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ANALYSIS OF SINGLE SERVER REPAIRABLE QUEUEING SYSTEM WITH MANY KINDS OF SERVICE, SINGLE WORKING VACATION AND STANDBY SERVER

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Abstract: We consider a single server queueing model in which the customers arrive according to a Markovian Arrival Process (MAP). We modeled this queueing system with many kinds of service where the customers can choose any kind from the server. Each kind of service follows Phase-type (PH) distribution. We analyzed the model with breakdown, repair, single working vacation and standby server. A standby server works only when the main server is breakdown while serving the customer during a busy or vacation period. Using the matrix-geometric approach, we examined the model. A few real-world examples have been given, and several graphical and numerical examples have also been provided

Keywords: Markovian arrival process, phase-type distribution, standby, breakdown, single working vacation.

MSC: 60K25, 60K30, 68M20, 90B22.

1. INTRODUCTION

Currently, queueing models with Markovian Arrival Process with phase-type distribution have been studied and analyzed with various service interruptions using the matrix-geometric method. Neuts [1] introduced Phase-type (PH) distribution, then developed matrix-geometric methods in queueing theory and popularized the algorithmic strategies to study stochastic models. Later he introduced the Markovian arrival process (MAP) and then Chakravarthy [2] deeply explained the concept of MAP. In his recent work [3], he analyzed the MAP queueing systems with phase-type service using the matrix-geometric method.

Server vacation queueing models are useful for queueing systems in which the server wants to utilize its time for different purposes. The vacation mechanism considered in this paper is termed as Single Working Vacation (SWV). We analyze $MAP/PH/1$ queueing model with single working vacations and service, in which the server provides service at lower rate, rather than stopping the service altogether during its vacation period. Such a vacation is called a Working Vacation and the vacation queueing model was deeply analyzed by servi and finn [4]. The working vacation of the server begin when the system is empty right after the completion of its services in busy period. If there are no customers in the system when a working vacation ends, then the server stays available and is ready to serve new arrivals. If the server returns from a working vacation and find the system as non-empty then the server immediately switches to the original service rate. This working vacation is called a SWV. Many researchers have looked into queueing models with server vacations ([5],[6],[7],[8],[9],[10]).

Using N-policy for vacations, Sreenivasan et al. [11] examined a $MAP/PH/1$ type queue. Goswami and Selvaraju [12] also evaluated a working vacation model that had two different priority customer kinds, the potential to interrupt vacations, and services that followed a phase-type distribution. The survey article by Chandrasekaran et al. [13] has a distinct section on working vacation models with Markovian Arrival Process (MAP). The $MAP/PH/1$ queueing structure with breakdown, instantaneous feedback, and server vacation has been examined by Ayyappan and Thilagavathy [14]. Choudhary et al. [15] looked at a single server queueing system with phase-type vacation and degraded service rate.

Especially in industrial, manufacturing, and communication systems, it can be unreasonable to assume that the service environment is completely reliable. The study of stochastic models with server breakdowns and repairs can result from the possibility of server breakdowns brought on by external shocks. We can use a standby (backup) server in the event that server fails. Chakravarthy and Agnihothri have examined the MAP and phase type service queueing model with the backup server in [16].

Yang and Wu [17] explored into the $M/M/1$ queueing system with working vacation and server failure under N-policy. They looked at various breakdown rates during working vacations, busy and idle time of the server. Working breakdown is a term that Kalidass and Kasturi [18] used to characterise the situation . The researchers also thought about having a backup server accessible to serve at a lower rate when the primary server is down. Later, Ye and Liu [19] investigated the $MAP/M/1$ type queue with working failures and repairs of the server. Deepa and Kalidass [20] studied the steady-state probability for a finite working vacation and working breakdown queue.

Sharma [21] explored the threshold N-policy for the $M^X/H_2/1$ queueing system with an unreliable server and vacations. Singh et al. [22] explored the $M/G/1$ queueing model with vacation and employed the generating function method to get various performance metrics. Wang et al. [23] concentrated on a single-arrival Erlangian service time queueing model with an unreliable server and extended the queueing model to the N-policy for the $M/H_2/1$ queueing system. The $M^X/H_k/1$ queueing model was developed by Jain et al. [24] using a server that was unreliable and multiple vacation policies.

The existing literature in this topic focuses mainly on a single server queueing model with multiple vacations. This would also encourage us to study the $MAP/PH/1$ queue with a single working vacation. We discovered the research gap in which the server only provides phase-type services with some disturbance. This helps us to develop the single server queueing model with a variety of services from which consumers can select based on their requirements. We investigated the single server queueing model with various kinds of service and standby servers as a backup for whenever the system breaks down during busy or working vacation periods. This type of queueing model is analyzed using Neuts' [1] well-known matrix-geometric approach to determine steady-state probability vectors.

This article's structure is as follows: Section 2 presents a discussion of the mathematical model as well as how the model's matrix was constructed. We outline the stability requirement in section 3 and we analyse the steady-state probability of the proposed model in section 4. The performance measures are evaluated in section 5. We discuss some special cases in section 6 and present a few real-world examples in section 7. Numerous numerical illustrations and graphical representations are included in section 8. The conclusion is provided in section 9. Throughout the work, the following conventional notations are used:

- The Kronecker product and Kronecker sum are denoted by the symbols $\otimes$ and $\oplus$, respectively.
- e is the column vector will be assigned the appropriate dimension its entries are all ones.
- The identity matrix of order m is denoted by the symbol I_m.
- The diagonal matrix with entries given in the parantheses is indicated by $\triangle(.)$
- e_1: $(3lmn)$ $times$ 1 vector with the first lmn component set to 1 and the remaining components set to 0.
- e_2: $(3lmn)$ $times$ 1 vector, where $(lmn + 1)$ to $2lmn$ components have a value of 1, while the rest have a value of 0.

- e_3: $(3lmn)$ *times* 1 vector, where $(2lmn+1)$ to $3lmn$ components have a value of 1, while the rest have a value of 0.

