Yugoslav Journal of Operations Research
34 (2024), Number 1, 23–50
DOI: https://doi.org/10.2298/YJOR230115014M

DECISION MAKING IN FUZZY CLOSED LOOP DUAL CHANNEL SUPPLY CHAIN

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Received: January 2023 / Accepted: May 2023

Abstract: This paper discusses the optimal pricing and collecting decisions in a fuzzy environment of the three closed-loop dual-channel supply chain models where a manufacturer produces the original product from fresh materials and unfashionable products. The manufacturer sells the original products to the customers using direct and retail channels (forward dual channel). In the reverse channel, the manufacturer, third-party collector and retailer individually take responsibility for unfashionable products acquired from the market in the three models. Remanufacturing cost, product collecting cost and market demand are treated as fuzzy variables. The optimal decisions wholesale price, direct price, retail price and collection rate are derived using fuzzy set theory and game theory approach in the three decision models. Finally, numerical illustrations clarify the impact of the fuzzy degree parameters on the decision variables and the expected profits of the players. It is found that the manufacturer decision model is superior for receiving the maximum expected profits of the players and also being more convenient from the customer's perspective.

Keywords: Closed loop dual channel supply chain, fuzzy set theory, game theory, remanufacturing.

MSC: 90B06, 90B50, 91A10, 91A86.

1. INTRODUCTION

With increasing environmental awareness, more people are looking for eco-friendly or recyclable products. For this, the customers are interested in paying a higher price. Corporations are embracing green technology to manufacture more environmentally friendly products to minimize pollution. In the literature [1, 2, 3], substantial research on green products has been conducted. On the other hand, most companies are leaning towards closed-loop supply chains. The most significant advantage of a closed-loop supply chain is waste minimization, and also it can help offset the expenses of implementing recycling and reusability programs by increasing sales. Thus, the necessity of a closed-loop supply chain has also been raised in the academic research area over the last two decades. Various research on product remanufacturing in closed-loop supply chains has been conducted over the years. Savaskan et al. [4] designed numerous product acquisition activities for remanufacturing in the closed-loop supply chain. Savaskan and Wassenhove [5] collaborated along the manufacturer's reverse channel selection to gather post-customer commodities and the strategic product pricing choice when competitive retailing in the forward channel. Choi et al. [6] looked at the performance of a closed-loop supply chain with a collector, a retailer and a manufacturer and the supply chain's achievement under different channel powers. Wei and Zhao [7] focused on the remanufacturing and pricing dilemma in two rival closed-loop supply chains. Yoo and Kim [8] described the relationship between the new and refurbished commodities in a three-tier supply chain that included a manufacturer, a refurbisher and a seller. They showed the importance of a supply chain that may control the market sales, like channel friction among new and refurbished products, increasing its gain by implementing the supply chain process and various its formation. Reimann et al. [9] highlighted the relationship between remanufacturing and the scope to base the changeable remanufacturing price on a novelty system in the closed-loop supply chain (CLSC).

Nowadays, the traditional single channel is no longer the only way to sell products. The rapid growth of modern technology and the increasing eagerness of customers to buy products via online channels directly is the other way to sell the product in the supply chain system [10, 11]. Hong et al. [12] focused on selecting the proper reverse channel format for collecting used products from consumers. Saha et al. [13] applied a new reward approach to collect used products for remanufacturing in a CLSC. They constructed the models for centralized and non-cooperative scenarios for characterizing the price rate and remanufacturing techniques that announce individual and overall supply chain players' performance. Giri et al. [14] focused on the utilized product collection and pricing decisions for a CLSC along two dual channels (forward and reverse dual) under five scenarios. Yang et al. [15] explained the remanufacturing techniques and up-gradation of products under the cap-and-trade rule in a dual-channel supply chain. Liu et al. [16] explained channel structure, the manufacturer's price decisions and the government's subsidy policy in a dual-channel closed-loop supply chain competing for new and remanufactured items.

Several unknown ways in the implementing world that impact the supply chain system for success still need to be addressed. These uncertainties are typically combined with market demand, product supply, etc. Zadeh [17] introduced the concept of fuzzy set theory. Liu [18, 19, 20] had a prominent influence on this field. A few researchers have used fuzzy set theory to solve supply chain issues. Zhao et al. [21] discussed the substitutable product's pricing problem by making a supply chain including a manufacturer and two rival retailers in a fuzzy uncertainty. Liu and Xu [22] analyzed pricing decisions under different strategies of a single manufacturer and two rival retailers for a single product. Zhao and Wang [23] explored pricing and retail service strategies in a fuzzy supply chain in three-game structures: manufacturer Stackelberg, retailer Stackelberg and vertical Nash. Khamesh et al. [24] investigated complementary product price decisions in a fuzzy supply chain involving two manufacturers and a single retailer. These articles [25, 26] show how soft computing approaches may handle scheduling problems in the manufacturing sector.

The above discourse does not highlight the remanufacturing and collection decision of used products in a fuzzy CLSC. Wei and Zhao [27, 28] explained how to set prices and collect returned products in a CLSC in fuzzy uncertainty. Fallah et al. [29] considered two CLSCs that involved recyclers, manufacturers and retailers in a fuzzy environment with the market demands of sensitive prices. The competitiveness between the two chains is investigated. The pricing of new products and the incentives given to buyers for unfashionable products are competitive issues. Alamdar et al. [30] investigated a fuzzy CLSC, including a single manufacturer, collector and retailer. They established six models and used game and fuzzy set theory to compare the best answers. They demonstrated how collaboration between retailer and manufacturer benefits consumers and the entire system. Collaboration between the collector and the manufacturer is an adequate model for acquiring used products. Ke et al. [31] discussed the problem of remanufacturing and pricing decisions within a manufacturer, two rival retailers and a third-party collector in a fuzzy closed-loop supply chain. They have proposed two game models under different power structures to formulate the pricing and remanufacturing decision problem. So far, minor work has been done on a dual-channel CLSC in a fuzzy environment. Karimabadi et al. [32] applied a fuzzy dual-channel supply chain constructing a manufacturer and a retailer for remanufacturing and pricing decisions in the centralized and decentralized scenario.

The research gap between proposed work and the existing literature are highlighted below.

−A lot of work is available in the literature where pricing and remanufacturing decisions are discussed in a deterministic environment by developing several closed-loop supply chains. When the market is fraught with uncertainty, the process of evaluating the pricing and collecting decisions of the products within the closed-loop supply chain models is deliberated in this work.

−Previous articles assumed that used items are collected by the manufacturer, 3PC, in the reverse chain. We have demonstrated that the manufacturer, retailer and 3PC take the collection choice via a reverse channel.

−Some researchers have discussed a closed-loop supply chain's pricing and collecting decisions using the dual channel. Based on the return product collection decision, three possible dual-channel closed-loop supply chains have been constructed in a fuzzy environment. We have tried to find out the possible best dual-channel CLSC for both the players and customers. Table 1 summarizes some current research as well as the contributions of this work.

The remaining part of the paper is organized as follows: Section 2 provides some preliminaries of fuzzy set theory. The list of notations are provided in section 3. In section 4, model formulation is described, and optimal decisions of the players in the three decision models are evaluated theoretically. Section 5 discusses outcomes of the problem numerically. Managerial implications are given in section 6. Conclusion is provided in section 7.

Table 1: Overview of relevent closed loop suppy chain's literature

Article	**Remanufacturing decision**			**Deterministic environment**		**Fuzzy environment**	
	M	R	3PC	Forward dual	Reverse	Forward dual	Reverse
Modak et al. [33]	✓	✓	✓		✓		
Arshad et al. [34]	✓		✓	✓	✓		
Chen et al. [35]	✓	✓	✓		✓		
Ji et al. [36]	✓	✓			✓		
Karimabadi et al. [32]	✓					✓	✓
Zhang et al. [37]	✓	✓	✓		✓		✓
Pal and Sana [38]	✓			✓	✓		
Mandal and Pal [39]			✓	✓	✓		
Present study	✓	✓	✓			✓	✓

"M-manufacturer, R-retailer, 3PC- third-party collector"

2. PRELIMINARIES

A "possibility space" is a triplet $(\Omega, \mathcal{P}(\Omega), Pos)$, where $\mathcal{P}(\Omega)$ is the power set of the non empty set Ω, and Pos is the "possibility measure". Every element of $\mathcal{P}(\Omega)$ is said to be a fuzzy event also for every event $\mathcal{A}$, $Pos\Big\{\mathcal{A}\Big\}$ denotes the possibility of the occuremce of $\mathcal{A}$. Nahmias [40] and Liu [41] provided the axioms which are given below

Axiom 1. $Pos\Big\{\Omega\Big\} = 1.$

Axiom 2. $Pos\Big\{\Phi\Big\} = 0$, *Φ is a non empty set.*

Axiom 3. $Pos\Big\{\bigcup_{i=1}^{m} \mathcal{A}_i\Big\} = \underset{1\le i\le m}{Sup}\, Pos\Big\{\mathcal{A}_i\Big\}$ *for any collection $\mathcal{A}_i$ in $\mathcal{P}(\Omega)$.*

Axiom 4. *Let Ω_i are nonempty sets, in which Pos_i is the "possibility measure" that satisfy the* ***Axiom 1,2,3****. $i = 1; 2; ...; n$ and $\Omega = \prod_{i=1}^{n} \Omega_i$. Then,*
$Pos\Big\{\mathcal{A}\Big\} = \underset{(\Omega_1,\Omega_2,...,\Omega_n)\in\mathcal{A}}{Sup} Pos_1\Big\{\Omega_1\Big\} \bigwedge Pos_2\Big\{\Omega_2\Big\} \bigwedge ... \bigwedge Pos_n\Big\{\Omega_n\Big\}$
for every $\mathcal{A} \in \mathcal{P}(\Omega)$. In such a case we write $Pos = \bigwedge_{i=1}^{n} Pos_i$.

Lemma 5. *[41] Let $(\Omega_i, \mathcal{P}(\Omega_i), Pos_i)$ be a "possibility space", $i = 1, 2, ..., n$. By* ***Axiom 4****,*
$(\prod_{i=1}^{n} \Omega_i, \mathcal{P}(\prod_{i=1}^{n} \Omega_i), \bigwedge_{i=1}^{n} Pos_i)$ is said to be the "product possibility space".

