similarity of two comparable systems defined on the common set of system states C is expressed in terms of the metric distance $d(\cdot,\cdot)$ between their BFs. To solve system problems in practice, it is necessary that the distance $d(\cdot,\cdot)$ satisfy the following requirements.

1. The distance $d(\cdot,\cdot)$ must satisfy the requirements of the metric. In particular, the distance $d(\cdot,\cdot): \mathcal{F} \times \mathcal{F} \to R$ will be a metric if $\forall x, y, z \in \mathcal{F}$ the following conditions are satisfied:

positive definiteness: $d(x, y) \geq 0$;

identity: if $d(x, y) = 0$, then $x = y$;

symmetry: $d(x, y) = d(y, x)$;

triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$.

2. Since the significant states of the system most fully describe the behavior of the system, the distance $d(\cdot,\cdot)$ should first of all take into account the sets of significant states, that is, the states of systems for which $f(c_i) \to max$.

3. For the constructive use of the metric, it is desirable that the limitation condition $d(\cdot,\cdot) \in [0,1]$ be satisfied.

4. Systems with similar behavior should have maximally close sets of significant states $A, B \subseteq C, A \cong B$.

5. The systems that maximally differ in behavior must have non-intersecting sets of significant states $A, B \subseteq C, A \cap B = \emptyset$.

Analysis of known approaches.

Today we know about a large number of different distances, a representative list of which is considered in the study [10]. Here we will consider distances that are most convenient for comparing BFs in system problems, where BFs are defined on the common discrete set of states $C = [0,1]^n$.

To determine the distance on the space $\mathcal{F}$, we can use metric functions based on the following formalisms:

norm metrics;

norm metrics defined on spaces with scalar product;

distances specified on a set of distribution laws;

symmetrical difference metrics;

distances between sets.

Consider the most suitable distances from the indicated groups in accordance with the requirements mentioned above.

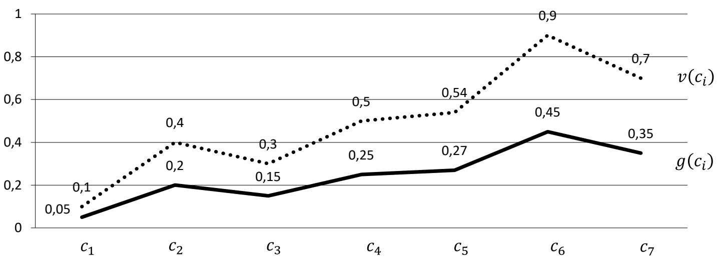
Norm's metrics [10]. The norm's metrics on a real vector space V are defined as: $\forall x, y \in V, \|x - y\|$, where $\|\cdot\|$ is the norm on V. The norm function $\|\cdot\|: V \to R$, satisfies the conditions: 1) $\forall x \in V, \|x\| \geq 0$ ($\|x\| = 0$ only when $x = 0$); 2) $\|a \cdot x\| = |a| \cdot \|x\|$, where a is a scalar; 3) $\forall x, y \in V, \|x + y\| \leq \|x\| + \|y\|$. A finite-dimensional normed vector space $(V, \|\cdot\|)$ is complete, that is, a Banach space. All norms defined on it are equivalent. If the space $\mathcal{F}$ is considered as a Banach space, then metrics of type l_p^n from the class of Minkowski distances [10-12] of the following form can be considered as a metric:

$$\forall g, v \in \mathcal{F}, l_p^n(g, v) = \left(\sum_{c_i \in C} |g(c_i) - v(c_i)|^p\right)^{1/p}, \qquad (2)$$

where $n = Card(C)$, $p \in [1, +\infty[$ is the distance function parameter.

If $p = 1$, dependence (2) describes the Hamming distance, if $p = 2$ – the Euclidean distance. If $p \to +\infty$, we have $l_p^n(g, v) = \max_{c_i \in C} |g(c_i) - v(c_i)|$. The l_p^n metrics can be normalized to 1 when divided by n. However, the use of these metrics entails an error when comparing systems with fuzzy measures of different modalities.

Example 1. Let two similar systems have BFs described using fuzzy measures (see Figure 1). These fuzzy measures have similar density functions $v(c_i) = 2 \cdot g(c_i), i = \overline{1, n}$, but different modalities: $\lambda_g \cong -0.79$,, $\lambda_v \cong 0.94$. In this case, the condition $l_p^n(g, v) = 0$ must be satisfied. As can be seen from metric values on Figure 1, this condition is not satisfied. The analysis showed that the use of l_p^n type metrics is admissible only for fuzzy measures of one modality. In addition, the l_p^n metric considers all states of systems as equivalent, that is, it does not take into account their significant states.



Metric	Value
$l_1^n(g, v)$	0.246
$l_2^n(g, v)$	0.104
$l_{10}^n(g, v)$	0.65

Figure 1: Density distribution of similar fuzzy measures and values of metrics of type $\mathrm{l}_\mathrm{p}^\mathrm{n}$ for different parameters p

Norm's metrics on spaces with scalar product. To compare two BFs, we can use the norm metrics, which are defined by the scalar product $\|x\| = \sqrt{\langle \cdot, \cdot \rangle}$, where: V is a real vector space; $\langle \cdot, \cdot \rangle: V \times V \to R$ is a function. For this function $\forall x, y, z \in V$ and scalars a, b, the following conditions must be satisfied: 1) $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0 \Rightarrow x = 0$; 2) $\langle x, y \rangle = \langle y, x \rangle$; 3) $\langle a \cdot x + b \cdot y, z \rangle = a \cdot \langle x, z \rangle + b \cdot \langle y, z \rangle$. In particular, we can use the angular half-metric [10] between the density vectors of discrete fuzzy measures $g, v \in \mathcal{F}$, which has the form:

$$d_{cos}(g, v) = \cos^{-1}\left(\frac{\langle g, v \rangle}{\|g\|_2 \cdot \|v\|_2}\right), \tag{3}$$

where $\cos^{-1}$ is the inverse cosine function;

$\|g\|_2 = \sqrt{\sum_i^n \left(g(c_i)\right)^2}$ is the Euclidean norm on C;

$\langle g, v \rangle = \sum_i^n \left(g(c_i) \cdot v(c_i)\right)$ is the scalar product of fuzzy measure density vectors.