2. THE MODEL DESCRIPTION

Customers are presumptively following the Markovian arrival process, or MAP as it was introduced by Neuts, when they access the system. Assume that the generator of MAP is $D = D_0 + D_1$, where D_0 governs transitions for no arrival and D_1 governs transitions for an arrival in the system, both of which are of order m. σ denotes the stationary vector of D, therefore we get

$$\sigma D = 0, \sigma e = 1. \tag{1}$$

Then, the average arrival rate is given by, $\lambda = \sigma D_1 e$.

The system contains a single server that offers many kinds of service and is referred to as the main server. Customers can select from a variety of services based on their needs from a list of l-kinds. Phase-type (PH) distribution is used for all service kinds. The main server takes a Single Working Vacation (SWV) while the system is supposed to be empty and offers service at a reduced rate when customers enter the system at that time. If the system is not empty after SWV is finished, the server switches to the busy period and serves the customer; otherwise, the server remains in the busy period and awaits the customers' arrival. An exponential distribution with rate γ has been used to estimate the length of working vacation. The main server breaks down and is sent for repair immediately which is exponentially distributed with rate ξ and η, respectively. The standby server immediately takes over the service at that point. Take in consideration that the backup server is not susceptible to failures. The standby server immediately surrenders the control of the service to the main server once the repair procedure is finished. We assume that the interarrival, service, breakdown, and vacation times are all independent of one another and that first come, first served (FCFS) is the service discipline.

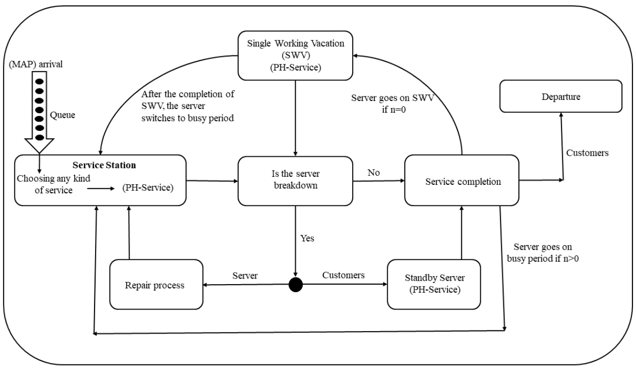

Figure 1: A Visual representation of the model

The service time for the model under consideration follows Hyper-(PH) distribution. A customer after entered into the server and choosing any one of the l kinds of service with rate $\theta_1, \theta_2, \cdots \theta_l$ and the probability $\alpha_1, \alpha_2, \cdots, \alpha_l$, respectively. This can be represented by (α, θ) where, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ and θ indicates the block diagonal matrix $\theta = \triangle(\theta_1, \theta_2, \theta_3, \ldots, \theta_l)$ with order l, where l indicates the total number of available kinds of service. We assume that each kinds of service follows PH-distribution with an irreducible representation (β, S) of order n. It is important to note that the service rate is expressed as μ where $\mu = [\beta(-S)^{-1}e]^{-1}$. In addition, we designate S^0 as the column vector satisfies $Se + S^0 = 0$. Whenever a customer $(i > 0)$ leaves any kind of service (k) i.e., (i, k), can take the system to any of the kinds $(i-1, k)$ at rate $\alpha_k \theta_k S$ for $k = 1, 2, \cdots, l$. Hence, the service distribution of our model follows Hyper-(PH) distribution and it can be represented by (ψ, T) with mutually independent PH-distributions:

$$\psi = (\alpha \otimes \beta) \text{ and } T = \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & T_3 & & \\ & & & \ddots & \\ & & & & T_k \end{bmatrix}$$

where, $T_k = \theta_k S$, T is the matrix with order ln and $T^0 = -Te$.
The model under study has three states such as busy, breakdown and working vacation. The service time of each states follows PH-distribution with representation as (ψ, T), $(\psi, \tau T)$ and $(\psi, \zeta T)$, $0 < \tau < \zeta < 1$ where, τ and ζ are the ratios of service speed when the server is in breakdown and working vacation, respectively. In Figure 1 provides a visual representation of the queueing model under investigation. The variables required to characterise the model under consideration are now defined.

- $N(t)$: the number of customers in the system at time t.
- $X_1(t)$ indicates the state of the server at time t.

$$X_1(t) = \begin{cases} 1, & \text{if the main server is available} \\ 2, & \text{if the main server is under repair and} \\ & \quad \text{offering service by standby server} \\ 3, & \text{if the main server is on working vacation} \end{cases}$$

- $X_2(t)$ indicates the phase of arrival process at time t.
- $X_3(t)$ indicates the phase of service process at time t.
- $X_4(t)$ indicates the kind of service at time t.

Then $\{(N(t), X_1(t), X_2(t), X_3(t), X_4(t)), t \geq 0\}$, is a continuous-time Markov chain with state space Ω as follows:

$$\Omega = \bigcup_{i=0}^{\infty} d(i)$$

where, for $i \geq 0$,

$$d(i) = \{(i, j, a, b, k) \colon 1 \leq j \leq 3;\ 1 \leq a \leq m;\ 1 \leq b \leq n;\ 1 \leq k \leq l\}.$$

Then the infinitesimal generator matrix of our model with following structure:

$$Q = \begin{bmatrix} B_{00} & B_{01} & \dots & & & \\ B_{10} & A_1 & A_0 & \dots & & \\ \vdots & A_2 & A_1 & A_0 & \dots & \\ & \vdots & A_2 & A_1 & A_0 & \dots \\ & & \vdots & \vdots & \vdots & \end{bmatrix}$$

where the block matrices in Q are as follows:

$$B_{00} = \begin{bmatrix} D_0 - \xi I_m & \xi I_m & 0 \\ \eta I_m & D_0 - 2\eta I_m & \eta I_m \\ \gamma I_m & \xi I_m & D_0 - (\gamma + \xi) I_m \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} \psi \otimes D_1 & 0 & 0 \\ 0 & \psi \otimes D_1 & 0 \\ 0 & 0 & \psi \otimes D_1 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} 0 & 0 & T^0 \otimes I_m \\ 0 & 0 & \zeta T^0 \otimes I_m \\ 0 & 0 & \tau T^0 \otimes I_m \end{bmatrix},$$

$$A_0 = \begin{bmatrix} I_{ln} \otimes D_1 & 0 & 0 \\ 0 & I_{ln} \otimes D_1 & 0 \\ 0 & 0 & I_{ln} \otimes D_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} (T \oplus D_0) - \xi I_{lmn} & \xi I_{lmn} & 0 \\ \eta I_{lmn} & (\zeta T \oplus D_0) - 2\eta I_{lmn} & \eta I_{lmn} \\ \gamma I_{lmn} & \xi I_{lmn} & (\tau T \oplus D_0) - (\xi + \gamma) I_{lmn} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} (\psi \otimes T^0) \otimes I_m & 0 & 0 \\ 0 & (\psi \otimes \zeta T^0) \otimes I_m & 0 \\ 0 & 0 & (\psi \otimes \tau T^0) \otimes I_m \end{bmatrix}$$