Definition 6. *[40] Let $(\Omega, \mathcal{P}(\Omega), Pos)$ be the "possibility space". A fuzzy variable is a function from $(\Omega, \mathcal{P}(\Omega), Pos)$ to the real numbers and its membership function is derived from the possibility by $\mu_\zeta(x) = Pos\{\psi \in \Omega|\zeta(\psi) = x\}, x \in \Re$.*

Definition 7. *[41] A fuzzy variable ζ is called non negative (or positive) if $Pos(\{\zeta < 0\}) = 0 (or Pos(\{\zeta \leq 0\}) = 0)$.*

Definition 8. *[41] Let $g : \Re^n \rightarrow \Re$ is a function also let ζ_i be the fuzzy variables on $(\Omega_i, \mathcal{P}(\Omega_i), Pos_i)$ for $i = 1, 2, ..., n$. Then $\zeta = g(\zeta_1, \zeta_2, ..., \zeta_n)$ be a "fuzzy variable" defined on the "product possibility space"*
$(\prod_{i=1}^{n} \Omega_i, \mathcal{P}(\prod_{i=1}^{n} \Omega_i), \bigwedge_{i=1}^{n} Pos_i)$ as
$\zeta(\psi_1, \psi_2, ..., \psi_n) = g(\zeta_1(\psi_1), \zeta_2(\psi_2), ..., \zeta_n(\psi_n))$ for any $(\psi_1, \psi_2, ..., \psi_n) \in \prod_{i=1}^{n} \Omega_i$.

Definition 9. *[41] The fuzzy variables $\zeta_1, \zeta_2, ..., \zeta_n$ are independent if*

$$Pos\{\zeta_i \in \mathcal{O}_i, i = 1, 2, ..., n\} = \underset{1\leq i\leq n}{Min} Pos\{\zeta_i \in \mathcal{O}_i\}$$

for any of the sets $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n$ of $\Re$.

Lemma 10. *[20] Let $g_i : \Re \rightarrow \Re$ be the function and ζ_i be the independent fuzzy variables for $i = 1, 2, ..., m$. then $g_1(\zeta_1), g_2(\zeta_2), ..., g_m(\zeta_m)$ are independent fuzzy variables.*

Definition 11. *[41] Let ζ be a fuzzy variable on $(\Omega, \mathcal{P}(\Omega), Pos)$, and $0 < \alpha \leq 1$. Then*

$$\zeta_\alpha^{\mathrm{L}} = \inf\{t|Pos\{\zeta \leq t\} \geq \alpha\} \quad , \quad \zeta_\alpha^{\mathrm{U}} = \sup\{t|Pos\{\zeta \geq t\} \geq \alpha\}$$

are said to be the "α-pessimistic" and "α-optimistic" value of ζ, respectively.

Example 12. *Let $\zeta = (u_1, u_2, u_3)$ be a "triangular fuzzy variable". The "α- pessimistic" and "α-optimistic" values of ζ are*

$$\zeta_\alpha^{\mathrm{L}} = u_2\alpha + u_1(1 - \alpha) \quad , \quad \zeta_\alpha^{\mathrm{U}} = u_2\alpha + u_3(1 - \alpha).$$

Lemma 13. *[18] Let ζ and μ be independent fuzzy variables. Then for any $0 < \alpha \leq 1$*

(a) $(\zeta+\mu)_\alpha^{\mathrm{L}} = \zeta_\alpha^{\mathrm{L}} + \mu_\alpha^{\mathrm{L}}$ *and* $(\zeta+\mu)_\alpha^{\mathrm{U}} = \zeta_\alpha^{\mathrm{U}} + \mu_\alpha^{\mathrm{U}}$,

(b) $(\zeta-\mu)_\alpha^{\mathrm{L}} = \zeta_\alpha^{\mathrm{L}} - \mu_\alpha^{\mathrm{L}}$ *and* $(\zeta-\mu)_\alpha^{\mathrm{U}} = \zeta_\alpha^{\mathrm{U}} - \mu_\alpha^{\mathrm{U}}$,

(c) $(\zeta\cdot\mu)_\alpha^{\mathrm{L}} = \zeta_\alpha^{\mathrm{L}} \cdot \mu_\alpha^{\mathrm{L}}$ *and* $(\zeta\cdot\mu)_\alpha^{\mathrm{U}} = \zeta_\alpha^{\mathrm{U}} \cdot \mu_\alpha^{\mathrm{U}}$.

Definition 14. *Let ζ and μ be positive independent "fuzzy variables", $\zeta > \mu$ if and only if for any $0 < \alpha \le 1$, $\zeta_\alpha^{\mathrm{L}} > \mu_\alpha^{\mathrm{L}}$ and $\zeta_\alpha^{\mathrm{U}} > \mu_\alpha^{\mathrm{U}}$.*

Lemma 15. *[42] Let ζ_i be the independent fuzzy variables defined on $(\Omega_i, \mathcal{P}(\Omega_i), Pos_i)$ having continuous membership function, for $i = 1, 2, \ldots, n$. Let $X \subset \Re^n$ and $g : X \to \Re$ is a measurable function. If $g(x_1, x_2, ..., x_n)$ is monotonic function with respect to x_i, then*

(a) $g_\alpha^{\mathrm{U}}(\zeta) = g(\zeta_{1\alpha}^{\vee}, \zeta_{2\alpha}^{\vee}, ..., \zeta_{n\alpha}^{\vee})$, *where* $\zeta_{i\alpha}^{\vee} = \zeta_{i\alpha}^{\mathrm{U}}$, *if* $g(x_1, x_2, ..., x_n)$ *is non decreasing with respect to* x_i, $\zeta_{i\alpha}^{\vee} = \zeta_{i\alpha}^{\mathrm{L}}$, *otherwise*

(b) $g_\alpha^{\mathrm{L}}(\zeta) = g(\zeta_{1\alpha}^{\overline{\vee}}, \zeta_{2\alpha}^{\overline{\vee}}, ..., \zeta_{n\alpha}^{\overline{\vee}})$, *where* $\zeta_{i\alpha}^{\overline{\vee}} = \zeta_{i\alpha}^{\mathrm{L}}$, *if* $g(x_1, x_2, ..., x_n)$ *is non decreasing with respect to* x_i, $\zeta_{i\alpha}^{\overline{\vee}} = \zeta_{i\alpha}^{\mathrm{U}}$, *otherwise. where* $g_\alpha^{\mathrm{U}}(\zeta)$ *and* $g_\alpha^{\mathrm{L}}(\zeta)$ *denote the "α-optimistic" value and the "α-pessimistic" value of the fuzzy variable* $g(\zeta)$, *respectively.*

Definition 16. *[18] Let $(\Omega, \mathcal{P}(\Omega), Pos)$ be a possibility space and $\mathcal{A}$ be a set in $\mathcal{P}(\Omega)$. The credibility measure of $\mathcal{A}$ is denoted by $Cr\{\mathcal{A}\}$ and is defined by $Cr\{\mathcal{A}\} = \frac{(1+Pos\{\mathcal{A}\}-Pos\{\mathcal{A}^c\})}{2}$ where $\mathcal{A}^c$ is the complement of $\mathcal{A}$.*

Definition 17. *[18] Let ζ be a "fuzzy variable". Then the expected value $E[\zeta]$ is defined by*

$$E[\zeta] = \int_0^{+\infty} Cr\{\zeta \ge x\}dx - \int_{-\infty}^{0} Cr\{\zeta \le x\}dx.$$

Provided at least one of the integrals is finite.

Example 18. *Let $\zeta = (u_1, u_2, u_3)$ be a "triangular fuzzy variable" and its expected value is*

$$E[\zeta] = \frac{u_1 + 2u_2 + u_3}{4}.$$

Definition 19. *[18] Let $g : \Re \to \Re$ be a function and ζ be a "fuzzy variable". Then the expected value $E[g(\zeta)]$ is defined by*

$$E[g(\zeta)] = \int_0^{+\infty} Cr\{g(\zeta) \ge x\}dx - \int_{-\infty}^{0} Cr\{g(\zeta) \le x\}dx.$$

Provided at least one of the integrals is finite.

Lemma 20. *[19] Let ζ be a "fuzzy variable" and the expected value of ζ is finite. Then* $E[\zeta] = \frac{1}{2}\int_0^1 (\zeta_\alpha^{\mathrm{L}} + \zeta_\alpha^{\mathrm{U}})d\alpha$.

Definition 21. *Let ζ and μ be positive independent "fuzzy variables" defined on $(\Omega, \mathcal{P}(\Omega), Pos)$ and $0 < \alpha \leq 1$, if $\zeta > \mu$ then,*
$E[\tilde{\zeta}] > E[\tilde{\mu}]$.

Lemma 22. *[19] Let ζ and μ be independent fuzzy variables and $E[\zeta]$, $E[\mu]$ are finite. Then $E[a\zeta + b\mu] = aE[\zeta] + bE[\mu]$ for any numbers a and b.*

Definition 23. *Let ζ and μ be "fuzzy variables" defined on $(\Omega, \mathcal{P}(\Omega), Pos)$ and $0 < \alpha \leq 1$, then we define*

$$E[\tilde{\zeta}\tilde{\mu}] = \frac{1}{2}\int_0^1 (\tilde{\zeta}_\alpha^{\mathrm{L}}\tilde{\mu}_\alpha^{\mathrm{L}} + \tilde{\zeta}_\alpha^{\mathrm{U}}\tilde{\mu}_\alpha^{\mathrm{U}})d\alpha \quad , \quad E[\tilde{\zeta}_\alpha^{\mathrm{L}}\tilde{\mu}_\alpha^{\mathrm{U}}] = \frac{1}{2}\int_0^1 (\tilde{\zeta}_\alpha^{\mathrm{L}}\tilde{\mu}_\alpha^{\mathrm{U}} + \tilde{\zeta}_\alpha^{\mathrm{U}}\tilde{\mu}_\alpha^{\mathrm{L}})d\alpha.$$

3. NOTATIONS

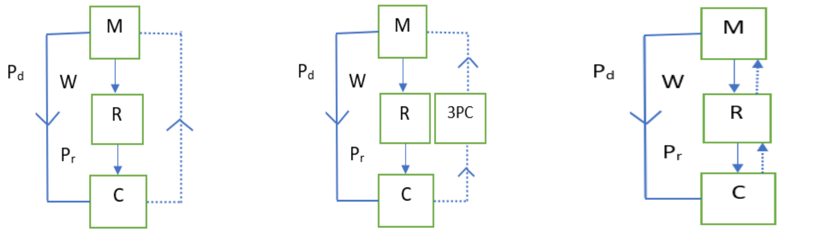
Table 2 presents a description of the notations used in this work.

Table 2: Notations of the mathematical terms

Notations	Definition
W	Unit wholesale price.
P_r	Unit retail price.
P_d	Unit direct price.
τ	Used product collection rate.
D_r	Retail channel demand.
D_d	Direct channel demand.
c_m	Unit manufacturing cost.
p_b	Unit transfering cost.
ρ	Share of demand goes to direct channel.
k	Scalling parameter.
$\tilde{c}_r$	Unit remanufacturing cost, a fuzzy variable.
$\tilde{p}_c$	Unit collecting cost, a fuzzy variable.
$\tilde{a}$	Primary market demand, a fuzzy variable.
$\tilde{\beta}$	Self-price sensitivity, a fuzzy variable.
$\tilde{\gamma}$	Cross-price sensitivity, a fuzzy variable.
$c(\tau)$	Total collecting cost.
Π_M	Profit of manufacturer.
Π_r	Profit of retailer.
Π_{3PC}	Profit of third party collector.
Π_C	System's total profit.

