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TOPSIS FOR MULTIPLE OBJECTIVE PROGRAMMING WITH ROUGH DECISION SET

Tarek ABOU-EL-ENIEN
Department of O.R. & D.S., Faculty of Computers and Artificial Intelligence, Cairo University, Giza 12613, Egypt
t.hanafy@fci-cu.edu.eg

Y. ABO-ELNAGA
Department of Basic Science, Higher Technological Institute, Tenth of Ramadan City, Egypt
yousria_naga@yahoo.com

Kamilia MOHAMMAD
Department of Physics and Engineering Mathematics, Faculty of Engineering, Tanta University, Tanta, Egypt
Kamelia.Aboelfadl@f-eng.tanta.edu.eg

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Abstract: Many optimization problems have competing objectives that require being optimized at the same time. These problems are called "multiple objective programming problems (MOPPs)". Real-world MOPPs may have some imprecision (roughness) in the decision set and/or the objective functions. These problems are known as "rough MOPPs (RMOPPs)". There is no unique method able to solve all RMOPPs. Accordingly, the decision maker (DM) should have more than one method for solving RMOPPs at his disposal so that he can select the most appropriate method. To contribute in this regard, we propose a new method for solving a specific class of RMOPPs in which all the objectives are precisely defined, but the decision set is roughly defined by its lower and upper approximations. Our proposed method is a modified version of the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS). TOPSIS was chosen as the foundation for our method because it is one of the most widely applied methods for solving MOPPs. The basic concept underlying TOPSIS is that the compromise solution is closer to the ideal solution while also being farther away from the anti-ideal solution. The conventional TOPSIS can only solve MOPPs with precise (crisp) definitions of the two main parts of the problem. We extend TOPSIS to optimize multiple precise objectives over an imprecise decision set. The proposed approach is depicted in a flowchart. A numerical example is given to demonstrate the effectiveness of our proposed method to solve RMOPPs with a rough decision set at different values of objectives' weights and using different L_p-metrics.

Keywords: TOPSIS, Rough programming, Rough decision set, Multiple objective programming.

MSC: 90C29.

1. INTRODUCTION

With the advancement of science and technology, decision-making has drawn the attention of many researchers from diverse fields of human activity. Making decisions in the quest for optimality is one of the globe's most fundamental principles. Accordingly, optimization is a significant scientific discipline. A situation in which the DM attempts to fulfill the ideal scenario under some circumstances is referred to as "an optimization problem." MOPPs are a typical class of optimization problems. In a MOPP, a set of conflicting crisp objectives is optimized simultaneously under a determined set of conditions [1].

In some realistic MOPPs, there is a lack of information required to precisely specify any part of the problem [2], [3], [4]. In other problems, the approximate descriptions of objective(s) and/or feasible set are more convenient and less expensive than the crisp descriptions [3], [4], [5]. Correspondingly, decisions are made based on imprecise information rather than precise information. These problems are named "RMOPPs". Rough set theory (RST) [6] provides an excellent mathematical tool to the DM for dealing with imprecision in RMOPPs. It is important to expand the study of the rough multi-objective programming field to address many real-world optimization problems that have multiple objectives to optimize over a rough decision set.

Youness [7] was the first to combine RST and optimization. He presented a new single objective programming problem with a crisp objective function and a rough decision set, called a "rough single objective programming (RSOP) problem". He also defined the term "rough optimal solution". Five years later, Osman et al. [8] stated that roughness may appear not only in the decision set but also in the objective function. Hence, they classified RSOP problems into three distinct classes according to the place of roughness. They talked about concepts like "rough feasibility", "rough optimality", and "rough optimal set". In the same manner, Atteya [4] introduced a concentrated study of the hybridization of multi-objective programming (MOP) and RST. He presented a novel MOPP in a rough environment and called it a "rough multiple objective programming problem". He proposed a classification of such problems based on the location of roughness. He modeled and solved problems of the 1^{st} class in which roughness exists just in the feasible set. Also, some research articles studied the appearance of rough intervals in MOPPs. For instance, TOPSIS was extended by El-Feky and Abou-El-Enien

[9] to solve MOPPs with rough interval parameters. In addition, the authors of [10] proposed a new methodology for addressing unbalanced multi-objective fixed-charge transportation problems whose decision variables and coefficients are rough intervals. Furthermore, an algorithm is introduced for solving multi-objective fractional transportation problems with parameters represented as rough intervals in [11].

Until now, there is only one method proposed to find a compromise solution to a RMOPP with a rough decision set. This method is introduced in [4], and it is based on the weighted-sum method. Despite the simplicity of the weighted-sum method, it is challenging to determine the weights of objectives when their magnitudes differ [12]. Moreover, this method is inefficient when solving non-convex problems [12]. This inspires us to devise a new method for dealing with RMOPPs whose decision sets are rough. We select TOPSIS to be the basis of our proposed method. TOPSIS was chosen because it is a simple and straightforward method that simulates DM's rational thinking [13]. In this paper, TOPSIS is modified to handle a particular type of RMOPPs where roughness exists only in the decision set. TOPSIS is one of the major MOP methods. Hwang and Yoon first presented it in [14] as a two-reference-point-based method for solving multiple-attribute decision-making (MADM) problems. TOPSIS was later expanded to tackle MOPPs [15]. Due to conflict between objectives, there is no solution to optimize all objectives simultaneously. Therefore, TOPSIS searches for a solution that is both closer to the best solution and further away from the worst solution. TOPSIS considers two extreme opposite poles as the best and worst reference points, which are referred to as "positive ideal solution (PIS)" and "negative ideal solution (NIS)," respectively [16]. To be more specific, TOPSIS transforms the MOPP into a bi-objective programming problem whose objectives are the distance to the PIS and the distance to the NIS. Since these two objectives are usually conflicting, the concept of a fuzzy membership function is used to represent the satisfaction level for both objectives. Eventually, a max-min operation [17] is used to obtain the most satisfactory (compromise) feasible solution.

