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A PYTHAGOREAN HESITANT FUZZY PROGRAMMING APPROACH AND ITS APPLICATION TO MULTI OBJECTIVE RELIABILITY OPTIMIZATION PROBLEM

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Abstract: Decision-making problems can often be effectively solved using traditional optimization methods that have a clearly defined configuration. However, in real-world scenarios, decision-makers frequently encounter doubt or hesitation, making it challenging to precisely specify certain parameters. As a result, they often seek input from different experts, leading to conflicting values and varying levels of satisfaction among decision-makers. This uncertainty and lack of crisp values make decision-making problems inherently non-deterministic. In this paper, a novel Pythagorean hesitant fuzzy (PHF) programming method is designed to address the challenges of optimization problems with multiple objectives. Here PHF aggregation operators are used to aggregate the PHF memberships and non-memberships of the objectives. Additionally, to account the uncertainties of the parameters of the optimization problem Parabolic Pythagorean fuzzy number is used and centroid method is applied for defuzzification. We solved an example of multi objective optimization problem of manufacturing system to demonstrate our proposed method and finally, presented a case study on reliability optimization model for Life Support Systems, where the primary objectives are to maximize system reliability and minimize cost. The result is compared with other existing methods by degree of closeness.

Keywords: Parabolic Pythagorean fuzzy number, Pythagorean hesitant fuzzy set, Pythagorean hesitant fuzzy aggregation operator, multi objective reliability optimization.

MSC: 03B52, 90C29, 90B25, 90B30.

1. INTRODUCTION

In today's complex and uncertain world, decision-making procedures frequently entail a number of competing goals that need to be optimized to achieve the most desirable outcomes. Such situations are prevalent in various disciplines including engineering, finance, transportation, and environmental management, where reliability is a critical concern. Although it might be challenging to find optimal solution that successfully addresses all goals, so decision makers (DMs) accept compromise solutions. Most of the time the goal of the DMs and the parameter in the objective and constraints are not precisely known. To deal with such issues like ambiguity and inaccuracy that inevitably arise in decision-making problems, Zadeh [1] invented fuzzy sets (FSs). Later on, Bellman and Zadeh [2] presented fuzzy decision set in uncertainty and ambiguity situations. Then to solve uncertain optimization problem, Zimmerman [3] introduced fuzzy programming approach. Various fuzzy programming methods have been proposed in literature to solve multi-objective optimization problem (MOOP) including transportation, supplier selection, inventory control, portfolio management, reliability optimization, etc. In order to find the best compromise solution to a multi-objective transportation problem, El-Wahed [4] introduced fuzzy programming approach. Parra et al. [5] found optimal portfolio of investors using fuzzy goal programming. Mahapatra et al. [6] solved reliability optimization model of series system by introducing max-min and max-additive operators based fuzzy goal geometric programming. Majumder et al. [7] used uncertainty theory to create three separate models allowed them to address the problem of multi-item fixed charge solid transportation. Further, Majumder et al. [8] used a chance constraint model and an expected value model to solve the multi-objective shortest route problem by treating the parameters as uncertain variables. An uncertain variable-based multi-objective minimal spanning tree problem is solved by Majumder et al. [9]. Afterword, Majumder et al. [10] solved the multi-objective mean entropy portfolio selection issue by making use of the investors' hazy securities.

In fuzzy environment, only membership degree of an element is considered which is not fulfill DMs choice every time. Most of the time non-membership degree with membership is required to express the uncertainty in optimization goal, parameters and solution set involved in the problem. To overcome this, Atanassov [11] invented the idea of intuitionistic fuzzy (IF) set (IFS) as generalized version of FS where the membership and non-membership degrees added together are less than or equal to one. After that, Angelov [12] proposed a programming approach in IF environment to solve MOOP considering IF decision set. Further, lots of work has been done to solve MOOP in IF environments. Dey and Roy [13] developed an IF programming approach to find the optimal design of multi objective plane truss structure. Garg et al. [14] solved reliability optimization problem (ROP) by IF optimization technique in interval environment. A optimization model of biological treatment process on industrial waste water is solved by Ghosh P. et al. [15] using IF goal geometric programming.

Reliability engineering is a crucial phase in the design and creation of a technological system. Considering availability of system resources effective technique to improve system reliability has always been the reliability engineer's top priority. The optimization model includes a variety of coefficients and parameters that are constantly inaccurate and ambiguous in nature due to the DM's uncertainty in everyday life. A fuzzy technique is used to analyze this type of nature in MOOP. Ravi et al. [16] complex system reliability optimization model as fuzzy MOOP. Sharma et al. [17] presented an analysis of system reliability in IF environment. Islam and Kundu [18] applied neutrosophic optimization technique to solve complex system reliability optimization model (ROM). A neutrosophic goal geometric programming approach is presented by Kundu and Islam [19] to solve multi objective ROP.

Yager [20] presented Pythagorean fuzzy (PF) set (PFS) as generalization of FS and IFS where the sum of the squares of the membership and non-membership degrees is ≤ 1. In last few years many decision-making problems has been solved in PF environment. Recently Fei et al. [21] solved a decision making problem considering Pythagorean fuzzy number (PFN) and interval valued PFN. Akram et al. [22] solved fully Pythagorean fuzzy linear programming problem considering triangular PFN. In a PF environment, Luqman et al. [23] designed digraph and matrix technique to assess risk in a Failure Mode and Effects Analysis. A minimal spanning tree agglomerative hierarchical clustering technique is proposed by Habib et al. [24] utilizing PF distance and similarity measurements. By taking into account linguistic capacities and flows, Akram et al. [25] created a novel Pythagorean fuzzy maximum flow method to handle various optimization problems using PF information. To handle the multi-attribute decision-making problem in a PF environment, Wan et al. [26] designed a Pythagorean fuzzy mathematical programming approach.

The main motivation of this study:

Most of the current approaches presented in the literature integrate multiple objectives into a single objective by using real-valued and utility functions. The utility functions can be expressed in many different ways, including the product form, max-min, weighted max-min and weighted sum. Typically, these approaches employ the preferences of the DM. Using a useful function, this choice is converted to a mathematical expression. For instance, the weighted sum method can't guarantee that the levels of achievement of fuzzy goals are consistent with desired relative weights or the expectations of the DM. Zimmermann [3] employed the "min" operator in his suggested fuzzy programming technique due to the ease of calculation, it does not guarantee a solution which is free of dominance. Li and Lai [27] developed some aggregation operators based on weighted root-power mean to solve the multi-objective transportation problem worldwide in order to get over the aforementioned limitation. Further, Liu et al. [28] used a modified s-curve membership function to address the supplier selection problem by weighted root mean power aggregation operator.

In the decision-making process, hesitation is actually a trivial concern. It describes the situation in which the decision-maker(s) are unsure of the precise values of the parameters, though there is considerable uncertainty around a few possible outcomes. It may be handled using a hesitant fuzzy (HF) set (HFS) [29] by assigning a set of varying membership degrees to an element in the set. A hesitant mathematical programming approach was created Zhang et al. [30] to address MCDM issues in the context of HF elements. Bharati [31] suggested a HF computational technique for the production planning problem. Additionally, aggregation procedures that were based on HFSs were expanded. A novel set of hesitant aggregation operators was addressed by Xia et al. [32] and their use in decision-making problems was provided. Additionally, Xia and Xu [33] established number of hesitant aggregations operators and demonstrated how to employ them to address decision-making issues. For hybrid multi-criteria group decision making with incomplete criterion weight information and HF membership degrees, Xu et al. [34]

suggested a new HF mathematical programming technique. In order to build a HF programming approach based on the linear programming methodology for multidimensional analysis of preference, Wan et al. [35] investigated a hybrid problem of multiple attribute decision making with unreliable attribute weight data. Recently F. Ahmad et al. [36] proposed a optimization method to solve optimization problem taking the advantage of HF aggregation operator.

Further, as an extension of hesitant FS, hesitant IFS, hesitant PFS are suggested to solve MOOP. Recently Bharti [37] developed an interactive optimization algorithm in hesitant IF environment to solve linear optimization problem. Adhami and Ahmad [38] solved transportation problem by introducing an interactive optimization algorithm in Pythagorean hesitant fuzzy (PHF) environment. Khan M.S.A et al. [39] introduced PHF aggregation operator (PHFAO) to solve multi criterion decision making problem. Later on, extension of PHFAO is established and applied to MCDM problem [40-41]. Best of our knowledge still there are no use of PHFAO to solve MOOP. Fulfilling this gap in this study we have presented an optimization algorithm using PHF weighted averaging (PHFWA) operator and PHF weighted geometric (PHFWG) operator. Also, here the parameters of the problem are taken as parabolic PFN (PPFN) and centroid method is used for defuzzification. To illustrate the validity and acceptability of the proposed method, a mathematical model of manufacturing system, a ROM for a life support system is solved, and the outcomes are compared to the existing approach using HF aggregation operator by closeness degree to ideal solution [42].

The main contribution of this study are as follows:

1) To handle the uncertain parameter of the optimization problem we presented PPFN.
2) A PHF programming approach is developed to solve MOOP by using the weighted sum of aggregated PHF membership and non-membership.
3) To demonstrate the effectiveness of the suggested approach, we addressed a practical manufacturing system and a real-world multi-objective ROM concerning the life support system within a space capsule. We then conducted a comparative analysis of the outcomes against established methods.
4) By varying the weight, sensitivity analysis of the results is shown and compare the results by degree of closeness to ideal solution.