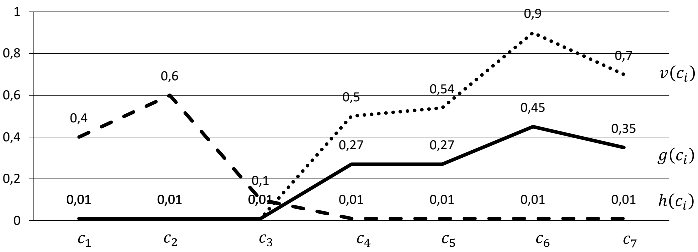
The half-metric $d_{cos}(g, v)$ is a positive-definite, symmetric, reflexive function that satisfies the triangle condition. This half-metric is also called the spectral angle of contrast [13].

For all strictly positive measures, the Hellinger metric [14, 15] can be used:

$$d_{Hel}(g, v) = 2 \cdot \left(\sum_{i=\overline{1,n}} \left(\sqrt{g(c_i)} - \sqrt{v(c_i)} \right)^2 \right)^{1/2}. \quad (4)$$

The metric (4) is used to compare distributions of measures defined on C. This metric belongs to special Riemannian metrics that are used in information theory [16].

Example 2. The values of metrics (3) and (4) for the case of three measure distributions are shown in Figure 2. As can be seen, the measures $g(\cdot)$ and $v(\cdot)$ are similar in accordance with the sets of significant states, and the measures $v(\cdot)$ and $h(\cdot)$ are maximally dissimilar $Supp(v(\cdot)) \cap Supp(h(\cdot)) = \emptyset$. The analysis showed that functions (3) and (4) are not limited to a unit interval if the measures are not probability measures. The Hellinger metric does not allow taking into account the similarity in behavior of systems, that is, the condition $d_{Hel}(g, v) \neq 0$ is not satisfied. In this sense, the angular half-metric (3) is more preferable. However, both functions do not take into account the significance of system states. This increases the influence of insignificant states of the system, which introduces a distortion in the estimate of the distance between BFs.



Metric	Value
$d_{Hel}(g, v)$	0.95
$d_{Hel}(v, h)$	3.36
$d_{cos}(g, v)$	0.01
$d_{cos}(v, h)$	1.53

Figure 2: Densities' distribution of fuzzy measures and metrics' values

The distances given on the set of distribution laws. The distances given on the set of distribution laws can be separated into a special group [10]. They are used primarily for probability distributions and reflect relative entropy or information error. Most of these distances are not symmetrical. The exception is the symmetric version of the Kullback-Leibler distance (Jeffrey divergence), which we will discuss below.

To compare probabilistic BFs, G. Klir proposed using the information distance based on the Shannon entropy [17]. This distance is the asymmetric information distance of the distribution $g(c_i)$ with respect to $v(c_i)$ and is defined as:

$$d_{CES}(g, v) = \frac{1}{\log_2|C|} \cdot \sum_{i=\overline{1,n}} g(c_i) \cdot \log_2 \left(\frac{g(c_i)}{v(c_i)} \right). \quad (5)$$

The distance is an estimate of the loss of information about the behavior of the system in the transition from BF $g(c_i)$ to BF $v(c_i)$. Distance reflects the directional relative difference between BFs and therefore may apply to not all system tasks.

To compare discrete BFs represented by probability measure distributions, Klir and Higashi proposed using an information distance of the form:

$$d_{DU}(g,v)=\sum_{i=1}^{n}(l_i-l_{i+1})\cdot\log_2\left(\frac{|A_{l_i}|}{|B_{l_i}|}\right), \tag{6}$$

where $A_{l_i}=\{c_i\in C|g(c_i)\geq l_i\in[0,1]\}$, $B_{l_i}=\{c_i\in C|v(c_i)\geq l_i\in[0,1]\}$, $l_1=1>l_2>\cdots>l_i>\cdots>l_n=0$.

It is possible that $l_i=\hat{g}(c_i)$, where $\hat{g}(c_i)$is the descending distribution of the possibility measure $g(\cdot)$, $\hat{g}(c_{n+1})=0$. However, this distance is also not symmetric, like (5).

As a rule, the distances specified on the set of distribution laws are not symmetrical. To fulfill the symmetry condition, you can use the symmetrical version of the Kullback-Leibler distance (Jeffrey divergence) [10, 18] of the form:

$$d_J(g,v)=\sum_{i=1,n}\left[g(c_i)\cdot\ln\left(\frac{g(c_i)}{v(c_i)}\right)+v(c_i)\cdot\ln\left(\frac{v(c_i)}{g(c_i)}\right)\right]. \tag{7}$$

Example 3. Figure 3 shows the distance values (5) and (7) for the BFs from Example 1. Jeffrey's divergence (7) is not normalized to 1 and does not provide the equality condition $d_J(g,v)=0$ for similar systems. Information distance (5) $d_{CES}(g,v)$ is not a metric. This distance is asymmetric $d_{CES}(h,v)\neq d_{CES}(v,h)$ and does not satisfy the requirements for comparing BFs represented by fuzzy measures.

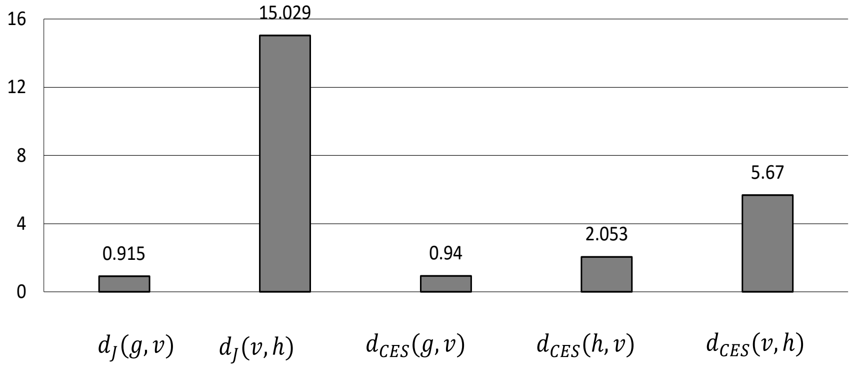
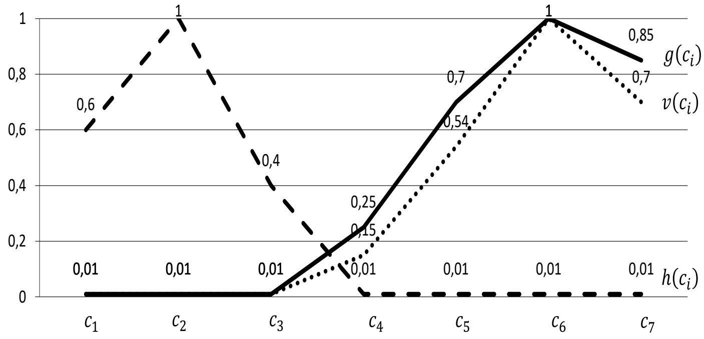