The main impacts of this paper are as follows:

1. We point out that when the feasible region is a rough set in a MOPP, the domain of the objective functions is either a set of simple elements from the fine universe or a set of equivalence classes from the coarse universe.
2. RST and granular computing have been used in modeling a RMOPP and characterizing its optimal sets.
3. Based on TOPSIS and RST, we propose a novel MOP algorithm for solving RMOPPs with a rough decision set and a search space composed of simple elements from the fine universe.
4. The validity of this algorithm is demonstrated by solving a RMOPP with it using different types of distance metrics. The results are compared to those of another algorithm.

The remainder of the paper is structured as follows. Section 1 goes over some essential concepts and notions of RST and GrC. In section 2, a model of a RMOPP with crisp functions and a rough decision set is created. Also, new concepts using RST and GrC are being developed along the lines of their crisp counterparts, such as "rough PIS", "rough NIS", "rough distance", and "rough compromise solution". In section 3, a modified TOPSIS is proposed to find the four optimal sets of compromise solutions to the problem above. Also, an illustrative example is provided to support the new method. Section 4 contains concluding remarks as well as some suggestions for future research.

2. ROUGH SET THEORY AND GRANULAR COMPUTING

Data of real-world problems may be very complex or may be vague. Hence, it is very complicated to solve such problems from a computational perspective. Therefore, Z. Pawlak proposed RST as a flexible mathematical means for representing vagueness and complexity [6]. The main notions of RST are the approximation space and the lower and upper approximations.

Let U denote a nonempty finite set of elements called a ground universe. Let $E \subseteq U \times U$, which is the mathematical representation of available information about elements of the universe, be an equivalence relation (*i.e.*, reflexive, symmetric, and transitive binary relation). Applying E on U partitions (divides) U into pairwise subsets $U/E = \{C_1, C_2, ..., C_m\}$ where $C_1, C_2, ..., C_m$ are the 'names' of the equivalence classes. The family U/E is called a quotient universe, and the pair $A = (U, E)$ is called an approximation space. That is, the equivalence relation interprets the universe as being a pair of a ground (fine) universe/space of elements and a quotient (coarse) universe/space of equivalence classes [18], [19].

The equivalence class containing an element $x \in U$ and all elements that have the same description as x in accordance with E is denoted by $[x]_E$ and is defined as $[x]_E = \{y \in U|xEy\}$. In other words, $[x]_E$, whose name belongs to the quotient universe, is a subset of the ground space (*i.e.*, $name([x]_E) \in U/E$) [20]. Thus, there are two representations of an equivalence class. An equivalence class can be expressed either as a set of elements in the fine universe or as a single granular element in the coarse universe.

In an approximation space $A = (U, E)$, an arbitrarily set $M \subseteq U$ is represented by its lower and upper approximations, M_L and M^U respectively [21]. The lower approximation contains all equivalence classes that are totally contained in the set, and the upper approximation contains all equivalence classes that overlap with the set. Both approximations can either be defined in terms of equivalence classes (equivalence class-oriented definition) or in terms of elements (elements-oriented definition) [22], as follows:

$$
\begin{aligned}
M_L &= \cup\{[x]_E \mid name([x]_E) \in U/E,\ [x]_E \subseteq M\} \\
&= \{x \mid x \in U,\ [x]_E \subseteq M\} \\
M^U &= \cup\{[x]_E \mid name([x]_E) \in U/E,\ [x]_E \cap M \neq \phi\} \\
&= \{x \mid x \in U,\ [x]_E \cap M \neq \phi\}
\end{aligned}
$$

Thus, $M_L \subseteq M \subseteq M^U$. The difference between the upper and the lower approximations is called the boundary region of M. It is denoted by M_{BN} and defined as $M_{BN} = M^U - M_L$. The set M is rough iff $M_{BN} \neq \phi$; otherwise it is crisp. More vagueness of a set means a larger boundary region of this set. An element $x \in U$ *surely belongs* to the set M iff $x \in M_L$. An element $x \in U$ *probably belongs* to the set M iff $x \in M^U$ [4], [23]. An element $x \in U$ *surely does not belong* to the set M iff $x \notin M^U$. An element $x \in U$ *may belong* to the set M iff $x \in M_{BN}$ [4].

Since M is characterized differently under different levels of granulation, the motive for presenting RST notions from GrC has emerged [24]. GrC is a general term for a multidisciplinary field that involves theories, methodologies, techniques, and tools that make use of granules in complex problem solving [3], [25]. There are two operators linking different representations of M [26].