The remaining portions of the paper are arranged as: section 2 describes the basic definitions, PPFN and PHFAO. In section 3, a MOOP using PHFAO is formulated. Application of the proposed method on multi objective ROP is shown in section 4. The result and discussion part are in section 5, The limitation of this article is highlighted in section 6 and at last the conclusion of the article is discussed in section 7.

2. PRELIMINARIES

2.1. Basic definitions

We have discussed some fundamental definitions of terms related to various fuzzy sets in this section.

Definition 1 (HFS [27]). *X be a fixed set, a HFS, A on X is defined as $A = \{\langle x, h_A(x)\rangle \mid x \in X\}$, where $h_A(x)$ is a collection of finite membership values in [0, 1]. $h_A(x)$ is referred to as a HF element also.*

Definition 2 [31]. *The intersection of two HFSs h_1 and h_2 in a given set X is defined as follows:*

$$h_1 \cap h_2 = \cup_{\alpha_1 \in h_1, \alpha_2 \in h_2} \min\{\alpha_1, \alpha_2\}$$

Definition 3. PFS [20]. *Given a set S, the following is the PFS, $\tilde{P}$ in S using ordered triplets:*

$$\tilde{P} = \{s, \mu_{\tilde{P}}(S), v_{\tilde{P}}(S) \mid s \in S\}$$

where $\mu_{\tilde{P}}(s)$,$v_{\tilde{P}}(s)$: $S \to [0,1]$ are membership and non-membership function of the element x into the set $\tilde{P}$, respectively, with $0 \leq \mu_{\tilde{P}}(s)^2 + v_{\tilde{P}}(s)^2 \leq 1$. Additionally, the value of uncertainty is indicated by $\sqrt{1 - \mu_{\tilde{P}}(s)^2 - v_{\tilde{P}}(s)^2}$. The PFS differs from IFS in that it places limits on the squares of the membership and non-membership degrees added together.

Definition 4. PHFS [38]. *Y be an arbitrary set. The following describes a PHFS on Y: $P_h = \{\langle y, \mu_{P_h}(y), v_{P_h}(y)\rangle \mid y \in Y\}$, where $\mu_{P_h}(y), v_{P_h}(y)$ are finite values in [0, 1], representing degrees of PHF membership and non-membership of y to P_h, with $0 \leq \alpha, \beta \leq 1$ and $0 \leq \alpha^2 + \beta^2 \leq 1$, where $\alpha \in \mu_{P_h}(y), \beta \in v_{P_h}(y)$ for all $y \in Y$. For simplicity, the $P_h(y) = \{\mu_{P_h}(y), v_{P_h}(y)\}$ is called a PHF element.*

Definition 5. [38]. *The union of two PHFSs P_{h_1} and P_{h_2} in on an arbitrary set Y can be defined as follows:*

$$P_{h_1} \cup P_{h_2} = \{\mu_h \in (\mu_{h_1} \cup \mu_{h_2}) \mid \mu_h \geq max(min\{\mu_{h_1} \cup \mu_{h_2}\}),$$
$$v_h \in (v_{h_1} \cup v_{h_2}) \mid v_h \leq min(max\{v_{h_1} \cup v_{h_2}\})\}$$

Definition 6. [38]. *The intersection of two PHFSs P_{h_1} and P_{h_2} in on an arbitrary set Y can be defined as follows:*

$$P_{h_1} \cap P_{h_2} = \{\mu_h \in (\mu_{h_1} \cap \mu_{h_2}) \mid \mu_h \leq min(max\{\mu_{h_1} \cap \mu_{h_2}\}),$$
$$v_h \in (v_{h_1} \cap v_{h_2}) \mid v_h \geq max(min\{v_{h_1} \cap v_{h_2}\})\}$$

2.2. Parabolic Pythagorean fuzzy number (PPFN) and defuzzification

Definition 7. *A PPFN, $\tilde{P} = \{(a, b, c); p, q\}$ is a PFS on $\mathbb{R}$, where p, q are maximum degree of membership and minimum degree of non-membership respectively, here membership ($\mu_{\tilde{P}}$) and non-membership($\vartheta_{\tilde{P}}$) are defined as:*

$$\mu_{\tilde{P}} = \begin{cases} p\left(\dfrac{x-a}{b-a}\right)^{\frac{1}{2}} & a \le x < b \\ p & x = b \\ p\left(\dfrac{c-x}{c-b}\right)^{\frac{1}{2}} & b < x \le c \\ 0 & otherwise \end{cases}$$

$$\vartheta_{\tilde{P}} = \begin{cases} \left[\dfrac{(b-x)+q^2(x-a)}{b-a}\right]^{\frac{1}{2}} & a \le x < b \\ q & x = b \\ \left[\dfrac{(x-b)+q^2(c-x)}{c-b}\right]^{\frac{1}{2}} & b < x \le c \\ 1 & otherwise \end{cases}$$

In particular, take $p = 1, q = 0$ then we get PPFN as $\tilde{P} = \{(a, b, c); (a', b', c')\}$ and membership and non-membership are shown in Figure 1.

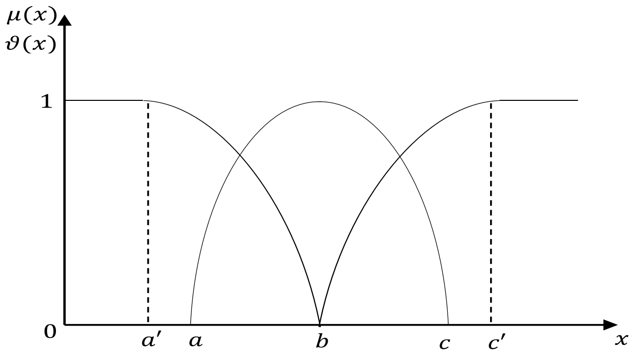

Figure1: Parabolic Pythagorean fuzzy number

$$\mu_{\tilde{P}}(x) = \begin{cases} \left(\dfrac{x-a}{b-a}\right)^{\frac{1}{2}} & a \le x < b \\ 1 & x = b \\ \left(\dfrac{c-x}{c-b}\right)^{\frac{1}{2}} & b < x \le c \\ 0 & otherwise \end{cases}$$

And $\vartheta_{\tilde{P}}(x) = \begin{cases} \left(\frac{b-x}{b-a'}\right)^{\frac{1}{2}} & a' \le x < b \\ 0 & x = b \\ \left(\frac{x-b}{c'-b}\right)^{\frac{1}{2}} & b < x \le c' \\ 1 & otherwise \end{cases}$

By COA method, here the Centre of area of membership and non-membership are calculated individually, i.e.

$$R_1(\tilde{P}) = \frac{\int_x x\mu_{\tilde{P}}(x)dx}{\int_x \mu_{\tilde{P}}(x)dx} \quad and \quad R_2(\tilde{P}) = \frac{\int_x x\vartheta_{\tilde{P}}(x)dx}{\int_x \vartheta_{\tilde{P}}(x)dx}$$

Now the defuzzied value of the PPFN is defined by, $R(\tilde{P}) = \frac{R_1(\tilde{P})+R_2(\tilde{P})}{2}$.

So, here from the membership function $R_1(\tilde{P}) = \frac{\int_a^b x\left(\frac{x-a}{b-a}\right)^{1/2}dx+\int_b^c x\left(\frac{c-x}{c-b}\right)^{1/2}dx}{\int_a^b\left(\frac{x-a}{b-a}\right)^{1/2}dx+\int_b^c\left(\frac{c-x}{c-b}\right)^{1/2}dx} = \frac{2a+b+2c}{5}$

And from non-membership function $R_2(\tilde{P}) = \frac{\int_a^b x\left(\frac{b-x}{b-a'}\right)^{1/2}dx+\int_b^c x\left(\frac{x-b}{c'-b}\right)^{1/2}dx}{\int_a^b\left(\frac{b-x}{b-a'}\right)^{1/2}dx+\int_b^c\left(\frac{x-b}{c'-b}\right)^{1/2}dx} = \frac{2a'+b+2c'}{5}$

Hence, $R(\tilde{P}) = \frac{(2a+b+2c)+(2a'+b+2c')}{10}$

2.3. PHF aggregation operators

Definition 8(PHFWA operator [39]). *Let $\hat{h}_i = \langle A_i, B_i\rangle$ be a collection of all PHFNs, and $w = (w_1, w_2, \dots, w_n)^T$ is weight vector of $\hat{h}_i$ with $w_i \geq 0$, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. Then, a mapping $\emptyset$: PHFN $^n \to$ PHFN is called PHFWA operator and defined as*

$$PHFWA(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n) = w_1\hat{h}_1 \oplus w_2\hat{h}_2 \oplus \dots \oplus w_n\hat{h}_n$$

$$= \bigcup_{\substack{\alpha_1\in A_1,\alpha_2\in A_2,\dots\dots\alpha_n\in A_n\\ \beta_1\in B_1,\beta_2\in B_2,\dots\dots\beta_n\in B_n}} \left\{\left\{\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2)^{w_i}}\right\}, \left\{\prod_{i=1}^{n}\beta_i^{w_i}\right\}\right\}$$

