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ON SOME THEORETICAL DEVELOPMENTS OF INTERVAL METRIC SPACE

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Abstract: The goal of this work is to introduce the concept of interval metric in a compact way by modifying the existing definitions of interval metric. Then, a result regarding the necessary and sufficient criterion for interval metric is established. Thereafter, to illustrate the idea of interval metric, a set of examples is provided. Then, several results regarding the formation of interval metric are derived. Also, the concept of interval diameter, boundedness of a set under interval metric and interval distance are introduced. All the theoretical results are illustrated with the help of some numerical examples. Finally, as an application of interval metric, all the theoretical developments of transformation of multi-objective interval optimization problem into interval single objective optimization problem by Global criterion method, Tchebycheff method and Weighted Tchebycheff method are established.

Keywords: Interval uncertainty, interval distance, interval diameter, interval product metric.

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1. INTRODUCTION

The idea of the usual metric has become very popular in the field of applied science and engineering to study different real-life problems viz. engineering design, remote controlling, data mining, multi-objective optimization problems, etc. However, due to the inexactness and randomness, most of the parameters involved in such real-life problems are not precise. As a result, the usual metric does not work to study the said real-life problems in imprecise circumstances. In these cases, the notion of imprecise metric is highly required. Several researchers introduced different types of metrics under uncertainty which can be categorized as probabilistic/statistical, fuzzy and interval metrics.

The notion of random distance was proposed in the last century and from that time onwards, a lot of developments in the area of probabilistic metric were accomplished by several researchers. Among those, some of the research works are mentioned here: Schweizer and Sklar [1] developed some results on statistical metric spaces. Egbert [2] derived the results on the products and quotients of probabilistic metric spaces. Tardiff [3] studied the topological structures on probabilistic metric spaces. Later a number of works on the fixed-point theorem on the probabilistic metric space were accomplished by Shisheng [4], Cho et al. [5], Vershik [6, 7], Schweizer and Sklar [8], Tiana et al. [9] and others. On the other hand, the journey of the research on fuzzy metric spaces was reached at a deep level. Some of the important developments on the fuzzy and generalized fuzzy metric spaces were done by Kaleva and Seikkala [10], George and Veeramani [11], Piera [12], Park [13], Melliani et al. [14], Riaz et al. [15] and Sabri et al. [16]. However, very few research works were carried out in the area of interval metric space. Trindade et al. [17] proposed a definition of interval metric in which a non-negativity assumption is missing. Bhunia and Samanta [18] also introduced a definition of interval metric in the centre and radius form of an interval. However, in their definition, the non-negativity characteristic of the distance was also missed out. On the other hand, Afravi et al. [19] had also given an idea about the interval metric in terms of lower and upper bounds forms which satisfies the non-negativity condition but it has some other limitations.

As an application, the interval mathematics, order relation and parametric approach play important role for mathematical modelling of real-life problems (especially, optimization problems) under uncertainty. As evidence, the use of this approach by several researchers in the area of inventory control can be mentioned here. Shaikh et al. [20] formulated a two-warehouse inventory model for deteriorating item with advance payment under interval uncertainty and studied the best-found policy using interval mathematics and soft-computing techniques. Succeeding them, Rahman et al. [21, 22] and Manna et al. [23] studied some inventory problems for perishable goods with some realistic features like, interval-valued demand rate, all-unit discount facility, prepayment policy in interval environment using interval parametric approach. On the other hand, Debnath and Ghosh [24] studied generalized-Hukuhara penalty method for interval optimization problem and applied it to study interval-valued portfolio optimization problems. Feng et al. [25] used interval field method to study the stability analysis of slopes in presence of uncertainties. Also, in mathematical biology/medical science, the interval approach was used by several researchers viz. Korczak and Jasiński [26], Ghosh et al. [27], Huang [28], Mondal et al. [29] and others. However, the application of interval metric was limited to very few researchers viz. Trindade et al. [17], Bhunia and Samanta [18] and Afravi et al. [19].

In this work, rectifying all the drawbacks of the existing definitions of interval metric proposed by Trindade et al. [17], Bhunia and Samanta [18] and Afravi et al. [19] are justified using some examples. Thereafter, introducing a partial modification of the existing definitions of interval metric, the concept of interval metric is defined precisely. Then, the necessary and sufficient criterion for interval metric is established in terms of lower and upper bounds form. To illustrate the basics of interval metric, a set of examples of interval metric are provided. Then the concept of the product of interval metrics, formation of interval metric, interval diameter of a set and boundedness of interval metric are proposed. Finally, an application of proposed interval metric is reported.

2. BASIC CONCEPTS

The closed interval $\tilde{A}=\left[\alpha_L,\alpha_U\right]$ can be represented in centre-radius form $\tilde{A}=\left\langle \alpha_c,\alpha_r\right\rangle$, where $\alpha_c=\dfrac{\alpha_U+\alpha_L}{2}$ & $\alpha_r=\dfrac{\alpha_U-\alpha_L}{2}$.

Let K_c be the set of all compact intervals of $\mathbb{R}$ i.e., $K_c=\left\{\left[\alpha_L,\alpha_U\right]:\alpha_L,\alpha_U\in\mathbb{R} \text{ and } \alpha_L\le\alpha_U\right\}$. The basic arithmetical operations viz. addition, subtraction, scalar multiplication, multiplication and division on K_c are given below.

2.1 Arithmetic operations on K_c

Definition 1. Let $\tilde{A}=\left[\alpha_L,\alpha_U\right]$ and $\tilde{B}=\left[\beta_L,\beta_U\right]\in K_c$. Then,

(i) Addition:

$$\tilde{A}+\tilde{B}=\left[\alpha_L,\alpha_U\right]+\left[\beta_L,\beta_U\right]=\left[\alpha_L+\beta_L,\alpha_U+\beta_U\right]$$

(ii) Subtraction:

$$\tilde{A}-\tilde{B}=\left[\alpha_L,\alpha_U\right]-\left[\beta_L,\beta_U\right]=\left[\alpha_L,\alpha_U\right]+\left[-\beta_U,-\beta_L\right]=\left[\alpha_L-\beta_U,\alpha_U-\beta_L\right]$$

(iii) Hukuhara difference: (Stefanini and Bede [30])

$$\begin{aligned}\tilde{A}\Theta_{gH}\tilde{B}&=\left[\alpha_L,\alpha_U\right]\Theta_{gH}\left[\beta_L,\beta_U\right]\\&=\left[\min\left\{\alpha_L-\beta_L,\alpha_U-\beta_U\right\},\max\left\{\alpha_L-\beta_L,\alpha_U-\beta_U\right\}\right]\end{aligned}$$

(iv) Scalar multiplication:

$$\lambda\tilde{A}=\lambda\left[\alpha_L,\alpha_U\right]=\begin{cases}\left[\lambda\alpha_L,\lambda\alpha_U\right] & \text{if } \lambda\ge 0\\ \left[\lambda\alpha_U,\lambda\alpha_L\right] & \text{if } \lambda<0,\end{cases}$$

for any real number $\lambda\in\mathbb{R}$.

(v) Multiplication:

$$\tilde{A}\tilde{B} = \left[\min\left(\alpha_L\beta_L, \alpha_L\beta_U, \alpha_U\beta_L, \alpha_U\beta_U\right), \max\left(\alpha_L\beta_L, \alpha_L\beta_U, \alpha_U\beta_L, \alpha_U\beta_U\right)\right]$$

Specially, $\tilde{A}\tilde{B} = \left[\alpha_L\beta_L, \alpha_U\beta_U\right]$ for $\alpha_L, \beta_L \geq 0$

(vi) Division:

$$\frac{\tilde{A}}{\tilde{B}} = \tilde{A}\times\left(\frac{1}{\tilde{B}}\right) = [\alpha_L, \alpha_U]\times\left[\frac{1}{\beta_U}, \frac{1}{\beta_L}\right], \text{ provided } 0 \notin [\beta_L, \beta_U].$$

Definition 2. (Ishibuchi and Tanaka [31])

Let $\tilde{A} = [\alpha_L, \alpha_U]$ and $\tilde{B} = [\beta_L, \beta_U]$ be two interval numbers. Then the interval order relation between $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$$\tilde{A} \leq_{LU} \tilde{B} \text{ iff } \alpha_L \leq \beta_L \text{ and } \alpha_U \leq \beta_U,$$
$$\tilde{A} <_{LU} \tilde{B} \text{ iff } \tilde{A} \leq_{LU} \tilde{B} \text{ and } \tilde{A} \neq \tilde{B}.$$

Definition 3.

The Moore distance [32] between two intervals is a mapping $D_M : K_c \times K_c \to \mathbb{R}$ defined by $D_M(\tilde{A}, \tilde{B}) = \max\{|\alpha_L - \beta_L|, |\alpha_U - \beta_U|\}$ where $\tilde{A} = [\alpha_L, \alpha_U]$ & $\tilde{B} = [\beta_L, \beta_U] \in K_c$.

Definition 4. Let $\tilde{A} = [\alpha_L, \alpha_U] \in K_c$. Then the module/norm of the interval $\tilde{A}$ is defined by $\left|\tilde{A}\right|_M = D_M(\tilde{A}, 0) = \max\{|\alpha_L|, |\alpha_U|\}$.