The first operator is called the 'zooming-in' operator. It enables us to expand the classes from coarse universe U/E into a subset of the fine universe U, and hence provide more details about these classes [27]. Let $ZI : 2^{U/E} \rightarrow 2^U$ be the 'zooming-in' operator. Then, the detailed view of equivalence class $C \in U/E$ is given by [26]:

$$ZI(C) = [x]_E \text{ where } C = name([x]_E)$$

By 'zooming-in' an arbitrary subset $H \subseteq U/E$, a unique subset $ZI(H) \subseteq U$ called the refinement of H is attained as follows [26]:

$$ZI(H) = \bigcup_{C \in H} \{ZI(C)\}$$

The second operator is called the 'zooming-out' operator. It enables us to coarsen elements from the fine universe U into some classes from the coarse universe U/E by ignoring certain details, which make indiscernible elements no longer discernible [27]. In contrast to 'zooming-in', 'zooming-out' on a subset of U may generate an imprecise representation in U/E. Let ZO be the 'zooming-out' operator. Then, the abstract view of a subset $M \subseteq U$ is defined by the 'zooming-out' of its lower and upper approximations as follows [26]:

$$ZO(M_L) \subseteq ZO(M) \subseteq ZO(M^U)$$

where

$$ZO(M_L) = \{C \in U/E \mid ZI(C) \subseteq M_L\}$$
$$ZO(M^U) = \{C \in U/E \mid ZI(C) \subseteq M^U\}$$

If $M \subset U$ is rough according to the approximation space A, then 'zooming-out' of its lower and upper approximations are not equal (*i.e.*, $ZO(M_L) \subset ZO(M) \subset ZO(M^U)$). If $M \subseteq U$ is crisp according to the approximation space A, then 'zooming-out' of its lower and upper approximations are equal, and the abstract view of M in the coarse universe is given by:

$$ZO(M) = \{C \in U/E \mid ZI(C) \subseteq M\}$$

3. MULTIPLE OBJECTIVE PROGRAMMING WITH ROUGH DECISION SET

In this section, we model a RMOPP in which the decision set is imprecisely defined, and all the objectives are precisely defined over a subset of the fine universe.

Let $A = (U, E)$ be an approximation space generated by applying an equivalence relation E on a universal set U, and let $U/E = \{C_1, C_2, ..., C_m\}$ be a coarse universe established by E on U.

Mathematically, a RMOP with crisp objectives and a rough decision set can be represented as follows:

$$\begin{aligned} \max \quad & [f_1(\mathbf{x}), f_2((\mathbf{x}), ..., f_k((\mathbf{x})] \\ \text{s.t.} \quad & \mathbf{x} \in M \\ & M_{\scriptscriptstyle L} \subset M \subset M^{\scriptscriptstyle U} \\ & M_{\scriptscriptstyle L}, M^{\scriptscriptstyle U} \subseteq U \\ & k > 1 \end{aligned} \tag{1}$$

where $M \subset U \subseteq \mathbb{R}^n$ is the decision set of the problem which is roughly defined by its lower and upper approximations, $M_{\scriptscriptstyle L}$ and $M^{\scriptscriptstyle U}$ respectively. $\mathbf{x}$ is an n-dimensional decision variable vector. $f_l : U \to R$, $l = 1, 2, ..., k$ are k scalar crisp functions.

Before we begin presenting our new approach to solving problem (1), we must first state some definitions.

Definition 1 (Rough positive ideal solution). *The positive ideal solution (PIS) $F^* = (f_1^*, f_2^*, ..., f_k^*)$ is defined by its lower and upper bounds, $\underline{F^*} = (\underline{f_1^*}, \underline{f_2^*}, ..., \underline{f_k^*})$ and $\overline{F^*} = (\overline{f_1^*}, \overline{f_2^*}, ..., \overline{f_k^*})$ respectively, such that:*

$$\underline{f_l^*} = \max\{a_l, b_l\} \ \textit{and} \ \overline{f_l^*} = \max\{a_l, c_l\}$$

where (assuming the existence of the solution of the following crisp problems)

$$a_l = \max_{\mathbf{x}\in M_L} f_l(\mathbf{x}), b_l = \max_{C\in ZO(M_{BN})} \min_{\mathbf{x}\in ZI(C)} f_l(\mathbf{x}), \ \textit{and} \ c_l = \max_{\mathbf{x}\in M_{BN}} f_l(\mathbf{x}), l = 1, 2, ..., k.$$

$\underline{F^*} = (\underline{f_1^*}, \underline{f_2^*}, ..., \underline{f_k^*})$ are a set of lower bounds of individual positive ideal solutions, and $\overline{F^*} = (\overline{f_1^*}, \overline{f_2^*}, ..., \overline{f_k^*})$ are a set of upper bounds of individual positive ideal solutions.

Definition 2 (Rough negative ideal solution). *The negative ideal solution (NIS) $F^- = (f_1^-, f_2^-, ..., f_k^-)$ is defined by its lower and upper bounds, $\underline{F^-} = (\underline{f_1^-}, \underline{f_2^-}, ..., \underline{f_k^-})$ and $\overline{F^-} = (\overline{f_1^-}, \overline{f_2^-}, ..., \overline{f_k^-})$ respectively, such that:*

$$\underline{f_l^-} = \min\{a_l, b_l\} \ \textit{and} \ \overline{f_l^-} = \min\{a_l, c_l\}$$

where (assuming the existence of the solution of the following crisp problems)

$$a_l = \min_{\mathbf{x}\in M_L} f_l(\mathbf{x}), b_l = \min_{C\in ZO(M_{BN})} \max_{\mathbf{x}\in ZI(C)} f_l(\mathbf{x}),\ \textit{and}\ c_l = \min_{\mathbf{x}\in M_{BN}} f_l(\mathbf{x}), l = 1, 2, ..., k.$$

$\underline{F^-} = (\underline{f_1^-}, \underline{f_2^-}, ..., \underline{f_k^-})$ are a set of lower bounds of individual negative ideal solutions, and $\overline{F^-} = (\overline{f_1^-}, \overline{f_2^-}, ..., \overline{f_k^-})$ are a set of upper bounds of individual negative ideal solutions.