Definition 9 (PHFWG operator) [39]. *Let $\hat{h}_i = \langle A_i, B_i\rangle$ be a collection of all PHFNs, and $w = (w_1, w_2, \dots, w_n)^T$ is weight vector of $\hat{h}_i$ with $w_i \geq 0$, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. Then, a mapping $\emptyset$: PHFN $^n \to$ PHFN is called PHFWG operator and defined as*

$$PHFWG(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n) = \hat{h}_1^{w_1} \otimes \hat{h}_2^{w_2} \otimes \dots\dots \otimes \hat{h}_n^{w_n}$$

$$= \bigcup_{\substack{\alpha_1\in A_1,\alpha_2\in A_2,\dots\dots\alpha_n\in A_n\\ \beta_1\in B_1,\beta_2\in B_2,\dots\dots\beta_n\in B_n}} \left\{\left\{\prod_{i=1}^{n}\alpha_i^{w_i}\right\}, \left\{\sqrt{1-\prod_{i=1}^{n}(1-\beta_i^2)^{w_i}}\right\}\right\}$$

3. MULTI OBJECTIVE OPTIMIZATION MODEL USING AGGREGATION OPERATORS BASED ON PHF SET

3.1. Multi objective optimization problem

A MOOP with l objectives, m constraints, and n decision variables are formulated as follows:

Minimize $(f_1, f_2, \dots\dots\dots..f_l)$

Subject to, $a_i(x) \le or \ge or = b_i,$ $\qquad i = 1, \dots \dots m.$

Where $x = (x_1, \dots \dots .. x_n)$ and $x_j \ge 0$ for all $j = 1, \dots \dots . n.$ (1)

Here f_l are objectives, a_i are real valued functions, b_i are real numbers, x_j are the decision variables.

Using weighted sum with weight vector $w^T = (w_1, w_2, \dots \dots .., w_l)$ and $\sum_{k=1}^{l} w_k = 1$, Problem (1) becomes,
Minimize $(w_1 f_1 + w_2 f_2 + \cdots + w_l f_l)$

Subject to, $a_i(x) \le or \ge or = b_i,$ $\qquad i = 1, \dots \dots m.$

Where $x = (x_1, \dots \dots .. x_n)$ and $x_j \ge 0$ for all $j = 1, \dots \dots . n.$ (2)

3.2. PHF membership and non-membership function

We have l solutions for the unique solution of each objective function: $(X^1, X^2, \dots ., X^l)$. Then, calculating value of each objective by all solutions we get the lower and upper bound of objectives:

$$L_k = \min[f_k(X^i)] \; and \; U_k = \max[f_k(X^i)] \; \forall i = 1,2,3, \dots, l \text{ for all } k = 1,2, \dots ., l.$$

The linear membership and non-membership have huge applications in uncertainty and a straightforward construction. Following is a definition of the linear membership function of minimization objective under PHFS (Figure 2):

$$\mu_{ph}^{E_n}(Z_k(x)) = \begin{cases} 1, & \text{if } f_k < L_k; \\ \delta_n \dfrac{U_k - f_k}{(U_k + \alpha) - L_k}, & \text{if } L_k \le f_k \le U_k + \alpha; \\ 0, & \text{if } f_k > U_k + \alpha \end{cases}$$

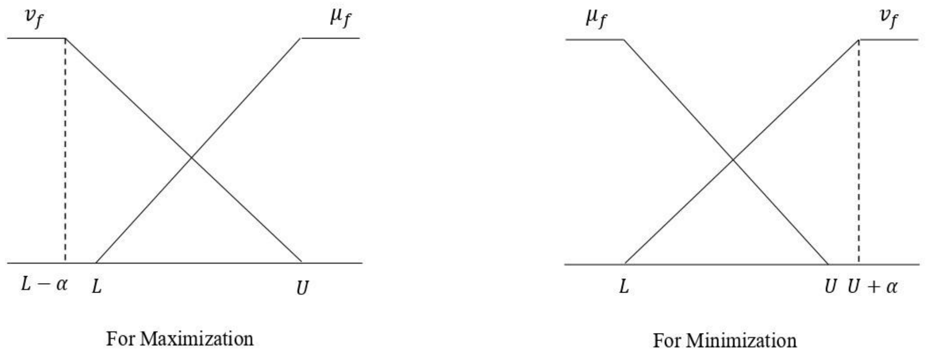

Figure 2: Linear membership and non-membership function

and linear non-membership under PHFS is:

$$\vartheta_{ph}^{E_n}(Z_k(x)) = \begin{cases} 0, & \text{if } f_k < L_k; \\ \gamma_n \dfrac{f_k - L_k}{(U_k + \alpha) - L_k}, & \text{if } L_k \leq f_k \leq U_k + \alpha; \\ 1, & \text{if } f_k > U_k + \alpha \end{cases}$$

where $\delta_n, \gamma_n \in [0,1]$ are assigned to expert (E_n).

Pareto optimal solution: pareto optimal solution to a MOOP is a basic feasible solution x^* in feasible space of the problem, iff $\nexists$ x s.t $f_i(x) \leq f_i(x^*)$ for all $i = 1, \dots . l$ and $f_i(x) < f_i(x^*)$ for at least one $i = 1, \dots . l$.

PHF pareto optimal solution: a point x^* in solution space(X) is called PHF pareto optimal solution , iff $\nexists$ x in X s.t $f_i(x) \leq f_i(x^*)$ for all $i = 1,2, \dots \dots l$ with $\mu_i^{E_k}(f_i(x)) \geq \mu_i^{E_k}(f_i(x^*))$ and $\vartheta_i^{E_k}(f_i(x)) \leq \vartheta_i^{E_k}(f_i(x^*))$ for all i and inequality holds for at least one i, where $\mu_i^{E_k}(x)$, $\vartheta_i^{E_k}(x)$ are PF membership and non-membership respectively.

3.3. Evaluation of MOOP using PHFAO

PHFAOs, which are previously mentioned, may serve as the foundation for the following type of utility function for MOOP:

$$\text{Max(PHFAO)}(\mu^{E_1}, \mu^{E_2}, \dots, \mu^{E_n}; \vartheta^{E_1}, \vartheta^{E_2}, \dots, \vartheta^{E_n}) = \sum_{k=1}^{l} W_k(\text{PHFAO})$$

Where, $W_k(\forall k = 1,2, \dots, l)$, such that $\sum_{k=1}^{l} W_k = 1$ is weight assigned by DMs to each objective. PHFAO denotes the family of PHF aggregation operators.

The fuzzy set, according to Bellman and Zadeh [2], consists of three notions: fuzzy goal (G), and fuzzy constraints (C), fuzzy decision (D). These concepts have been integrated into numerous real-world decision-making applications within fuzzy environments. A fuzzy decision set is formally defined as: $D = G \cap C = \cup_{\alpha_n \in G, \beta_n \in C} \, min\{(\alpha_n, \beta_n)\}$.

Now, the PHF decision set $D_{ph}$is defined as follows, using PHF constraints (C) and goals (Z):

$$\begin{aligned} D_{ph} = Z \cap C \quad &= \left(\cap_{k=1}^{l} Z_k\right) \cap \left(\cap_{j=1}^{m} C_j\right) \\ &= \begin{Bmatrix} x, \cup \min\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n)\}(x), \\ \max\{(\theta_1, \rho_1), (\theta_2, \rho_2), \dots, (\theta_n, \rho_n)\}(x) \mid x \in X \end{Bmatrix} \\ &= \{x, \{\mu^{E_1}, \mu^{E_2}, \dots, \mu^{E_n}; \vartheta^{E_1}, \vartheta^{E_2}, \dots, \vartheta^{E_n}\}(x) \mid x \in X\} \\ where, \mu^{E_1} \quad &= \min(\alpha_1, \beta_1);\ \vartheta^{E_1} = max(\theta_1, \rho_1) \\ \mu^{E_2} \quad &= \min(\alpha_2, \beta_2);\ \vartheta^{E_2} = max(\theta_2, \rho_2) \\ &\vdots \\ \mu^{E_n} \quad &= \min(\alpha_n, \beta_n);\ \vartheta^{E_n} = max(\theta_n, \rho_n) \end{aligned}$$

where, μ^{E_n}, ϑ^{E_n} are PHF membership and non-membership by nth experts.

Now MOOP in PHF environment become as:

Max $\mathrm{PHFAO}(\mu^{E_1}, \mu^{E_2}, \dots, \mu^{E_n};\ \vartheta^{E_1}, \vartheta^{E_2}, \dots, \vartheta^{E_n}) = \mathrm{Max} \sum_{k=1}^{l} W_k(\mathrm{PHFAO})$
subject to

$$\mu^{E_1}\big(f_k(x)\big) \geq \alpha_{k1};\quad \vartheta^{E_1}\big(f_k(x)\big) \leq \beta_{k1}$$
$$\mu^{E_2}\big(f_k(x)\big) \geq \alpha_{k2};\quad \vartheta^{E_2}\big(f_k(x)\big) \leq \beta_{k2}$$
$$\dots \dots ..$$
$$\mu^{E_n}(f_k(x)) \geq \alpha_{kn},\quad \vartheta^{E_n}\big(f_k(x)\big) \leq \beta_{kn}$$
$$a_i(x) \leq or = or \geq b_i \qquad i = 1,2, \dots \dots . m$$
$$x \geq 0\ , 0 \leq\ \alpha_{kr}^2 + \beta_{ks}^2 \leq 1\ \ and\ \alpha_{kr} \geq \beta_{ks} \qquad r, s\ \in \{1, \dots \dots . n\} \tag{3}$$

$W_k(\forall k = 1,2, \dots, l)$, such that $\sum_{k=1}^{l} W_k = 1$ are weights given to the objective by experts. $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight vector allocated to each PHF element, $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1. \alpha_{kn}, \beta_{kn}$ are minimum membership and maximum non-membership to each PHF membership function.