Figure 3: The distance values of the Jeffrey divergence (7) and the modified version of the Shannon entropy (5)

Figure 4 shows BFs represented by possibility distributions and distances (6) between them.



Distance	Value
$d_{DU}(g,v)$	0.258
$d_{DU}(v,g)$	-0.258
$d_{DU}(v,h)$	0.2
$d_{DU}(h,v)$	-0.2

Figure 4: BFs represented by possibility distributions and distances (6) between them

The use of distance (6) is not always possible. This distance is not a metric. The function $d_{DU}(g,v)$ is asymmetric. This leads to the appearance of its negative values. In addition, the use of distance (6) requires additional conditions for the BFs [1], which are not satisfied when the BFs is represented by arbitrary fuzzy measures.

Metrics of symmetric difference. These metrics [10, 19] are defined on the space $(C,\mathcal{A},\mu)$ with measure, where $\mathcal{A}$ is the set of all subsets of the set C, $\forall A\in\mathcal{A},\mu(A)<\infty$. In a particular case, the measure $\mu(\cdot)$ can be a cardinal number, area, or volume. The metric of the symmetric difference on the set $\mathcal{A}$ can be determined based on the following half-metric:

$$d_\Delta(A,B)=\mu(A\Delta B),A\Delta B=(A\cup B)\backslash(A\cap B). \quad (8)$$

If A and B are equal almost everywhere, then $d_\Delta(A,B)=0$. If we identify A and B under the condition $d_\Delta(A,B)=0$, then we obtain the metric of the symmetric difference (the Fréchet–Nikodym–Aronszyan metric [20]).

The modified metric of the symmetric difference is the Steinhaus metric [21]:

$$d_{Sh}(A,B)=\frac{\mu(A\Delta B)}{\mu(A\cup B)}=1-\frac{\mu(A\cap B)}{\mu(A\cup B)},\mu(A\cup B)>0. \quad (9)$$

The use of these metrics for solving system problems requires the definition of sets A and B based on the distributions of fuzzy measures $g(\cdot)$ and $v(\cdot)$. For arbitrary fuzzy measures, level sets $A_l=\{c_i\in C|g(c_i)\geq l\in[0,1]\}$, $B_l=\{c_i\in C|v(c_i)\geq l\in[0,1]\}$. The metrics can be normalized to the cardinality of the state set C:

$$d_\Delta^l(g,v)=\frac{d_\Delta(A_l,B_l)}{Card(C)},d_{Sh}^l(g,v)=\frac{d_{Sh}(A_l,B_l)}{Card(C)}. \quad (10)$$

Example 4. Figure 5 shows the values of the symmetric difference metrics for three fuzzy measures (see Figure 2). Metric values are calculated for levels $l\in\{0.2;0.3;0.5;0.6\}$ provided that $\mu(\cdot)=|\cdot|$ is a cardinal number.

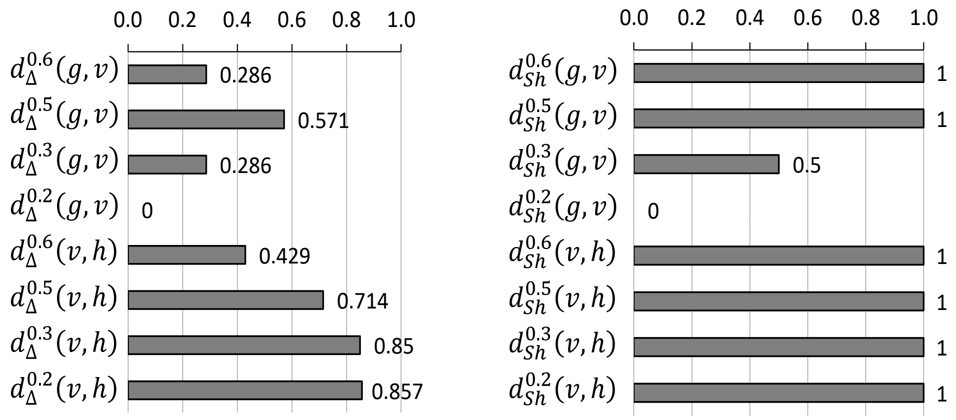


Figure 5: The values of the metrics of the symmetric difference for three fuzzy measures from Figure 2: the Fréchet–Nikodym–Aronszyan and the Steinhaus metric

The analysis shows that the metrics of the symmetric difference are normalized to 1. These metrics make it possible to measure the distance between BFs, but they strongly depend on the level l, which determines the compared sets A_l and B_l. The use of the metrics of the symmetric difference can cause distance estimation errors in the case of comparing fuzzy measures that have different modalities. For example, if $\lambda_g \gg \lambda_v$, then $g(c_i) \ll v(c_i)$, hence the subsets A_l and B_l are very different, which causes a significant increase in $A\Delta B$ even in the case of systems with similar behavior.

Distances between sets. A separate group of metrics includes metrics that are based on an estimate of the distance between sets [10]. In particular, this is the Hausdorff L_p-distance [22] , which has the form:

$$\forall g, v \in \mathcal{F}, d_{Haus}(g,v) = \left(\sum_{c_i \in C} \left| d\big(g(c_i), v(\cdot)\big) - d\big(g(\cdot), v(c_i)\big) \right|^p \right)^{1/p}, \quad (11)$$

where $d\big(g(c_i), v(\cdot)\big) = \min_{c_j \in C} d\left(g(c_i), v(c_j)\right)$ is the distance between the point and the set, in the special case $d\left(g(c_i), v(c_j)\right) = \left|g(c_i) - v(c_j)\right|$.