2.2 Some preliminaries on real metric space

Definition 5. Let S be a nonempty set. A function $\rho : S \times S \to \mathbb{R}$ is said to be a real metric on S if ρ satisfies the following properties:

(i) $\rho(s,t) \geq 0 \ \forall s,t \in S$ and $\rho(s,t) = 0$ iff $s = t$.

(ii) $\rho(s,t) = \rho(t,s), \ \forall s,t \in S$.

(iii) $\rho(s,t) \leq \rho(s,w) + \rho(w,t), \forall s,t,w \in S$.

Then the pair (S, ρ) is called a real metric space.

Example 1. Let $S = \mathbb{R}^n$, the set of all ordered n-tuples of real numbers. Then the followings are some standard real metrics on $\mathbb{R}^n$:

(i) $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\rho_1(s,t) = \sum_{i=1}^{n} |s_i - t_i|$,

(ii) $\rho_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\rho_2(s,t) = \left(\sum_{i=1}^{n} |s_i - t_i|^2\right)^{1/2}$,

(iii) $\rho_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\rho_\infty(s,t) = \max_{1\leq i\leq n}\{|s_i - t_i|\}$,

for all $s = (s_1, s_2, ..., s_n)$ & $t = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$.

Example 2. Let $S = C[a,b]$, the set of all real-valued continuous functions on $[a,b]$. Then the followings are some standard real metrics on $C[a,b]$:

(i) $\sigma_1(u,v) = \int_a^b |u(t) - v(t)| dt,\ u\ \&\ v \in C[a,b].$

(ii) $\sigma_2(u,v) = \left(\int_a^b |u(t) - v(t)|^2\ dt \right)^{1/2},\ u\ \&\ v \in C[a,b].$

(iii) $\sigma_\infty(u,v) = \underset{t \in [a,b]}{Sup} \{|u(t) - v(t)|\},\ u\ \&\ v \in C[a,b].$

Example 3. Moore distance D_M on K_c defined in **Definition 3** is a real metric on K_c.

Example 4. Let $S = K_c^n$, the set of all n-tuples of interval numbers. Then followings are some real metrics on K_c^n :

(i) $D_{M1}(\tilde{A},\tilde{B}) = \sum_{i=1}^{n} \{| \alpha_{iL} - \beta_{iL} | + | \alpha_{iU} - \beta_{iU} |\}$

(ii) $D_{M2}(\tilde{A},\tilde{B}) = \left(\sum_{i=1}^{n} \left\{| \alpha_{iL} - \beta_{iL} |^2 + | \alpha_{iU} - \beta_{iU} |^2 \right\} \right)^{1/2}$

(iii) $D_{M\infty}(\tilde{A},\tilde{B}) = \max_{1 \le i \le n} \{| \alpha_{iL} - \beta_{iL} |, | \alpha_{iU} - \beta_{iU} |\}$,

$\forall\ \tilde{A} = ([\alpha_{iL}, \alpha_{iU}] : i = 1,...,n)\ \&\ \tilde{B} = ([\beta_{iL}, \beta_{iU}] : i = 1,...,n) \in K_c^n.$

Proof.

(i)

$(i.a)$ Let $\tilde{A} = ([\alpha_{iL}, \alpha_{iU}] : i = 1,...,n)\ \&\ \tilde{B} = ([\beta_{iL}, \beta_{iU}] : i = 1,...,n) \in K_c^n.$

Clearly, $D_{M1}(\tilde{A},\tilde{B}) \ge 0$ and

$$
\begin{aligned}
& D_{M1}(\tilde{A},\tilde{B}) = 0 \\
& \Leftrightarrow \sum_{i=1}^{n} \{| \alpha_{iL} - \beta_{iL} | + | \alpha_{iU} - \beta_{iU} |\} = 0 \\
& \Leftrightarrow | \alpha_{iL} - \beta_{iL} | = 0\ \&\ | \alpha_{iU} - \beta_{iU} | = 0 \text{ for each } i = 1,...,n \\
& \Leftrightarrow \alpha_{iL} = \beta_{iL}\ \&\ \alpha_{iU} = \beta_{iU} \text{ for each } i = 1,...,n \\
& \Leftrightarrow [\alpha_{iL}, \alpha_{iU}] = [\beta_{iL}, \beta_{iU}] \text{ for each } i = 1,...,n \\
& \Leftrightarrow \tilde{A} = \tilde{B}.
\end{aligned}
$$

$(i.b)$ Also,

$$
\begin{aligned}
D_{M1}(\tilde{A},\tilde{B}) &= \sum_{i=1}^{n} \{| \alpha_{iL} - \beta_{iL} | + | \alpha_{iU} - \beta_{iU} |\} \\
&= \sum_{i=1}^{n} \{| \beta_{iL} - \alpha_{iL} | + | \beta_{iU} - \alpha_{iU} |\} \\
&= D_{M1}(\tilde{B},\tilde{A}).
\end{aligned}
$$

Thus, D_{M1} is symmetric.

($i.c$)

Let $\tilde{A}=\left([\alpha_{iL},\alpha_{iU}]:i=1,...,n\right),\tilde{B}=\left([\beta_{iL},\beta_{iU}]:i=1,...,n\right),\tilde{C}=\left([\gamma_{iL},\gamma_{iU}]:i=1,...,n\right)\in K_c^n$

Now,

$$
\begin{aligned}
D_{M1}(\tilde{A},\tilde{B}) &= \sum_{i=1}^{n}\left\{|\alpha_{iL}-\beta_{iL}|+|\alpha_{iU}-\beta_{iU}|\right\} \\
&\le \sum_{i=1}^{n}\left\{|\alpha_{iL}-\gamma_{iL}|+|\gamma_{iL}-\beta_{iL}|+|\alpha_{iU}-\gamma_{iU}|+|\gamma_{iU}-\beta_{iU}|\right\} \\
&= \sum_{i=1}^{n}\left\{|\alpha_{iL}-\gamma_{iL}|+|\alpha_{iU}-\gamma_{iU}|\right\}+\sum_{i=1}^{n}\left\{|\gamma_{iL}-\beta_{iL}|+|\gamma_{iU}-\beta_{iU}|\right\} \\
&= D_{M1}(\tilde{A},\tilde{C})+D_{M1}(\tilde{C},\tilde{B}).
\end{aligned}
$$

Hence $\left(K_c^n, D_{M1}\right)$ is a metric space.

(ii)

Let $\tilde{A}=\left([\alpha_{iL},\alpha_{iU}]:i=1,...,n\right),\tilde{B}=\left([\beta_{iL},\beta_{iU}]:i=1,...,n\right),\tilde{C}=\left([\gamma_{iL},\gamma_{iU}]:i=1,...,n\right)\in K_c^n$

($ii.a$) Clearly, $D_{M2}(\tilde{A},\tilde{B})\ge 0$ and

$D_{M2}(\tilde{A},\tilde{B})=0$

$$\Leftrightarrow \left(\sum_{i=1}^{n}\left\{|\alpha_{iL}-\beta_{iL}|^2+|\alpha_{iU}-\beta_{iU}|^2\right\}\right)^{1/2}=0$$

$\Leftrightarrow |\alpha_{iL}-\beta_{iL}|=0$ & $|\alpha_{iU}-\beta_{iU}|=0$ for each $i=1,...,n$

$\Leftrightarrow \alpha_{iL}=\beta_{iL}$ & $\alpha_{iU}=\beta_{iU}$ for each $i=1,...,n$

$\Leftrightarrow [\alpha_{iL},\alpha_{iU}]=[\beta_{iL},\beta_{iU}]$ for each $i=1,...,n$

$\Leftrightarrow \tilde{A}=\tilde{B}.$

(ii.b) Clearly, $D_{M2}(\tilde{A},\tilde{B})=D_{M2}(\tilde{B},\tilde{A}).$

($ii.c$) If $a_1,...,a_n;\ b_1,...,b_n;\ c_1,...,c_n$ & $d_1,...,d_n$ are real numbers, then using Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^{n}\left\{(a_i+c_i)^2+(b_i+d_i)^2\right\}\right]^{1/2}\le\left[\sum_{i=1}^{n}(a_i{}^2+b_i{}^2)\right]^{1/2}+\left[\sum_{i=1}^{n}(c_i{}^2+d_i{}^2)\right]^{1/2}.$$

Putting $a_i=\alpha_{iL}-\gamma_{iL}, b_i=\alpha_{iU}-\gamma_{iU}, c_i=\gamma_{iL}-\beta_{iL}$ & $d_i=\gamma_{iU}-\beta_{iU}$, $i=1,...,n$ in above inequality, we get

$$
\begin{aligned}
\left[\sum_{i=1}^{n}\left\{(\alpha_{iL}-\beta_{iL})^2+(\alpha_{iU}-\beta_{iU})^2\right\}\right]^{1/2} &\le \left[\sum_{i=1}^{n}\left\{(\alpha_{iL}-\gamma_{iL})^2+(\alpha_{iU}-\gamma_{iU})^2\right\}\right]^{1/2} \\
&+\left[\sum_{i=1}^{n}\left\{(\gamma_{iL}-\beta_{iL})^2+(\gamma_{iU}-\beta_{iU})^2\right\}\right]^{1/2}
\end{aligned}
$$

i.e., $D_{M2}(\tilde{A},\tilde{B}) \le D_{M2}(\tilde{A},\tilde{C}) + D_{M2}(\tilde{C},\tilde{B})$.