Here, $\mu^{E_i}(f_k(x))$: membership to the k^{th} objective by i^{th} expert.

$\vartheta^{E_i}(f_k(x))$: non-membership to the k^{th} objective by i^{th} expert.

Theorem1: *For non-zero objective weight vector, a unique optimal solution (x^*, α^*, β^*) of Problem (3) is a pareto optimal solution (x^*) of problem (1). Where, $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots \dots, \alpha_l^*)$, $\beta^* = (\beta_1^*, \beta_2^*, \dots \dots, \beta_l^*)$ and $\alpha_i^* = (\alpha_{i1}^*, \alpha_{i2}^*, \dots \dots, \alpha_{in}^*)$,and $\beta_i^* = (\beta_{i1}^*, \beta_{i2}^*, \dots \dots, \beta_{in}^*)$.*

Proof: (for PHFWA operator)

Consider $f(\alpha, \beta) = \sum_{k=1}^{l} W_k\Big(\sqrt{1 - \prod_{i=1}^{n} (1 - \alpha_{ki}^2)^{\omega_i}} - \prod_{i=1}^{n} \beta_{ki}^{\omega_i}\Big)$
Let (x^*, α^*, β^*) is optimal solution of problem (3) then $f(\alpha^*, \beta^*) > f(\alpha, \beta)$ for all (x, α, β) in the feasible space of the problem (3).

Suppose x^* is not a pareto optimal solution of problem (1) then there exist x^{**} in feasible space (X) such that $f_i(x^{**}) \leq f_i(x^*)$ for all $i = 1,2, \dots . l$ and $f_i(x^{**}) < f_i(x^*)$ for at least one $i = 1,2, \dots . l$. (i)

So, $\frac{U_k - f_k(x^{**})}{U_k - L_k} \geq \frac{U_k - f_k(x^*)}{U_k - L_k}$ for all $k = 1,2 \dots . . l$ and $\frac{U_k - f_k(x^{**})}{U_k - L_k} > \frac{U_k - f_k(x^*)}{U_k - L_k}$ for at least one $k = 1,2 \dots . . l$
Now, $\delta_i \frac{U_k - f_k(x^{**})}{U_k - L_k} \geq \delta_i \frac{U_k - f_k(x^*)}{U_k - L_k}$ for all $k = 1,2 \dots . . p$ and $\delta_i \frac{U_k - f_k(x^{**})}{U_k - L_k} > \delta_i \frac{U_k - f_k(x^*)}{U_k - L_k}$ for at least one $k = 1,2 \dots . . l$ and for all $i = 1,2, \dots . , n$.

That implies, $Max_{k,i}\, \delta_i \frac{U_k - f_k(x^{**})}{U_k - L_k} \geq Max_{k,i} \delta_i \frac{U_k - f_k(x^*)}{U_k - L_k}$ for all $k = 1,2 \ldots.. l$ and $Max_{k,i}\, \delta_i \frac{U_k - f_k(x^{**})}{U_k - L_k} > Max_{k,i}\, \delta_i \frac{U_k - f_k(x^*)}{U_k - L_k}$ for at least one $k = 1,2 \ldots.. l$ and for all $i = 1,2, \ldots. n$.

Consider, $Max_{k,i}\, \delta_i \frac{U_k - f_k(x^{**})}{U_k - L_k} = \alpha_{ki}^{**}$ and $Max_{k,i}\, \delta_i \frac{U_k - f_k(x^*)}{U_k - L_k} = \alpha_{ki}^*$

$\alpha_{ki}^{**} \geq\ \alpha_{ki}^*$ for all $k = 1,2 \ldots.. l$ and for all $i = 1,2, \ldots n$, the inequality occurs for at least one $k$and i.

$$\prod_{i=1}^{n} \left(1 - \alpha_{ki}^{**2}\right)^{\omega_i} < \prod_{i=1}^{n} \left(1 - \alpha_{ki}^{**2}\right)^{\omega_i}$$

$$\Rightarrow \sqrt{1 - \prod_{i=1}^{n} \left(1 - \alpha_{ki}^{**2}\right)^{\omega_i}} > \sqrt{1 - \prod_{i=1}^{n} \left(1 - {\alpha_{ki}^*}^2\right)^{\omega_i}} \qquad \text{(ii)}$$

Similarly, from Eq. (i) $\gamma_i \frac{f_k(\mathrm{x}^{**}) - L_k}{U_k + \alpha - L_k} \leq \gamma_i \frac{f_k(\mathrm{x}^*) - L_k}{U_k + \alpha - L_k}$ for all $k = 1,2 \ldots.. l$ and $\gamma_i \frac{f_k(\mathrm{x}^{**}) - L_k}{U_k + \alpha - L_k} < \gamma_i \frac{f_k(\mathrm{x}^*) - L_k}{U_k + \alpha - L_k}$ for at least one $k$and i.

That implies, $Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^{**}) - L_k}{U_k + \alpha - L_k} \leq Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^*) - L_k}{U_k + \alpha - L_k}$ for all k and i and $Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^{**}) - L_k}{U_k + \alpha - L_k} < Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^*) - L_k}{U_k + \alpha - L_k}$ for at least one $k$and i.

Consider, $Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^{**}) - L_k}{U_k + \alpha - L_k} = \beta_{ki}^{**}$and $Min_{k,i}\, \gamma_i \frac{f_k(\mathrm{x}^*) - L_k}{U_k + \alpha - L_k} = \beta_{ki}^*$

So, $\beta_{ki}^{**} \leq \beta_{ki}^*$ for all $k$and i. and $\beta_{ki}^{**} < \beta_{ki}^*$ for at least one k and i.

$$\prod_{i=1}^{n} {\beta_{ki}^{**}}^{\omega_i} < \prod_{i=1}^{n} {\beta_{ki}^*}^{\omega_i} \qquad \text{(iii)}$$

Eqs. (ii) and (iii) $\Rightarrow \sqrt{1 - \prod_{i=1}^{n} \left(1 - \alpha_{ki}^{**2}\right)^{\omega_i}} - \prod_{i=1}^{n} {\beta_{ki}^{**}}^{\omega_i} > \sqrt{1 - \prod_{i=1}^{n} \left(1 - {\alpha_{ki}^*}^2\right)^{\omega_i}} - \prod_{i=1}^{n} {\beta_{ki}^*}^{\omega_i}$

$$\Rightarrow \sum_{k=1}^{l} W_k \left[\sqrt{1 - \prod_{i=1}^{n} \left(1 - \alpha_{ki}^{**2}\right)^{\omega_i}} - \prod_{i=1}^{n} {\beta_{ki}^{**}}^{\omega_i}\right] > \sum_{k=1}^{l} W_k \left[\sqrt{1 - \prod_{i=1}^{n} \left(1 - {\alpha_{ki}^*}^2\right)^{\omega_i}} - \prod_{i=1}^{n} {\beta_{ki}^*}^{\omega_i}\right]$$

Hence, $f(\alpha^{**}, \beta^{**}) >\ f(\alpha^*, \beta^*)$, which contradict (x^*, α^*, β^*) is unique optimal solution of Problem (3). So x^* is a pareto optimal solution of Problem (1).

(For PHFWG operator) Similar as above.

Theorem2: *For non-zero weight vector, a pareto optimal solution* (x^*) *of Problem (1) is an efficient solution of problem (3) if* f_i *are convex.*

Proof: x^* is a pareto optimal solution of Problem 1.

Suppose, x^* is not an efficient solution of Problem 3. Then there exists x^{**} in feasible space such that $f(\alpha^*, \beta^*) <\ f(\alpha^{**}, \beta^{**})$

$$\Rightarrow \sum_{k=1}^{l} W_k\left(\sqrt{1-\prod_{i=1}^{n}\,(1-\alpha_{ki}^{*2})^{\omega_i}}-\prod_{i=1}^{n}\beta_{ki}^{*\omega_i}\right)$$
$$< \sum_{k=1}^{l} W_k\left(\sqrt{1-\prod_{i=1}^{n}\,(1-\alpha_{ki}^{**2})^{\omega_i}}-\prod_{i=1}^{n}\beta_{ki}^{**\omega_i}\right)$$

Since, $\delta_i\frac{U_k-f_k(x^*)}{U_k-L_k}=\alpha_{ki}^*,\ \gamma_i\frac{f_k(\mathrm{x}^*)-L_k}{(U_k+\alpha)-L_k}=\beta_{ki}^*\ and\ \delta_i\frac{U_k-f_k(x^{**})}{U_k-L_k}=\alpha_{ki}^*\ and\ \gamma_i\frac{f_k(\mathrm{x}^{**})-L_k}{(U_k+\alpha)-L_k}=\beta_{ki}^*$

If f_i are convex, then for all $k=1,2,\dots l$

$$\left(\sqrt{1-\prod_{i=1}^{n}\,(1-\alpha_{ki}^{*2})^{\omega_i}}-\prod_{i=1}^{n}\beta_{ki}^{*\omega_i}\right)<\sqrt{1-\prod_{i=1}^{n}\,(1-\alpha_{ki}^{**2})^{\omega_i}}-\prod_{i=1}^{n}\beta_{ki}^{**\omega_i}$$