The following annotations are used throughout the work for ease of computation:

$p_1 = E[\tilde{\beta}]$, $p_2 = E[\tilde{\gamma}]$, $p_7 = E[\tilde{a}]$, $p_3 = E[\tilde{c_r}\tilde{\beta}] = \frac{1}{2}\int\limits_0^1(\tilde{c_r}_{\alpha}^{\mathrm{L}}\tilde{\beta}_{\alpha}^{\mathrm{L}} + \tilde{c_r}_{\alpha}^{\mathrm{U}}\tilde{\beta}_{\alpha}^{\mathrm{U}})d\alpha$,
$p_4 = E[\tilde{p_c}\tilde{\beta}] = \frac{1}{2}\int\limits_0^1(\tilde{p_c}_{\alpha}^{\mathrm{L}}\tilde{\beta}_{\alpha}^{\mathrm{L}} + \tilde{p_c}_{\alpha}^{\mathrm{U}}\tilde{\beta}_{\alpha}^{\mathrm{U}})d\alpha$, $p_5 = E[\tilde{c_r}_{\alpha}^{\mathrm{L}}\tilde{\gamma}_{\alpha}^{\mathrm{U}}] = \frac{1}{2}\int\limits_0^1(\tilde{c_r}_{\alpha}^{\mathrm{L}}\tilde{\gamma}_{\alpha}^{\mathrm{U}} + \tilde{c_r}_{\alpha}^{\mathrm{U}}\tilde{\gamma}_{\alpha}^{\mathrm{L}})d\alpha$,
$p_6 = E[\tilde{p_c}_{\alpha}^{\mathrm{L}}\tilde{\gamma}_{\alpha}^{\mathrm{U}}] = \frac{1}{2}\int\limits_0^1(\tilde{p_c}_{\alpha}^{\mathrm{L}}\tilde{\gamma}_{\alpha}^{\mathrm{U}} + \tilde{p_c}_{\alpha}^{\mathrm{U}}\tilde{\gamma}_{\alpha}^{\mathrm{L}})d\alpha$, $p_8 = E[\tilde{c_r}_{\alpha}^{\mathrm{L}}\tilde{a}_{\alpha}^{\mathrm{U}}] = \frac{1}{2}\int\limits_0^1(\tilde{c_r}_{\alpha}^{\mathrm{L}}\tilde{a}_{\alpha}^{\mathrm{U}} + \tilde{c_r}_{\alpha}^{\mathrm{U}}\tilde{a}_{\alpha}^{\mathrm{L}})d\alpha$,
$p_9 = E[\tilde{p_c}_{\alpha}^{\mathrm{L}}\tilde{a}_{\alpha}^{\mathrm{U}}] = \frac{1}{2}\int\limits_0^1(\tilde{p_c}_{\alpha}^{\mathrm{L}}\tilde{a}_{\alpha}^{\mathrm{U}} + \tilde{p_c}_{\alpha}^{\mathrm{U}}\tilde{a}_{\alpha}^{\mathrm{L}})d\alpha$.



(a) Manufacturer decision (b) 3PC decision (c) Retailer decision

Figure 1: Schematic diagram of the models

4. MODEL FORMULATION

In this study, we assume the three closed-loop dual-channel supply chain decision models (manufacturer decision, third-party collector decision and retailer decision): which are constituted by one manufacturer, one third-party collector and one retailer. The manufacturer produces original goods from new materials and used products by a unit manufacturing cost of c_m and a unit remanufacturing cost of $\tilde{c}_r$, a fuzzy variable. In each model, the manufacturer wholesales the original products to the retailer via the retail channel at a unit wholesale price W and sells the product to customers using the direct channel with unit direct pricing P_d, a decision variable. The retailer offers it to customers for the unit retail price P_r, a decision variable. The manufacturer, third-party collector and retailer collect the unfashionable products from the customers in the three decision models through the reverse channel. $\tau(0 \leq \tau \leq 1)$ be the used product collection rate. And similar to Savaskan et al. [4] collecting cost $c(\tau) = k\tau^2 + \tilde{p}_c\tau D$, where $k\tau^2$ signifies the fixed amount cost provided to the customers who return unfashionable products, k is a scaling parameter. $\tilde{p}_c$ be the unit collecting cost which is a fuzzy variable. τD be the total amount of unfashionable return products. In the manufacturer decision model, the manufacturer gives the customers the cost $c(\tau)$. In the third-party collector decision model, the third-party collector gives the cost $c(\tau)$ to the customers and handover the unfashionable product to the manufacturer for remanufacturing. And manufacturer provides a unit transfer price p_b to the third-party collector. In the retailer decision model, the retailer takes responsibility for the third-party collector's place in the third-party collector decision model. Figure 1 describes the structure of the models. Retail channel and direct channel demands are the following:

$$D_r = (1-\rho)\tilde{a} - \tilde{\beta}P_r + \tilde{\gamma}P_d, \tag{1}$$

$$D_d = \rho\tilde{a} - \tilde{\beta}P_d + \tilde{\gamma}P_r, \tag{2}$$

$$D = \tilde{a} - \tilde{\beta}(P_r + P_d) + \tilde{\gamma}(P_r + P_d). \tag{3}$$

D be the total market demand, where $\tilde{a}$ be the primary market demand, $\tilde{\beta}$ be the product's self-price sensitivity to the price of another channel, $\tilde{\gamma}$ be the cross-price sensitivity which is all fuzzy variable. ρ is the share of demand that goes to the direct channel. In this paper, we consider some assumptions as follows:

(1) The fuzzy variables $\tilde{a}, \tilde{\beta}, \tilde{\gamma}, \tilde{c}_r, \tilde{p}_c$ are non negative and mutualy independent. The channel's own price has a more substantial influence on channel demand than the price of the rival channel i.e., $E[\tilde{\beta}] > E[\tilde{\gamma}]$.

(2) For the sake of simplicity, the collecting rate τ of used products is the same in all models, and we ignore the transportation cost.

(3) Manufacturer, retailer and third-party have complete information about the demand of the dual channel and price of the market.

(4) The problem is discussed in a single period.

4.1. Manufacturer collection decision

Manufacturer announces first direct price (P_d), wholesale price (W) and collection rate (τ). After receiving the manufacturer's decision, the retailer sets the retail price (P_r) decision. The retailer's profit function is given by

$$\Pi_r(P_r) = (P_r - W)D_r. \tag{4}$$

Retailer maximizes the expected profit using Eq.(4) and Lemma 22

$$\begin{aligned} E[\Pi_r(P_r)] &= (P_r - W)\Big((1-\rho)E[\tilde{a}] - P_rE[\tilde{\beta}] + P_dE[\tilde{\gamma}]\Big) \\ &= (P_r - W)\Big((1-\rho)p_7 - P_rp_1 + P_dp_2\Big). \end{aligned} \tag{5}$$

Proposition 24. *Under the manufacturer collection decision, the retail price is given by*

$$P_r^* = \frac{(1-\rho)p_7 + Wp_1 + P_dp_2}{2p_1}. \tag{6}$$

Proof. From Eq.(5), first and second-order derivatives with respect to retail price are given by

$$\frac{\partial E[\Pi_r(P_r)]}{\partial P_r} = (1-\rho)p_7 - 2P_r p_1 + P_d p_2 + W p_1, \tag{7}$$

$$\frac{\partial^2 E[\Pi_r(P_r)]}{\partial P_r^2} = -2p_1 < 0. \tag{8}$$

From Eq.(8), $E[\Pi_r(P_r)]$ is concave. Solving Eq.(7) equal to zero, **Proposition 24** is proved. □

Manufacturer's profit function is given by

$$\Pi_M(W, P_d, \tau) = WD_r + P_d D_d - Dc_m(1-\tau) - D\tilde{c_r}\tau - c(\tau). \tag{9}$$

Manufacturer maximizes his expected profit using Eqs.(6), (9) and Lemmas 15, 22 and Definitions 11, 23

$$\begin{aligned} E[\Pi_M(W, P_d, \tau)] &= W((1-\rho)E[\tilde{a}] - P_r^* E[\tilde{\beta}] + P_d E[\tilde{\gamma}]) + P_d \\ &\quad (\rho E[\tilde{a}] - E[\tilde{\beta}]P_d + E[\tilde{\gamma}]P_r^*) - c_m(1-\tau) \\ &\quad (E[\tilde{a}] - E[\tilde{\beta}](P_r^* + P_d) + E[\tilde{\gamma}](P_r^* + P_d)) - \\ &\quad \tau(E[\tilde{c_r}{}_\alpha^{\mathrm{L}}\tilde{a}_\alpha^{\mathrm{U}}] + E[\tilde{p_c}{}_\alpha^{\mathrm{L}}\tilde{a}_\alpha^{\mathrm{U}}] - (E[\tilde{c_r}\tilde{\beta}] + E[\tilde{p_c}\tilde{\beta}]) \\ &\quad (P_r^* + P_d) + (E[\tilde{c_r}{}_\alpha^{\mathrm{L}}\tilde{\gamma}_\alpha^{\mathrm{U}}] + E[\tilde{p_c}{}_\alpha^{\mathrm{L}}\tilde{\gamma}_\alpha^{\mathrm{U}}])(P_r^* + P_d)) \\ &\quad -k\tau^2 \\ &= W((1-\rho)p_7 - P_r^* p_1 + P_d p_2) + P_d(\rho p_7 - \\ &\quad P_d p_1 + P_r^* p_2) - c_m(1-\tau)(p_7 - p_1(P_r^* + P_d) \\ &\quad + p_2(P_r^* + P_d)) - \tau(p_8 + p_9 - (p_3 + p_4) \\ &\quad (P_r^* + P_d) + (p_6 + p_5)(P_r^* + P_d)) - k\tau^2. \end{aligned} \tag{10}$$

Proposition 25. *If* $k > \frac{(p_1+p_2)(3p_1+p_2)(p_5+p_6-p_3-p_4+c_m(p_1-p_2))^2}{8p_1(p_1^2-p_2^2)}$ *holds,*

(i) $E[\Pi_M(W, P_d, \tau)]$ *is concave in* W, P_d *and* τ.