The main goal of TOPSIS is to find the feasible solution with the least distance from the PIS and with the greatest distance from the NIS. Therefrom, a formulation of a rough weighted normalized distance (d_p-metric) in a k-dimensional space is required.

Definition 3 (Rough distance to the PIS). *The distance between point* $\mathbf{x}$ *and the PIS is denoted by* $d_p^{PIS}(\mathbf{x})$*, and it is defined by its least and most probable distances to the PIS,* $(d_p^{PIS}(\mathbf{x}))_L$ *and* $(d_p^{PIS}(\mathbf{x}))^U$ *respectively, such that:*

$$\left(d_p^{PIS}(\mathbf{x})\right)_L = \left[\sum_{\substack{l=1\\ \overline{f_l^-}\le f_l(\mathbf{x})\le \underline{f_l^*}}}^{k} w_l^p \cdot \left(\frac{\underline{f_l^*} - f_l(\mathbf{x})}{\underline{f_l^*} - \overline{f_l^-}}\right)^p\right]^{\frac{1}{p}}$$

$$\left(d_p^{PIS}(\mathbf{x})\right)^U = \left[\sum_{l=1}^{k} w_l^p \cdot \left(\frac{\overline{f_l^*} - f_l(\mathbf{x})}{\overline{f_l^*} - \underline{f_l^-}}\right)^p\right]^{\frac{1}{p}}$$

The DM selects the values of $0 < w_l < 1,\ l = 1, 2, ..., k$ that represent the relative weights of objectives, as well as the value of $p \geq 1$ that represents the balancing factor between the group utility and maximal individual regret.

Definition 4 (Rough distance to the NIS). *The distance between point* $\mathbf{x}$ *and the NIS is denoted by* $d_p^{NIS}(\mathbf{x})$*, and it is defined by its least and most probable distances to the NIS,* $(d_p^{NIS}(\mathbf{x}))_L$ *and* $(d_p^{NIS}(\mathbf{x}))^U$ *respectively, such that:*

$$\left(d_p^{NIS}(\mathbf{x})\right)_L = \left[\sum_{\substack{l=1\\ \overline{f_l^-}\le f_l(\mathbf{x})\le \underline{f_l^*}}}^{k} w_l^p \cdot \left(\frac{f_l(\mathbf{x}) - \overline{f_l^-}}{\underline{f_l^*} - \overline{f_l^-}}\right)^p\right]^{\frac{1}{p}}$$

$$\left(d_p^{NIS}(\mathbf{x})\right)^U = \left[\sum_{l=1}^{k} w_l^p \cdot \left(\frac{f_l(\mathbf{x}) - \underline{f_l^-}}{\overline{f_l^*} - \underline{f_l^-}}\right)^p\right]^{\frac{1}{p}}$$

Definition 5 (Surely-compromise solution). *The solution* $\acute{\mathbf{x}}$ *is a* surely-compromise *solution iff* $\acute{\mathbf{x}}$ *has a smaller most probable distance to the PIS and, at the same time, a larger most probable distance to the NIS (see Figure 1).*

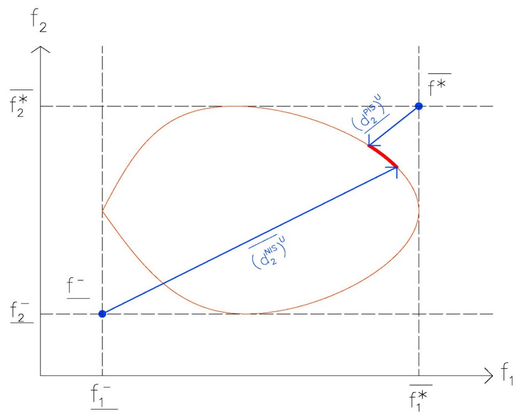

Figure 1: The surely-compromise solution exists in the thick curve, when p = 2

Definition 6 (Probably-compromise solution). *The solution* $\acute{\mathbf{x}}$ *is a* probably-compromise *solution iff* $\acute{\mathbf{x}}$ *has a smaller least probable distance to the PIS and, at the same time, a larger least probable distance to the NIS (see Figure 2).*

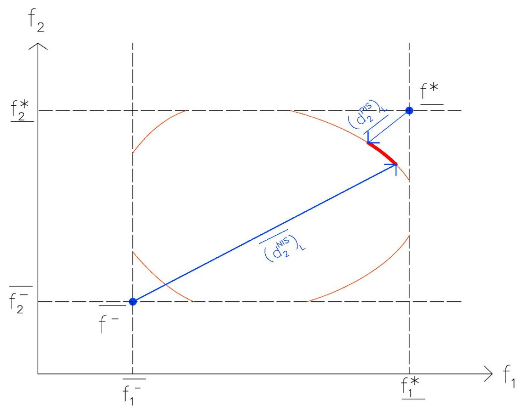

Figure 2: The probably-compromise solution exists in the thick curve, when p = 2

Definition 7 (Surely-feasible solution). *The solution* $\acute{\mathbf{x}}$ *is a* surely-feasible *solution iff* $\acute{\mathbf{x}} \in M_L$ *[8].*

Definition 8 (Probably-feasible solution). *The solution* $\acute{\mathbf{x}}$ *is a* probably-feasible *solution iff* $\acute{\mathbf{x}} \in M^U$ *[8].*

4. THE PROPOSED TOPSIS FOR MULTIPLE OBJECTIVE PROGRAMMING WITH A ROUGH DECISION SET

Suppose we have the following RMOPP:

$$\begin{aligned} \max/\min \quad & [f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_k(\mathbf{x})] \\ \text{s.t.} \quad & \mathbf{x} \in M \\ & M_L \subset M \subset M^U \\ & M_L, M^U \subseteq U \\ & k > 1 \end{aligned} \tag{2}$$

where
$f_i(\mathbf{x})$ =Benefit objective for maximization, $i \in I$.
$f_j(\mathbf{x})$ =Cost objective for minimization, $j \in J$.