Taking, $\omega_1=1,\omega_2=\omega_3=\cdots.=\omega_n=0$

$$\sqrt{1-(1-\alpha_{k1}^{*2})}-\beta_{k1}^*<\sqrt{1-(1-\alpha_{k1}^{**2})}-\beta_{k1}^{**}$$
$$\Rightarrow \alpha_{k1}^*-\beta_{k1}^*<\alpha_{k1}^{**}-\beta_{k1}^{**}$$
$$\Rightarrow \delta_1\frac{U_k-f_k(x^*)}{U_k-L_k}-\gamma_1\frac{f_k(\mathrm{x}^*)-L_k}{(U_k+\alpha)-L_k}<\delta_1\frac{U_k-f_k(x^{**})}{U_k-L_k}-\gamma_1\frac{f_k(\mathrm{x}^{**})-L_k}{(U_k+\alpha)-L_k}$$
$$\Rightarrow -(\delta_1+\gamma_1)f_k(x^*)<-(\delta_1+\gamma_1)f_k(x^{**})$$
$$\Rightarrow f_k(x^{**})<f_k(x^*)\ for\ all\ k=1,2,\dots..l.$$

Which contradict that x^* is pareto optimal solution of Problem (1)
Hence, x^* is efficient solution of Problem (3)
The proposed method to solve MOOP is classified by two categories by two PHFAOs, which are PHFWA, PHFWG operators. The following is the formulation of MOOP using PHFAOs (weighted averaging and weighted geometric).

Model 1: by PHFWA operator

$$\text{Max PHFWA}(\mu^{E_1},\mu^{E_2},\dots,\mu^{E_n};\ \vartheta^{E_1},\vartheta^{E_2},\dots,\vartheta^{E_n})$$
$$=\text{Max}\sum_{k=1}^{l} W_k\left(\sqrt{1-\prod_{i=1}^{n}\,(1-\alpha_{ki}^{2})^{\omega_i}}-\prod_{i=1}^{n}\beta_{ki}^{\omega_i}\right).$$

Subject to the constraints in Problem (3).

Model 2: by PHFWG operator

$$\text{Max PHFWG}(\mu^{E_1}, \mu^{E_2}, \dots, \mu^{E_n};\ \vartheta^{E_1}, \vartheta^{E_2}, \dots, \vartheta^{E_n})$$
$$= \text{Max} \sum_{k=1}^{l} W_k \left(\prod_{i=1}^{n} \alpha_{ki}^{\omega_i} - \sqrt{1 - \prod_{i=1}^{n} (1 - \beta_{ki}^2)^{\omega_i}} \right).$$

Subject to the constraints in Problem (3).

4. APPLICATION OF THE METHOD TO MULTI OBJECTIVE OPTIMIZATION PROBLEM

A) Manufacturing system:

We have taken an example of multi objective optimization problem of manufacturing system described in Singh and Yadav [43]. A manufacturing factory is planning to produce three types of products: A, B, and C over a period of one month. To produce each product, the factory requires three types of resources: R1, R2, and R3.

There are around 30, 20 and 20 units of tolerance is allowed for resource R1, R2, R3 respectively by manager. Unit cost and sale price of the product are c_A, c_B, c_C and $S_A = \frac{s_A}{x_1^{\frac{1}{a1}}}$, $S_B = \frac{s_B}{x_2^{\frac{1}{a2}}}$, $S_C = \frac{s_C}{x_1^{\frac{1}{a3}}}$ respectively. Where a_1, a_2, a_3 are positive real number. Company wants to maximize the profit and minimize the time required for production.

	A	B	C	
R1	2	4	3	≤ 325
R2	4	2	2	≤ 360
R3	3	2	3	≥ 360
Time(h)	4	5	6	

There are around 30, 20 and 20 units of tolerance is allowed for resource R1, R2, R3 respectively by manager. Unit cost and sale price of the product are c_A, c_B, c_C and $S_A = \frac{s_A}{x_1^{\frac{1}{a1}}}$, $S_B = \frac{s_B}{x_2^{\frac{1}{a2}}}$, $S_C = \frac{s_C}{x_1^{\frac{1}{a3}}}$ respectively. Where a_1, a_2, a_3 are positive real number. Company wants to maximize the profit and minimize the time required for production.

To handling the uncertainty, all parameters are taken as PPFN.

$$s_A = \widetilde{100} = (95,100,103; 95,100,104), R(s_A) = 99.4$$

$$s_B = \widetilde{120} = (118,120,122; 115,120,123), R(s_B) = 119.6$$

$$s_C = \widetilde{95} = (94,95,96; 93,95,97), R(s_C) = 95$$

$$c_A = \widetilde{7.5} = (7.5,7.5,8.5; 7,7.5,9), R(c_A) = 7.9$$

$$c_B = \widetilde{10} = (9,10,10.5; 9,10,11), R(c_B) = 9.9$$

$$c_C = \tilde{8} = (7.5,8,8.5; 7.5,8,9), R(c_C) = 8.1$$

$$\widetilde{325} = (325,325,340; 325,325,345), R\left(\widetilde{325}\right) = 332$$

$$\widetilde{360} = (360, 360, 375; 360, 360, 380), R\left(\widetilde{360}\right) = 367$$

$$\widetilde{365} = (350, 365, 365; 345, 365, 365), R\left(\widetilde{365}\right) = 358$$

$$\tilde{2} = (1.5, 2, 2.3; 1.2, 2, 2.5), R\left(\tilde{2}\right) = 1.9$$

$$\tilde{3} = (2.5, 3, 3.4; 2.3, 3, 3.7), R\left(\tilde{3}\right) = 2.98$$

$$\tilde{4} = (3.5, 4, 4.3; 3.2, 4, 4.5), R\left(\tilde{4}\right) = 3.9$$

$$\tilde{5} = (4.8, 5, 5.6; 4.5, 5, 5.8), R\left(\tilde{5}\right) = 5.14$$

$$\tilde{6} = (5.8, 6, 6.2; 5.5, 6, 6.3), R\left(\tilde{6}\right) = 5.96$$

$$a_1 = a_2 = 2, a_3 = \frac{3}{2}$$

The optimization problem can be written as

$$maximize \;\; f_1 = \widetilde{100}x_1^{1-1/a_1} - \widetilde{7.5}x_1 + \widetilde{120}x_1^{1-1/a_2} - \widetilde{10}x_2 + \widetilde{95}x_1^{1-1/a_1} - \tilde{8}x_3$$

$$minimize \; f_2 = \tilde{4}x_1 + \tilde{5}x_2 + \tilde{6}x_3$$

$Subject\ to$

$$\tilde{2}x_1 + \tilde{4}x_2 + \tilde{3}x_3 \leq \widetilde{325}$$

$$\tilde{4}x_1 + \tilde{2}x_2 + \tilde{2}x_3 \leq \widetilde{360}$$

$$\tilde{3}x_1 + \tilde{2}x_2 + \tilde{3}x_3 \geq \widetilde{365}$$

$$x_1, x_2, x_3 \geq 0.$$

The equivalent crisp multi objective optimization problem is

$$maximize \;\; f_1 = 99.4x_1^{\frac{1}{2}} - 7.9x_1 + 119.6x_2^{\frac{1}{2}} - 9.9x_2 + 95x_3^{\frac{1}{3}} - 8.1x_3$$

$$minimize \; f_2 = 3.9x_1 + 5.14x_2 + 5.96x_3$$

$Subject\ to$

$$1.9x_1 + 3.9x_2 + 2.98x_3 \leq 332$$

$$3.9x_1 + 1.9x_2 + 1.9x_3 \leq 367$$

$$2.98x_1 + 1.9x_2 + 2.98x_3 \geq 358$$

$$x_1, x_2, x_3 \geq 0.$$

Pay-off matrix is:

	X^1	X^2
f_1	606.4352	220.4427
f_2	615.3611	573.0927

So, $U_1 = 606.4352, L_1 = 220.4427, U_2 = 615.3611, L_2 = 573.0927$

Here three experts assign their hesitant values $\delta_1 = .96$, $\delta_2 = .98$, $\delta_3 = 1$ for membership function of objective functions and hesitant values $\gamma_1 = .96, \gamma_2 = .98$, and $\gamma_3 = 1$ for non-membership function. Tolerance $\alpha_{f_1} = 50$ and $\alpha_{f_2} = 50$. Take $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$, Then we can formulate the optimization problem in PHF environment as follows:

Using PHFWA operator:

$$\text{Max PHFWA}(\mu^{E_1}, \mu^{E_2}, \mu^{E_3};\ \vartheta^{E_1}, \vartheta^{E_2}, \vartheta^{E_3})$$
$$= \text{Max} \sum_{k=1}^{2} W_k \left(\sqrt{1 - \prod_{i=1}^{3} (1 - \alpha_{ki}^2)^{\frac{1}{3}}} - \prod_{i=1}^{3} \beta_{ki}^{\frac{1}{3}} \right).$$