Hence $\left(K_c^n, D_{M2}\right)$ is a real metric space.

(iii) To show $D_{M\infty}$ is a metric on K_c^n, let $\tilde{A} = \left([\alpha_{iL}, \alpha_{iU}] : i = 1,...,n\right)$, $\tilde{B} = \left([\beta_{iL}, \beta_{iU}] : i = 1,...,n\right)$, $\tilde{C} = \left([\gamma_{iL}, \gamma_{iU}] : i = 1,...,n\right) \in K_c^n$.

$(iii.a)$ Clearly, $D_{M\infty}(\tilde{A},\tilde{B}) \ge 0$ and

$D_{M\infty}(\tilde{A},\tilde{B}) = 0$

$\Leftrightarrow \max\limits_{1\le i\le n}\left\{|\alpha_{iL} - \beta_{iL}|, |\alpha_{iU} - \beta_{iU}|\right\} = 0$

$\Leftrightarrow |\alpha_{iL} - \beta_{iL}| = 0$ & $|\alpha_{iU} - \beta_{iU}| = 0$ for each $i = 1,...,n$

$\Leftrightarrow \alpha_{iL} = \beta_{iL}$ & $\alpha_{iU} = \beta_{iU}$ for each $i = 1,...,n$

$\Leftrightarrow [\alpha_{iL}, \alpha_{iU}] = [\beta_{iL}, \beta_{iU}]$ for each $i = 1,...,n$

$\Leftrightarrow \tilde{A} = \tilde{B}.$

$(iii.b)$ Also,

$$\begin{aligned} D_{M\infty}(\tilde{A},\tilde{B}) &= \max_{1\le i\le n}\left\{|\alpha_{iL} - \beta_{iL}|, |\alpha_{iU} - \beta_{iU}|\right\} \\ &= \max_{1\le i\le n}\left\{|\beta_{iL} - \alpha_{iL}|, |\beta_{iU} - \alpha_{iU}|\right\} \\ &= D_{M\infty}(\tilde{B},\tilde{A}). \end{aligned}$$

$(iii.c)$ Now, for $i = 1,...,n,$

$$\begin{aligned} |\alpha_{iL} - \beta_{iL}| &\le |\alpha_{iL} - \gamma_{iL}| + |\gamma_{iL} - \beta_{iL}| \\ &\le \max_{1\le i\le n}\left\{|\alpha_{iL} - \gamma_{iL}|, |\alpha_{iU} - \gamma_{iU}|\right\} + \max_{1\le i\le n}\left\{|\gamma_{iL} - \beta_{iL}|, |\gamma_{iU} - \beta_{iU}|\right\} \\ &= D_{M\infty}(\tilde{A},\tilde{C}) + D_{M\infty}(\tilde{C},\tilde{B}). \end{aligned}$$

Similarly, for $i = 1,...,n,$ $|\alpha_{iU} - \beta_{iU}| \le D_{M\infty}(\tilde{A},\tilde{C}) + D_{M\infty}(\tilde{C},\tilde{B})$.

Therefore, $\max\limits_{1\le i\le n}\left\{|\alpha_{iL} - \beta_{iL}|, |\alpha_{iU} - \beta_{iU}|\right\} \le D_{M\infty}(\tilde{A},\tilde{C}) + D_{M\infty}(\tilde{C},\tilde{B})$

which implies, $D_{M\infty}(\tilde{A},\tilde{B}) \le D_{M\infty}(\tilde{A},\tilde{C}) + D_{M\infty}(\tilde{C},\tilde{B})$.

Hence, $D_{M\infty}$ is a real metric on K_c^n.

Result 1. The real metrics on $\mathbb{R}^n$ defined in **Example 1,** have the following relation: $\frac{1}{n}\rho_1(s,t) \le \frac{1}{\sqrt{n}}\rho_2(s,t) \le \rho_\infty(s,t) \le \rho_2(s,t) \le \rho_1(s,t),\ \forall s,t \in \mathbb{R}^n$.

Result 2. If $\rho_1, \rho_2, ..., \rho_n$ are metrics on a nonempty set S, then $a_1\rho_1 + a_2\rho_2 + ... + a_n\rho_n$ is also a metric on S, where $a_i \ge 0$ & not all $a_i = 0$, $i = 1,...,n.$

Result 3. If ρ is a metric on S and $c(>0)$ be any real number, then $c\cdot\rho$ is also a metric on S.

Result 4. Let (S,ρ) be a metric space and $\tilde{\rho} : S \times S \to \mathbb{R}$ be a function defined by

$\tilde{\rho}(s,t) = \min\{1, \rho(s,t)\}, \forall s,t \in S.$ Then $\tilde{\rho}$ is a metric on S.

Result 5 (**Product metric space**). Let (S,ρ_1) and (T,ρ_2) be two metric spaces. Then followings are the metrics on the product set $S \times T$:

(*i*) $\varsigma_1\left((s_1,t_1),(s_2,t_2)\right) = \rho_1(s_1,s_2) + \rho_2(t_1,t_2),\ (s_1,t_1)\ \&\ (s_2,t_2) \in S \times T.$

(*ii*) $\varsigma_2\left((s_1,t_1),(s_2,t_2)\right) = \sqrt{\left(\rho_1(s_1,s_2)\right)^2 + \left(\rho_2(t_1,t_2)\right)^2},\ (s_1,t_1)\ \&\ (s_2,t_2) \in S \times T.$

(*iii*) $\varsigma_3\left((s_1,t_1),(s_2,t_2)\right) = \max\left\{\rho_1(s_1,s_2), \rho_2(t_1,t_2)\right\},\ (s_1,t_1)\ \&\ (s_2,t_2) \in S \times T.$

3. EXISTING INTERVAL METRICS AND DRAWBACKS

In this section, we have discussed the existing definitions of interval metric proposed by Trindade et al. [17], and Bhunia and Samanta [18]. Also, we have justified the limitations of their definitions with examples. Thereafter, we have discussed the actual idea of interval metric using the concept proposed by Afravi et al. [19].

3.1 Interval metric proposed by Trindade et al. [17]

Trindade et al. [17] gave the definition of interval metric in the following:

Definition 6. Let S be any non-empty set. A function $\bar{\rho}: S \times S \to K_c$ is said to be an interval metric if for all $U,V,W \in S$, $\bar{\rho}$ satisfies the following properties:

(i) $0 \in \bar{\rho}(U,U)$ (Reflexivity)

(ii) $\left|\bar{\rho}(U,V)\right|_M \le \left|\bar{\rho}(U,W)\right|_M + \left|\bar{\rho}(W,V)\right|_M$ (Triangular inequality)

(iii) $\bar{\rho}(U,V) = \bar{\rho}(V,U)$ (Symmetry)

(iv) If $0 \in \bar{\rho}(U,V) = \bar{\rho}(U,U) = \bar{\rho}(V,V)$ then $U = V$ (Indiscernible identity)

Now, we justify with the following example that **Definition 6** cannot be an interval metric:

Example 5. Let $\bar{\rho}: K_c \times K_c \to K_c$ be an interval-valued function defined by $\bar{\rho}(U,V) = [-2,-1] \cdot D_M\left(U,V\right),$ where D_M is Moore distance on K_c; U $\&\ V \in K_c$.

Now, we show that $\bar{\rho}$ satisfies all the properties (i)-(iv) of **Definition 6**.

(i) Let $U \in K_c$.

Then,

$$\begin{aligned}\bar{\rho}(U,U) &= [-2,-1] \cdot D_M\left(U,U\right)\\ &= \left[-2,-1\right] \cdot 0\ \left(\because D_M \text{ is real metric on } K_c,\ D_M\left(U,U\right) = 0\right)\\ &= [0,0]\end{aligned}$$

$\Rightarrow 0 \in \bar{\rho}(U,U).$

This shows that $\bar{\rho}$ satisfies reflexive property.

(ii) Let $U,V,W \in K_c$.

Now,

$$\begin{aligned}\left|\bar{\rho}(U,V)\right|_M &= \left|\left[-2,-1\right]\cdot D_M(U,V)\right|_M\\ &= \left|\left[-2D_M(U,V),-D_M(U,V)\right]\right|_M\\ &= 2D_M(U,V)\\ &\le 2\left\{D_M(U,W)+D_M(W,V)\right\}\\ &= 2D_M(U,W)+2D_M(W,V)\\ &= \left|\bar{\rho}(U,W)\right|_M+\left|\bar{\rho}(W,V)\right|_M.\end{aligned}$$

Thus, $\bar{\rho}$ satisfies the triangular inequality.