Example 5. Figure 6 shows the values of L_p-distances between BFs shown in Figure 2 for different values of the distance function parameter $p \in [1, +\infty[$.

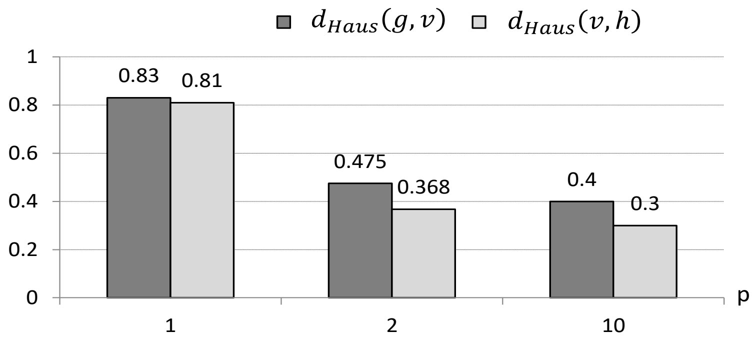


Figure 6: Values of L_p-distances between BFs (see Figure 2) for $p \in \{1,2,10\}$

The use of the distance $d\big(g(c_i), v(\cdot)\big)$ between a point and a set in formula (11) reduces the sensitivity of the Hausdorff L_p-distance for the case of fuzzy measures. The distance $d_{Haus}(g, v)$ does not distinguish between significant and insignificant states. This leads to errors when comparing the behavior of systems.

An analysis of the possibility of using known distance functions and metrics to compare BFs showed that:

1. Known distances and metrics do not fully meet the requirements for comparing BFs represented by fuzzy measures.

2. The most productive metrics for comparing BFs are the angular half-metric $d_{cos}(g, v)$, as well as the metric of the symmetric difference of Fréchet–Nikodym–Aronszyan $d_{\Delta}^{l}(g, v)$. The half-metric $d_{cos}(g, v)$ makes it possible to take into account the similarity of systems, but requires normalization and does not take into account the priority of significant states of the system. The metric $d_{\Delta}^{l}(g, v)$ takes into account the similarity of systems and the priority of significant states of the system, but strongly depends on the choice of the level $l \in [0,1]$, which determines the compared sets.

Thus, in order to eliminate these shortcomings, it is necessary to develop a new distance function on the set F BFs represented by arbitrary fuzzy measures. This function must satisfy the properties of the metric, and must also be normalized and take into account the priority of significant states of a system.

3. RESEARCH PROBLEM

It is necessary to develop a new metric $d_{FM}(\cdot,\cdot)$ for the space $\mathcal{F}$ of BFs, which are represented by Sugeno fuzzy measures. These measures are defined on the set C and can have different modalities $\lambda \in [-1, +\infty[$. The metric must be normalized: $\forall g, v \in \mathcal{F}, d_{FM}(g, \mathrm{v}) \in [0,1]$. The metric should also take into account the priority of meaningful sets of system states that are being compared. In addition, it is necessary to develop an algorithm for calculating a new metric and demonstrate its performance.

4. NEW METRIC BASED ON THE CARTESIAN PRODUCT OF FUZZY MEASURES

The main idea in constructing the metric $d_{FM}(\cdot,\cdot)$ is to take into account the sets of significant states when comparing fuzzy measures. To determine these sets, we proposed to use the measure concentration function. Based on this, in order to solve the research problem, it is necessary:

develop the mechanism for determining the significant states of the system, which do not depend on the modality of fuzzy measures;

justify the procedure for comparing subsets of significant states of the system and build a metric for comparing fuzzy measures;

develop an algorithm for calculating the metric.

Determination of the measure concentration function. Let $g(\cdot): 2^C \to [0,1]$ be the fuzzy Sugeno measure [8], which models a parameter-invariant constraint on the state set C of the system. The measure $g(\cdot)$ describes the BF of the system. Moreover, within the set of states C, there are areas of significant states. Let's denote this area as $A \subseteq C$. The confidence in the appearance of states from A is much higher than that of states from $\bar{A}$. Taking into account the monotonicity of fuzzy measures [23], we can state that $\forall c_i \in A, c_j \notin A, g(c_i) \geq g(c_j)$. Let us order the density function of the fuzzy measure $g(c_i)$ in descending order. Then $\forall c_i \in C$ splits the set of states into two subsets $A(c_i) = \{c_j \in C, j \leq i\}$ and its complement $\bar{A}(c_i)$ so that $A(c_i) \cup \bar{A}(c_i) = C, A(c_i) \cap \bar{A}(c_i) = \emptyset$. Let's define a function of the form:

$$\delta_g(c_i) = g\big(A(c_i)\big) - g\big(\bar{A}(c_i)\big), \forall c_i \in C, \delta_g(c_i) \in [-1,1]\,. \tag{12}$$

Taking into account the normalization condition of the fuzzy measure [23], we can write:

$$\delta_g(c_i) = g\big(A(c_i)\big) - g\big(\bar{A}(c_i)\big) = g\big(A(c_i)\big) - \frac{1 - g\big(A(c_i)\big)}{1 + \lambda_g \cdot g\big(A(c_i)\big)}.$$

For the case when the fuzzy measure is the probability measure $g(\cdot) = Pr(\cdot)$, we have $\lambda_g = 0$. Then $g\big(A(c_i)\big) = 0.5 \cdot \big(\delta_g(c_i) + 1\big)$.

We define the measure concentration function as:

$$Conc_g(c_i) = 0.5 \cdot \big(\delta_g(c_i) + 1\big). \tag{13}$$

Note that the function $Conc_g(c_i)$ satisfies the following condition: $\forall g(\cdot) \in \mathcal{F}, Conc_g(c_i) \in [0,1]$. This function shows how much more confidence is concentrated in the subset of states $A(c_i) \subseteq C$ than in its complement. In other words, the function $Conc_g(c_i)$ displays the confidence concentration on the subset $A(c_i) = \{c_j \in C, j \leq i\}$ for a fuzzy measure.