Subject to,

$$0.96 \frac{f_1 - 220.4427}{606.4352 - 220.4427} \geq \alpha_{11};\ 0.96 \frac{606.4352 - f_1}{606.4352 - 170.4427} \leq \beta_{11}$$
$$0.98 \frac{f_1 - 220.4427}{606.4352 - 220.4427} \geq \alpha_{12}; 0.98 \frac{606.4352 - f_1}{606.4352 - 170.4427} \leq \beta_{12}$$
$$\frac{f_1 - 220.4427}{606.4352 - 220.4427} \geq \alpha_{13}; \frac{606.4352 - f_1}{606.4352 - 170.4427} \leq \beta_{13}$$
$$0.96 \frac{615.3611 - f_2}{615.3611 - 573.0927} \geq \alpha_{21};\ 0.96 \frac{f_2 - 573.0927}{665.3611 - 573.0927} \leq \beta_{21}$$
$$0.98 \frac{615.3611 - f_2}{615.3611 - 573.0927} \geq \alpha_{22};\ 0.98 \frac{f_2 - 573.0927}{665.3611 - 573.0927} \leq \beta_{22}$$
$$\frac{615.3611 - f_2}{615.3611 - 573.0927} \geq \alpha_{23};\ \frac{f_2 - 573.0927}{665.3611 - 573.0927} \leq \beta_{23}$$

Where $0 \leq\ \alpha_{kr}^2 + \beta_{ks}^2 \leq 1 \quad \alpha_{kr}^2 \geq \beta_{ks}^2\ \ \forall k = 1,2$ and $r, s \in \{1,2,3\}$

$$1.9x_1 + 3.9x_2 + 2.98x_3 \leq 332$$
$$3.9x_1 + 1.9x_2 + 1.9x_3 \leq 367$$
$$2.98x_1 + 1.9x_2 + 2.98x_3 \geq 358$$

$x_1, x_2, x_3 \geq 0.$

$W_1 + W_2 = 1.$ (4)

Using PHFWG operator:

$$\text{Max PHFWG}(\mu^{E_1}, \mu^{E_2}, \mu^{E_3};\ \vartheta^{E_1}, \vartheta^{E_2}, \vartheta^{E_3})$$
$$= \text{Max} \sum_{k=1}^{2} W_k \left(\prod_{i=1}^{3} \alpha_{ki}^{\frac{1}{3}} - \sqrt{1 - \prod_{i=1}^{3} (1 - \beta_{ki}^2)^{\frac{1}{3}}} \right).$$

Subject to, the same constraints as Problem (4). (5)

B) Life support system (LSS):

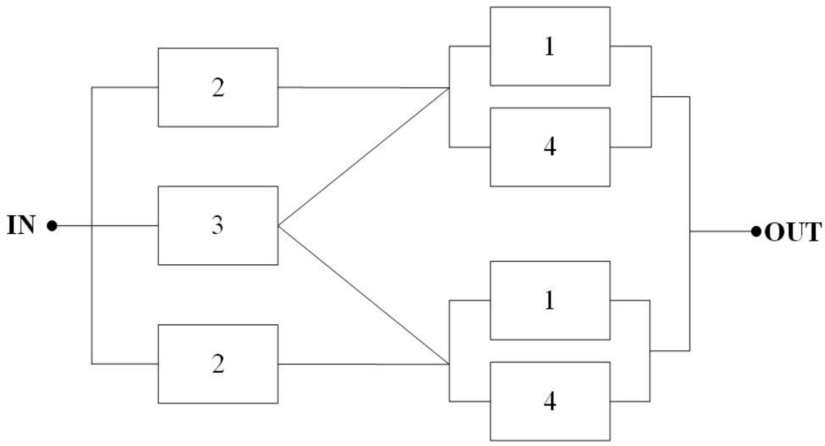

Figure 3: Life support system

The Figure 3 presents the Life support system, here the system cost (C_s),system reliability (R_S) are presented by:

$$R_S = 1 - r_3[(1 - r_1)(1 - r_4)]^2 - (1 - r_3)[1 - r_2\{1 - (1 - r_1)(1 - r_4)\}]^2$$
$$C_S = 2K_1.r_1^{a_1} + 2K_2.r_2^{a_2} + K_3.r_3^{a_3} + 2K_4.r_4^{a_4}$$

The goal of this problem is to reduce system cost while maintaining component and system reliability and taking cost coefficient as PPFN. Therefore, the problem is described as:

Maximize R_S, Minimize $\widetilde{C_s}$

Subject to

$$.5 \leq r_i \leq 1 - 10^{-6} \qquad i = 1,2,3,4 \tag{6}$$

Where, r_i is reliability of i^{th} component.

Here $K_1 = 100, K_2 = 100, K_3 = 200, K_4 = 150\ and\ a_i = 0.6\ for\ i = 1,2,3,4.$

Here cost coefficients K_1, K_2, K_3, K_4 are taken as parabolic fuzzy number and the crisp value is calculated in Table 1.

Table 1: Cost coefficient as parabolic Pythagorean fuzzy number

Parameter	Parabolic Pythagorean fuzzy number $\{(a,b,c),(a',b,c')\}$	Crisp value $\left[\frac{(2a+b+2c)+(2a'+b+2c')}{10}\right]$
K_1	$\{(98,100,102),(96,100,103)\}$	99.8
K_2	$\{(99,100,102),(98,100,103)\}$	100.4
K_3	$\{(198,200,201),(197,200,202)\}$	199.6
K_4	$\{(147,150,152),(147,150,153)\}$	149.8

Pay-off matrix is:

	R^1	R^2
R_s	0.999999	0.7734375
C_s	1099.199	725.2015

So, $U_1 = .999999, L_1 = 0.773437, U_2 = 1099.199, L_2 = 725.2015$

Here three experts assign their hesitant values $\delta_1 = .96$, $\delta_2 = .98$, $\delta_3 = 1$ for membership function of objective functions and hesitant values $\gamma_1 = .96, \gamma_2 = .98$, and $\gamma_3 = 1$ for non-membership function. Tolerance $\alpha_{R_s} = .1$ and $\alpha_{C_s} = 50$. Take $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$, Then we can formulate the optimization problem in PHF environment as follows:

Using PHFWA operator:

$$\text{Max PHFWA}(\mu^{E_1}, \mu^{E_2}, \mu^{E_3};\ \vartheta^{E_1}, \vartheta^{E_2}, \vartheta^{E_3})$$
$$= \text{Max} \sum_{k=1}^{2} W_k \left(\sqrt{1 - \prod_{i=1}^{3} (1-\alpha_{ki}^2)^{\frac{1}{3}}} - \prod_{i=1}^{3} \beta_{ki}^{\frac{1}{3}} \right).$$

Subject to,

$$0.96\,\frac{\mathrm{R_s(r)} - 0.7734375}{0.999999 - .7734375} \geq \alpha_{11};\ 0.96\,\frac{0.999999 - \mathrm{R_s(r)}}{0.999999 - 0.6734375} \leq \beta_{11}$$
$$0.98\,\frac{\mathrm{R_s(r)} - 0.7734375}{0.999999 - .7734375} \geq \alpha_{12};\ 0.98\,\frac{0.999999 - \mathrm{R_s(r)}}{0.999999 - 0.6734375} \leq \beta_{12}$$
$$\frac{\mathrm{R_s(r)} - 0.7734375}{0.999999 - .7734375} \geq \alpha_{13};\ \frac{0.999999 - \mathrm{R_s(r)}}{0.999999 - 0.6734375} \leq \beta_{13}$$

$$0.96\frac{1099.199 - C_s(r)}{1099.199 - 725.2015} \geq \alpha_{21};\ 0.96\frac{C_s(r) - 725.2015}{1149.199 - 725.2015} \leq \beta_{21}$$

$$0.98\frac{1099.199 - C_s(r)}{1099.199 - 725.2015} \geq \alpha_{22};\ 0.98\frac{C_s(r) - 725.2015}{1.199 - 725.2015} \leq \beta_{22}$$

$$\frac{1099.199 - C_s(r)}{1099.199 - 725.2015} \geq \alpha_{23};\ \frac{C_s(r) - 725.2015}{1099.199 - 725.2015} \leq \beta_{23}$$

Where $0 \leq \alpha_{kr}^2 + \beta_{ks}^2 \leq 1 \quad \alpha_{kr}^2 \geq \beta_{ks}^2 \ \forall k = 1,2$ and $r, s \in \{1,2,3\}$

$$0.5 \leq r_j \leq 1 - 10^{-6} \qquad j = 1,2,3,4.$$

$$W_1 + W_2 = 1. \tag{7}$$

Using PHFWG operator:

$$\text{Max PHFWG}(\mu^{E_1}, \mu^{E_2}, \mu^{E_3};\ \vartheta^{E_1}, \vartheta^{E_2}, \vartheta^{E_3}) = \text{Max} \sum_{k=1}^{2} W_k \left(\prod_{i=1}^{3} \alpha_{ki}^{\frac{1}{3}} - \sqrt{1 - \prod_{i=1}^{3} (1 - \beta_{ki}^2)^{\frac{1}{3}}} \right).$$

Subject to, the same constraints as Problem (7). (8)

Degree of closeness to Ideal solution:

A pivotal factor in ascertaining the compromise solution to demonstrate the effectiveness of various solution strategies is how near the solutions are to the ideal ones. Multi-objective optimization problem solution a variety of distance function families exist, and these families may be used to evaluate the effectiveness of different solution methods. In this section, we have considered a family of distance functions that may be described as follows:

$$D_e(\gamma, l) = \left[\sum_{k=1}^{l} \gamma_k^e (1 - r_k)^e \right]^{\frac{1}{e}}$$

where r_k is the fraction of compromise solution to the ideal solution of k^{th} objective and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_l)$ is the weight vector assigned to objectives.