(iii) Let $U,V\in K_c$.
Then

$$\begin{aligned}\bar{\rho}(U,V) &= \left[-2,-1\right]\cdot D_M(U,V)\\ &= \left[-2,-1\right]\cdot D_M(V,U)\\ &= \bar{\rho}(V,U)\end{aligned}$$

Therefore, the symmetry property is satisfied.

(iv) Let $U,V\in K_c$ and $0\in\bar{\rho}(U,V)=\bar{\rho}(U,U)$.
Then $\bar{\rho}(U,V)=\bar{\rho}(U,U)$

$\Rightarrow\left[-2,-1\right]\cdot D_M(U,V)=\left[-2,-1\right]\cdot D_M(U,U)=\left[0,0\right]$

$\Rightarrow D_M(U,V)=0$

$\Rightarrow U=V$.

Thus, indiscernible identity is satisfied.

Hence, $\bar{\rho}$ satisfies all the properties (i)-(iv) of **Definition 6**. However, $\bar{\rho}$ cannot be interval metric because in this definition the notion of positive set is not clearly defined. Therefore, $\bar{\rho}$ does not preserve the actual sense of distance.

3.2 Interval metric proposed by Bhunia and Samanta [18]

Bhunia and Samanta [18] proposed another definition of interval metric which is given below:

Definition 7. Let S be any non-empty set. A function $\bar{\rho}:S\times S\to K_c$ is said to be an interval metric if for all $U,V,W\in M$, $\bar{\rho}$ satisfies the following properties:

(i) $\bar{\rho}(U,U)_c=0\in\bar{\rho}(U,U)$ (Reflexivity)

(ii) $\bar{\rho}(U,V)\le^{\min}\bar{\rho}(U,W)+\bar{\rho}(W,V)$ (Triangular inequality)

(iii) $\bar{\rho}(U,V)=\bar{\rho}(V,U)$ (Symmetry)

(iv) If $0\in\bar{\rho}(U,V)=\bar{\rho}(U,U)=\bar{\rho}(V,V)$ then $U=V$ (Indiscernible identity)

where $\le^{\min}$ be the interval order relation proposed by Bhunia and Samanta [18] as follows: Let $U=\left[u_L,u_U\right]=\left\langle u_c,u_r\right\rangle$ & $V=\left[v_L,v_U\right]=\left\langle v_c,v_r\right\rangle$. Then

$$U \leq^{\min} V \Leftrightarrow \begin{cases} u_c \leq v_c, \text{ when } u_c \neq v_c \\ u_r \leq v_r, \text{ when } u_c = v_c. \end{cases}$$

Now, we justify with the **Example 6** that **Definition 7** cannot be an interval metric.

Example 6. Let $\bar{\rho}: K_c \times K_c \to K_c$ be an interval-valued function defined by $\bar{\rho}(U,V) = [-2,2] \cdot D_M(U,V)$, where D_M is Moore distance on K_c; $U \,\&\, V \in K_c$.

First of all, we show that $\bar{\rho}$ satisfies all the properties (i)-(iv) of **Definition 7**.

(i) Let $U \in K_c$.

Then,

$$\begin{aligned} \bar{\rho}(U,U) &= [-2,2] \cdot D_M(U,U) \\ &= [-2,2] \cdot 0 \ (\because D_M \text{ is real metric on } K_c,\ D_M(U,U) = 0) \\ &= [0,0] \end{aligned}$$

$\Rightarrow 0 \in \bar{\rho}(U,U)$.

This shows that $\bar{\rho}$ satisfies reflexive property.

(ii) Let $U,V,W \in K_c$.

Now,

$$\begin{aligned} \bar{\rho}(U,V)_c &= 0 \cdot D_M(U,V) = 0 \\ &= 0 + 0 \\ &= \bar{\rho}(U,W)_c + \bar{\rho}(W,V)_c \end{aligned}$$

and

$$\begin{aligned} \bar{\rho}(U,V)_r &= 2D_M(U,V) \\ &\leq 2\{D_M(U,W) + D_M(W,V)\} \\ &= 2D_M(U,W) + 2D_M(W,V) \\ &= \bar{\rho}(U,W)_r + \bar{\rho}(W,V)_r \end{aligned}$$

Thus, combining above, we can say that

$\bar{\rho}(U,V) \leq^{\min} \bar{\rho}(U,W) + \bar{\rho}(W,V)$.

Thus, $\bar{\rho}$ satisfies the triangular inequality.

Hence, in this case also, $\bar{\rho}$ satisfies all the properties (i)-(iv) of **Definition 7**. However, $\bar{\rho}$ cannot be the interval metric because the notion of positive set is not clearly mentioned.

3.3 Interval metric proposed by Afravi et al. [19]

Definition 8 [Afravi et al. [19]]. An interval-valued metric or simply interval metric on a non-empty set S is a function that relates each pair $(s,t) \in S \times S$ to an interval $[\rho_L(s,t), \rho_U(s,t)]$ with following properties:

(i) $\left[\rho_L(s,s),\rho_U(s,s)\right]=[0,0]$ & $\rho_U(s,t)>0$ for $s\neq t$

(ii) $\rho_L(s,t)=\rho_L(t,s)$ & $\rho_U(s,t)=\rho_U(t,s)$

(iii) For any finite chain $s_1,s_2,s_3,...,s_n$,

$\rho_L(s_1,s_n)\leq\rho_U(s_1,s_2)+\rho_U(s_2,s_3)+...+\rho_U(s_{n-1},s_n)$

Based on this definition, Afravi et al. [19] derived a proposition as follows:

Proposition 1. For any interval metric $\left[\rho_L(\cdot),\rho_U(\cdot)\right]$ on S, there exists a real metric ρ on S such that $\rho(s,t)\in\left[\rho_L(s,t),\rho_U(s,t)\right]$.

However, with the help of **Example 7**, it can be verified that **Proposition 1** cannot be derived from **Definition 8**.

Example 7. Let $S=\Re[0,1]$, the set of all Riemann integrable functions. Let $\bar{\rho}:\Re[0,1]\times\Re[0,1]\to K_c$ be an interval-valued function defined by

$$\bar{\rho}(u,v)=\left[\rho_L(u,v),\rho_U(u,v)\right]=[1,2]\cdot\int_0^1\left|u(t)-v(t)\right|dt,\ u\ \&\ v\in\Re[0,1].$$

Here, $\rho_L(u,v)=\int_0^1\left|u(t)-v(t)\right|dt$ & $\rho_U(u,v)=2\int_0^1\left|u(t)-v(t)\right|dt.$

Clearly, $\left[\rho_L(\cdot),\rho_U(\cdot)\right]$ satisfies all the properties (i)-(iii) of **Definition 8**.

Now, any element $\rho(u,v)\in\bar{\rho}(u,v)$ can be written as

$$\rho_\lambda(u,v)=(1-\lambda)\rho_L(u,v)+\lambda\rho_U(u,v)=(1+\lambda)\int_0^1\left|u(t)-v(t)\right|dt,\ \lambda\in[0,1].$$

Take $u(t)=\begin{cases}1, t\in[0,1]\setminus\left\{\frac{1}{2}\right\}\\ 2, t=\frac{1}{2}\end{cases}$ & $v(t)=1, t\in[0,1]$

Then $u\neq v$ but $\rho_\lambda(u,v)=0,\forall\lambda\in[0,1]$.

Therefore, no $\rho(u,v)\in\bar{\rho}(u,v)$ be a real metric.

Hence, **Proposition 1** is not valid in general.

4. PROPOSED INTERVAL METRIC WITH CONSEQUENT RESULTS

Keeping the limitations of the definitions of existing interval metric in mind, in this section, we have proposed a new definition of interval metric. We have also proposed a definition of interval order relation, which is used in the tri-angle inequality of the proposed interval metric. Thereafter, some properties of interval metric have been discussed.

Definition 9. The gH-order relation between two intervals $\tilde{A}=[\alpha_L,\alpha_U]$ and $\tilde{B}=[\beta_L,\beta_U]$ is as follows:

$$\tilde{A} \geq_{gH} \tilde{B} \text{ iff } \tilde{A}\Theta_{gH}\tilde{B} \in K_c^{\#},$$
$$\tilde{A} >_{gH} \tilde{B} \text{ iff } \tilde{A} \geq_{gH} \tilde{B} \text{ and } \tilde{A} \neq \tilde{B},$$

where $K_c^{\#} = \{[\alpha_L, \alpha_U] : \alpha_L \geq 0\}.$

Proposition 2. The order relations $\geq_{LU}$ and $\geq_{gH}$ between two intervals are equivalent.