3. THE STABILITY ANALYSIS

Assume that the matrix A is defined as $A = A_0 + A_1 + A_2$. Let π represent the steady-state probability vector of A, which satisfies the condition [25]

$$\pi A = 0, \pi e = 1. \tag{2}$$

The vector π is partitioned by $\pi = (\pi_0, \pi_1, \pi_2)$, where π_0, π_1 and π_2 are of dimension lmn.

$$A = \begin{bmatrix} U & \xi I_{lmn} & 0 \\ \eta I_{lmn} & V & \eta I_{lmn} \\ \gamma I_{lmn} & \xi I_{lmn} & W \end{bmatrix}$$

where,

$$U = (T + \psi \otimes T^0) \oplus D - \xi I_{lmn}, \tag{3}$$

$$V = \zeta(T + \psi \otimes T^0) \oplus D - 2\eta I_{lmn}, \tag{4}$$

$$W = \tau(T + \psi \otimes T^0) \oplus D - (\gamma + \xi) I_{lmn}. \tag{5}$$

By resolving the following equations, the vector π can be found. The steady-state equations that correspond to (2) are expressed as

$$\pi_0((T + \psi \otimes T^0) \oplus D - \xi I_{lmn}) + \pi_1 \eta I_{lmn} + \pi_2 \gamma I_{lmn} = 0 \tag{6}$$

$$\pi_0(\xi I_{lmn}) + \pi_1((T + \psi \otimes \zeta T^0) \oplus D - 2\eta I_{lmn}) + \pi_2 \xi I_{lmn} = 0 \tag{7}$$

$$\pi_1 \eta I_{lmn} + \pi_2((T + \psi \otimes \tau T^0) \oplus D - (\gamma + \xi) I_{lmn}) = 0 \tag{8}$$

$$\pi_0 e + \pi_1 e + \pi_2 e = 1 \tag{9}$$

and using the condition $\pi A_0 e < \pi A_2 e$, we obtain the stability condition for our proposed model i.e.,

$$(\pi_0+\pi_1+\pi_2)(e_n \otimes D_1 e_m) < \pi_0((\psi \otimes T^0) \otimes I_m) + \pi_1((\psi \otimes \zeta T^0) \otimes I_m) + \pi_2((\psi \otimes \tau T^0) \otimes I_m) \tag{10}$$

4. THE STEADY-STATE PROBABILITY VECTOR

Let we define $P = (p_0, p_1, p_2, p_3, \cdots)$ as the steady-state probability vector of Q, where p_0 is of dimension $3m$ and $p_1, p_2, \cdots$ is of dimension $3lmn$, then P satisfies the following condition.

$$PQ = 0, Pe = 1 \tag{11}$$

where e is an appropriate dimension column vector of 1s [25].
The following equations from (11) can be used to generate the subvectors of P if the stability criterion is satisfied.

$$p_0 B_{00} + p_1 B_{10} = 0 \tag{12}$$

$$p_0 B_{01} + p_1 A_1 + p_2 A_2 = 0 \tag{13}$$

$$p_{i-1} A_0 + p_i A_1 + p_{i+1} A_2 = 0 \quad \text{for } i \geq 1 \tag{14}$$

$$p_i = p_1 R^{(i-1)} \quad \text{for } i \geq 2 \tag{15}$$

where R is the rate matrix is the minimal non-negative solution of the matrix quadratic equation.

$$R^2 A_2 + R A_1 + A_0 = 0 \tag{16}$$

Substituting the (15) in (13) we have

$$p_0 B_{01} + p_1(A_1 + RA_2) = 0 \tag{17}$$

and the normalizing condition is subjected to

$$p_0 e + p_1(I - R)^{-1} e = 1 \tag{18}$$

We partition the subvectors p_i for $i \geq 0$ as $p_0 = (p_{0,1}, p_{0,2}, p_{0,3})$, $p_1 = (p_{1,1}, p_{1,2}, p_{1,3})$ and $p_i = (p_{i,1}, p_{i,2}, p_{i,3})$, $i \geq 2$, such that $p_{0,1}$, $p_{0,2}$ and $p_{0,3}$ are of dimension m and $p_{i,1}$, $p_{i,2}$ and $p_{i,3}$ are of dimension lmn. The following provides the steady-state vector interpretations:

- $p_{0,1}$, the main server offers l kinds of service The arrival is in one of the m phases and there are no customers in the system.
- $p_{0,2}$, the main server is under repair and the standby server is available with l kinds of service. There are no customers in the system and the arrival is in one of m phases.
- $p_{0,3}$, the main server is providing l kinds of service while it is on working vacation. The arrival is in one of the m phases and there are no customers in the system.
- $p_{i,1}$, the system has exactly i consumers, and it is busy providing services with l kinds. The services and arrival are both in various phases.
- $p_{i,2}$, the system has exactly i consumers, the main server is undergoing repair, and the standby server is serving l kinds. Services and arrival are both in various phases.
- $p_{i,3}$, the system has exactly i customers, the main server is on working vacation and serving with l kinds. The arrival and services are in various phases.