$$D_1(\gamma, l) = 1 - \sum_{k=1}^{l} \gamma_k r_k,$$

$$D_2(\gamma, l) = \left[\sum_{k=1}^{l} \gamma_k^2 (1 - r_k)^2 \right]^{\frac{1}{2}}$$

$$D_\infty(\gamma, l) = \max_k [\gamma_k (1 - r_k)]$$

In minimization type objective function,

$$r_k = \frac{\text{the ideal solution of } f_k}{\text{the desired compromise solution of } f_k}$$

In maximization type objective function,

$$r_k = \frac{\text{the desired compromise solution of } f_k}{\text{the ideal solution of } f_k}$$

5. RESULTS AND DISCUSSION

The compromise solution of manufacturing system and life support system has been achieved through the utilization of PHF linear membership functions and non-membership functions under two distinct models, as illustrated in Table 2 and Table 3, respectively. In each model, different crisp weights have been employed. The weights assigned to the first objective have been selected randomly within the range of 0 to 1. These weights are determined by the decision-maker(s) based on their level of satisfaction. Simultaneously, the weight assigned to the second objective is the complement of weight to the first objective function. The weight of PHF element in aggregation operator are taken maintaining same weightage. Then the single objective optimization models are solved by LINGO 18.0. To assess the sensitivity of the results, a thorough sensitivity analysis was performed by modifying the weight parameter. To show the efficiency of the solution degree of closeness to ideal solution are calculated and compared with solution obtained by the existing method using HF weighted averaging (HFWA) and HF weighted geometric (HFWG) operators in Table 5 and Table 6.

A. Compromise solutions of manufacturing system

In Table 2, by HFWA operator, for weight $(.1, .9)$ to the objective function the optimal solution is $(220.7964, 573.0927)$, for weight $(.3, .7)$ the solution is $(347.7069, 575.7889)$, for weight $(.5, .5)$ the solution is $(453.2292, 583.6776)$, for weight $(.7, .3)$ solution is $(567.2987, 603.2049)$ and for $(.9, .1)$ is $(606.4351, 615.3611)$.

So, when the weight of the first objective is increased then the solution goes from worst to best solution and when the weight of second objective is decreased the solution goes from best to the worst solution.

By PHFWA operator, for weight $(.1, .9)$ to the objective function the optimal solution is $(292.2183, 573.9100)$, for weight $(.3, .7)$ the solution is $(427.8848, 581.1482)$, for weight $(.5, .5)$ the solution is $(543.0254, 597.5321)$, for weight $(.7, .3)$ solution is $(606.4313, 615.3597)$ and for $(.9, .1)$ is $(606.4332, 615.3604)$.

So, when the weight allocated to the first objective is increased, the solution transitions from a relatively poorer solution to a better one. Conversely, when the weight allocated to the second objective is decreased, the solution moves from a better solution to a relatively poorer one.

The result obtained by using PHFWA operator is different from the result obtained by using HFWAO, the comparison is presented by degree of closeness to ideal solution in Table 5.

By HFWG operator, for weight $(.1, .9)$ to the objective function the optimal solution is $(260.2183, 573.3294)$, for weight $(.3, .7)$ the solution is $(349.8160, 575.8869)$, for weight $(.5,.5)$ the solution is $(453.3863, 583.6948)$, for weight $(.7,.3)$ solution is $(566.1693, 602.9149)$ and for $(.9,.1)$ is $(606.4351, 615.3611)$.

So, when the weight allocated to the first objective is increased, the solution progresses from its worst state to the best possible solution. Conversely, when the weight allocated to the second objective is reduced, the solution regresses from its best state to the worst possible solution.

Table 2: Optimal compromise objective values for manufacturing system

weight	Objective values $[(f_1^*, f_2^*)]$			
(W_1, W_2)	HFWA	PHFWA	HFWG	PHFWG
$(.1,.9)$	$(220.7964, 573.0927)$	$(292.9689, 573.9100)$	$(260.2183, 573.3294)$	$(270.9738, 573.4794)$
$(.2,.8)$	$(296.2601, 573.9894)$	$(362.3822, 576.5138)$	$(303.2744, 574.1721)$	$(323.4829, 574.8055)$
$(.3,.7)$	$(347.7069, 575.7889)$	$(427.8848, 581.1482)$	$(349.8160, 575.8869)$	$(377.7452, 577.3857)$
$(.4,.6)$	$(399.2316, 578.8155)$	$(488.4911, 588.0708)$	$(399.9206, 578.8656)$	$(433.2288, 581.6428)$
$(.5,.5)$	$(453.2292, 583.6776)$	$(543.0254, 597.5321)$	$(453.3863, 583.6948)$	$(488.9955, 588.1421)$
$(.6,.4)$	$(509.6103, 591.2903)$	$(590.1488, 609.7415)$	$(509.4388, 591.2621)$	$(543.4850, 597.6295)$
$(.7,.3)$	$(567.2987, 603.2049)$	$(606.4313, 615.3597)$	$(566.1693, 602.9149)$	$(606.4351, 615.3610)$
$(.8,.2)$	$(606.4351, 615.3611)$	$(606.4322, 615.3600)$	$(606.4351, 615.3611)$	$(606.4351, 615.3610)$
$(.9,.1)$	$(606.4351, 615.3611)$	$(606.4332, 615.3604)$	$(606.4351, 615.3611)$	$(606.4351, 615.3610)$

By PHFWG operator, for weight $(.1,.9)$ to the objective function the optimal solution is $(270.9738, 573.4794)$, for weight $(.3,.7)$ the solution is $(377.7452, 577.3857)$, for weight $(.5,.5)$ the solution is $(488.9955, 588.1421)$, for weight $(.7,.3)$ solution is $(606.4351, 615.3610)$ and for $(.9,.1)$ is $(606.4351, 615.3610)$.

It is observed that as the weights assigned to the first objective function are increased, the solution values progressively move closer to their best possible outcome. Similarly, As the weights assigned to the second objective function decrease, the solution values tend to converge toward their least favorable state.

The results obtained by PHFWG and HFWG operators are different; in Table 5, the comparison is shown by how closely the results match the ideal solution.

B. Compromise solutions of reliability optimization model of Life support system

In Table 3, by HFWA operator, at weight $(.1,.9)$ assigned to the objective functions the optimal solution is $(.773437, 725.2015)$, for weight $(.3,.7)$ the solution is $(.955693, 805.6241)$, at weight $(.5,.5)$ the solution is $(.999999, 861.3603)$, for weight $(.7,.3)$ solution is $(.999999, 861.3603)$ and for $(.9,.1)$ is $(.999999, 861.3603)$.

In summary, it can be concluded that increasing the weight of the first objective brings the solution closer to the best outcome, while decreasing the weight of the second objective moves the solution towards the worst possible result and vice versa.

By PHFWA operator, for weight $(.1,.9)$ to the objective function the optimal solution is $(.868483, 759.8243)$, for weight $(.3,.7)$ the solution is $(.938986, 794.4539)$, for weight $(.5,.5)$ the solution is $(.992998, 840.6600)$, for weight $(.7,.3)$ solution is $(.999999, 861.3583)$ and for $(.9,.1)$ is $(.999999, 861.3583)$.

It is observed that altering the weight assigned to the first (second) objective leads the solution closer to the best (worst) possible outcome.

By HFWG operator, for weight $(.1,.9)$ to the objective function the optimal solution is $(.773440, 725.2023)$, for weight $(.3,.7)$ the solution is $(.955632, 805.5805)$, for weight

$(.5, .5)$ the solution is $(.993971, 842.2107)$, for weight $(.7, .3)$ solution is $(.998981, 853.6873)$, and for $(.9, .1)$ is $(.999934, 859.4861)$.

The results obtained by PHFWA and HFWA operators are different from one other, and Table 6 compares them according to how closely they get to the ideal optimal solution. So, when the weight allocated to the first objective is increased, the solution progresses from its worst state to the best possible solution. Conversely, when the weight allocated to the second objective is reduced, the solution regresses from its best state to the worst possible solution.

By PHFWG operator, for weight $(.1, .9)$ to the objective function the optimal solution is $(.868482, 759.8240)$, for weight $(.3, .7)$ the solution is $(.939301, 794.6547)$, for weight $(.5, .5)$ the solution is $(.992432, 839.8040)$, for weight $(.7, .3)$ solution is $(.998735, 852.7920)$ and for $(.9, .1)$ is $(.999919, 859.2686)$.

The result produced by PHFWG operator differs from the result produced by HFWG operator; the comparison is shown in Table 6 by the degree of similarity to the ideal solution.

It is seen that the solution values gradually approach their ideal conclusion when the weights applied to the first goal function are increased. Similar to this, when the weight allocated to second objective is reduced, the solution values go to worse.