Figure 7 shows the concentration function for the fuzzy measure $g(\cdot)$ in Figure 1.

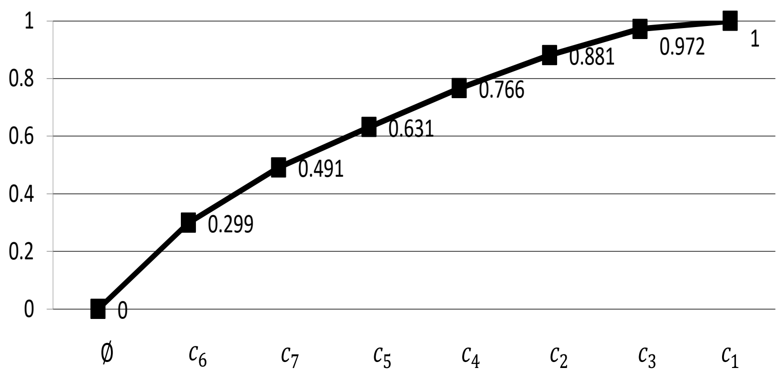


Figure 7: Function $Conc_g(c_i)$ for the measure $g(\cdot)$ in Figure 1

Comparison of two fuzzy measures. Denote by $A_l = \{c_i \in C \big| Conc_g(c_i) \le l \in [0,1], g(c_i) \neq 0\}$ the concentration set of the measure $g(\cdot)$ at the level l. This expression means that the set $A_l \subseteq C$ with confidence l concentrates in itself all significant states of the system. Similarly, for the measure $v(\cdot)$, we define the concentration set as follows: $B_l = \{c_i \in C | Conc_v(c_i) \le l \in [0,1], v(c_i) \neq 0\}$. Obviously, in the general case $A_l \neq B_l$.

The Fréchet–Nikodym–Aronszyan metric can be used to compare two sets $A_l, B_l \subseteq C$. For comparison, we use the relation $\mu(A\Delta B), A\Delta B = (A \cup B)\backslash(A \cap B)$. To take into account the values of fuzzy measures $g(\cdot)$ and $v(\cdot)$ we define the measure $\mu(\cdot)$ as the Cartesian product [24] of fuzzy measures in the form: $\mu(\cdot) = g(\cdot) \times v(\cdot) = \langle g, v\rangle(\cdot)$. Let $D_l = A_l \Delta B_l$. Let's define a subset $H_l = D_l \times D_l \subseteq C \times C$. Then the measure of the Cartesian product $\langle g, v\rangle(\cdot)$ for H_l will be determined by the expression:

$$\langle g, v\rangle(H_l) = (s)\int_{D_l} g(D_l) \circ v(\cdot) = g(D_l) \wedge v(D_l), \tag{14}$$

where $(s)\int$ – designation of Sugeno integral.

The measure $\langle g, v\rangle(H_l)$ depends on the value of the level l. To compare fuzzy measures, we have to consider $n = Card(C)$ levels for concentration functions. Therefore, we define the level l_i using the relation: $l_i = Conc_g(x_i) \vee Conc_v(y_i)$, where $x_i, y_i \in C$. We also denote the set of levels as $L = \{l_i, i = \overline{1,n}, n = Card(C)\}$. Then the distance between the fuzzy measures $g(\cdot)$ and $v(\cdot)$ will be determined by the expression:

$$d_{FM}(g, v) = \sup_{l_i \in L}\{l_i \wedge \langle g, v\rangle(H_{l_i})\}, \tag{15}$$

where $\langle g, v\rangle(H_{l_i}) = g(A_{l_i}\Delta B_{l_i}) \wedge v(A_{l_i}\Delta B_{l_i})$, $A_{l_i} = \{c_i \in C \big| Conc_g(c_i) \le l_i \in [0,1], g(c_i) \neq 0\}$, $B_{l_i} = \{c_i \in C | Conc_v(c_i) \le l_i \in [0,1], v(c_i) \neq 0\}$, $l_i = Conc_g(x_i) \vee Conc_v(y_i), x_i, y_i \in C$, $Conc_g(x_i)$ и $Conc_v(y_i)$ – concentration functions of the form (13) for the measures $g(\cdot)$ and $v(\cdot)$, respectively.

Below we will show that expression (15) satisfies all the requirements for metrics and, therefore, is a metric distance.

An algorithm for comparing two BFs represented by fuzzy measures. In view the expressions presented above, the algorithm for comparing two fuzzy measures $g(\cdot)$ and $v(\cdot)$ consists in performing the following steps:

Step 1. Determining the concentration functions $Conc_g(c_i)$ and $Conc_v(c_i)$ for the fuzzy measures $g(\cdot)$ and $v(\cdot)$. The following steps must be performed for each fuzzy measure:

Step 1.1. Descending ordering of the fuzzy measure density function $g(\cdot)$ so that $\forall c_i \in C, g(c_i) \geq g(c_{i+1}), i = \overline{1,n}, n = Card(C)$.

Step 1.2. Construction of the function $\delta_g(c_i) = g\big(A(c_i)\big) - g\big(\bar{A}(c_i)\big)$, where $A(c_i) \cup \bar{A}(c_i) = C, A(c_i) \cap \bar{A}(c_i) = \emptyset$, and $A(c_i) = \{c_j \in C, j \leq i\}$.

Step 1.3. Determination of the concentration function for the fuzzy measure $g(\cdot)$ by expression (13).

Step 1.4. Construction of the concentration function for the measure $v(\cdot)$ by performing steps 1.1-1.3.

Step 2. Determining the set of levels $L = \{l_i, i = \overline{1,n}, n = Card(C)\}$, where $l_i = Conc_g(x_i) \vee Conc_v(y_i), x_i, y_i \in C$.

Step 3. Determining the concentration sets of fuzzy measures $g(\cdot)$ and $v(\cdot)$ at levels $l_i \in L$ in the form: $A_{l_i} = \{c_i \in C \mid Conc_g(c_i) \leq l_i \in [0,1], g(c_i) \neq 0\}$, $B_{l_i} = \{c_i \in C \mid Conc_v(c_i) \leq l_i \in [0,1], v(c_i) \neq 0\}$.