Proof. Let $\tilde{A} = [\alpha_L, \alpha_U], \tilde{B} = [\beta_L, \beta_U] \in K_c$ and $\tilde{A} \geq_{LU} \tilde{B}.$ Then $\alpha_L \geq \beta_L$ and $\alpha_U \geq \beta_U$.
Therefore, $\tilde{A}\Theta_{gH}\tilde{B} = \left[\min\{\alpha_L - \beta_L, \alpha_U - \beta_U\}, \max\{\alpha_L - \beta_L, \alpha_U - \beta_U\}\right] \in K_c^{\#}$
$\Rightarrow \tilde{A} \geq_{gH} \tilde{B}.$

Conversely, let $\tilde{A} \geq_{gH} \tilde{B}.$ Then
$\tilde{A}\Theta_{gH}\tilde{B} = \left[\min\{\alpha_L - \beta_L, \alpha_U - \beta_U\}, \max\{\alpha_L - \beta_L, \alpha_U - \beta_U\}\right] \in K_c^{\#}$
$\Rightarrow \min\{\alpha_L - \beta_L, \alpha_U - \beta_U\} \geq 0$
$\Rightarrow \alpha_L - \beta_L \geq 0$ and $\alpha_U - \beta_U \geq 0$
$\Rightarrow \alpha_L \geq \beta_L$ and $\alpha_U \geq \beta_U \Rightarrow \tilde{A} \geq_{LU} \tilde{B}.$

Definition 10. Let S be a non-empty set. Then the interval-valued function $D_I : S \times S \to K_c$ is said to be an interval metric if it satisfies the following properties:

(i) $D_I(s,t) \in K_c^{\#}, \ \forall s,t \in S$ and $D_I(s,t) = [0,0]$ iff $s = t.$

(ii) $D_I(s,t) = D_I(t,s), \ \forall s,t \in S$

(iii) $D_I(s,w) + D_I(w,t) \geq_{gH} D_I(s,t), \forall s,t,w \in S.$

Then the pair (S, D_I) is called interval metric space.

Example 8. Let (S,ρ) be a real metric space and $[c_L, c_U]$ be an interval with $c_L > 0.$ Then $D_I : S \times S \to K_c$ defined by $D_I(s,t) = [c_L, c_U]\rho(s,t),$ is an interval metric.

Proof. (i) Let $s,t \in S.$
Then $D_I(s,t) = [c_L, c_U]\rho(s,t) = [c_L\rho(s,t), c_U\rho(s,t)]$
where $c_L\rho(s,t) \geq 0$ $[\because \rho$ is metric and $c_L > 0]$.
Thus, $D_I(s,t) \in K_c^{\#}.$
Again,
$D_I(s,t) = [0,0]$
$\Leftrightarrow [c_L, c_U]\rho(s,t) = [0,0]$
$\Leftrightarrow \rho(s,t) = 0$ $[\because c_L > 0]$
$\Leftrightarrow s = t$ $[\because \rho$ is a metric$]$
(ii) Let $s,t \in S.$
Then, $D_I(s,t) = [c_L, c_U]\rho(s,t) = [c_L, c_U]\rho(t,s) = D_I(t,s).$
(iii) Let $s,t,w \in S.$ Then $\rho(s,t) \leq \rho(s,w) + \rho(w,t) [\because \rho$ is a metric on $S]$

$\Rightarrow c_L\left\{\rho(s,w)+\rho(w,t)-\rho(s,t)\right\}\geq 0$ and $c_U\left\{\rho(s,w)+\rho(w,t)-\rho(s,t)\right\}\geq 0$

$\Rightarrow \left\{D_I(s,w)+D_I(w,t)\right\}\Theta_{gH} D_I(s,t)\in K_c^{\#}$

$\Rightarrow D_I(s,w)+D_I(w,t)\geq_{gH} D_I(s,t).$

Therefore, according to **Definition 10**, D_I is an interval metric on S.

Proposition 3. Let S be a non-empty set and $D_I : S\times S\rightarrow K_c$ be an interval-valued function of the form $D_I(s,t)=\left[\rho_L(s,t),\rho_U(s,t)\right]$, for $s,t\in S$. Then (S,D_I) is an interval metric space iff (S,ρ_L) and (S,ρ_U) are real metric spaces.

Proof. First of all, let (S,D_I) be an interval metric space. Then D_I satisfies all the properties of **Definition 10**. Thus, from **Definition 10**, we get

(i) $D_I(s,t)\in K_c^{\#}$, $\forall s,t\in S.$

$\Rightarrow \left[\rho_L(s,t),\rho_U(s,t)\right]\in K_c^{\#}$

$\Rightarrow \rho_L(s,t)\geq 0$ and $\rho_U(s,t)\geq 0$ $\quad\left[\because \rho_U(s,t)\geq \rho_L(s,t)\right]$

Again, $D_I(s,t)=[0,0]$ iff $s=t$

$\Rightarrow \left[\rho_L(s,t),\rho_U(s,t)\right]=\left[0,0\right]$ iff $s=t$

$\Rightarrow \rho_L(s,t)=0$ iff $s=t$ & $\rho_U(s,t)=0$ iff $s=t.$

Therefore, we get the following conditions:

$\rho_L(s,t)\geq 0$, $\forall s,t\in S$ & $\rho_L(s,t)=0$ iff $s=t,$

and $\rho_U(s,t)\geq 0,\ \forall s,t\in S$ & $\rho_U(s,t)=0$ iff $s=t$ (1)

(ii) $D_I(s,t)=D_I(t,s),\ \forall s,t\in S.$

i.e., $\left[\rho_L(s,t),\rho_U(s,t)\right]=\left[\rho_L(t,s),\rho_U(t,s)\right],\ \forall s,t\in S$

$\Rightarrow \rho_L(s,t)=\rho_L(t,s),\ \forall s,t\in S.$

and $\rho_U(s,t)=\rho_U(t,s),\ \forall s,t\in S$ (2)

(iii) $D_I(s,w)+D_I(w,t)\geq_{gH} D_I(s,t), \forall s,t,w\in S.$

Then $\left\{D_I(s,w)+D_I(w,t)\right\}\Theta_{gH} D_I(s,t)\in K_c^{\#}$

$\Rightarrow \rho_L(s,w)+\rho_L(w,t)-\rho_L(s,t)\geq 0$ and $\rho_U(s,w)+\rho_U(w,t)-\rho_U(s,t)\geq 0$

$\Rightarrow \rho_L(s,t)\leq \rho_L(s,w)+\rho_L(w,t), \forall s,t,w\in S$

and $\rho_U(s,t)\leq \rho_U(s,w)+\rho_U(w,t), \forall s,t,w\in S$ (3)

Hence, from the conditions (1)-(3), it can be concluded that both ρ_L and ρ_U are metrics on S.

Conversely, let (S,ρ_L) and (S,ρ_U) are metric spaces.

(i) Then, $\rho_L(s,t)\geq 0$ and $\rho_U(s,t)\geq 0,\ \forall s,t\in S.$

$\Rightarrow \left[\rho_L(s,t), \rho_U(s,t)\right] \in K_c^{\#},\ \forall s,t \in S.$

$\Rightarrow D_I(s,t) \in K_c^{\#},\ \forall s,t \in S.$

Again, $\rho_L(s,t) = 0$ iff $s = t$ and $\rho_U(s,t) = 0$ iff $s = t$

$\Rightarrow \left[\rho_L(s,t), \rho_U(s,t)\right] = [0,0]$ iff $s = t$

i.e., $D_I(s,t) = [0,0]$ iff $s = t.$

(ii) The symmetry property of ρ_L & ρ_U gives

$\rho_L(s,t) = \rho_L(t,s)$ & $\rho_U(s,t) = \rho_U(t,s),\ \forall s,t \in S.$

Hence, $\left[\rho_L(s,t), \rho_U(s,t)\right] = \left[\rho_L(t,s), \rho_U(t,s)\right],\ \forall s,t \in S$

i.e., $D_I(s,t) = D_I(t,s),\ \forall s,t \in S.$

(iii) The triangle inequality of ρ_L & ρ_U gives

$\rho_L(s,t) \le \rho_L(s,w) + \rho_L(w,t)$ & $\rho_U(s,t) \le \rho_U(s,w) + \rho_U(w,t), \forall s,t,w \in S$

$\Rightarrow \rho_L(s,w) + \rho_L(w,t) - \rho_L(s,t) \ge 0$ & $\rho_U(s,w) + \rho_U(w,t) - \rho_U(s,t) \ge 0,\ \forall s,t,w \in S$

$\Rightarrow \left\{D_I(s,w) + D_I(w,t)\right\} \Theta_{gH} D_I(s,t) \in K_c^{\#},\ \forall s,t,w \in S$

$\Rightarrow D_I(s,w) + D_I(w,t) \ge_{gH} D_I(s,t), \forall s,t,w \in S.$

Therefore, according to **Definition 10**, D_I is an interval metric on S.

Example 9. Let $S = \mathbb{R}^n$. Then some interval metrics on $\mathbb{R}^n$ are given below:

(i) $D_{I_1}(s,t) = \left[\rho_2(s,t), \rho_1(s,t)\right]$

(ii) $D_{I_2}(s,t) = \left[\rho_\infty(s,t), \rho_2(s,t)\right]$

(iii) $D_{I_3}(s,t) = \left[\rho_\infty(s,t), \rho_1(s,t)\right]$

(iv) $D_{I_4}(s,t) = \left[\rho_1(s,t), n\rho_2(s,t)\right]$

(v) $D_{I_5}(s,t) = \left[\rho_2(s,t), \sqrt{n}\rho_\infty(s,t)\right]$

where ρ_1, ρ_2 and ρ_∞ are defined in **Example 1**.