The steady-state equations (11) can thus be expressed as

$$p_{0,1}(D_0 - \xi I_m) + p_{0,2}\eta I_m + p_{0,3}\gamma I_m = 0 \tag{19}$$

$$p_{0,1}(\xi I_m) + p_{0,2}(D_0 - 2\eta I_m) + p_{0,3}\xi I_m = 0 \tag{20}$$

$$p_{0,2}\eta I_m + p_{0,3}(D_0 - (\gamma + \xi) I_m) + p_{1,1}(T^0 \otimes I_n) + p_{1,2}(\zeta T^0 \otimes I_n) + p_{1,3}(\tau T^0 \otimes I_n) = 0 \tag{21}$$

$$p_{0,1}(\psi \otimes D_1) + p_{1,1}((T \oplus D_0) - \xi I_{lmn}) + p_{1,2}(\eta I_{lmn}) + p_{1,3}(\gamma I_{lmn}) + p_{2,1}((\psi \otimes T^0) \otimes I_m) = 0 \tag{22}$$

$$p_{0,2}(\psi \otimes D_1) + p_{1,1}(\xi I_{lmn}) + p_{1,2}((\zeta T \oplus D_0) - 2\eta I_{lmn}) + p_{1,3}(\xi I_{lmn}) + p_{2,2}((\psi \otimes \zeta T^0) \otimes I_m) = 0 \tag{23}$$

$$p_{0,3}(\psi \otimes D_1) + p_{1,2}(\eta I_{lmn}) + p_{1,3}((\tau T \oplus D_0) - (\xi + \gamma) I_{lmn}) + p_{2,3}((\psi \otimes \tau T^0) \otimes I_m) = 0 \tag{24}$$

$$p_{i-1,1}(I_n \otimes D_1) + p_{i,1}((T \oplus D_0) - \xi I_{lmn}) + p_{i,2}(\eta I_{lmn}) + p_{i,3}(\gamma I_{lmn}) + p_{i+1,1}((\psi \otimes T^0) \otimes I_m) = 0 \quad (25)$$

$$p_{i-1,2}(I_n \otimes D_1) + p_{i,1}(\xi I_{lmn}) + p_{i,2}((\zeta T \oplus D_0) - 2\eta I_{lmn}) + p_{i,3}(\xi I_{lmn}) + p_{i+1,2}((\psi \otimes \zeta T^0) \otimes I_m) = 0 \quad (26)$$

$$p_{i-1,3}(I_n \otimes D_1) + p_{i,2}(\eta I_{lmn}) + p_{i,3}((\tau T \oplus D_0) - (\xi + \gamma) I_{lmn}) + p_{i+1,3}((\psi \otimes \tau T^0) \otimes I_m) = 0 \quad (27)$$

along with the normalization condition

$$\sum_{i=0}^{\infty}(p_{i,1} + p_{i,2} + p_{i,3})e = 1 \quad (28)$$

by substitute the block matrices in the equations (12), (13), (14) and (18), we get (19) to (28). The matrix R has computed using the iterative method.

5. PERFORMANCE MEASURES

We examine our model's qualitative behaviour at steady-state.

- The probability that the system is available with no customers $(P_{inactive})=P_0 e$
- The probability that the main server is available $(P_{available})=P_1(I-R)^{-1}e_1$
- The probability that the standby server is busy while the main server is under repair $(P_{standby})=P_1(I-R)^{-1}e_2$
- The probability that the main server is busy on working vacation $(P_{WV})=P_1(I-R)^{-1}e_3$
- Expected system size $E_{System}=P_1(I-R)^{-2}e$

6. SPECIAL CASES

6.1. M/M/1 queueing model

We reduce our proposed model into $M/M/1$, by taking inter-arrival time and service times are exponentially distributed and substitute $l = 1$, $\theta_1 = 1$, $\beta = 1$, $n = 1$ and $m = 1$. Then $D_0 = [-\lambda]$, $D_1 = [\lambda]$, $\alpha = [1]$, $\beta = [1]$ and $S = [-\mu]$ The infinitesimal generator becomes

$$Q = \begin{bmatrix} B_{00} & B_{01} & \dots & & & \\ B_{10} & A_1 & A_0 & \dots & & \\ \vdots & A_2 & A_1 & A_0 & \dots & \\ & \vdots & A_2 & A_1 & A_0 & \dots \\ & & \vdots & \vdots & \vdots & \end{bmatrix}$$

where

$$B_{00} = \begin{bmatrix} -\lambda-\xi & \xi & 0 \\ \eta & -\lambda-2\eta & \eta \\ \gamma & \xi & -\lambda-\gamma-\xi \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, B_{10} = \begin{bmatrix} 0 & 0 & -\mu \\ 0 & 0 & -\zeta\mu \\ 0 & 0 & -\tau\mu \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, A_2 = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \zeta\mu & 0 \\ 0 & 0 & \tau\mu \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu+\xi) & \xi & 0 \\ \eta & -(\lambda+\zeta\mu+2\eta) & \eta \\ \gamma & \xi & -(\lambda+\tau\mu+\xi+\gamma) \end{bmatrix}$$

The generator matrix A as $A_0 + A_1 + A_2$, that is,

$$A = \begin{bmatrix} -\xi & \xi & 0 \\ \eta & -2\eta & \eta \\ \gamma & \xi & -(\gamma+\xi) \end{bmatrix}$$

To obtain the stability condition, it should satisfy $\pi A = 0$ and $\pi e = 1$ and the steady-state probability vector $\pi = (\pi_0, \pi_1, \pi_2)$. From the conditions, we obtain the following equations,

$$-\xi\pi_0 + \eta\pi_1 + \gamma\pi_2 = 0 \tag{29}$$

$$\xi\pi_0 - 2\eta\pi_1 + \xi\pi_2 = 0 \tag{30}$$

$$\eta\pi_1 - (\gamma+\xi)\pi_2 = 0 \tag{31}$$

$$\pi_0 + \pi_1 + \pi_2 = 0 \tag{32}$$

from (31), we get

$$\pi_1 = \pi_2 \frac{\gamma+\xi}{\eta} \tag{33}$$

substituting (33) in (29) then,

$$\pi_0 = \pi_2 \frac{2\gamma+\xi}{\xi} \tag{34}$$

by substituting π_0 and π_1 in normalization condition (32), we get π_2

$$\pi_2 = \left[\frac{\xi(\gamma+\xi) + \eta(2\gamma+\xi)}{\eta\xi} + 1 \right]^{-1} \tag{35}$$

then, we obtain the stability condition by using $\pi A_0 e < \pi A_2 e$ i.e.,

$$\lambda < \mu \left[\frac{(2\gamma+\xi)\eta + \zeta(\gamma+\xi)\xi + \tau\xi\eta}{\xi(\gamma+\xi) + \eta(2\gamma+\xi) + \eta\xi} \right] \tag{36}$$

6.1.1.