(ii) Wholesale price (W^*), *direct channel price* (P_d^*) *and collecting rate* (τ^*) *are given by*

$$W^* = \frac{A_1 + A_2 c_m + A_3 c_m^2}{2(p_1+p_2)(B_1 + B_2 c_m - A_9 c_m^2)}, \tag{11}$$

$$P_d^* = \frac{A_4 + A_5 c_m + A_6 c_m^2}{2(p_1+p_2)(B_1 + B_2 c_m - A_9 c_m^2)}, \tag{12}$$

$$\tau^* = -\frac{A_7 + A_8 c_m + A_9 c_m^2}{B_1 + B_2 c_m - A_9 c_m^2}. \tag{13}$$

Where $A_1, A_2, ..., A_9$, B_1 *and* B_2 *are given in the Appendix.*

Proof. From Eq.(10), the first order derivatives of
$E[\pi_M(W, P_d, \tau)]$ with respect to W, P_d and τ are respectively

$$\begin{aligned}\frac{\partial E[\Pi_M(W,P_d,\tau)]}{\partial W} &= -Wp_1 + P_d p_2 + \frac{\tau}{2}((c_m(p_2-p_1)+p_3+p_4 \\ &-p_5-p_6) + \frac{1}{2}((1-\rho)p_7 + c_m(p_1-p_2)), \end{aligned} \quad (14)$$

$$\begin{aligned}\frac{\partial E[\Pi_M(W,P_d,\tau)]}{\partial P_d} &= Wp_2 + P_d(\frac{p_2^2}{p_1} - 2p_1) + \frac{\tau}{2}((p_3+p_4-p_5 \\ &-p_6)(\frac{p_2}{p_1}+2) - c_m(2p_1 - p_2 - \frac{p_2^2}{p_1})) + p_7 \\ &(\rho + (1-\rho)\frac{p_2}{2p_1} + c_m(p_1 - \frac{p_2}{2} - \frac{p_2^2}{2p_1}), \end{aligned} \quad (15)$$

$$\begin{aligned}\frac{\partial E[\Pi_M(W,P_d,\tau)]}{\partial \tau} &= \frac{W}{2}(c_m(p_2-p_1)+p_3+p_4-p_5-p_6) + \frac{P_d}{2} \\ &((p_3+p_4-p_5-p_6)(\frac{p_2}{p_1}+2) - c_m(2p_1-p_2 \\ &-\frac{p_2^2}{p_1})) - 2k\tau + c_m\frac{p_7}{2}((1+\rho)+(1-\rho)\frac{p_2}{p_1}) \\ &+(1-\rho)\frac{p_7}{2p_1}(p_3+p_4-p_5-p_6) - p_8 - p_9. \end{aligned} \quad (16)$$

The Hessian matrix is obtained by

$$H_1 = \begin{pmatrix} \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial W^2} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial W \partial P_d} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial W \partial \tau} \\ \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial W \partial P_d} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial {P_d}^2} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial \tau \partial P_d} \\ \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial \tau \partial W} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial \tau \partial P_d} & \frac{\partial^2 E[\Pi_M(W,P_d,\tau)]}{\partial \tau^2} \end{pmatrix}$$

$$= \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_4 & G_5 \\ G_3 & G_5 & G_6 \end{pmatrix}.$$

The values of $G_1, G_2, ..., G_6$ are given in the Appendix. The principal minors $G_1 < 0$, $G_1G_4 - G_2^2 = 2(p_1^2 - p_2^2) > 0$ is obvious. The Hessian matrix is negative definite if Det $H_1 < 0$ i.e., $E[\Pi_M(W, P_d, \tau)]$ is concave in W, P_d and τ if $k > \frac{(p_1+p_2)(3p_1+p_2)(p_5+p_6-p_3-p_4+c_m(p_1-p_2))^2}{8p_1(p_1^2-p_2^2)}$.
Setting Eqs.(14) - (16) equal to zero and solve simultaneously we get Eqs.(11) - (13). □

Proposition 26. *Under manufacturer collection decision, if* $k > \frac{(p_1+p_2)(3p_1+p_2)(p_5+p_6-p_3-p_4+c_m(p_1-p_2))^2}{8p_1(p_1^2-p_2^2)}$ *holds, then optimal retail price* (P_r^{**}) *is given by*

$$P_r^{**} = \frac{(1-\rho)p_7 + W^* p_1 + P_d^* p_2}{2p_1}. \quad (17)$$

Proof. Using **propositions 24** and **25**, we have **proposition 26**. □

4.2. Third party collection decision

In this decision model, the manufacturer first sets the wholesale and direct prices; then, the retailer determines the retail price, and a third-party collector determines the collection rate. We first provide the retailer's profit function, then the third-party collector's profit function. So, retailer's profit is

$$\Pi_r(P_r) = (P_r - W)D_r. \quad (18)$$

Retailer maximize the expected profit using Eq.(18) and Lemma 22

$$\begin{aligned} E[\Pi_r(P_r)] &= (P_r - W)((1-\rho)E[\tilde{a}] - P_r E[\tilde{\beta}] + P_d E[\tilde{\gamma}]) \\ &= (P_r - W)((1-\rho)p_7 - P_r p_1 + P_d p_2). \end{aligned} \quad (19)$$

Proposition 27. *Under third-party collection decision, the retail price* (P_r^*) *is given by*

$$P_r^* = \frac{(1-\rho)p_7 + W p_1 + P_d p_2}{2p_1}. \quad (20)$$

Proof. From Eq.(19), the first order derivatives of $E[\Pi_r(P_r)]$ with respect to P_r are

$$\frac{\partial E[\Pi_r(P_r)]}{\partial P_r} = (1-\rho)p_7 - 2P_r p_1 + P_d p_2 + W p_1, \quad (21)$$

$$\frac{\partial^2 E[\Pi_r(P_r)]}{\partial P_r^2} = -2p_1 < 0. \quad (22)$$

Solving Eq.(21) equal to zero, **Proposition 27** is proved. □

Third-party collector's profit is given by

$$\Pi_{3PC}(\tau) = p_b \tau D - c(\tau). \quad (23)$$

Expected profit of 3PC using Eqs.(20), (23) and Lemmas 15, 22 and Definitions 11, 23 is

$$\begin{aligned}
E[\Pi_{3PC}(\tau)] &= p_b\tau(E[\tilde{a}] - E[\tilde{\beta}](P_r^* + P_d) + E[\tilde{\gamma}](P_r^* + P_d)) - \tau \\
&(E[\tilde{p_c}_\alpha^{\mathrm{L}}\tilde{a}_\alpha^{\mathrm{U}}] - E[\tilde{p_c}\tilde{\beta}](P_r^* + P_d) + E[\tilde{p_c}_\alpha^{\mathrm{L}}\tilde{\gamma}_\alpha^{\mathrm{U}}](P_r^* + P_d)) \\
&-k\tau^2 \\
&= p_b\tau p_7 + p_b\frac{\tau}{2}p_2(P_d + W) - p_b\tau p_1(P_d + \frac{W}{2}) + \tau p_4 \\
&(P_d + \frac{W}{2} + \frac{P_d p_2}{2p_1} + \frac{(1-\rho)p_7}{2p_1}) - \tau p_6(P_d + \frac{W}{2} + \\
&\frac{P_d p_2}{2p_1}) + \frac{p_b\tau p_2}{2p_1}((1-\rho)p_7 + P_d p_2) - \tau p_9 - k\tau^2. \qquad (24)
\end{aligned}$$

Proposition 28. *Under third-party collection decision, collection rate* (τ^*) *is given by*

$$\begin{aligned}
\tau^* &= \frac{1}{2k}(\frac{W}{2}(p_b(p_2 - p_1) + p_4 - p_6) + P_d((1 + \frac{p_2}{2p_1})((p_4 - p_6) \\
&+(p_2 - p_1)p_b)) + (\frac{p_7}{2p_1}(1-\rho)((p_4 - p_6) + p_2 p_b) + \frac{p_7}{2p_1}p_b \\
&(1+\rho) - p_9)). \qquad (25)
\end{aligned}$$

Proof. By Eq.(24), first and second-order derivatives of $E[\Pi_{3PC}(\tau)]$ with respect to τ are given by

$$\begin{aligned}
\frac{\partial E[\Pi_{3PC}(\tau)]}{\partial\tau} &= p_b p_7 + p_b\frac{1}{2}p_2(P_d + W) - p_b p_1(P_d + \frac{W}{2}) + p_4(P_d + \frac{W}{2} \\
&+\frac{P_d p_2}{2p_1} + \frac{(1-\rho)p_7}{2p_1}) - p_6(P_d + \frac{W}{2} + \frac{P_d p_2}{2p_1}) + \frac{p_b p_2}{2p_1} \\
&((1-\rho)p_7 + P_d p_2) - p_9 - 2k\tau, \qquad (26)
\end{aligned}$$

$$\frac{\partial^2 E[\Pi_{3PC}(\tau)]}{\partial\tau^2} = -2k < 0. \qquad (27)$$

So, $E[\Pi_{3PC}(\tau)]$ is concave function in τ. We have **proposition 28**, setting Eq.(26) equal to zero. □

Knowing the reactions of the retailer and third-party collector, the manufacturer employs them to optimize the expected profit

$$\Pi_M(W, P_d) = WD_r + P_d D_d - Dc_m(1-\tau) - D\tilde{c_r}\tau - p_b\tau D. \qquad (28)$$

Expected profit of manufacturer using Eqs.(20), (25), (28) and Lemmas 15, 22 and Definitions 11, 23 is

$$\begin{aligned} E[\Pi_M(W,P_d)] &= W((1-\rho)E[\tilde{a}] - P_r^* E[\tilde{\beta}] + P_d E[\tilde{\gamma}]) + P_d(\rho E[\tilde{a}] \\ &-E[\tilde{\beta}]P_d + E[\tilde{\gamma}]P_r^*) - c_m(E[\tilde{a}] - E[\tilde{\beta}](P_r^* + P_d) + \\ &E[\tilde{\gamma}](P_r^* + P_d)) + (c_m - p_b)(E[\tilde{a}] - E[\tilde{\beta}](P_r^* + P_d) \\ &+E[\tilde{\gamma}](P_r^* + P_d))\tau^* - \tau^*(E[\tilde{c_r}_\alpha^{\mathrm{L}} \tilde{a}_\alpha^{\mathrm{U}}] - E[\tilde{c_r}\tilde{\beta}] \\ &(P_r^* + P_d) + E[\tilde{c_r}_\alpha^{\mathrm{L}} \tilde{\gamma}_\alpha^{\mathrm{U}}](P_r^* + P_d)) \\ &= W((1-\rho)p_7 - P_r^* p_1 + P_d p_2) + P_d(\rho p_7 - P_d p_1 \\ &+P_r^* p_2) - c_m(p_7 - p_1(P_r^* + P_d) + p_2(P_r^* + P_d)) + \\ &(c_m - p_b)(p_7 - p_1(P_r^* + P_d) + p_2(P_r^* + P_d))\tau^* - \\ &\tau^*(p_8 - p_3(P_r^* + P_d) + p_5(P_r^* + P_d)). \end{aligned} \tag{29}$$

Proposition 29. *Under third-party collection decision, if* $k > max\left\{\frac{\nabla_1}{4p_1}, \frac{\nabla_2}{4p_1(p_1-p_2)}\right\}$, *the optimal wholesale price* (W^*) *and direct price* (P_d^*) *of manufacturer's are given by*

$$W^* = \frac{C_3C_4 - C_2C_5}{C_2^2 - C_1C_4}, \tag{30}$$

$$P_d^* = \frac{C_1C_5 - C_2C_3}{C_2^2 - C_1C_4}. \tag{31}$$

Where $\nabla_1, \nabla_2, C_1, C_2, ..., C_5$ *are given in the Appendix.*

Proof. From Eq.(29), the first order derivatives of $E[\Pi_M(W,P_d)]$ with respect to W and P_d are

$$\frac{\partial E[\Pi_M(W,P_d)]}{\partial W} = C_1W + C_2P_d + C_3, \tag{32}$$

$$\frac{\partial E[\Pi_M(W,P_d)]}{\partial P_d} = C_2W + C_4P_d + C_5. \tag{33}$$