Step 4. Calculation of the metric value $d_{FM}(g, v)$ in accordance with expression (15).

4. NEW METRIC BASED ON THE CARTESIAN PRODUCT OF FUZZY MEASURES

So, above we have developed a new metric, as well as an algorithm for its calculation. In an effort to ensure the completeness of the research, below we want to discuss three questions:

is expression (15) a metric distance?

what are the advantages and disadvantages of the proposed distance (15) in comparison with the known distances?

is expression (15) a productive tool that can be used to solve system problems?

Question 1. Is relation (15) a metric distance?

As mentioned above, for a function to be a metric, the conditions [10] of positive definiteness, identity, symmetry, and the triangle inequality must be satisfied. Let us show that the proposed distance (15) satisfies these conditions and is therefore a metric.

1. The proposed metric $d_{FM}(g, v)$ is positive definite. Since the value of the measure concentration function belongs to the unit interval, then $\forall l_i \in L, l_i \in [0,1]$. Because of the fuzzy measure axiomatics, the measure $\langle g, v\rangle\big(H_{l_i}\big)$ is a positive definite set function whose value belongs to the unit interval. Therefore, expression (15) for $d(g, v)$ will also be positive definite, that is, $\forall g, v \in \mathcal{F}, d_{FM}(g, v) \geq 0$. Moreover, since $l_i \in [0,1]$ and $\langle g, v\rangle\big(H_{l_i}\big) \in [0,1]$, then $d_{FM}(g, v) \in [0,1]$ is a limited metric.

2. Two similar in behavior systems have similar BFs. If BFs are described by fuzzy measures, then for such systems the following condition is satisfied: $\forall c_i \in C, v(c_i) \in$

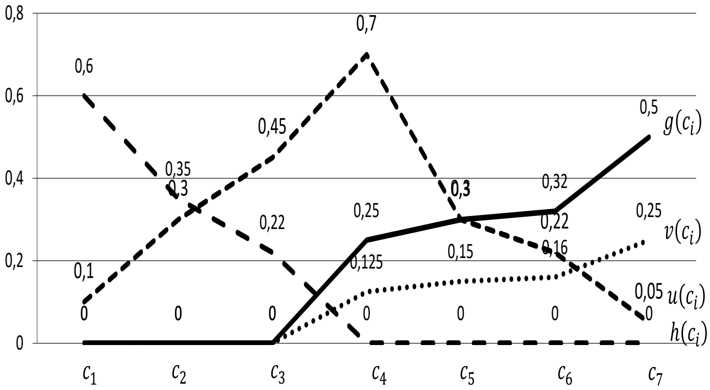
$[0,1], v(c_i) = \alpha \cdot g(c_i)$, where α is the similarity coefficient. Then, $\forall c_i \in C$, the condition $A_{l_i} = B_{l_i}$ is satisfied. Therefore, the symmetric difference $A_{l_i} \Delta B_{l_i} = \emptyset$ and $\forall l_i, \langle g, v \rangle (H_{l_i}) = 0$. Obviously, then for systems similar in behavior, the identity condition $d_{FM}(g, v) = 0$ is satisfied.

3. Symmetric operators are used in constructing the metric. In particular, the symmetric operation $\vee$ is used to determine l_i. The operation $A_{l_i} \Delta B_{l_i}$ is symmetric by definition. The measure of a Cartesian product is also a symmetric operation [24]. Therefore, the proposed metric satisfies the symmetry condition $d_{FM}(g, v) = d_{FM}(v, g)$.

4. The validity of the triangle inequality for the developed metric follows from metric transformations [10]. In particular, to construct the metric $d_{FM}(g, v)$ at the level l_i, the metric of the symmetric difference of Fréchet–Nikodym–Aronszyan is used. Transformations of the form $\min(t, d(x, y))$ and $\max(d_1(x, y), d_2(x, y))$ are metric and form a metric as a result. Therefore, the successive application of the indicated metric transformations preserves the properties of the metric of the symmetric difference in the developed metric.

Thus, for $d_{FM}(g, v)$ all conditions of the metric will be satisfied.

Example 6. Let's show an example that $d_{FM}(\cdot,\cdot)$ satisfies the properties of the metric and the conditions, which are necessary for comparing BFs systems. Figure 8 shows the density distributions of fuzzy measures that describe several BFs and the values of the metric $d_{FM}(\cdot,\cdot)$ for all possible pair combinations of fuzzy measures.



Metric	Value	Metric	Value
$d_{FM}(g, v)$	0	$d_{FM}(g, u)$	0.817
$d_{FM}(g, h)$	1	$d_{FM}(u, g)$	0.817
$d_{FM}(v, h)$	1	$d_{FM}(h, u)$	0.763

Figure 8: The density distributions of fuzzy measures that describe several BFs and the values of the metric $d_{FM}(\cdot,\cdot)$ for all possible pair combinations of fuzzy measures

Calculations show that the proposed metric is positive definite, normalized $\forall f_1, f_2 \in \mathcal{F}, d_{FM}(f_1, f_2) \in [0,1]$ and symmetric, for example $d_{FM}(g, u) = d_{FM}(u, g) = 0.817$. The proposed metric effectively takes into account the priority of significant states of the system even in the case of different modalities. In particular, systems with BFs $g(c_i)$ and $v(c_i)$ are similar $\forall c_i \in C, v(c_i) \in [0,1], v(c_i) = 0.5 \cdot g(c_i), \lambda_g = -0.619, \lambda_v = 1.556$, as evidenced by $d_{FM}(g, v) = 0$. The proposed metric satisfies the triangle condition $\forall f_1, f_2, f_2 \in \mathcal{F}, d_{FM}(f_1, f_2) \le d_{FM}(f_1, f_3) + d_{FM}(f_3, f_2)$. For example, $d_{FM}(g, h) = 1$, $d_{FM}(g, u) = 0.817$, $d_{FM}(u, h) = 0.763$. And then $d_{FM}(g, u) + d_{FM}(u, h) = 1.580 > d_{FM}(g, h)$.