$\zeta = 0$, $\eta = 1$ and $\xi = 0$ if there is no backup and breakdown, then our model coincide with [26].

6.1.2.

$\gamma \to \infty$, $\zeta = 0$, $\eta = 1$ and $\xi = 0$ if there is no vacation, no breakdown and no backup, then our model synchronize with [27].

6.2. MAP/PH/1 queueing model

We reduce our model into $MAP/PH/1$ with same service rate for working vacation and standby service, omitting the breakdown and repair rate of working vacation and also ignore the kinds of service then, we obtain the model in [28].

7. APPLICATIONS

Our proposed model is motivated by various realistic systems. In manufacturing industries, call centers, healthcare, etc., the customers receive the appropriate service based on their needs. For example:

1. In manufacturing industries, a machine used to produce a variety of items can be modeled as a single server queue with phase service. A customers can choose any item from that varieties available. The items may be different types juice in a juice machine, variety of pizza from pizza making machine, various types of service in CNC machine etc. While serving the customer, the server get breaks down at any time. The standby (backup) server takes over instantly and provides service in the same manner as main machine. when there are no customers in the system, the service provider will do some other works such as machine adjustment, cleaning, making pre-required items etc. while doing the other works if the customer enter into the system, then the service provider will serve the customer with lower rate. Hence this situation is coincide with a SWV. While the server is on working vacation, a breakdown may occur only because of providing slow service. This is caused when the machine (server) loses its energy and causes more heat. As a result, whenever the server breaks down, the standby server takes over.
2. Similarly, at a call center, the server serves the customers by requesting them to select a number based on their wants and offers services in phases based on their demands. In this situation, the service provider may be stuck due to technical or human issues. This refers to a breakdown and the service provider will use a standby server as an alternative. Such a scenario coincides with our model. if there is no call, the service provider may perform some other work such as, documentation or data-collection. During that time, the server may receive the calls. At that time the attender will to both works simultaneously and hence it cause the slow service rate. At the completion of the other documentation works, the service provider switches to the regular working process. The server waits to serve the customer until they receive

their call if the system is empty or starts to attend to the calls from the customer if the server receives more calls.

3. In a mechanic shop, the server gives service to customers to repair and maintain their products. The service provider may know how to rectify issues, such as bike engine issues, water washing, brake problem, etc., not only in the bike but even in the cycle, car, etc. Hence the mechanic will get customers who need those kinds of service. The kinds of service increase as many customers may gather in the same place for different issues. The mechanic repairing the item as step by step procedure. Hence it represent the phase service. The mechanic might do some other work such as cleaning his place and rearranging the materials around them, etc. when there are no customers. if the customer arrive at that time, the mechanic will serve the customer and also continuing the cleaning process which may cause the slow service rate. After the completion of cleaning process, he will wait for the customers arrival and serve them in actual rate. Hence it represent the SWV. Whenever the mechanic is unavailable, there may be some other person to provide service. This is the standby mechanic whose service rate may be slower than that of the mechanic. The unavailability of the mechanic occurs if the mechanic suffers any issues while handling the product or due to some personal emergency. Such type of model also coincides with our queueing model.

8. SENSITIVE ANALYSIS

In this section, we provide a few numerical examples to help the reader grasp the qualitative features of the working vacation, breakdown-repair, standby queueing model under consideration. We examine several hypotheses by altering the model's parameters, such as the arrival processes and service time distributions.

The following three sets of values for D_0 and D_1, which are used as input data in many research papers, are taken into consideration for the arrival procedure (see [14],[28],[2],[3]). For our convenient, we display them here.

- **Expo-A** (Exponential Arrival): The standard exponential distribution is as follows.

  $D_0 = (-1)$, $D_1 = (1)$.

- **Erlang-A** (Erlang Arrival): This is order 2 of the Erlang distribution.

  $$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

- **Hyper-A** (Hyper-exponential Arrival): Two exponentials have been combined to get this.

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1.71 & 0.19 \\ 0.171 & 0.019 \end{bmatrix}$$

We take into consideration the following three PH-distributions for the service times with representation $(\beta,\ S)$. To obtain the desired μ, these representations will be normalised.

- **Expo-S** (Exponential Service): The standard exponential distribution is as follows.

  $\beta = 1,\ S = (-1)$.

- **Erlang-S** (Erlang Service): This is order 2 of the Erlang distribution.

  $\beta = \begin{bmatrix} 1 & 0 \end{bmatrix},\ S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$

- **Hyper-S** (Hyper-exponential Service): Two exponentials have been combined to get this.

  $\beta = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix},\ S = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$

8.1. Illustration 1:

In figures 2, 3 and 4, we investigate the behaviour of arrival rate λ on different scenarios. We fix the parameters as per the stability condition is satisfied, $\mu = 4$, $l = 3$, $\eta = 2.8$, $\xi = 2.5$, $\zeta = 0.5$, $\tau = 0.7$, $\alpha = (0.5, 0.3, 0.2)$, $\theta = \triangle(0.9, 0.7, 0.5)$ and $\gamma = 2$.

- The expected system size increases for different service and arrival distributions as the arrival rate (λ) rises. This is due to the fact that an increase in arrival rates results in more customers of the system when there is a fixed service rate.
- The expected system size increases more quickly in Hyper-A and more slowly in Erlang-A when we correlate the above data for distinctive arrival distribution. Additionally, the expected system size grows gradually during Erlang-S and swiftly during Hyper-S compared to Expo-S.