The Hessian matrix is given by

$$H_2 = \begin{pmatrix} \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W^2} & \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W \partial P_d} \\ \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W \partial P_d} & \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial {P_d}^2} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_4 \end{pmatrix}.$$

The principal minors $G_7 < 0$ if $k > \frac{\nabla_1}{4p_1}$, and Det $H_2 > 0$ if $k > \frac{\nabla_2}{4p_1(p_1-p_2)}$. If $k > max\left\{\frac{\nabla_1}{4p_1}, \frac{\nabla_2}{4p_1(p_1-p_2)}\right\}$ holds, then concavity conditions of H_2 satisfy. Setting Eqs.(32) - (33) equal to zero and solve simultaneously, we get Eqs.(30) - (31). □

Proposition 30. *If $k > max\left\{\frac{\nabla_1}{4p_1}, \frac{\nabla_2}{4p_1(p_1-p_2)}\right\}$, optimal retail price ($P_r^{**}$) of retailer and collection rate (τ^{**}) of third-party collector are given by*

$$P_r^{**} = \frac{(1-\rho)p_7 + W^* p_1 + P_d^* p_2}{2p_1}, \tag{34}$$

$$\begin{aligned}\tau^{**} &= \frac{1}{2k}(\frac{W^*}{2}(p_b(p_2-p_1)+p_4-p_6) + P_d{}^*((1+\frac{p_2}{2p_1})\\ &((p_4-p_6)+(p_2-p_1)p_b)) + (\frac{p_7}{2p_1}(1-\rho)((p_4-p_6)\\ &+p_2p_b) + \frac{p_7}{2p_1}p_b(1+\rho)-p_9)).\end{aligned} \tag{35}$$

Proof. Using **propositions 28** and **29**, **proposition 30** is proved. □

4.3. Retailer collection decision

In this decision model, the manufacturer first sets the direct and wholesale prices, then the retailer decides the retail price and collecting rate. So retailer's profit is given by

$$\Pi_r(P_r, \tau) = (P_r - W)D_r + p_b\tau D - c(\tau). \tag{36}$$

Retailer maximizes his expected profit using Eq.(36) and Lemmas 15, 22 and Definitions 11, 23 is

$$\begin{aligned}E[\Pi_r(P_r,\tau)] &= (P_r-W)((1-\rho)E[\tilde{a}] - P_rE[\tilde{\beta}] + P_dE[\tilde{\gamma}]) + \\ &p_b\tau(E[\tilde{a}] - E[\tilde{\beta}](P_r+P_d) + E[\tilde{\gamma}](P_r+P_d)) - \\ &\tau(E[\tilde{p_c}{}^{\mathrm{L}}_{\alpha}\tilde{a}^{\mathrm{U}}_{\alpha}] - E[\tilde{p_c}\tilde{\beta}](P_r+P_d) + E[\tilde{p_c}{}^{\mathrm{L}}_{\alpha}\tilde{\gamma}^{\mathrm{U}}_{\alpha}] \\ &(P_r+P_d)) - k\tau^2 \\ &= (P_r-W)((1-\rho)p_7 - P_rp_1 + P_dp_2) + p_b\tau \\ &(p_7 - p_1(P_r+P_d) + p_2(P_r+P_d)) - \tau(p_9 - \\ &p_4(P_r+P_d) + p_6(P_r+P_d)) - k\tau^2.\end{aligned} \tag{37}$$

Proposition 31. *Under retailer collection decision, if $k > \frac{(p_b(p_2-p_1)+p_4-p_6)^2}{4p_1}$, then retail price ($P_r^*$) and collecting rate ($\tau^*$) are respectively*

$$\begin{aligned}P_r^* &= \frac{1}{4kp_1 - (p_b(p_1-p_2)-p_4+p_6)^2}[2kp_1W + \\ &((p_b(p_1-p_2)-p_4+p_6)^2 + 2kp_2)P_d + 2k(p_7-p_7\rho) \\ &-(p_b(p_1-p_2)-p_4+p_6)(p_bp_7-p_9)],\end{aligned} \tag{38}$$

$$\begin{aligned}\tau^* &= \frac{1}{4kp_1 - (p_b(p_1-p_2)-p_4+p_6)^2}[p_1((p_4-p_6) + \\ &p_b(p_2-p_1))W - (2p_1+p_2)(p_b(p_1-p_2)-p_4+p_6)P_d \\ &+(1-\rho)p_7(p_4-p_6) + p_bp_7(p_1(1+\rho)+p_2(1-\rho)) - \\ &2p_1p_9].\end{aligned} \tag{39}$$

Proof. From Eq.(37), we get the first order derivatives of $E[\Pi_r(P_r,\tau)]$ with respect to P_r and τ

$$\begin{aligned}\frac{\partial E[\Pi_r(P_r,\tau)]}{\partial P_r} &= -2P_r p_1 + \tau(p_b(p_2-p_1)+p_4-p_6)+Wp_1\\ &+P_d p_2 + (1-\rho)p_7, \end{aligned} \quad (40)$$

$$\begin{aligned}\frac{\partial E[\Pi_r(P_r,\tau)]}{\partial \tau} &= P_r(p_b(p_2-p_1)+p_4-p_6)-2k\tau+P_d\\ &(p_b(p_2-p_1)+p_4-p_6)+p_b p_7 - p_9, \end{aligned} \quad (41)$$

$\frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial P_r^2} = -2p_1, \quad \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial \tau^2} = -2k,$
$\frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial P_r \partial \tau} = \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial \tau \partial P_r} = p_b(p_2-p_1)+p_4-p_6.$
The Hessian matrix is given by

$$H_3 = \begin{pmatrix} \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial P_r^2} & \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial P_r \partial \tau} \\ \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial \tau \partial P_r} & \frac{\partial^2 E[\Pi_r(P_r,\tau)]}{\partial \tau^2} \end{pmatrix}$$

$$= \begin{pmatrix} -2p_1 & p_b(p_2-p_1)+p_4-p_6 \\ p_b(p_2-p_1)+p_4-p_6 & -2k \end{pmatrix}.$$

$-2p_1 < 0$, $DetH_3 = 4kp_1 - (p_b(p_2-p_1)+p_4-p_6)^2 > 0$ if $k > \frac{(p_b(p_2-p_1)+p_4-p_6)^2}{4p_1}$. So H_3 is negative definite if $k > \frac{(p_b(p_2-p_1)+p_4-p_6)^2}{4p_1}$ holds. Setting Eqs.(40) and (41) equal to zero and solve simultaneously, we get Eqs.(38) - (39). □

Manufacturer's profit is given by

$$\Pi_M(W,P_d) = WD_r + P_d D_d - Dc_m(1-\tau) - D\tilde{c_r}\tau - p_b\tau D. \quad (42)$$

Knowing the information of retailer, the manufacturer employs them to maximize the expected profit using Eqs.(42), (38), (39) and Lemmas 15, 22 and Definitions 11, 23 is

$$\begin{aligned} E[\Pi_M(W,P_d)] &= W((1-\rho)E[\tilde{a}] - P_r^* E[\tilde{\beta}] + P_d E[\tilde{\gamma}]) + P_d\\ &(\rho E[\tilde{a}] - E[\tilde{\beta}]P_d + E[\tilde{\gamma}]P_r^*) - c_m(E[\tilde{a}] - E[\tilde{\beta}]\\ &(P_r^* + P_d) + E[\tilde{\gamma}](P_r^* + P_d)) + (c_m - p_b)(E[\tilde{a}]\\ &-E[\tilde{\beta}](P_r^* + P_d) + E[\tilde{\gamma}](P_r^* + P_d))\tau^* - \tau^*\\ &(E[\tilde{c_r}{}^{\mathrm{L}}_{\alpha}\tilde{a}^{\mathrm{U}}_{\alpha}] - E[\tilde{c_r}\tilde{\beta}](P_r^* + P_d) + E[\tilde{c_r}{}^{\mathrm{L}}_{\alpha}\tilde{\gamma}^{\mathrm{U}}_{\alpha}]\\ &(P_r^* + P_d))\\ &= W((1-\rho)p_7 - P_r^* p_1 + P_d p_2) + P_d(\rho p_7 -\\ &P_d p_1 + P_r^* p_2) - c_m(p_7 - p_1(P_r^* + P_d) + p_2\\ &(P_r^* + P_d)) + (c_m - p_b)(p_7 - p_1(P_r^* + P_d) +\\ &p_2(P_r^* + P_d))\tau^* - \tau^*(p_8 - p_3(P_r^* + P_d) + p_5\\ &(P_r^* + P_d)). \end{aligned} \quad (43)$$

Proposition 32. *Under retailer collection decision, if* $k > max\left\{\frac{\nabla_3}{4p_1}, \frac{\nabla_4+\nabla_5}{-8p_1^2(p_1-p_2)}, \frac{\nabla_6+\nabla_5}{8p_1^2(p_1-p_2)}\right\}$*, then manufacturer's optimal decisions wholesale price* (W^*) *and direct price* (P_d^*) *are respectively*

$$W^* = \frac{C_8C_9 - C_7C_{10}}{C_7^2 - C_6C_9}, \tag{44}$$

$$P_d^* = \frac{C_6C_{10} - C_7C_8}{C_7^2 - C_6C_9}. \tag{45}$$

Where $\nabla_3, \nabla_4, \nabla_5, \nabla_6$ *and* $C_6, C_7, ..., C_{10}$ *are given in the Appendix.*

Proof. From Eq.(43), the first order derivatives of $E[\Pi_M(W, P_d)]$ with respect to W and P_d are

$$\frac{\partial E[\Pi_M(W,P_d)]}{\partial W} = C_6W + C_7P_d + C_8, \tag{46}$$

$$\frac{\partial E[\Pi_M(W,P_d)]}{\partial P_d} = C_7W + C_9P_d + C_{10}. \tag{47}$$

The Hessian matrix is given by

$$H_4 = \begin{pmatrix} \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W^2} & \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W \partial P_d} \\ \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial W \partial P_d} & \frac{\partial^2 E[\Pi_M(W,P_d)]}{\partial {P_d}^2} \end{pmatrix} = \begin{pmatrix} C_6 & C_7 \\ C_7 & C_9 \end{pmatrix}.$$

$C_6 < 0$ if, $k > \frac{\nabla_3}{4p_1}$ and Det $H_4 > 0$ if, $k > max\left\{\frac{\nabla_4+\nabla_5}{-8p_1^2(p_1-p_2)}, \frac{\nabla_6+\nabla_5}{8p_1^2(p_1-p_2)}\right\}$. So, from these two conditions together, we get if $k > max\left\{\frac{\nabla_3}{4p_1}, \frac{\nabla_4+\nabla_5}{-8p_1^2(p_1-p_2)}, \frac{\nabla_6+\nabla_5}{8p_1^2(p_1-p_2)}\right\}$ then, H_4 is negative definite. We get Eqs.(44), (45) by setting Eqs.(46) and (47) equal to zero and solving them simultaneously. □