Proof. Proofs of (i)-(v) follow from the **Result 1** and **Proposition 3**.

Proposition 4. Let $\rho_1,\ \rho_2$ be two real metrics on S and $\left[\alpha_{1L}, \alpha_{1U}\right], \left[\alpha_{2L}, \alpha_{2U}\right] \in K_c$ with $\alpha_{1L} > 0, \alpha_{2L} > 0.$ Then the interval-valued function $D_I : S \times S \to K_c$ defined by $D_I(s,t) = \left[\alpha_{1L}, \alpha_{1U}\right]\rho_1(s,t) + \left[\alpha_{2L}, \alpha_{2U}\right]\rho_2(s,t)$ is an interval metric on S.

Proof. $D_I(s,t)$ can be written as

$$\begin{aligned} D_I(s,t) &= \left[\alpha_{1L}, \alpha_{1U}\right]\rho_1(s,t) + \left[\alpha_{2L}, \alpha_{2U}\right]\rho_2(s,t) \\ &= \left[\alpha_{1L}\rho_1(s,t), \alpha_{1U}\rho_1(s,t)\right] + \left[\alpha_{2L}\rho_2(s,t), \alpha_{2U}\rho_2(s,t)\right] \ \left[\because \rho_1(s,t), \rho_2(s,t) \ge 0\right] \\ &= \left[\alpha_{1L}\rho_1(s,t) + \alpha_{2L}\rho_2(s,t), \alpha_{1U}\rho_1(s,t) + \alpha_{2U}\rho_2(s,t)\right] \end{aligned}$$

Now, by the **Result 2**, $\alpha_{1L}\rho_1 + \alpha_{2L}\rho_2$ and $\alpha_{1U}\rho_1 + \alpha_{2U}\rho_2$ both are real metrics on S and also $\alpha_{1L}\rho_1(s,t) + \alpha_{2L}\rho_2(s,t) \le \alpha_{1U}\rho_1(s,t) + \alpha_{2U}\rho_2(s,t),\ \forall s,t \in S.$

Hence by **Proposition 3,** (S, D_I) is an interval metric space.

Remark 1.

In general, if $\rho_1, \rho_2, ..., \rho_n$ be real metrics on S and $[\alpha_{1L}, \alpha_{1U}]$, $[\alpha_{2L}, \alpha_{2U}], ..., [\alpha_{nL}, \alpha_{nU}] \in K_c$ where $\alpha_{1L} > 0, \alpha_{2L} > 0, ..., \alpha_{nL} > 0,$ then the interval-valued function $D_I : S \times S \to K_c$ given by $D_I(s,t) = \sum_{i=1}^{n} [\alpha_{iL}, \alpha_{iU}] \rho_i(s,t), \forall s,t \in S$ is interval metric on S.

Theorem 1. Let $S \neq \phi$ and D_{I_1}, D_{I_2} be two interval metrics on S and $\bar{c} = [c_L, c_U]$, $c_L > 0.$ Then

(i) $D_{I_1} + D_{I_2}$ is an interval metric on S.

(ii) $\bar{c}D_{I_1}$ is an interval metric on S.

Proof. (i) Let $D_{I_1}(s,t) = \left[\rho_{L_1}(s,t), \rho_{U_1}(s,t)\right]$ and $D_{I_2}(s,t) = \left[\rho_{L_2}(s,t), \rho_{U_2}(s,t)\right]$.

$\therefore D_{I_1}(s,t) + D_{I_2}(s,t) = \left[\rho_{L_1}(s,t) + \rho_{L_2}(s,t), \rho_{U_1}(s,t) + \rho_{U_2}(s,t)\right].$

Since D_{I_1} and D_{I_2} are interval metrics on S, then by **Proposition 3**, $\rho_{L_1}, \rho_{U_1}, \rho_{L_2}, \rho_{U_2}$ are metrics on S. Also, by **Result 2**, we can say that $\rho_{L_1} + \rho_{L_2}$ and $\rho_{U_1} + \rho_{U_2}$ are metrics on S.

Therefore, again by **Proposition 3**, $D_{I_1} + D_{I_2}$ is an interval metric on S.

(ii) Since D_{I_1} is an interval metric on S, by **Proposition 3**, one can say that ρ_{L_1} and ρ_{U_1} are metrics on S.

Now $\bar{c}D_{I_1}(s,t) = [c_L, c_U]\left[\rho_{L_1}(s,t), \rho_{U_1}(s,t)\right]$

$$= \left[c_L \rho_{L_1}(s,t), c_U \rho_{U_1}(s,t)\right] \ \text{[since } c_L > 0, \rho_{L_1}(s,t) \geq 0]$$

Also, by **Result 3**, one can say that $c_L \rho_{L_1}, c_U \rho_{U_1}$ are metrics on S.

Therefore, by **Proposition 3**, it can be concluded that $\bar{c}D_{I_1}$ is an interval metric on S.

Remark 2. In general, if $D_{I_1}, D_{I_2}, ..., D_{I_n}$ are interval metrics on S and $\bar{c}_i = [c_{iL}, c_{iU}] \in K_c$ & $c_{iL} > 0,\ i = 1, ..., n,$ then $\bar{c}_1 D_{I_1} + \bar{c}_2 D_{I_2} + ... + \bar{c}_n D_{I_n}$ is an interval metric on S.

Example 10.

Let (S, D_I) be an interval metric space where $D_I(s,t) = [\rho_L(s,t), \rho_U(s,t)],\ s,t \in S.$ Then the function $D_{I_{\min}} : S \times S \to K_c$ defined by

$D_{I_{\min}}(s,t) = \left[\min\{1, \rho_L(s,t)\}, \min\{1, \rho_U(s,t)\}\right]$ is also an interval metric on S.

Proof. Proof follows from **Result 4** and **Proposition 3**.

Definition 11. Let (S, D_I) be an interval metric space and P be a non-empty subset of S. Let $u \in S$. Then the interval distance of u from P with respect to D_I is denoted by $D_I(u,P)$ and is defined by $D_I(u,P) = \{\rho(u,P) : \rho \in D_I\}$.

Remark 3. For any $\rho \in D_I$ and $P \subseteq S$, $\rho(u,P) = \inf_{p \in P}\{\rho(u,p)\}$.

Proposition 5. Let us suppose (S, D_I) as an interval metric space with $D_I(s,t) = [\rho_L(s,t), \rho_U(s,t)]$, $s,t \in S$. Then $D_I(u,P) \subseteq [\rho_L(u,P), \rho_U(u,P)]$, where $\rho_L(u,P)$ and $\rho_U(u,P)$ are distances of u from P with respect to ρ_L and ρ_U respectively.

Proof. Let $s \in D_I(u,P)$. Then $s = \rho(u,P) = \inf_{p \in P}\{\rho(u,p)\}$, for some $\rho \in D_I$.

For every $p \in P$, $\rho_L(u,P) \le \rho_L(u,p) \le \rho(u,p)$.

Hence, $\rho_L(u,P) \le \inf_{p \in P}\{\rho(u,p)\}$ i.e., $\rho_L(u,P) \le s$ (4)

Again, $s = \inf_{p \in P}\{\rho(u,p)\} \le \rho(u,p) \le \rho_U(u,p)$, for every $p \in P$

$\Rightarrow s \le \inf_{p \in P}\{\rho_U(u,p)\}$ i.e., $s \le \rho_U(u,P)$ (5)

Therefore, from (4) and (5), we get

$\rho_L(u,P) \le s \le \rho_U(u,P)$

$\Rightarrow s \in [\rho_L(u,P), \rho_U(u,P)]$

Hence, $D_I(u,P) \subseteq [\rho_L(u,P), \rho_U(u,P)]$.

Definition 12. Let (S, D_I) be an interval metric space and P, Q be two non-empty subsets of S. Then the interval distance between P and Q with respect to D_I is denoted by $D_I(P,Q)$ and is defined by

$$D_I(P,Q) = \left\{\inf_{p \in P}\{d(p,Q)\} : \rho \in D_I\right\}.$$

Proposition 6. Suppose (S, D_I) as an interval metric space with $D_I(s,t) = [\rho_L(s,t), \rho_U(s,t)]$, for $s,t \in S$. Then $D_I(P,Q) \subseteq [\rho_L(P,Q), \rho_U(P,Q)]$, where $\rho_L(P,Q)$ and $\rho_U(P,Q)$ are distances between the subsets P and Q with respect to ρ_L and ρ_U respectively.

Proof. The proof is similar as **Proposition 5**.

Definition 13. Let (S, D_I) be an interval metric space and H be a non-empty subset of S. Then the diameter of H with respect to D_I denoted by $\delta_I(H)$ and is defined by

$$\delta_I(H) = \left\{\sup_{u,v \in H}\{\rho(u,v)\} : \rho \in D_I\right\}.$$

Proposition 7. Suppose (S, D_I) as an interval metric space with $D_I(s,t) = [\rho_L(s,t), \rho_U(s,t)]$, for $s,t \in S$ and $H(\neq \phi) \subseteq S$. Then $\delta_I(H) \subseteq [\delta_L(H), \delta_U(H)]$, where $\delta_L(H)$ and $\delta_U(H)$ are diameters of H with respect to ρ_L and ρ_U respectively.