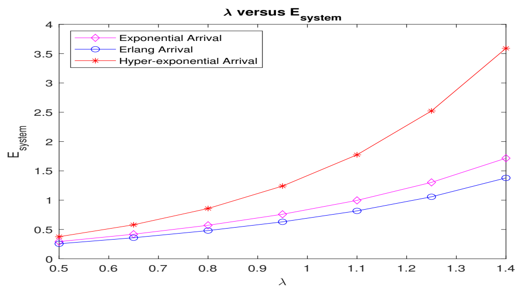

Figure 2: Arrival rate versus the expected system size for all arrivals distribution with Exponential service

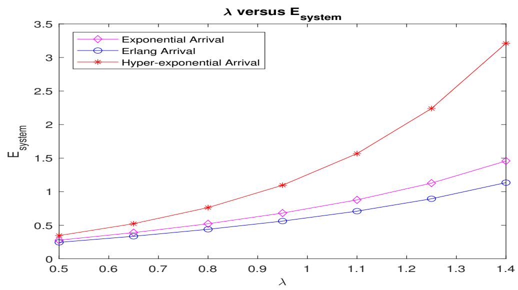

Figure 3: Arrival rate versus the expected system size for all arrivals distribution with Erlang phase service

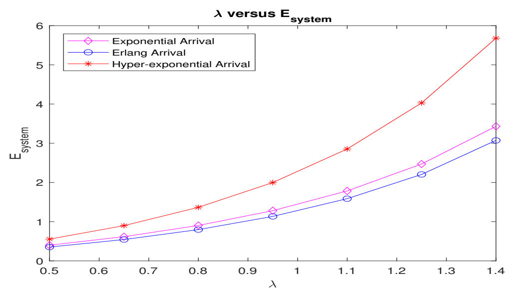

Figure 4: Arrival rate versus the expected system size for all arrivals distribution with Hyper-exponential phase service

8.2. Illustration 2:

In this figures 5, 6 and 7, We look into the impact of service rate μ on expected system size under various scenarios. We fix the parameters as $\lambda = 1$, $l = 3$, $\eta = 2.8$, $\xi = 2.5$, $\zeta = 0.5$, $\tau = 0.7$, $\alpha = (0.5, 0.3, 0.2)$, $\theta = \triangle(0.9, 0.7, 0.5)$ and $\gamma = 2$ as per the stability condition is satisfied.

- From the figures 5, 6 and 7, We can see that as the service rate *mu* grows, the expected system size also grows since an increase in service rate results in a decrease in service time, which raises server availability. The estimated system size is reduced as a result of the speedy customer service.
- Comparing the figures 5, 6 and 7, we can see that the expected system size for Hyper-A and Erlang-A are both quick. Additionally, Erl-S is slower while Hyp-Exp-S is faster.

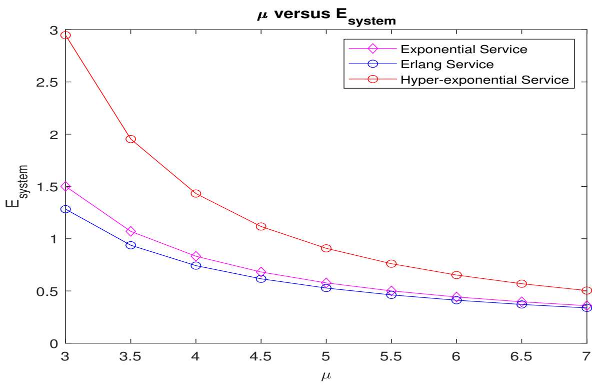

Figure 5: Service rate versus the expected system size for all service distribution with Exponential arrival

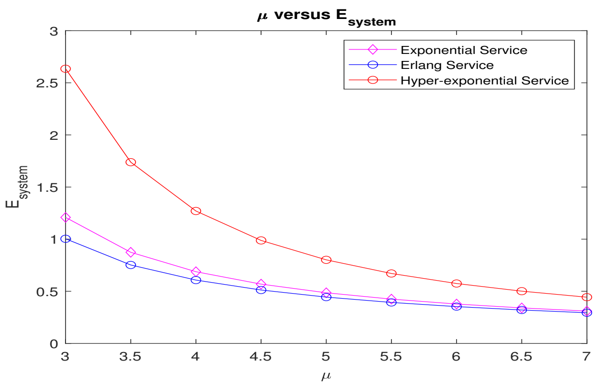

Figure 6: Service rate versus the expected system size for all service distribution with Erlang arrival

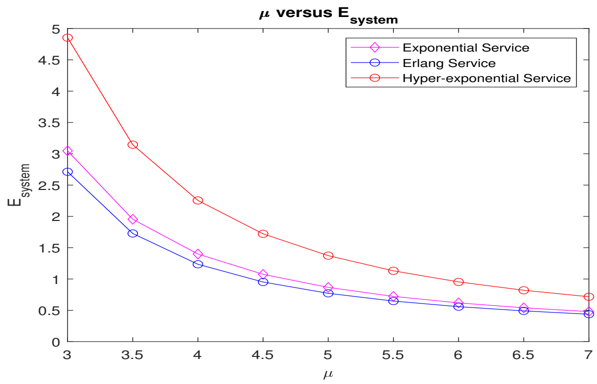

Figure 7: Service rate versus the expected system size for all service distribution with Hyper-exponential arrival

8.3. Illustration 3:

In 3D figures 8, 9, 10 and 11, we visualise the breakdown rate's (η) and repair rate's (ξ) impact of the server on the expected system size under different scenarios. We fix $\lambda = 1$, $\mu = 4$, $l = 3$, $\zeta = 0.5$, $\tau = 0.7$, $\alpha = (0.5, 0.3, 0.2)$, $\theta = \triangle(0.9, 0.7, 0.5)$ and $\gamma = 2$ as per the stability condition is satisfied.