Proposition 33. *Under retailer collection decision if* $k > max\left\{\frac{\nabla_3}{4p_1}, \frac{\nabla_4+\nabla_5}{-8p_1^2(p_1-p_2)}, \frac{\nabla_6+\nabla_5}{8p_1^2(p_1-p_2)}\right\}$ *then, retailer's optimal decisions retail price* (P_r^{**}) *and collecting rate* τ^{**} *are respectively*

$$\begin{aligned} P_r^{**} = & \frac{1}{4kp_1 - (p_b(p_1 - p_2) - p_4 + p_6)^2}[2kp_1W^* + \\ & ((p_b(p_1 - p_2) - p_4 + p_6)^2 + 2kp_2)P_d^* + 2k \\ & (p_7 - p_7\rho) - (p_b(p_1 - p_2) - p_4 + p_6)(p_bp_7 - p_9)], \end{aligned} \tag{48}$$

$$\begin{aligned} \tau^{**} = & \frac{1}{4kp_1 - (p_b(p_1 - p_2) - p_4 + p_6)^2}[p_1((p_4 - p_6) + \\ & p_b(p_2 - p_1))W^* - (2p_1 + p_2)(p_b(p_1 - p_2) - p_4 + p_6) \\ & P_d^* + (1 - \rho)p_7(p_4 - p_6) + p_bp_7(p_1(1 + \rho) + p_2 \\ & (1 - \rho)) - 2p_1p_9]. \end{aligned} \tag{49}$$

Proof. Using **Propositions 31** and **32**, we get **Proposition 33.** □

5. RESULTS AND DISCUSSIONS

This section provides numerical illustrations to analyze the optimal outcomes of three decision models. Considering some of the "triangular fuzzy variables" values, same as that of [28], the relationships between "triangular fuzzy variable" and "linguistic expression" have been obtained, which is shown in Table 3. We assume the case when $k = 2000, c_m = 20, p_b = 7, \rho = 0.2$ and collecting cost per unit ($\tilde{p_c}$) is M (about 4), remanufacturing cost per unit ($\tilde{c_r}$) is M (about 10), primary market demand ($\tilde{a}$) is S (about 200), self price elasticity ($\tilde{\beta}$) is S′ (about 0.8) and cross price elasticity ($\tilde{\gamma}$) is S′ (about 0.4). Using Table 3, we have the α pessimistic and α optimistic values of fuzzy variables listed in Table 4. So, we get $p_1 = 0.8$, $p_2 = 0.4$, $p_3 = 8.2667$, $p_4 = 3.33333$, $p_5 = 3.73333$, $p_6 = 1.46667$, $p_7 = 200$, $p_8 = \frac{5800}{3}$, $p_9 = \frac{2300}{3}$ by applying the values of Table 4. Also, it explored how modifying the fuzzy degree parameters affects the optimum variables, expected profit of all players and whole supply chains' expected profit ($E[\tilde{\Pi}_c]$) in three decision models.

Table 3: Relation between "linguistic expression" and "triangular fuzzy variable"

	"Linguistic expression"	"Triangular fuzzy variable"
	Low (L)	$(1, 3, 5)$
unit collecting cost($\tilde{p_c}$)	Medium (M)	$(2, 4, 6)$
	High (H)	$(4, 6, 8)$
	Low (L)	$(4, 6, 8)$
unit remanufacturing cost($\tilde{c_r}$)	Medium (M)	$(6, 10, 14)$
	High (H)	$(12, 15, 18)$
primary market demand($\tilde{a}$)	Small (S)	$(150, 200, 250)$
	Large (L)	$(250, 300, 350)$
self price elasticity($\tilde{\beta}$)	Sensitive (S')	$(0.6, 0.8, 1.0)$
	Very sensitive (V)	$(1.5, 2.0, 2.5)$
cross price elasticity($\tilde{\gamma}$)	Sensitive (S')	$(0.2, 0.4, 0.6)$
	Very sensitive (V)	$(1.0, 1.5, 2.0)$

Table 4: "α - pessimistic" and "α - optimistic" value of fuzzy variables

$\tilde{\beta}^{\mathrm{L}}_{\alpha} = 0.8\alpha + 0.6(1-\alpha)$	$\tilde{\beta}^{\mathrm{U}}_{\alpha} = 0.8\alpha + 1.0(1-\alpha)$
$\tilde{\gamma}^{\mathrm{L}}_{\alpha} = 0.4\alpha + 0.2(1-\alpha)$	$\tilde{\gamma}^{\mathrm{U}}_{\alpha} = 0.4\alpha + 0.6(1-\alpha)$
$\tilde{a}^{\mathrm{L}}_{\alpha} = 200\alpha + 150(1-\alpha)$	$\tilde{a}^{\mathrm{U}}_{\alpha} = 200\alpha + 250(1-\alpha)$
$\tilde{c_r}^{\mathrm{L}}_{\alpha} = 10\alpha + 6(1-\alpha)$	$\tilde{c_r}^{\mathrm{U}}_{\alpha} = 10\alpha + 14(1-\alpha)$
$\tilde{p_c}^{\mathrm{L}}_{\alpha} = 4\alpha + 2(1-\alpha)$	$\tilde{p_c}^{\mathrm{U}}_{\alpha} = 4\alpha + 6(1-\alpha)$

Table 5: Optimal values of decision variables

collection decision	W^*	P_d^*	P_r^{**}	τ^{**}
Manufacturer decision	159.603	109.603	207.203	0.19828
Third party collector decision	160.358	110.358	207.769	0.265039
Retailer decision	200.215	149.988	237.377	0.124266

Table 6: Expected profits of the players

collection decision	$E[\tilde{\Pi}_M]$	$E[\tilde{\Pi}_r]$	$E[\tilde{\Pi}_{3PC}]$	$E[\tilde{\Pi}_c]$
Manufacturer decision	8548.52	1812.54	–	10361.06
Third party collector decision	8493.99	1798.2	45.0776	10337.2676
Retailer decision	5695.75	1149.2	–	6844.95

Table 7: Change of fuzzy degree parameter $\tilde{p}_c$ vs optimal values of decision variables

collection decision	$\tilde{p}_c$	W^*	P_d^*	P_r^{**}	τ^{**}
Manufacturer decision	$(1,4,7)$	159.61	109.61	207.207	0.213003
	$(2,4,6)$	159.603	109.603	207.203	0.19828
	$(3,4,5)$	159.602	109.602	207.202	0.183555
Third party collector decision	$(1,4,7)$	160.387	110.387	207.79	0.279776
	$(2,4,6)$	160.358	110.358	207.769	0.265039
	$(3,4,5)$	160.33	110.33	207.748	0.250308
Retailer decision	$(1,4,7)$	202.341	152.099	238.953	0.138764
	$(2,4,6)$	200.215	149.988	237.377	0.124266
	$(3,4,5)$	198.155	147.945	235.853	0.109922

Table 8: Change of fuzzy degree parameter $\tilde{p}_c$ vs Expected profits of the players

collection decision	$\tilde{p}_c$	$E[\tilde{\Pi}_M]$	$E[\tilde{\Pi}_r]$	$E[\tilde{\Pi}_{3PC}]$	$E[\tilde{\Pi}_c]$
Manufacturer decision	$(1,4,7)$	8560.63	1812.43	–	10373.06
	$(2,4,6)$	8548.52	1812.54	–	10361.06
	$(3,4,5)$	8537.27	1812.56	–	10349.83
Third party collector decision	$(1,4,7)$	8495.33	1797.66	55.83	10348.82
	$(2,4,6)$	8493.99	1798.2	45.0776	10337.2676
	$(3,4,5)$	8492.65	1798.73	35.197	10326.577
Retailer decision	$(1,4,7)$	5471.89	1125.03	–	6596.92
	$(2,4,6)$	5695.75	1149.2	–	6844.95
	$(3,4,5)$	5928.87	1173.77	–	7102.64

Table 9: Change of fuzzy degree parameter $\tilde{c}_r$ vs optimal values of decision variables

collection decision	$\tilde{c}_r$	W^*	P_d^*	P_r^{**}	τ^{**}
Manufacturer decision	$(5,10,15)$	159.609	109.609	207.207	0.213
	$(6,10,14)$	159.603	109.603	207.203	0.19828
	$(7,10,13)$	159.602	109.602	207.202	0.183551
Third party collector decision	$(5,10,15)$	160.349	110.349	207.761	0.265052
	$(6,10,14)$	160.358	110.358	207.769	0.265039
	$(7,10,13)$	160.368	110.368	207.776	0.265027
Retailer decision	$(5,10,15)$	199.687	149.458	236.979	0.124946
	$(6,10,14)$	200.215	149.988	237.377	0.124266
	$(7,10,13)$	200.75	150.523	237.779	0.123577

Table 10: Change of fuzzy degree parameter $\tilde{c}_r$ vs Expected profits of the players

collection decision	$\tilde{c}_r$	$E[\tilde{\Pi}_M]$	$E[\tilde{\Pi}_r]$	$E[\tilde{\Pi}_{3PC}]$	$E[\tilde{\Pi}_c]$
	$(5, 10, 15)$	8560.63	1812.43	−	10373.06
Manufacturer decision	$(6, 10, 14)$	8548.52	1812.54	−	10361.06
	$(7, 10, 13)$	8537.27	1812.56	−	10349.83
	$(5, 10, 15)$	8509.64	1798.38	45.0865	10353.1065
Third party collector decision	$(6, 10, 14)$	8493.99	1798.2	45.0776	10337.2676
	$(7, 10, 13)$	8478.33	1798.01	45.0687	10321.4087
	$(5, 10, 15)$	5865.81	1157.45	−	7023.26
Retailer decision	$(6, 10, 14)$	5695.75	1149.2	−	6844.95
	$(7, 10, 13)$	5526.25	1140.9	−	6667.15

Table 11: Change of fuzzy degree parameter $\tilde{a}$ vs optimal values of decision variables

collection decision	$\tilde{a}$	W^*	P_d^*	P_r^{**}	τ^{**}
	$(140, 200, 260)$	159.593	109.593	207.195	0.203287
Manufacturer decision	$(150, 200, 250)$	159.603	109.603	207.203	0.19828
	$(160, 200, 240)$	159.613	109.613	207.21	0.193273
	$(140, 200, 260)$	160.349	110.349	207.762	0.266718
Third party collector decision	$(150, 200, 250)$	160.358	110.358	207.769	0.265039
	$(160, 200, 240)$	160.368	110.368	207.776	0.263361
	$(140, 200, 260)$	200.3	150.069	237.436	0.125829
Retailer decision	$(150, 200, 250)$	200.215	149.988	237.377	0.124266
	$(160, 200, 240)$	200.131	149.906	237.317	0.122703

Table 12: Change of fuzzy degree parameter $\tilde{a}$ vs Expected profits of the players

collection decision	$\tilde{a}$	$E[\tilde{\Pi}_M]$	$E[\tilde{\Pi}_r]$	$E[\tilde{\Pi}_{3PC}]$	$E[\tilde{\Pi}_c]$
	$(140, 200, 260)$	8552.54	1812.73	−	10365.27
Manufacturer decision	$(150, 200, 250)$	8548.52	1812.54	−	10361.06
	$(160, 200, 240)$	8544.6	1812.35	−	10356.95
	$(140, 200, 260)$	8497.7	1798.37	45.2856	10341.3286
Third party collector decision	$(150, 200, 250)$	8493.99	1798.2	45.0776	10337.2676
	$(160, 200, 240)$	8490.33	1798.02	43.9079	10332.2579
	$(140, 200, 260)$	5707.1	1148.68	−	6855.78
Retailer decision	$(150, 200, 250)$	5695.75	1149.2	−	6844.95
	$(160, 200, 240)$	5685.31	1149.74	−	6835.05

From Table 5, the manufacturer decision model is better than the third-party collector decision (3PC) model, followed by the retailer decision model. From the customer's viewpoint, this relationship is convenient as the manufacturer and the retailer of the manufacturer's decision model sell the products to the customers at a cheaper rate than the third-party collector decision (3PC) model and retailer decision model's players. The third-party collector decision (3PC) model is better than the manufacturer and retailer decision models for used product collecting decisions sequentially.