Now, our new modified TOPSIS algorithm (see Figure 3) to generate a compromise solution to problem (2) is presented.

The algorithm

Step (1): Determine the lower bound of the PIS $\underline{F^*} = (\underline{f_1^*}, \underline{f_2^*}, ..., \underline{f_k^*})$, where:

$$\underline{f_i^*} = \max\{\max_{\mathbf{x}\in M_L} f_i(\mathbf{x}), \max_{C\in ZO(M_{BN})} \min_{\mathbf{x}\in ZI(C)} f_i(\mathbf{x})\},\ \forall i \in I$$

$$\underline{f_j^*} = \min_{\mathbf{x}\in M^U} f_j(\mathbf{x}),\ \forall j \in J$$

Step (2): Determine the upper bound of the PIS $\overline{F^*} = (\overline{f_1^*}, \overline{f_2^*}, ..., \overline{f_k^*})$, where:

$$\overline{f_i^*} = \max_{\mathbf{x}\in M^U} f_i(\mathbf{x}),\ \forall i \in I$$

$$\overline{f_j^*} = \min\{\min_{\mathbf{x}\in M_L} f_j(\mathbf{x}), \min_{C\in ZO(M_{BN})} \max_{\mathbf{x}\in ZI(C)} f_j(\mathbf{x})\},\ \forall j \in J$$

Step (3): Determine the lower bound of the NIS $\underline{F^-} = (\underline{f_1^-}, \underline{f_2^-}, ..., \underline{f_k^-})$, where:

$$\underline{f_i^-} = \min_{\mathbf{x}\in M^U} f_i(\mathbf{x}),\ \forall i \in I$$

$$\underline{f_j^-} = \max\{\max_{\mathbf{x}\in M_L} f_j(\mathbf{x}), \max_{C\in ZO(M_{BN})} \min_{\mathbf{x}\in ZI(C)} f_j(\mathbf{x})\},\ \forall j \in J$$

Step (4): Determine the upper bound of the NIS $\overline{F^-} = (\overline{f_1^-}, \overline{f_2^-}, ..., \overline{f_k^-})$, where:

$$\overline{f_i^-} = \min\{\min_{\mathbf{x}\in M_L} f_i(\mathbf{x}), \min_{C\in ZO(M_{BN})} \max_{\mathbf{x}\in ZI(C)} f_i(\mathbf{x})\},\ \forall i \in I$$

$$\overline{f_j^-} = \max_{\mathbf{x}\in M^U} f_j(\mathbf{x}),\ \forall j \in J$$

Step (5): Construct the least and most probable distances to the PIS, $(d_p^{PIS}(\mathbf{x}))_L$ and $(d_p^{PIS}(\mathbf{x}))^U$ respectively. They can be expressed as:

$$\left(d_p^{PIS}(\mathbf{x})\right)_L = \left[\sum_{\substack{i\in I \\ \overline{f_i^-}\le f_i(\mathbf{x})\le \underline{f_i^*}}} w_i^p \cdot \left(\frac{\underline{f_i^*} - f_i(\mathbf{x})}{\underline{f_i^*} - \overline{f_i^-}}\right)^p + \sum_{\substack{j\in J \\ \overline{f_j^*}\le f_j(\mathbf{x})\le \underline{f_j^-}(\mathbf{x})}} w_j^p \cdot \left(\frac{f_j(\mathbf{x}) - \overline{f_j^*}}{\underline{f_j^-} - \overline{f_j^*}}\right)^p\right]^{\frac{1}{p}}$$

$$\left(d_p^{PIS}(\mathbf{x})\right)^U = \left[\sum_{i\in I} w_i^p \cdot \left(\frac{\overline{f_i^*} - f_i(\mathbf{x})}{\overline{f_i^*} - \underline{f_i^-}}\right)^p + \sum_{j\in J} w_j^p \cdot \left(\frac{f_j(\mathbf{x}) - \underline{f_j^*}}{\overline{f_j^-} - \underline{f_j^*}}\right)^p\right]^{\frac{1}{p}}$$

$$where\, p = 1, 2, ..., \infty, \sum_{l=1}^{k} w_l = 1.$$

Step (6): Construct the least and most probable distances to the NIS, $(d_p^{NIS}(\mathbf{x}))_L$ and $(d_p^{NIS}(\mathbf{x}))^U$ respectively. They can be expressed as:

$$\left(d_p^{NIS}(\mathbf{x})\right)_L = \left[\sum_{\substack{i\in I \\ \overline{f_i^-}\le f_i(\mathbf{x})\le \underline{f_i^*}}} w_i^p \cdot \left(\frac{f_i(\mathbf{x}) - \overline{f_i^-}}{\underline{f_i^*} - \overline{f_i^-}}\right)^p + \sum_{\substack{j\in J \\ \overline{f_j^*}\le f_j(\mathbf{x})\le \underline{f_j^-}(\mathbf{x})}} w_j^p \cdot \left(\frac{\underline{f_j^-} - f_j(\mathbf{x})}{\underline{f_j^-} - \overline{f_j^*}}\right)^p\right]^{\frac{1}{p}}$$