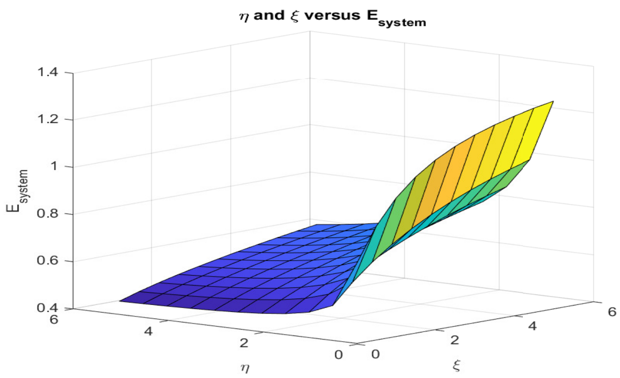

Figure 8: Breakdown rate(η) and repair rate(ξ) versus expected system size with $E_m/E_n/1$

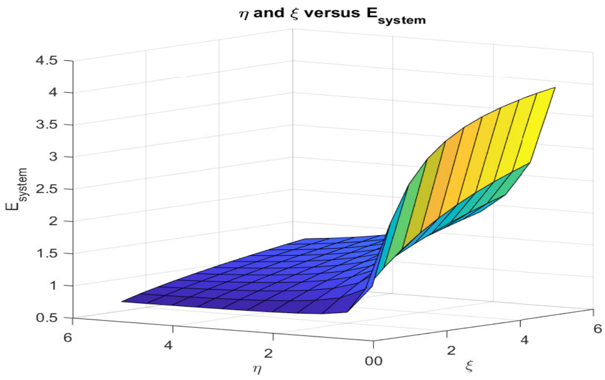

Figure 9: Breakdown rate(η) and repair rate(ξ) versus expected system size with $E_m/H_n/1$

An overviewing of figures 8, 9, 10 and 11 reveals that

- Maximize the repair rate cause the expected system size decrease for all arrivals and service distributions. Increase the repair rate means the faster the server repaired and get into the system sooner, so the server will be availabile as soon as the server gets repaired. It cause the lower in expected system size.
- Maximize the breakdown rate cause the expected system size higher. Because increase in breakdown means the server face the breakdown frequently. So the server gets unavailablity many times and it cause the system size higher at lower repair rate.
- This influence makes the expected system size increasing slowly in $E_m/E_n/1$, rapidly in $H_m/H_n/1$ and gradually increase in $E_m/H_n/1$ and $H_m/E_n/1$

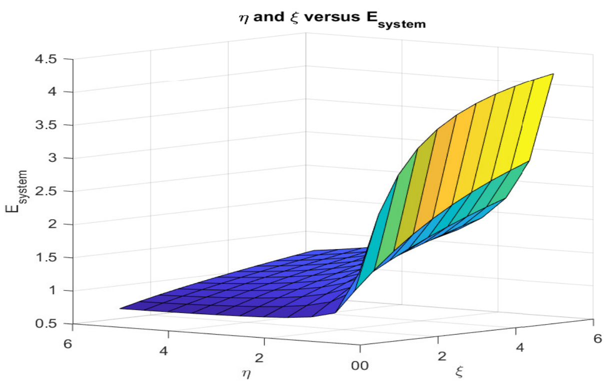

Figure 10: Breakdown rate(η) and repair rate(ξ) versus expected system size with $H_m/E_n/1$

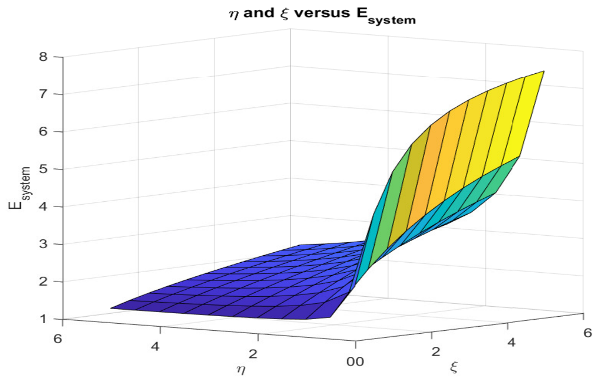

Figure 11: Breakdown rate(η) and repair rate(ξ) versus expected system size with $H_m/H_n/1$

8.4. Illustration 4

In table 1,2,3, we investigate the influence of kinds of service rate (l) on expected system size. We fix the parameters as $\lambda = 1$, $\mu = 4$, $\eta = 2.8$, $\xi = 2.5$, $\zeta = 0.5$, $\tau = 0.7$ and $\gamma = 2$. In case 1,2 and 3, we extract the interpretation of the kinds of service on expected system size for various θ as decreasing, increasing and random order, respectively in accordance with the satisfaction of the stability condition.

8.4.1. Case 1

For $l = 1 : \theta = \triangle(0.9)$ and $\alpha = (1)$
For $l = 2 : \theta = \triangle(0.9, 0.7)$ and $\alpha = (0.6, 0.4)$
For $l = 3 : \theta = \triangle(0.9, 0.7, 0.5)$ and $\alpha = (0.5, 0.3, 0.2)$
For $l = 4 : \theta = \triangle(0.9, 0.7, 0.5, 0.3)$ and $\alpha = (0.4, 0.3, 0.2, 0.1)$
For $l = 5 : \theta = \triangle(0.9, 0.7, 0.5, 0.3, 0.1)$ and $\alpha = (0.35, 0.25, 0.2, 0.15, 0.05)$

Table 1: Effect of l on expected system size

	Expo-A			Erlang-A			Hyper-A		
l	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S
1	0.5687	0.5220	0.8836	0.4779	0.4390	0.7772	0.8523	0.7605	1.3454
2	0.6773	0.6139	1.1017	0.5639	0.5094	0.9717	1.0706	0.9492	1.7105
3	0.8322	0.7423	1.4321	0.6877	0.6075	1.2702	1.3993	1.2356	2.2547
4	1.1621	1.0082	2.1838	0.9569	0.8137	1.9603	2.1203	1.8714	3.4456
5	3.3819	2.7197	7.7632	2.9076	2.2569	7.2698	6.3888	5.6112	11.1150