From Table 6, the manufacturer decision model gets more advantage from the decision maker's point of view as all the players of the manufacturer decision model gain more expected profits and total expected profit than two decision models third-party collector decision (3PC) model followed by the retailer decision model.

From Table 7, it is observed that in the three decision models, when the fuzzy degree parameter unit collecting cost ($\tilde{p}_c$) decreases, all the decision variables like-wholesale price, direct price, retail price and collecting rate decrease.

From Table 8, when fuzzy degree parameter unit collecting cost ($\tilde{p}_c$) decreases, the retailer decision model will be more beneficial than the other two decisions models, as all the players of the retailer decision model receive their increasing expected profits and total expected profits. In the manufacturer decision model, the manufacturer's expected profit decreases, whereas the retailer's expected profit increases slightly, and the total expected profit reduces. In the third-party collector decision model (3PC), the manufacturer and retailer's expected profits comply with the manufacturer decision model's players. Also, the expected profit of 3PC decreases, and the total expected profit reduces.

From Table 9, when fuzzy degree parameter unit remanufacturing cost ($\tilde{c}_r$) decreases, all the decision variables' value in the manufacturer decision model reduce, on the other side, all the decision variables' value increase except collecting rate in both the 3PC decision and retailer decision models.

From Table 10, when fuzzy degree parameter unit remanufacturing cost ($\tilde{c}_r$) decreases, in the manufacturer decision model, the expected profits of all the players and total expected gains are not affected, as in the case with fuzzy degree parameter $\tilde{p}_c$. The individual and total expected profits of all the players in the 3PC and retailer decision models decrease. When $\tilde{c}_r$ decreases, the retailer in the manufacturer's decision model benefits as profits increase slightly.

From Table 11, when fuzzy degree parameter primary market demand $\tilde{a}$ decreases, all the decision variables W^*, P_r^{**}, P_d^* increase and collection rate τ^{**} decreases in both the decision models manufacturer and 3PC. But in the retailer decision model, all decision variables decrease slightly.

From Table 12, when fuzzy degree parameter $\tilde{a}$ decreases, all the player's expected profits and total expected profit decrease of the manufacturer decision and 3PC decision models. But in the retailer decision model, the retailer's expected profit slightly increases, the manufacturer's expected profit decreases, and total expected profits decrease.

6. MANAGERIAL IMPLICATIONS

We have shed light on product pricing and collecting decisions in fuzzy environments by simulating three dual closed-loop supply chain models. Most research has focused on dual channels in supply chains in deterministic contexts, concluding that customers prefer direct (online) channels. The sensitivity analysis reveals that, even in a fuzzy environment, the direct channel has a more significant influence than the retail channel. It has been observed that the retailer decision model is not ideal for interacting with customers. Compared to the other two models, the players in this model decide the product's pricing at a high rate, the used product's collecting rate is low, and their expected earnings are meager. It is visible that manufacturer and 3PC decision models have competed against one another. The numerical study indicates that the players' incomes in these two models are comparable, and there is not much difference in the price at which they sell the products to the customer. Regarding the used product collection, the 3PC model surpasses the manufacturer decision model. Reviewing this study, the decision-makers can choose the best model with the highest chance of success.

7. CONCLUSION

Environmental contamination is becoming more severe by the day. Polluting activities are widespread in the energy, agricultural and industrial sectors. Pollution may be reduced in the industrial sector by altering a production process to create less trash, recycling items such as containers and pallets rather than discarding them as garbage, and so on. In these circumstances, the closed-loop supply chain is crucial as it minimizes waste by recycling used products. Nowadays, people lead different lives than in the past. People find going to the market in today's hectic society uncomfortable and time-consuming. As days go by, companies are advancing. Currently, most companies are using online channels along with retail channels so that customers get better service. To reduce pollution and increase customer satisfaction, a closed-loop dual-channel supply chain has been developed.

In this paper, three closed-loop dual-channel decision models are considered where in the first model manufacturer takes responsibility for used product collection; in the second model third-party collector (3PC); and in the third model retailer takes responsibility for collecting the used product from the customer. The manufacturer sells the original products through the dual-channel (direct and retail channels). The products collector collects the unfashionable products from the market and sent back to the manufacturer for remanufacturing through the reverse channel. Collecting cost, remanufacturing cost and market demand are "triangular fuzzy variables". The three decision models are solved using fuzzy uncertainty theory and game theory, and optimal direct price, retail price, collection rate and expected profit of the players are derived. Numerical illustrations are given to demonstrate the solution of the models. Also, the effect of fuzzy degree parameters on the players' optimal decision variables and expected profits are discussed. As a result, customers are more flexible in dealing with the manufacturer decision model followed by the third-party decision model (3PC) than the retailer decision model. The players of the manufacturer decision model receive their maximum expected profits and total supply chain expected profits, followed by the third-party decision model (3PC) and retailer decision model. The results of this study will assist the players in choosing the best decision-making model to handle clients.

There are several limitations to this study. The activity of the direct route for used goods collecting is not considered. In future studies, the direct channel can be used for unfashionable product collections for the three models, which makes the CLSC perfect as the forward and reverse flow of the products occurs smoothly.

Further, we opted for the manufacturer's Stackelberg power strategy in this paper. To ascertain the results of the models and establish which model is most benevolent to both chain members and customers, the other power strategies, such as retailer power and Nash strategies, may also be applied in future work. It is assumed that the remanufactured products are sold at the same price as the original products. So, One can use different selling prices for remanufactured products in this problem to extend this work. This issue can be explored and compared with deterministic and fuzzy situations. In addition, fuzzy numbers such as trapezoidal and Gaussian can be considered in place of triangular fuzzy numbers. Governmental incentives might be added to this issue to boost player efficacy, opening up a new line of inquiry.

Acknowledgement. The first Author is thankful to Council of Scientific and Industrial Research(CSIR), India for financial assistance in the form of Senior Research Fellowship (File no: 09/1007(0006)/2019-EMR-I).

The authors would like to sincerely thank the anonymous reviewers for their insightful comments those enabled to improve the technical quality of this article.

Funding. No funding is supported.

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Appendix

$$G_1 = -p_1,\ G_2 = p_2,\ G_3 = \frac{1}{2}((c_m(p_2 - p_1) + (p_3 + p_4) - (p_5 + p_6)),$$
$$G_4 = \frac{p_2^2}{p_1} - 2p_1,\ G_5 = c_m(\frac{p_2}{2}(\frac{p_2}{p_1} + 1) - p_1) + (p_3 + p_4 - p_5 - p_6)(\frac{p_2}{2p_1} + 1),$$
$$G_6 = -2k,\ \nabla_1 = (p_3 - p_5 - p_b(p_2 - p_1) + c_m(p_2 - p_1))(p_4 - p_6 + p_b(p_2 - p_1)),$$

$$\begin{aligned}
\nabla_2 &= (3p_1 + p_2)((p_4 - p_6)(p_3 - p_5) - (p_b(p_1 - p_2))^2) - (3p_1^2 - \\
&2p_1p_2 - p_2^2)(c_m(p_4 - p_6 + p_b(p_2 - p_1)) - p_b(p_4 - p_6 - p_3 + p_5)),\\
\nabla_3 &= (p_b(p_1 - p_2) - p_4 + p_6)^2 - c_m((p_b - 1)(p_1 - p_2) + p_3 - p_5))\\
&(p_b(p_1 - p_2) - p_4 + p_6),
\end{aligned}$$

$$\begin{aligned}
\nabla_4 &= p_1(p_b(p_1 - p_2) - p_4 + p_6)(c_m(3p_1 + p_2)((p_b - 1)p_1\\
&+p_2 - p_bp_2 + p_3 - p_5) - (p_1 - p_2)(p_b(p_1 - p_2) - p_4 + p_6)),
\end{aligned}$$

$$\begin{aligned}
\nabla_5 &= ({p_1}^2(3p_1 + p_2)(p_b(p_1 - p_2) - p_4 + p_6)^2({c_m}^2(3p_1\\
&+p_2)((p_b - 1)p_1 + p_2 - p_bp_2 + p_3 - p_5)^2 - 2c_m\\
&(p_1 - p_2)((p_b - 1)p_1 + p_2 - p_bp_2 + p_3 - p_5)(p_b\\
&(p_1 - p_2) - p_4 + p_6) + (p_1 - p_2)\\
&(p_b(p_1 - p_2) - p_4 + p_6)^2))^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\nabla_6 &= -c_mp_1(3p_1 + p_2)((p_b - 1)p_1 + p_2 - p_bp_2 + p_3 - p_5)\\
&(p_b(p_1 - p_2) - p_4 + p_6) + p_1(p_1 - p_2)\\
&(p_b(p_1 - p_2) - p_4 + p_6)^2,
\end{aligned}$$

$$
\begin{aligned}
A_1 \quad &= 8kp_1p_7(p_1 - p_1\rho + p_2\rho) + (p_3 + p_4 - p_5 - p_6) \\
&(p_2(p_3p_7 + p_4p_7 - p_5p_7 - p_6p_7 - 4p_1p_8 - 4p_1p_9) \\
&-2p_1(2p_1(p_8 + p_9) + (-p_3 - p_4 + p_5 + p_6)p_7\rho)),
\end{aligned}
$$