$$\left(d_p^{NIS}(\mathbf{x})\right)^U = \left[\sum_{i\in I} w_i^p \cdot \left(\frac{f_i(\mathbf{x}) - \underline{f_i^-}}{\overline{f_i^*} - \underline{f_i^-}}\right)^p + \sum_{j\in J} w_j^p \cdot \left(\frac{\overline{f_j^-} - f_j(\mathbf{x})}{\overline{f_j^-} - \underline{f_j^*}}\right)^p\right]^{\frac{1}{p}}$$

$$where\, p = 1, 2, ..., \infty, \sum_{l=1}^{k} w_l = 1.$$

Step (7): Determine the lower and upper bounds of both the least and most probable distances to the PIS, $\underline{(d_p^{PIS})_L}$, $\underline{(d_p^{PIS})^U}$, $\overline{(d_p^{PIS})_L}$, and $\overline{(d_p^{PIS})^U}$ respectively, and determine the lower and upper bounds of both the least and most probable distances to the NIS, $\underline{(d_p^{NIS})_L}$, $\underline{(d_p^{NIS})^U}$, $\overline{(d_p^{NIS})_L}$, and $\overline{(d_p^{NIS})^U}$ respectively. They can be expressed as:

$$\begin{aligned}
\underline{(d_p^{PIS})_L} = \min \quad & (d_p^{PIS}(\mathbf{x}))_L \\
\text{s.t.} \quad & \mathbf{x} \in M^U \\
& \overline{f_i^-} \le f_i(\mathbf{x}) \le \underline{f_i^*},\ i \in I \\
& \overline{f_j^*} \le f_j(\mathbf{x}) \le \underline{f_j^-},\ j \in J
\end{aligned}$$

$\underline{(d_p^{NIS})_L} = (d_p^{NIS}(\mathbf{x}_{PL}))_L$; $\mathbf{x}_{PL}$ is the solution to the above problem

$$\overline{(d_p^{NIS})_L} = \max \quad (d_p^{NIS}(\mathbf{x}))_L$$
$$\text{s.t.} \quad \mathbf{x} \in M^U$$
$$\overline{f_i^-} \leq f_i(\mathbf{x}) \leq \underline{f_i^*},\ i \in I$$
$$\overline{f_j^*} \leq f_j(\mathbf{x}) \leq \underline{f_j^-},\ j \in J$$

$\overline{(d_p^{PIS})_L} = (d_p^{PIS}(\mathbf{x}_{NL}))_L$; $\mathbf{x}_{NL}$ is the solution to the above problem

$$\underline{(d_p^{PIS})^U} = \min \quad (d_p^{PIS}(\mathbf{x}))^U$$
$$\text{s.t.} \quad \mathbf{x} \in M^U$$

$\underline{(d_p^{NIS})^U} = (d_p^{NIS}(\mathbf{x}_{PU}))^U$; $\mathbf{x}_{PU}$ is the solution to the above problem

$$\overline{(d_p^{NIS})^U} = \max \quad (d_p^{NIS}(\mathbf{x}))^U$$
$$\text{s.t.} \quad \mathbf{x} \in M^U$$

$\overline{(d_p^{PIS})^U} = (d_p^{PIS}(\mathbf{x}_{NU}))^U$; $\mathbf{x}_{NU}$ is the solution to the above problem

Step (8): Construct fuzzy membership functions representing the individual optima of the most probable distance to the PIS and the most probable distance to the NIS, μ_{PU} and μ_{NU} respectively. They can be expressed as:

$$\mu_{PU}(\mathbf{x}) = \begin{cases} 1 & (d_p^{PIS}(\mathbf{x}))^U < \underline{(d_p^{PIS})^U} \\ \frac{(d_p^{PIS}(\mathbf{x}))^U - \overline{(d_p^{PIS})^U}}{\underline{(d_p^{PIS})^U} - \overline{(d_p^{PIS})^U}} & \underline{(d_p^{PIS})^U} \leq (d_p^{PIS}(\mathbf{x}))^U \leq \overline{(d_p^{PIS})^U} \\ 0 & (d_p^{PIS}(\mathbf{x}))^U > \overline{(d_p^{PIS})^U} \end{cases}$$

$$\mu_{NU}(\mathbf{x}) = \begin{cases} 1 & (d_p^{NIS}(\mathbf{x}))^U > \overline{(d_p^{NIS})^U} \\ \frac{(d_p^{NIS}(\mathbf{x}))^U - \underline{(d_p^{NIS})^U}}{\overline{(d_p^{NIS})^U} - \underline{(d_p^{NIS})^U}} & \underline{(d_p^{NIS})^U} \leq (d_p^{NIS}(\mathbf{x}))^U \leq \overline{(d_p^{NIS})^U} \\ 0 & (d_p^{NIS}(\mathbf{x}))^U < \underline{(d_p^{NIS})^U} \end{cases}$$

Step (9): Solve the following problem to obtain the set of surely-feasible, surely-compromise solutions F_sC_s and the set of probably-feasible, surely-compromise solutions F_pC_s:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{PU}(\mathbf{x}) \geq \alpha \\ & \mu_{NU}(\mathbf{x}) \geq \alpha \\ & \mathbf{x} \in M^U \\ & \alpha \in [0,1] \end{aligned} \tag{3}$$

where α is the satisfactory level for both minimizing the most probable distance to the PIS and maximizing the most probable distance to the NIS.