Proof. Let $s \in \delta_I(H)$. Then $s = \sup\limits_{u,v \in H} \{\rho(u,v)\}$, for some $\rho \in D_I$.

Now for every $u,v \in H$, $\rho_L(u,v) \le \rho(u,v) \le \rho_U(u,v)$.

So, $\sup\limits_{u,v \in H} \{\rho_L(u,v)\} \le \sup\limits_{u,v \in H} \{\rho(u,v)\} \le \sup\limits_{u,v \in H} \{\rho_U(u,v)\}$

$\Rightarrow \delta_L(H) \le s \le \delta_U(H)$

$\Rightarrow s \in [\delta_L(H), \delta_U(H)]$

Hence, $\delta_I(H) \subseteq [\delta_L(H), \delta_U(H)]$.

Definition 14. Let us suppose (S, D_I) as an interval metric space and P be a subset of S. Then P is said to be bounded with respect to D_I if for each $\rho \in D_I, \exists\, k_\rho > 0$ such that $\rho(u,v) \le k_\rho, \forall u,v \in P$. Otherwise, P is said to be unbounded with respect to D_I.

In particular, the interval metric space (S, D_I) is said to be bounded or unbounded according as the set S is bounded or unbounded with respect to D_I. If (S, D_I) is bounded then D_I is said to be a bounded interval metric on S.

Observation 1. Let $(S, [\rho_L(\cdot), \rho_U(\cdot)])$ be a bounded interval metric space.

Then for each $\rho \in D_I, \exists\, k_\rho > 0$ such that $\rho(u,v) \le k_\rho, \forall u,v \in S$.

In particular, for $\rho = \rho_L$ and $\rho = \rho_U$, we can find $k_L > 0$ and $k_U > 0$ respectively such that

$$\rho_L(u,v) \le k_L, \forall u,v \in S$$
$$\rho_U(u,v) \le k_U, \forall u,v \in S.$$

This shows that (S, ρ_L) and (S, ρ_U) are bounded metric spaces.

Hence, if $(S, [\rho_L(\cdot), \rho_U(\cdot)])$ is a bounded interval metric space then (S, ρ_L) and (S, ρ_U) both are bounded metric spaces.

Again, let (S, ρ_U) be a bounded metric space. Then $\exists k > 0$ such that

$$\rho_U(u,v) \le k, \forall u,v \in S.$$

Now, for each $\rho \in D_I$, $\rho_L(u,v) \le \rho(u,v) \le \rho_U(u,v), \forall u,v \in S$

$\Rightarrow \rho(u,v) \le \rho_U(u,v) \le k, \forall u,v \in S$

$\Rightarrow (S, [\rho_L(\cdot), \rho_U(\cdot)])$ is an bounded interval metric space.

So, if $\left(S,\rho_U\right)$ is a bounded metric space then $\left(S,\left[\rho_L(\cdot),\rho_U(\cdot)\right]\right)$ is an bounded interval metric space.

From the earlier mentioned observation, we get the following proposition:

Proposition 8. Let us suppose $\left(S,\left[\rho_L(\cdot),\rho_U(\cdot)\right]\right)$ as an interval metric space. Then $\left(S,\left[\rho_L(\cdot),\rho_U(\cdot)\right]\right)$ is a bounded interval metric space iff $\left(S,\rho_U\right)$ is a bounded metric space.

Remark 4. Boundedness of an interval metric space $\left(S,\left[\rho_L(\cdot),\rho_U(\cdot)\right]\right)$ depends only on the metric ρ_U.

Example 11. Let us suppose $\left(S,D_I\right)$ as an interval metric space with $D_I(s,t)=\left[\rho_L(s,t),\rho_U(s,t)\right],\ s,t\in S.$ Then the interval metric $D_{I_{\min}}:S\times S\to K_c$ defined by $D_{I_{\min}}(s,t)=\left[\min\left\{1,\rho_L(s,t)\right\},\min\left\{1,\rho_U(s,t)\right\}\right]$ is a bounded interval metric on S.

5. APPLICATION OF PROPOSED INTERVAL METRIC

In this section, we shall discuss the application of proposed interval metric. We shall use our proposed interval metric to convert interval multi-objective optimization problem into interval single objective optimization problem.

The general form of multi-objective optimization problem with interval objectives can be written as

Minimize $\left\{A_1(x),A_2(x),...,A_k(x)\right\}$

subject to $x\in S\subset\mathbb{R}^n$

where $A_i:\mathbb{R}^n\to K_c$ given by $A_i(x)=\left[f_{iL}(x),f_{iU}(x)\right],\quad i=1,2,...,k$ and $S=\left\{x:g_j(x)\le 0,\ j=1,2,...,m\right\}.$

Bhunia and Samanta [18] solved the interval multi-objective optimization problem by converting it into interval single objective optimization problem with the help of their proposed interval metric. Our proposed interval metric can also be used to formulate the problem into single objective optimization problem.

In this connection, we shall define Pareto optimality, ideal objectives and different interval metrics on K_c.

Definition 15. A decision vector $x^*\in S$ is Pareto optimal if there does not exist another decision vector $x\in S$ such that $A_i(x)\le_{gH} A_i(x^*)$ for all $i=1,2,...,k$ and $A_j(x)<_{gH} A_j(x^*)$ for at least one index j.

Definition 16. A decision vector $x^* \in S$ is weakly Pareto optimal if there does not exist another decision vector $x \in S$ such that $A_i(x) <_{gH} A_i(x^*)$ for all $i = 1, 2, ..., k.$

Definition 17. An objective vector minimizing each of the objective functions is called an ideal objective vector.

Definition 18. Let $X = [x_L, x_U], Y = [y_L, y_U] \in K_c$. Define two metrics $D_1 : K_c \times K_c \to \mathbb{R}$ and $D_2 : K_c \times K_c \to \mathbb{R}$ as follows:

$D_1(X,Y) = | x_L - y_L | + | x_U - y_U |$ and $D_2(X,Y) = \left(| x_L - y_L |^2 + | x_U - y_U |^2 \right)^{1/2}$.

Now, define two interval valued mappings $D_{I_{M2}} : K_c \times K_c \to K_c$ and $D_{I_{21}} : K_c \times K_c \to K_c$ by $D_{I_{M2}}(X,Y) = [D_M(X,Y), D_2(X,Y)]$ and $D_{I_{21}}(X,Y) = [D_2(X,Y), D_1(X,Y)]$, where D_M is the Moore distance given in **Definition 3**. Here, $D_{I_{M2}}$ and $D_{I_{21}}$ are interval metrics on K_c, by **Proposition 3**.

Bhunia and Samanta [18] solved the interval multi-objective optimization problems by different techniques. Some of these techniques are as follows:

(i) Global criterion method
(ii) Tchebycheff method
(iii) Weighted Tchebycheff method.

Now we shall construct all these problems in another way by applying our proposed interval metric.

5.1 Global Criterion Method

In this method, the auxiliary problem is as follows:

$$\text{Minimize } \sum_{i=1}^{k} D_{I_{M2}}\left(A_i(x), z_i^* \right) \tag{6}$$

subject to $x \in S$, where z_i^* be the *i*-th component of ideal objective vector.

Now, using **Definition 18**, the auxiliary problem (6) can be written as:

$$\text{Minimize } \sum_{i=1}^{k} \left[\max\left\{ \left| f_{iL}(x) - z_{iL}^* \right|, \left| f_{iU}(x) - z_{iU}^* \right| \right\}, \left(\left| f_{iL}(x) - z_{iL}^* \right|^2 + \left| f_{iU}(x) - z_{iU}^* \right|^2 \right)^{1/2} \right]$$

subject to $x \in S$.

Remark 5. Another auxiliary problem can be defined as follows:

$$\text{Minimize } \sum_{i=1}^{k} D_{I_{21}}\left(A_i(x), z_i^* \right) \tag{7}$$

subject to $x \in S$, where z_i^* be the *i*-th component of ideal objective vector.

Using **Definition 18**, the auxiliary problem (7) can be written as:

$$\text{Minimize } \sum_{i=1}^{k} \left[\left(\left| f_{iL}(x) - z_{iL}^* \right|^2 + \left| f_{iU}(x) - z_{iU}^* \right|^2 \right)^{1/2}, \left| f_{iL}(x) - z_{iL}^* \right| + \left| f_{iU}(x) - z_{iU}^* \right| \right]$$

subject to $x \in S$.

Theorem 2. The solution of the problem (6) is Pareto optimal.

Proof. Let $x^* \in S$ be a solution of the problem (6). Let us assume that x^* is not Pareto optimal. Then there exists a point $x \in S$ such that $A_i(x) \leq_{gH} A_i(x^*)$ for all $i = 1, 2, ..., k$ and $A_j(x) <_{gH} A_j(x^*)$ for at least one index j.