Note that the entire set of systems similar in behavior form a class of systems equivalent in behavior, since $\forall f_1, f_2 \in \mathcal{F}, d_{FM}(f_1, f_2) = 0$. The distance from an arbitrary BF to any BFs of a system from the class of equivalent systems will be preserved. For example, $d_{FM}(g, h) = d_{FM}(v, h)$. In this sense, the proposed metric is similar to the metric of the symmetric difference of Fréchet–Nikodym–Aronszyan.

The proposed metric takes on a maximum value for systems that have completely different behavior, i.e. when $Supp(g(c_i)) \cap Supp(h(c_i)) = \emptyset$. On Figure 8 these are systems with BFs $g(c_i)$ and $h(c_i)$. For them $d_{FM}(g, h) = 1$.

We see that the calculations also confirm the conclusion that expression (15) is a metric and satisfies the conditions considered above, which are necessary for comparing the BFs of systems.

Question 2. What are the advantages and disadvantages of the proposed distance (15) in comparison with the known distances?

In order to compare the proposed distance (15) with the distances discussed above, we use the requirements listed at the beginning of section 2. We also use the additional constraint “Processability of the calculation”, which reflects the need to specify additional data for the metrics calculation. The comparison results are presented in Table 1.

Table 1: Distance comparison results

Distance / Constraints	Norm’s metrics (2)	Norm's metrics on spaces with scalar product (3), (4)	The distances given on the set of distribution laws (5)-(7)	Metrics of symmetric difference (8)-(10)	Distances between sets (11)	Proposed metric (15)
Positive definiteness	+	+	-,+	+	+	+
Identity	+	+	+	+	+	+
Symmetry	+	+	-,+	+	+	+
Triangle inequality	+	+	-,+	+	+	+
Taking into account the sets of significant states	-	-	-	+	-	+
The limitation condition	-	-	+	+	-	+
Matching similar BFs	-	+	-	+	-	+
Manufacturability of the calculation	+	+	-	-	+	+

As you can see, most of the distances indicated do not take into account the priority of significant states of the system. In accordance with the listed properties, the proposed metric is closest to the metric of the symmetric difference.

Question 3. Is relation (15) a productive tool that can be used to solve system problems?

Let's consider an example that demonstrates the performance of using the proposed metric in solving practical problems of systems research.

Example 7. The problem of choosing a goal-oriented system.

The problem is to choose an effective marketing tool that is used to change the preferences (behavior) of the target group of consumers (system F_1), in other words, to increase trust in the brand. This is a classic marketing problem.

The system state is specified on the set of three variables $C = V_1 \times V_2 \times V_3$:

v_1 – "Attitude to the company's trademark". This variable is defined on the values set $v_1 \in V_1 = \{0,1\}$, where $v_1 = 0$ means "Positive attitude" and $v_1 = 1$ means "Not always positive attitude".

v_2 – "Visit to a specialized store of the company". This variable is defined on the values set $v_2 \in V_2 = \{0,1\}$, where $v_2 = 0$ means "Yes", $v_2 = 1$ means "No".

v_3 – "Financial viability of consumer". This variable is defined on the values set $v_3 \in V_3 = \{0,1\}$, where $v_3 = 0$ means "Insolvent consumer", $v_3 = 1$ means "Wealthy comsumer".

As a result of observing the behavior of the system, we constructed BF as a density distribution of the fuzzy measure $f_{123}(c): C \to [0,1]$ (see Table 2).

Table 2: BF of system F_1

$\#c, c \in C$	v_1	v_2	v_3	$f_{123}(c)$
1	0	0	0	0.023
2	0	0	1	0.08
3	0	1	0	0.018
4	0	1	1	0.014
5	1	0	0	0.023
6	1	0	1	0.065
7	1	1	0	0.043
8	1	1	1	0.044

The company plans to conduct a marketing action to increase the trust in the brand from the side of wealthy customers. In other words, the goal of a marketing action is to move from the current behavior of the system to the desired behavior, which is described by the target BF in the form of a density distribution of a fuzzy measure $f_{13}^*(c): Proj_{13}C \to [0,1]$, where $Proj_{13}C = V_1 \times V_3$ (see Table 3).

Table 3: Target BF of system

$\#c, c \in Proj_{13}C$	v_1	v_3	$f_{13}^*(c)$
1	0	0	0.0230
2	0	1	0.0870
3	1	0	0.0180
4	1	1	0.0140

The company considers two options for marketing actions, which can be described by an additional variable v_4 – "Marketing actions". This value is specified on a set of values: $v_4 \in V_4 = \{0,1\}$, where $v_4 = 0$ – "Performing a PR-action in the mass media”, $v_4 = 1$ – “Performing a PR-action in the stores”.

Based on marketing research, the expected behavior of the system after marketing actions (system F_2) will be determined by BF with the fuzzy measure density $f_{1234}(c): C \times V_4 \to [0,1]$ (see Table 4).

Table 4: BF of the system F_2

$\#c, c \in C \times V_4$	v_1	v_2	v_3	v_4	$f_{1234}(c)$
1	0	0	0	0	0.25
2	0	0	0	1	0.22
3	0	0	1	0	0.6
4	0	0	1	1	0.6
5	0	1	0	0	0.28
6	0	1	0	1	0.28
7	0	1	1	0	0.14
8	0	1	1	1	0.18
9	1	0	0	0	0.15
10	1	0	0	1	0.12
11	1	0	1	0	0.25
12	1	0	1	1	0.25
13	1	1	0	0	0.34
14	1	1	0	1	0.35
15	1	1	1	0	0.14
16	1	1	1	1	0.15

In terms of systems theory, our problem is to determine the degree of purposefulness of the new system F_2 in comparison with the purposefulness of the original system F_1 relative to the target system (see Table 3). The purposefulness of the system can be defined as the distance between the BF of the target system and the BF of the projection of the original system onto the set of states of the target system. Here, the projection of the original system is used to provide the condition of the common set of states.