$$F_sC_s = \{\mathbf{x} \in M_L \mid \mathbf{x} \text{ solves problem (3)}\}$$
$$F_pC_s = \{\mathbf{x} \in M^U \mid \mathbf{x} \text{ solves problem (3)}\}$$

Step (10): Construct fuzzy membership functions representing the individual optima of the least probable distance to the PIS and the least probable distance to the NIS, μ_{PL} and μ_{NL} respectively:

$$\mu_{PL}(\mathbf{x}) = \begin{cases} 1 & (d_p^{PIS}(\mathbf{x}))_L < \underline{(d_p^{PIS})_L} \\ \frac{(d_p^{PIS}(\mathbf{x}))_L - \overline{(d_p^{PIS})_L}}{\underline{(d_p^{PIS})_L} - \overline{(d_p^{PIS})_L}} & \underline{(d_p^{PIS})_L} \leq (d_p^{PIS}(\mathbf{x}))_L \leq \overline{(d_p^{PIS})_L} \\ 0 & (d_p^{PIS}(\mathbf{x}))_L > \overline{(d_p^{PIS})_L} \end{cases}$$

$$\mu_{NL}(\mathbf{x}) = \begin{cases} 1 & (d_p^{NIS}(\mathbf{x}))_L > \overline{(d_p^{NIS})_L} \\ \frac{(d_p^{NIS}(\mathbf{x}))_L - \underline{(d_p^{NIS})_L}}{\overline{(d_p^{NIS})_L} - \underline{(d_p^{NIS})_L}} & \underline{(d_p^{NIS})_L} \leq (d_p^{NIS}(\mathbf{x}))_L \leq \overline{(d_p^{NIS})_L} \\ 0 & (d_p^{NIS}(\mathbf{x}))_L < \underline{(d_p^{NIS})_L} \end{cases}$$

Step (11): Solve the following problem to obtain the set of surely-feasible, probably-compromise solutions F_sC_p and the set of probably-feasible, probably-compromise solutions F_pC_p:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{PL}(\mathbf{x}) \geq \alpha \\ & \mu_{NL}(\mathbf{x}) \geq \alpha \\ & \mathbf{x} \in M^U \\ & \overline{f_i^-} \leq f_i(\mathbf{x}) \leq \underline{f_i^*}, i \in I \\ & \overline{f_j^*} \leq f_j(\mathbf{x}) \leq \underline{f_j^-}, j \in J \\ & \alpha \in [0,1] \end{aligned} \tag{4}$$

where α is the satisfactory level for both minimizing the least probable distance to the PIS and maximizing the least probable distance to the NIS.

$$F_sC_p = \{\mathbf{x} \in M_L \,|\mathbf{x} \text{ solves problem (4)}\}$$
$$F_pC_p = \{\mathbf{x} \in M^U \mid \mathbf{x} \text{ solves problem (4)}\}$$

The purpose of using RST in the algorithm is to only describe and handle the roughness that appears in the problem. Hence, the limitations of our proposed algorithm are the same limitations of the conventional TOPSIS. Our algorithm assumes that the coefficients are described as exact values. Unfortunately, in some actual problems, the coefficients are provided by the DM as interval numbers, fuzzy numbers, hesitant fuzzy sets, etc. Furthermore, our algorithm is inappropriate for situations in which the DM is concerned only with finding a compromise solution that is near the ideal solution, whatever the distance between this compromise solution and the anti-ideal solution [13].

Let us consider the next numerical example to illustrate our proposed algorithm.