Now, $A_i(x) \leq_{gH} A_i(x^*)$ implies $A_i(x^*) \Theta_{gH} A_i(x) \in K_c^{\#}$

$\Rightarrow f_{iL}(x) \leq f_{iL}(x^*)$ and $f_{iU}(x) \leq f_{iU}(x^*)$

Therefore, $f_{iL}(x) - z_{iL}^* \leq f_{iL}(x^*) - z_{iL}^*$ and $f_{iU}(x) - z_{iU}^* \leq f_{iU}(x^*) - z_{iU}^*$

$\Rightarrow \left| f_{iL}(x) - z_{iL}^* \right| \leq \left| f_{iL}(x^*) - z_{iL}^* \right|$

and $\left| f_{iU}(x) - z_{iU}^* \right| \leq \left| f_{iU}(x^*) - z_{iU}^* \right| \quad \left[\because z_i^* \leq_{gH} A_i(x) \text{ for all } x \right]$ (8)

From (8), we have

$$\max\left\{ \left| f_{iL}(x) - z_{iL}^* \right|, \left| f_{iU}(x) - z_{iU}^* \right| \right\} \leq \max\left\{ \left| f_{iL}(x^*) - z_{iL}^* \right|, \left| f_{iU}(x^*) - z_{iU}^* \right| \right\} \tag{9}$$

and $$\left(\left| f_{iL}(x) - z_{iL}^* \right|^2 + \left| f_{iU}(x) - z_{iU}^* \right|^2 \right)^{1/2} \leq \left(\left| f_{iL}(x^*) - z_{iL}^* \right|^2 + \left| f_{iU}(x^*) - z_{iU}^* \right|^2 \right)^{1/2} \tag{10}$$

Combining (9) and (10), we have

$D_{I_{M2}}\left(A_i(x), z_i^* \right) \leq_{gH} D_{I_{M2}}\left(A_i(x^*), z_i^* \right)$ for all $i = 1, 2, ..., k.$

Similarly, $A_j(x) <_{gH} A_j(x^*) \Rightarrow D_{I_{M2}}\left(A_j(x), z_j^* \right) <_{gH} D_{I_{M2}}\left(A_j(x^*), z_j^* \right)$

Therefore, $\sum_{i=1}^{k} D_{I_{M2}}\left(A_i(x), z_i^* \right) <_{gH} \sum_{i=1}^{k} D_{I_{M2}}\left(A_i(x^*), z_i^* \right)$, which contradicts our assumption that $x^* \in S$ is a solution of the problem (6).

Hence, x^* is Pareto optimal. This completes the proof.

Theorem 3. The solution of the problem (7) is Pareto optimal.

Proof. Proof is similar to **Theorem 2**.

5.2 Tchebycheff Method

In this method, the auxiliary problem is as follows:

$$\text{Minimize } \underset{i=1,2,...,k}{Max}\, D_{I_{M2}}\left(A_i(x), z_i^* \right) \tag{11}$$

subject to $x \in S,$ where z_i^* be the *i*-th component of ideal objective vector.

Remark 6. Another auxiliary problem can be defined as:

$$\text{Minimize } \underset{i=1,2,...,k}{Max}\, D_{I_{21}}\left(A_i(x), z_i^* \right) \tag{12}$$

subject to $x \in S,$ where z_i^* be the *i*-th component of ideal objective vector.

Theorem 4. The solution of the problem (11) is weakly Pareto optimal.

Proof. Let $x^* \in S$ be a solution of the problem (11). Let us assume that x^* is not weakly Pareto optimal. Then there exists a point $x \in S$ such that $A_i(x) <_{gH} A_i(x^*)$ for all $i = 1, 2, ..., k.$

Now, $A_i(x) <_{gH} A_i(x^*)$ implies $A_i(x) \leq_{gH} A_i(x^*)$ and $A_i(x) \neq A_i(x^*)$.
Here two cases may arise. These are:
Case I. $f_{iL}(x) < f_{iL}(x^*)$ and $f_{iU}(x) \leq f_{iU}(x^*)$
Case II. $f_{iL}(x) \leq f_{iL}(x^*)$ and $f_{iU}(x) < f_{iU}(x^*)$.
We consider Case I only. The other case is similar to Case I.
Case I. Let $f_{iL}(x) < f_{iL}(x^*)$ and $f_{iU}(x) \leq f_{iU}(x^*)$.
Then $f_{iL}(x) - z_{iL}^* < f_{iL}(x^*) - z_{iL}^*$ and $f_{iU}(x) - z_{iU}^* \leq f_{iU}(x^*) - z_{iU}^*$
$\Rightarrow \left| f_{iL}(x) - z_{iL}^* \right| < \left| f_{iL}(x^*) - z_{iL}^* \right|$
and $\left| f_{iU}(x) - z_{iU}^* \right| \leq \left| f_{iU}(x^*) - z_{iU}^* \right| \quad \left[\because z_i^* \leq_{gH} A_i(x) \text{ for all } x \right]$ (13)
From (13), we have

$$\max\left\{ \left| f_{iL}(x) - z_{iL}^* \right|, \left| f_{iU}(x) - z_{iU}^* \right| \right\} \leq \max\left\{ \left| f_{iL}(x^*) - z_{iL}^* \right|, \left| f_{iU}(x^*) - z_{iU}^* \right| \right\} \tag{14}$$

and $$\left(\left| f_{iL}(x) - z_{iL}^* \right|^2 + \left| f_{iU}(x) - z_{iU}^* \right|^2 \right)^{1/2} < \left(\left| f_{iL}(x^*) - z_{iL}^* \right|^2 + \left| f_{iU}(x^*) - z_{iU}^* \right|^2 \right)^{1/2} \tag{15}$$

Combining (14) and (15), we have
$D_{I_{M2}}\left(A_i(x), z_i^*\right) <_{gH} D_{I_{M2}}\left(A_i(x^*), z_i^*\right)$ for all $i = 1, 2, ..., k$
$\Rightarrow \underset{i=1,2,...,k}{Max}\, D_{I_{M2}}\left(A_i(x), z_i^*\right) <_{gH} \underset{i=1,2,...,k}{Max}\, D_{I_{M2}}\left(A_i(x^*), z_i^*\right)$, which contradicts our assumption that $x^* \in S$ is a solution of the problem (11).
Hence, x^* is weakly Pareto optimal. This completes the proof.
Theorem 5. The solution of the problem (12) is weakly Pareto optimal.
Proof. Proof is similar to **Theorem 4**.

5.3 Weighted Tchebycheff Method

In this method, the auxiliary problem is as follows:

$$\text{Minimize} \quad \underset{i=1,2,...,k}{Max}\, \omega_i\, D_{I_{M2}}\left(A_i(x), z_i^*\right) \tag{16}$$

subject to $\sum_{i=1}^{k} \omega_i = 1$, $\omega_i > 0$ and $x \in S$, where z_i^* be the *i*-th component of ideal objective vector.
Remark 7. Another auxiliary problem can be defined as:

$$\text{Minimize} \quad \underset{i=1,2,...,k}{Max}\, \omega_i\, D_{I_{21}}\left(A_i(x), z_i^*\right) \tag{17}$$

subject to $\sum_{i=1}^{k} \omega_i = 1$, $\omega_i > 0$ and $x \in S$, where z_i^* be the *i*-th component of ideal objective vector.
Theorem 6. The solution of the problem (16) is weakly Pareto optimal.
Proof. With the help of **Theorem 4** it can easily be proved.
Theorem 7. The solution of the problem (17) is weakly Pareto optimal.

Proof. Proof is similar to **Theorem 6**.

6. MERITS AND DEMERITS OF THE WORK

In the present work, the notion of interval distance has been introduced in an alternative way by rectifying all the drawbacks of the existing interval distances. Though, this work has been done from the theoretical point of view, it has a number of advantages which are given below:

(i) The idea of interval metric can be used to measure the distance between two points in a space under uncertain/inexact/imprecise circumstances.

(ii) Using the proposed interval metric, the topological structure, sequential property of a space and fixed point theory can be easily introduced.

(iii) The most notable merits of this interval distance are that the idea of usual distance with all relevant properties can be derived from it as a particular case.

Apart from the above mentioned merits, the interval metric has few limitations also. These are as follows:

(i) Since, the space of interval is not totally ordered set, interval diameter, interval distance between two sets cannot be determined explicitly.

(ii) Also, the notion of equivalent interval metric does not make the usual sense due to the lack of total interval order relation.

7. CONCLUDING REMARKS

In this work, all the limitations of the existing interval metrics have been sorted out by introducing proposed interval metric. Then, the necessary and sufficient conditions for interval metric have been established. Every theoretical derivation of this work regarding interval diameter, boundedness of a set and interval distance between two sets has made a significant impact to move forward the theory of interval metric. In application of proposed interval metric, multi-objective interval optimization problem is converted to single objective interval optimization problem which can be solved by any metaheuristic or hybrid metaheuristic algorithm with interval fitness and interval order relations.

With the introduction of interval metric, in this area, for future investigation, there are lot of scopes to enrich this work. In this regard, one can extend the findings of topological properties for interval metric. Also, the notion of convergence may be incorporated in the interval metric space. Finally, this concept can also be generalized by taking Type-2 interval uncertainty (Rahman et al. [33, 34]).

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