8.4.2. Case 2

For $l = 1 : \theta = \triangle(0.9)$ and $\alpha = (1)$
For $l = 2 : \theta = \triangle(0.5, 0.9)$ and $\alpha = (0.6, 0.4)$
For $l = 3 : \theta = \triangle(0.5, 0.7, 0.9)$ and $\alpha = (0.5, 0.3, 0.2)$
For $l = 4 : \theta = \triangle(0.3, 0.5, 0.7, 0.9)$ and $\alpha = (0.4, 0.3, 0.2, 0.1)$

For $l = 5 : \theta = \triangle(0.3, 0.4, 0.5, 0.7, 0.9)$ and $\alpha = (0.35, 0.25, 0.2, 0.15, 0.05)$

Table 2: Effect of l on expected system size

	Expo-A			Erlang-A			Hyper-A		
l	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S
1	0.5687	0.5220	0.8836	0.4779	0.4390	0.7772	0.8523	0.7605	1.3454
2	1.1648	1.0135	2.1695	0.9487	0.8087	1.9343	2.1820	1.9305	3.5073
3	1.1622	1.0122	2.1591	0.9445	0.8061	1.9220	2.1869	1.9358	3.5076
4	3.1828	2.5875	7.1216	2.5416	1.9609	6.4584	7.4214	6.6945	11.7619
5	3.8917	3.1321	8.9161	3.0713	2.3275	8.0728	9.5461	8.6536	14.9819

8.4.3. Case 3

For $l = 1 : \theta = \triangle(0.9)$ and $\alpha = (1)$
For $l = 2 : \theta = \triangle(0.7, 0.9)$ and $\alpha = (0.6, 0.4)$
For $l = 3 : \theta = \triangle(0.6, 0.9, 0.8)$ and $\alpha = (0.5, 0.3, 0.2)$
For $l = 4 : \theta = \triangle(0.4, 0.9, 0.5, 0.7)$ and $\alpha = (0.4, 0.3, 0.2, 0.1)$
For $l = 5 : \theta = \triangle(0.3, 0.7, 0.4, 0.9, 0.6)$ and $\alpha = (0.35, 0.25, 0.2, 0.15, 0.05)$

Table 3: Effect of l on expected system size

	Expo-A			Erlang-A			Hyper-A		
l	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S
1	0.5687	0.522	0.8836	0.4779	0.4390	0.7772	0.8523	0.7605	1.3454
2	0.735	0.6625	1.2198	0.6091	0.5460	1.0768	1.1936	1.0563	1.9126
3	0.8577	0.7641	1.4817	0.7061	0.6225	1.3121	1.4633	1.2923	2.3533
4	1.5987	1.3587	3.1895	1.2914	1.0639	2.8621	3.2610	2.9037	5.2133
5	2.9352	2.3965	6.5001	2.3540	1.8293	5.8975	6.7036	6.0343	10.6652

- The expected system size will increase as we enhance the kinds of service that are available. Because additional service options draw a wider range of customers to one location. The expected system size varies depending on the duration of each kinds of service and the probability that customers will select the service.
- According to cases 1,2 and 3, the variations in the service rates of kinds affect the system size. Still, they have a similarity: an increase in expected system size when the kinds l increase.
- Because of the outcomes for our predetermined parameters and the expansion of the services offered, a variety of customers of all kinds begin to accumulate in the system. As a result, the expected system size grows.
- When we take into account the data from tables 1,2,3, for distinct arrival times, the expected system size increases more quickly in the duration of

Hyper-A and more slowly in the duration of Erlang-A. Additionally, in Erlang-S and Hyper-S, the predicted system size grows substantially and quickly, respectively.

8.5. Illustration 5

We look at how the working vacation rate γ affects the expected size of the system in table 4. We use the following values to determine whether the stability criterion is met: $\lambda = 1$, $\mu = 4$, $\eta = 2.8$, $\xi = 2.5$, $\zeta = 0.5$, $\tau = 0.7$ and $l = 3$.

Table 4: Effect of γ on expected system size

	Expo-A			Erlang-A			Hyper-A		
γ	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S	Expo-S	Erlang-S	Hyper-S
1	0.8696	0.7742	1.5059	0.7181	0.6326	1.3368	1.4783	1.3060	2.3776
1.5	0.8485	0.7563	1.4639	0.7010	0.6186	1.2989	1.4334	1.2660	2.3075
2	0.8322	0.7423	1.4321	0.6877	0.6075	1.2702	1.3993	1.2356	2.2547
2.5	0.8192	0.7310	1.4073	0.6771	0.5985	1.2477	1.3726	1.2116	2.2134
3	0.8086	0.7217	1.3872	0.6683	0.5910	1.2296	1.3510	1.1922	2.1803
3.5	0.7997	0.7140	1.3708	0.6611	0.5848	1.2148	1.3332	1.1763	2.1531
4	0.7923	0.7074	1.3570	0.6549	0.5795	1.2024	1.3183	1.1628	2.1303

- The expected system size decreases for a given service and arrival rate as the vacation rate γ rises. The length of vacation time is reduced as a result of an increase in the vacation rate. This cause the server rapidly resumes normal service rate. So, since the server is available for longer to provide the service, the expected system size reduces.
- The expected system size increases immediately for Hyper-A and progressively for Erlang-A when we correlate the enlisted values for particular arrival times. Also, with Erlang-S and Hyper-S, the expected system size is reduced significantly and fast, respectively.

9. CONCLUSION

We investigated the single server MAP queueing system with many kinds of service, in the framework of a server going on a single working vacation, a server breaking down due to some issues or while changing the types frequently, a server being repaired, and a server being assisted by a standby server during breakdowns that occur during working vacation or busy periods. Our system's stability requirement has been satisfied. Additionally, certain important performance metrics have been provided and MATLAB is used to offer numerical demonstrations. Investigation has been done on how the expected system size is affected by the arrival rate and service rate. Additionally, with the help of 3D graphs, the effect of the server's breakdown and repair rates on the expected system size has been depicted. Additionally, the interpretation of the kinds of service on the expected system size has been tabulated. Furthermore, by using $BMAP$ for the arrival process, one can expand the study by carrying out a cost analysis for our model. These are currently being investigated.

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