$$
\begin{aligned}
A_2 \quad &= 2(4k({p_1}^3 - p_1{p_2}^2) + {p_2}^2(p_3 + p_4 - p_5 - p_6)p_7 + 2{p_1}^3 \\
&(p_8 + p_9) - 2{p_1}^2(p_3 + p_4 - p_5 - p_6)p_7(-1 + \rho) - p_1p_2 \\
&(p_5p_7 + p_6p_7 + 2p_2p_8 + 2p_2p_9 + 2p_5p_7\rho + 2p_6p_7\rho - p_3 \\
&(p_7 + 2p_7\rho) - p_4(p_7 + 2p_7\rho)))c_m,
\end{aligned}
$$

$$
A_3 \quad = (p_1 - p_2)p_7(-{p_2}^2 + 2{p_1}^2(-2 + \rho) - p_1p_2(3 + 2\rho)),
$$

$$
\begin{aligned}
A_4 \quad &= -8kp_1p_7(p_2 + p_1\rho - p_2\rho) + (p_3 + p_4 - p_5 - p_6) \\
&(4{p_1}^2(p_8 + p_9) + 2p_2(p_3 + p_4 - p_5 - p_6)p_7(-1 + \rho) + \\
&p_1(3p_5p_7 + 3p_6p_7 + 4p_2p_8 + 4p_2p_9 - 4p_5p_7\rho - 4p_6p_7\rho \\
&+p_3p_7(-3 + 4\rho) + p_4p_7(-3 + 4\rho))),
\end{aligned}
$$

$$
\begin{aligned}
A_5 \quad &= -2(4k({p_1}^3 - p_1{p_2}^2) + 2{p_1}^3(p_8 + p_9) - 2{p_2}^2(p_3 + p_4 \\
&-p_5 - p_6)p_7(-1 + \rho) + {p_1}^2(p_3 + p_4 - p_5 - p_6)p_7 \\
&(-1 + 4\rho) - p_1p_2(3p_5p_7 + 3p_6p_7 + 2p_2p_8 + 2p_2p_9 - \\
&2p_5p_7\rho - 2p_6p_7\rho + p_3p_7(-3 + 2\rho) + p_4p_7(-3 + 2\rho))),
\end{aligned}
$$

$$
\begin{aligned}
A_6 \quad &= (p_1 - p_2)p_7(p_1p_2(5 - 2\rho) - 2{p_2}^2(-1 + \rho) + {p_1}^2 \\
&(1 + 4\rho)),
\end{aligned}
$$

$$
\begin{aligned}
A_7 \quad &= -4{p_1}^2(p_8 + p_9) + p_2(p_3 + p_4 - p_5 - p_6)p_7(-1 + \rho) + \\
&p_1(-3p_5p_7 - 3p_6p_7 + 4p_2p_8 + 4p_2p_9 - p_3p_7(-3 + \rho) - \\
&p_4p_7(-3 + \rho) + p_5p_7\rho + p_6p_7\rho),
\end{aligned}
$$

$$
\begin{aligned}
A_8 \quad &= (p_1 - p_2)(p_2(p_3 + p_4 - p_5 - p_6 + p_7 - p_7\rho) + p_1 \\
&(3p_3 + 3p_4 - 3p_5 - 3p_6 + p_7 + p_7\rho)),
\end{aligned}
$$

$$
A_9 \quad = -(p_1 - p_2)^2(3p_1 + p_2),
$$

$$
B_1 \quad = -8kp_1(p_1 - p_2) + (3p_1 + p_2)(p_3 + p_4 - p_5 - p_6)^2,
$$

$$
B_2 \quad = -2(3{p_1}^2 - 2p_1p_2 - {p_2}^2)(p_3 + p_4 - p_5 - p_6),
$$

$$
\begin{aligned}
C_1 \quad &= -p_1 + \frac{1}{4k}\left(p_3 - p_5 + \left(p_2 - p_1\right)\left(c_m - p_b\right)\right) \\
&\left(p_4 - p_6 + \left(p_2 - p_1\right)p_b\right),
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= p_2 + \frac{1}{4k}(1 + \frac{p2}{2p1})(p_3 - p_5 + (p_2 - p_1)(c_m - p_b)) \\
&(p_4 - p_6 + (p_2 - p_1)p_b) + \frac{1}{4k}(p_4 - p_6 + (p_2 - p_1)p_b) \\
&((1 + \frac{p_2}{2p_1})(p_3 - p_5) + (p_2(1 + \frac{p_2}{2p_1}) - (p_1 + \frac{p_2}{2})) \\
&(c_m - p_b)),
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \frac{1}{2}((1 - \rho)p_7 - (p_2 - p_1)c_m) + \frac{1}{4k}(p_3 - p_5 + (p_2 - p_1) \\
&(c_m - p_b))(-p_9 + \frac{p_7}{2p_1}((1 + \rho)p_b + (1 - \rho)(p_4 - p_6 + \\
&p_2 p_b))) + \frac{1}{4k}(p_4 - p_6 + (p_2 - p_1)p_b)(-p_8 + \frac{p_7(1 - \rho)}{2p_1} \\
&(p_3 - p_5) + (p_7 - \frac{1}{2}p_7(1 - \rho) + \frac{p_2 p_7(1 - \rho)}{2p_1})(c_m - p_b)),
\end{aligned}
$$

$$
\begin{aligned}
C_4 &= -2p_1 + \frac{p_2^2}{p_1} + \frac{1}{k}(1 + \frac{p_2}{2p_1})(p_4 - p_6 + (p_2 - p_1)p_b) \\
&((1 + \frac{p_2}{2p_1})(p_3 - p_5) + (p_2(1 + \frac{p_2}{2p_1}) - (p_1 + \frac{p_2}{2})) \\
&(c_m - p_b)),
\end{aligned}
$$

$$
\begin{aligned}
C_5 &= \frac{p_2 p_7(1 - \rho)}{2p_1} + p_7\rho - (p_2(1 + \frac{p_2}{2p_1}) - (p_1 + \frac{p_2}{2}))c_m \\
&+ \frac{1}{2k}((1 + \frac{p_2}{2p_1})(p_3 - p_5) + (p_2(1 + \frac{p_2}{2p_1}) - (p_1 + \frac{p_2}{2})) \\
&(c_m - p_b))(-p_9 + \frac{p_7}{2p_1}((1 + \rho)p_b + (1 - \rho)(p_4 - p_6 + \\
&p_2 p_b))) + \frac{1}{2k}(1 + \frac{p_2}{p_1})(p_4 - p_6 + (p_2 - p_1)p_b)(-p_8 + \\
&\frac{p_7(1 - \rho)}{2p_1}(p_3 - p_5) + (p_7 - \frac{1}{2}p_7(1 - \rho) + \frac{p_2 p_7(1 - \rho)}{2p_1}) \\
&(c_m - p_b)),
\end{aligned}
$$

$$
\begin{aligned}
C_6 &= -\frac{1}{(-4kp_1 + (p_b(p_1 - p_2) - p_4 + p_6)^2)^2}[4k{p_1}^2 \\
&(4kp_1 + (p_b(p_2 - p_1) + p_4 + c_m((p_b - 1)(p_1 - p_2) + \\
&p_3 - p_5) - p_6)(p_b(p_1 - p_2) - p_4 + p_6))],
\end{aligned}
$$

$$
\begin{aligned}
C_7 &= \frac{1}{(-4kp_1 + (p_b(p_1 - p_2) - p_4 + p_6)^2)^2}[16k^2{p_1}^2 p_2 \\
&-4c_m k p_1(2p_1 + p_2)((p_b - 1)(p_1 - p_2) + p_3 - p_5) \\
&(p_b(p_1 - p_2) - p_4 + p_6) - 4kp_1(p_1 + 2p_2) \\
&(p_b(p_1 - p_2) - p_4 + p_6)^2 + (p_1 + p_2) \\
&(p_b(p_1 - p_2) - p_4 + p_6)^4],
\end{aligned}
$$

$$
\begin{aligned}
C_8 \quad &= p_7(1-\rho) - \frac{1}{-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2} \\
&[2c_m kp_1(p_1-p_2) + p_1((p_b(p_1-p_2) - p_4 + p_6) \\
&(p_b p_7 - p_9) + 2kp_7(\rho - 1))] + \\
&\frac{1}{(-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2)^2} \\
&[2c_m kp_1((p_b - 1)(p_1 - p_2) + p_3 - p_5)(p_6 p_7(\rho - 1) \\
&+p_b p_7(p_1 + p_2 + \rho(p_1 - p_2)) + p_4 p_7(1-\rho) - 2p_1 p_9) \\
&-c_m p_1(p_b(p_1-p_2) - p_4 + p_6)((p_b(p_1-p_2) - p_4 + p_6) \\
&((p_4 - p_6)(p_7 - p_8) - (p_1 - p_2 - p_3 + p_5)p_9 + p_b \\
&((-p_3 - p_4 + p_5 + p_6)p_7 + (p_1 - p_2)(p_8 + p_9))) + 2k \\
&(((p_b - 1)p_2 - p_3 + p_5) \\
&p_7(\rho - 1) + p_1(-2p_8 - (p_b - 1)p_7(1+\rho))))],
\end{aligned}
$$

$$
\begin{aligned}
C_9 \quad &= -\frac{1}{(-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2)^2}[2(8k^2 p_1 \\
&(2{p_1}^2 - {p_2}^2) + (p_1 + p_2)(p_b(p_1-p_2) - p_4 + p_6)^4 + \\
&2k(p_b(p_1-p_2) - p_4 + p_6)(c_m(2p_1 + p_2)^2((p_b - 1) \\
&(p_1 - p_2) + p_3 - p_5) - (4{p_1}^2 + 2p_1 p_2 - {p_2}^2) \\
&(p_b(p_1 - p_2) - p_4 + p_6)))],
\end{aligned}
$$

$$
\begin{aligned}
C_{10} \quad &= p_7\rho - \frac{2c_m k({p_1}^2 - p_1 p_2 - {p_2}^2)}{-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2} + \\
&\frac{1}{-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2}[p_2((p_b(p_1 - p_2) \\
&-p_4 + p_6)(p_b p_7 - p_9) - 2kp_7(1-\rho))] - \\
&\frac{1}{(-4kp_1 + (p_b(p_1-p_2) - p_4 + p_6)^2)^2}[2c_m k(2p_1 + p_2) \\
&((p_b - 1)(p_1 - p_2) + p_3 - p_5)(p_7(p_4 - p_6)(\rho - 1) \\
&+2p_1 p_9 + p_b p_7(p_2(\rho - 1) - p_1(1+\rho))) + c_m \\
&(2p_1 + p_2)(p_b(p_1 - p_2) - p_4 + p_6)((p_b(p_1 - p_2) - \\
&p_4 + p_6)((p_4 - p_6)(p_7 - p_8) - (p_1 - p_2 - p_3 + p_5) \\
&p_9 + p_b((-p_3 - p_4 + p_5 + p_6)p_7 + (p_1 - p_2)(p_8 + p_9))) \\
&+2k(((p_b - 1)p_2 - p_3 + p_5)p_7(\rho - 1) + p_1(-2p_8 - \\
&(p_b - 1)p_7(1+\rho))))].
\end{aligned}
$$