The projection BF is considered as a projection of the measure [25] onto some subset. Based on this, the purposefulness of the system is determined by the expression [1]:

$$\omega(F_2, F_1) = d_{FM}\left(f_{13}^{F_1}(c), f_{13}^{*}(c)\right) - d_{FM}\left(f_{13}^{F_2}(c), f_{13}^{*}(c)\right), \tag{16}$$

where: $d_{FM}(\cdot,\cdot)$ – proposed metric, defined on $\mathcal{F}$;

$f_{13}^{F_1}(c), f_{13}^{F_2}(c)$ – projections BFs of systems F_1 and F_2 onto the common set of states $Proj_{13}C$, which for fuzzy measures can be determined by the following expressions:

$$f_{13}^{F_1}(c) = \frac{1}{\lambda_1} \cdot \left(\prod_{x \in A_c} \left(1 + \lambda_1 \cdot f_{123}(x)\right) - 1 \right), \tag{17}$$

where $A_c = \{x \in C | c = Proj_{13}x\}$;

λ_1 – normalization parameter of fuzzy measure $f_{123}(x): C \to [0,1]$,

$$f_{13}^{F_2}(c) = \frac{1}{\lambda_2} \cdot \left(\prod_{y \in B_c} \left(1 + \lambda_2 \cdot f_{1234}(y)\right) - 1 \right), \tag{18}$$

where $B_c = \{y \in C \times V_4 | c = Proj_{13}y\}$;

λ_2 – normalization parameter of fuzzy measure $f_{1234}(y): C \times V_4 \to [0,1]$.

The higher the purposefulness, the better suited the system is to achieve the goal of the behavior. A negative value of purposefulness indicates that the new F_2 system is worse than the original F_1 system in terms of the purpose of the behavior. Table 5 shows the calculation results of BFs $f_{13}^{F_2}(c)$ and $f_{13}^{F_1}(c)$.

Table 5: Calculation of BFs $f_{13}^{F_2}(c)$ and $f_{13}^{F_1}(c)$

$\#c, c \in Proj_{13}C$	v_1	v_3	$f_{13}^{F_2}(c)$	$f_{13}^{F_1}(c)$
1	0	0	0.6989	0.0447
2	0	1	0.8911	0.1040
3	1	0	0.6810	0.0749
4	1	1	0.5902	0.1346

Figure 9 shows the density of fuzzy measures that are used in calculating the purposefulness of the system. For clarity Figure 9 instead of $f_{13}^{F_2}(c)$ shows $f_{13}^{F_2'}(c) = 0.07 \cdot f_{13}^{F_2}(c)$.

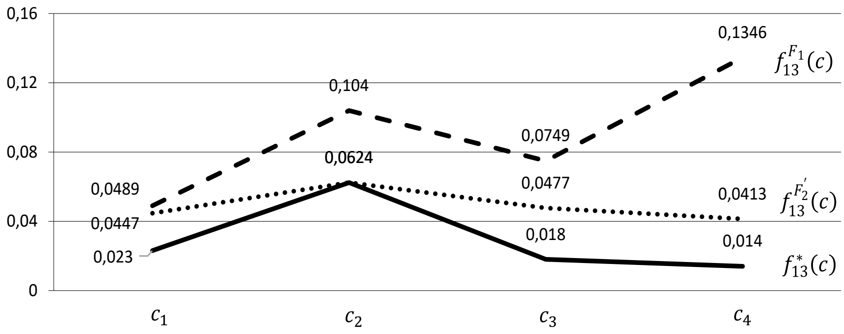


Figure 9: Densities of fuzzy measures that are used in determining the purposefulness of a system

A preliminary analysis of the distributions in Figure 9 shows that $f_{13}^{F_2}(c)$ more closely describes the target system in terms of significant states. Therefore, we can expect the new system F_2 to be more targeted than the system F_1.

Let's compare the use of the proposed metric $d_{FM}(\cdot,\cdot)$ and the previously described Minkowski metric l_1^n to calculate the purposefulness of the system F_2 in relation to the original system F_1. The calculation results are reflected in Table. 5.

Table 5: Values of metrics and estimations of the purposefulness of the system

$d_{FM}\left(f_{13}^{F_1}(c), f_{13}^{*}(c)\right)$	$d_{FM}\left(f_{13}^{F_2}(c), f_{13}^{*}(c)\right)$	$\omega_{FM}(F_2, F_1)$
0.1779	0.0140	0.1639
$l_1^n\left(f_{13}^{F_1}(c), f_{13}^{*}(c)\right)$	$l_1^n\left(f_{13}^{F_2}(c), f_{13}^{*}(c)\right)$	$\omega_{l_1^n}(F_2, F_1)$
0.0309	0.3885	-0.3576

Analysis of the obtained results shows that the proposed metric $d_{FM}(\cdot,\cdot)$ gives a positive estimation of purposefulness. That is, the new system provides an increase in the purposefulness of the behavior of the original system. Therefore, the planned marketing actions are effective, what we expected.

However, the calculation result in accordance with the Minkowski metric is negative, which does not correspond to the expected result from the view-point of significant system states. It can be assumed that in this example, the insignificant states of the system formed a systematic error, which led to an incorrect conclusion. Therefore, the use of the Minkowski metric to determine the difference between BFs described by fuzzy measures can lead to erroneous decisions in practice.

7. CONCLUSIONS

Thus, the proposed distance allows us to use BFs described by Sugeno fuzzy measures to solve system problems in which it is necessary to evaluate the difference in the behavior of systems. The advantage of the proposed distance is that it meets all the requirements that apply to metrics. In particular, the proposed distance is positive, symmetrical, identical, normalized, and subject to the triangle inequality condition. Consequently, the space of BFs described using Sugeno fuzzy measures is a metric space. We can establish an order relation on this space and solve many system problems that require the use of a measure of difference in the behavior of systems. In turn, Sugeno fuzzy measures make it possible to use estimates with different modalities to describe the behavior of systems and to explore the dynamics of systems in the context of modality. Unlike known metrics and distances, the proposed metric gives priority to significant system states over insignificant states. The use of well-known metrics in relation to Sugeno fuzzy measures (in particular, Minkowski distances) can lead to incorrect solutions to system problems.

The limitations of the proposed metric follows from the properties of the fuzzy measure, which must be specified on the set of all subsets of system states. This significantly increases the amount of assessments required, as well as the amount of work to ensure they are adequate and consistent.

This research is the basis for starting work on the definition of an event as one of the fundamental categories of science. In particular, as an event, we consider an impact that causes a change in the behavior of a complex system. In turn, the proposed metric can be considered both as a measure of the strength of an event and as a measure of the system's resistance to external influences.

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