Table 3: Optimal compromise objective values for Life support system (LSS)

weight	Objective values $[(f_1^*, f_2^*)]$			
(W_1, W_2)	HFWA	PHFWA	HFWG	PHFWG
$(.1, .9)$	$(.773437, 725.2015)$	$(.868483, 759.8243)$	$(.773440, 725.2023)$	$(.868482, 759.8240)$
$(.2, .8)$	$(.773437, 725.2015)$	$(.868483, 759.8243)$	$(.873537, 761.8927)$	$(.868482, 759.8240)$
$(.3, .7)$	$(.955693, 805.6241)$	$(.938986, 794.4539)$	$(.955632, 805.5805)$	$(.939301, 794.6547)$
$(.4, .6)$	$(.985855, 831.4092)$	$(.981391, 826.6951)$	$(.985179, 830.6577)$	$(.981096, 826.4021)$
$(.5, .5)$	$(.999999, 861.3603)$	$(.992998, 840.6600)$	$(.993971, 842.2107)$	$(.992432, 839.8040)$
$(.6, .4)$	$(.999999, 861.3603)$	$(.997634, 849.5475)$	$(.997451, 849.0868)$	$(.996825, 847.6161)$
$(.7, .3)$	$(.999999, 861.3603)$	$(.999999, 861.3583)$	$(.998981, 853.6873)$	$(.998735, 852.7920)$
$(.8, .2)$	$(.999999, 861.3603)$	$(.999999, 861.3583)$	$(.999661, 856.9925)$	$(.998581, 856.4895)$
$(.9, .1)$	$(.999999, 861.3603)$	$(.999999, 861.3583)$	$(.999934, 859.4861)$	$(.999919, 859.2686)$

To summarize, increasing the weight of an objective enhances its influence, enabling the solution to reach its optimal performance in that objective. Conversely, decreasing the weight of an objective weakens its impact, potentially leading to a deterioration in performance in that aspect while aiming for a better balance with other objectives. This approach allows decision-makers to control the trade-offs between objectives and tailor the solution to their preferences and priorities.

Using classical weighted sum method, the solution obtained for manufacturing system and life support system are listed in Table 4. In this context, the solution falls within the same coverage spectrum as the solution obtained through the proposed method.

Table 4: Optimal value of the objective by classical weighted sum method

weight	Objective values $[(f_1^*, f_2^*)]$ by weighted sum method	
(W_1, W_2)	Manufacturing system	Life support system
(.1, .9)	(455.3876, 5839156)	(.773437, 725.2015)
(.2, .8)	(563.5913,602.2633)	(.773437, 725.2015)
(.3, .7)	(.606.4352,615.3611)	(.773437, 725.2015)
(.4, .6)	(606.4352,615.3611)	(.773437, 725.2015)
(.5, .5)	(606.4352,615.3611)	(.773437, 725.2015)
(.6, .4)	(606.4352,615.3611)	(.773437, 725.2015)
(.7, .3)	(606.4352,615.3611)	(.773437, 725.2015)
(.8, .2)	(606.4352,615.3611)	(.773437, 725.2015)
(.9, .1)	(606.4352,615.3611)	(.773437, 725.2015)

It is shown that, in the weighted sum method, the DMs needs to provide fixed weights for each objective in advance. These fixed weights may not reflect the DMs true preferences accurately and may lead to biased or suboptimal solutions.

In Table 5, for each weight combination the closeness degree to ideal solution of solutions obtained by PHFWA operator is better than the solution obtained by HFWA operator, similarly the solution derived from the PHFWG operator is more proximate to the ideal solution than the solution obtained by HFWG operator.

Table 5: Comparison of the optimal solution of manufacturing system by degree of closeness to ideal solution

Weight	Degree of closeness to ideal solution	HFWA	PHFWA	HFWG	PHFWG
(.1,.9)	D_1	0.0636	0.0529	0.0574	0.0559
	D_2	0.0636	0.0517	0.0571	0.0553
	D_∞	0.0636	0.0516	0.0571	0.0553
(.2,.8)	D_1	0.1035	0.0823	0.1015	0.0957
	D_2	0.1023	0.0806	0.0999	0.0933
	D_∞	0.1023	0.0806	0.0998	0.0933
(.3,.7)	D_1	0.1313	0.0980	0.1303	0.1183
	D_2	0.1280	0.0888	0.1269	0.1132
	D_∞	0.1279	0.0832	0.1269	0.1131
(.4,.6)	D_1	0.1426	0.0930	0.1422	0.1230
	D_2	0.1368	0.0792	0.1363	0.1145
	D_∞	0.1366	0.0777	0.1362	0.1142
(.5,.5)	D_1	0.1354	0.0727	0.1353	0.1096
	D_2	0.1266	0.0561	0.1265	0.0976
	D_∞	0.1263	0.0522	0.1261	0.0968
(.6,.4)	D_1	0.1081	0.0401	0.1082	0.0787
	D_2	0.0966	0.0289	0.0967	0.0644
	D_∞	0.0958	0.0240	0.0959	0.0622
(.7,.3)	D_1	0.0602	0.0206	0.0613	0.0206
	D_2	0.0476	0.0206	0.0488	0.0206
	D_∞	0.0451	0.0206	0.0464	0.0206
(.8,.2)	D_1	0.0137	0.0137	0.0137	0.0137
	D_2	0.0137	0.0137	0.0137	0.0137
	D_∞	0.0137	0.0137	0.0137	0.0137
(.9,.1)	D_1	0.0069	0.0069	0.0069	0.0069
	D_2	0.0069	0.0069	0.0069	0.0069
	D_∞	0.0069	0.0069	0.0069	0.0069

To show the efficiency of the proposed method closeness degree to ideal solution is calculated and listed in Table 6. From this table, except the weight $(.1,.9)$ the closeness degree to ideal solution of the obtained solution by PHFWA operator is better than the closeness of the solution obtained by HFWA operator and closeness degree of the solution obtained by PHFWG operator is better than the solution obtained by HFWG operator.

Table 6: Comparison of the optimal solution of LSS by degree of closeness to ideal solution

Weight	Degree of closeness to ideal solution	HFWA	PHFWA	HFWG	PHFWG
(.1,.9)	D_1	0.0226	0.0541	0.0226	0.0541
	D_2	0.0226	0.0430	0.0226	0.0430
	D_∞	0.0226	0.0410	0.0226	0.0410
(.2,.8)	D_1	0.0453	0.0627	0.0638	0.0627
	D_2	0.0453	0.0449	0.0460	0.0449
	D_∞	0.0453	0.0364	0.0385	0.0364
(.3,.7)	D_1	0.0832	0.0793	0.0832	0.0794
	D_2	0.0711	0.0637	0.0711	0.0638
	D_∞	0.0698	0.0610	0.0698	0.0611
(.4,.6)	D_1	0.0823	0.0811	0.0821	0.0810
	D_2	0.0768	0.0740	0.0764	0.0738
	D_∞	0.0766	0.0736	0.0761	0.0734
(.5,.5)	D_1	0.0790	0.0721	0.0724	0.0720
	D_2	0.0790	0.0687	0.0695	0.0683
	D_∞	0.0790	0.0687	0.0694	0.0682
(.6,.4)	D_1	0.0632	0.0631	0.0598	0.0598
	D_2	0.0632	0.0631	0.0584	0.0581
	D_∞	0.0632	0.0631	0.0583	0.0581
(.7,.3)	D_1	0.0474	0.0473	0.0459	0.0458
	D_2	0.0474	0.0473	0.0451	0.0450
	D_∞	0.0474	0.0473	0.0451	0.0450
(.8,.2)	D_1	0.0316	0.0316	0.0310	0.0310
	D_2	0.0316	0.0316	0.0307	0.0307
	D_∞	0.0316	0.0316	0.0307	0.0307
(.9,.1)	D_1	0.0158	0.0158	0.0157	0.0157
	D_2	0.0158	0.0158	0.0156	0.0156
	D_∞	0.0158	0.0158	0.0156	0.0156

6. LIMITATION OF OUR STUDY

We have identified certain limitations in our approach. While our method seeks to obtain the Pareto optimal solution, it's important to note that the reverse may not hold true. This is because we employ a weighted sum of the aggregated PHF membership and non-membership as a utility function, which may not effectively explore solutions in the non-convex region of the Pareto front, if one exists. To address this limitation and find Pareto optimal solutions in the non-convex portion of the Pareto front, alternative approaches such as metaheuristic methods or ϵ-constraint methods with suitable ϵ bounds on the objective function are necessary. By explicitly discussing these limitations, we aim to offer a transparent and balanced assessment of our work and provide guidance for future research in this field.

7. CONCLUSIONS

In this paper, a multi-objective optimization approach is presented in PHF environment. To convert the uncertain multi-objective optimization problem to single objective optimization problem the PHFWA and PHFWG operators are used. So, here PHF environments gives the option to choose memberships and non-memberships in much range of area than FS and IFS to decision making experts. For uncertainty of the parameters of objective functions, that are taken as PPFN. This method conclude that the result obtained by using PHFWA and PHFWG operator is better than the result obtained by HFWA and HFWG operator by degree of closeness to ideal solution. In order to demonstrate the practicality of the suggested approach a numerical example of manufacturing system and a real-life multi-objective ROM of LSS is solved. A sensitivity analysis of the optimal solution is presented by different weights of the objectives and giving the same weight to all DM's preferences. A comparison of the solutions obtained by proposed method and existing method is presented by closeness degree to ideal solution. The proposed method is also useful when several experts are involved in decision making process and they are confused about the exact value of parameters and goal of the optimization problem.

In future, multi objective optimization model of various field such as engineering, finance, inventory control, transportation, environmental management etc. can be solved by the proposed method. We can make this method more reliable by using the extension of PHF aggregation operator. At the place of linear membership, we can use non-linear membership and non-membership functions (exponential, hyperbolic, etc.). Various metaheuristic algorithm can be applied to the proposed model for more insightful result in future.

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