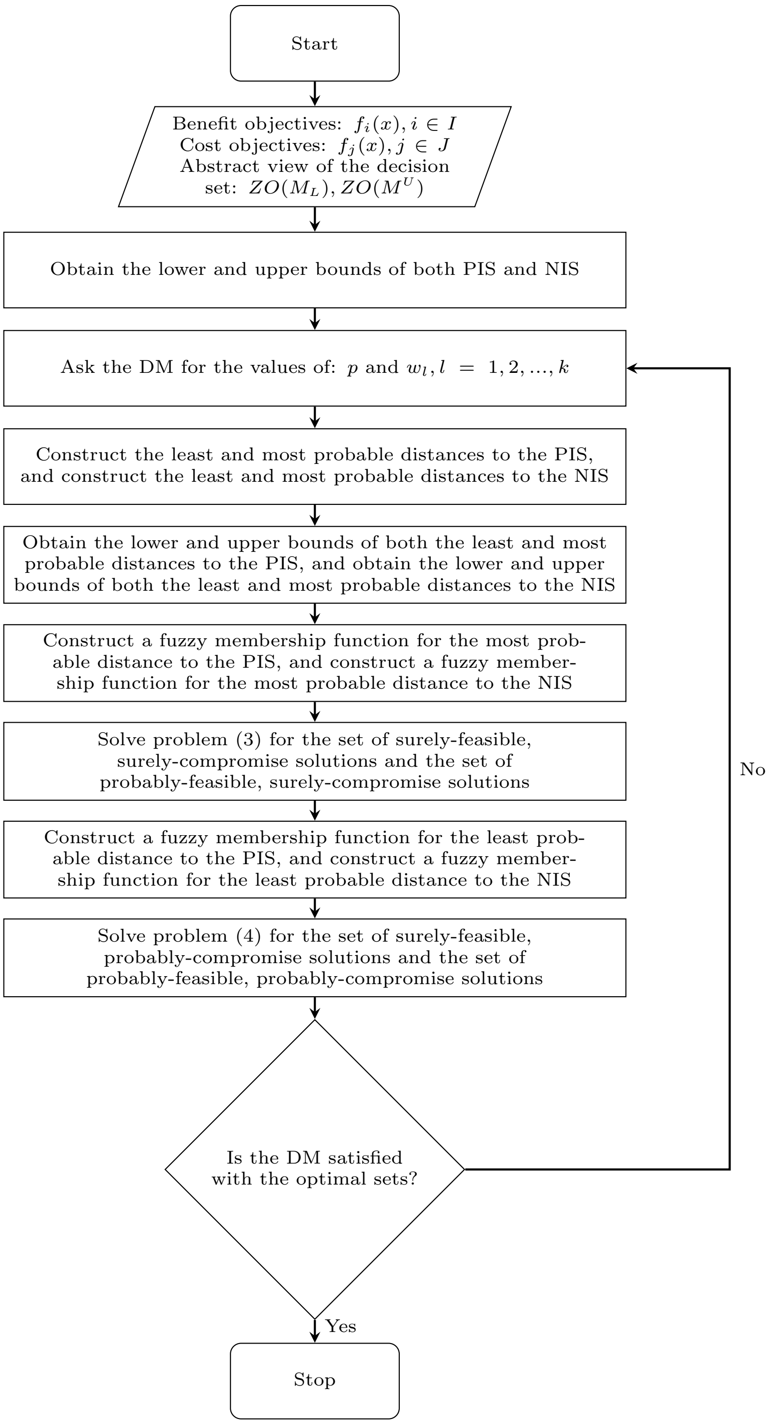

Figure 3: Flowchart of the proposed TOPSIS to solve a MOPP with rough decision set

Example 1. Suppose U is a universe defined by:
$U = \{\mathbf{x} \in R^2 \mid x_1 \in [-5, 5],\ x_2 \in [-5, 5]\}$ where $\mathbf{x} = (x_1, x_2)$.
Let E be an equivalence relation that generates a partition $U/E = \{C_1, C_2, C_3, C_4, C_5\}$ such that:
$ZI(C_1) = \{\mathbf{x} \in U \mid -5x_1 + 4x_2 \leq 16,\ x_1 \leq 0,\ x_2 \geq 0\}$
$ZI(C_2) = \{\mathbf{x} \in U \mid -5x_1 + 4x_2 \leq 16,\ x_1 + 4x_2 \geq -8,\ x_1 \leq 0,\ x_2 < 0\}$
$ZI(C_3) = \{\mathbf{x} \in U \mid x_1 > 0,\ x_1 \leq 2,\ x_2 \geq 0,\ x_2 \leq 4\}$
$ZI(C_4) = \{\mathbf{x} \in U \mid x_1 > 0,\ x_1 \leq 2,\ x_2 \geq -2,\ x_2 < 0\}$
$ZI(C_5) = \{\mathbf{x} \in U \mid \mathbf{x} \notin \bigcup_{n=1}^{4}\{ZI(C_n)\}\}$

Consider the following RMOPP:

$$\begin{aligned} \max \quad & f_1(\mathbf{x}) = 2x_1 - 3x_2 \\ \min \quad & f_2(\mathbf{x}) = x_1 + 5x_2 \\ \text{s.t.} \quad & \mathbf{x} \in M \\ & M_L \subset M \subset M^U \\ & ZO(M_L) = \{C_1, C_2\} \\ & ZO(M^U) = \{C_1, C_2, C_3, C_4\} \end{aligned} \tag{5}$$

Solution

The boundary region of the feasible set in the coarse universe is:

$$ZO(M_{BN}) = ZO(M^U) - ZO(M_L) = \{C_3, C_4\}$$

Step (1): The lower bound of the PIS is $\underline{F^*} = (\underline{f_1^*}, \underline{f_2^*})$, where:

$$\underline{f_1^*} = \max\{\max_{\mathbf{x}\in M_L} f_1(\mathbf{x}), \max_{C\in ZO(M_{BN})} \min_{\mathbf{x}\in ZI(C)} f_1(\mathbf{x})\} = \max\{6, 0^+\} = 6$$

$$\underline{f_2^*} = \min_{\mathbf{x}\in M^U} f_2(\mathbf{x}) = -10$$

Step (2): The upper bound of the PIS is $\overline{F^*} = (\overline{f_1^*}, \overline{f_2^*})$, where:

$$\overline{f_1^*} = \max_{\mathbf{x}\in M^U} f_1(\mathbf{x}) = 10$$

$$\overline{f_2^*} = \min\{\min_{\mathbf{x}\in M_L} f_2(\mathbf{x}), \min_{C\in ZO(M_{BN})} \max_{\mathbf{x}\in ZI(C)} f_2(\mathbf{x})\} = \min\{-10, 4^-\} = -10$$

Step (3): The lower bound of the NIS is $\underline{F^-} = (\underline{f_1^-}, \underline{f_2^-})$, where:

$$\underline{f_1^-} = \min_{\mathbf{x}\in M^U} f_1(\mathbf{x}) = -12$$

$$\underline{f_2^-} = \max\{\max_{\mathbf{x}\in M_L} f_2(\mathbf{x}), \max_{C\in ZO(M_{BN})} \min_{\mathbf{x}\in ZI(C)} f_2(\mathbf{x})\} = \max\{20, 0^+\} = 20$$

Step (4): The upper bound of the NIS is $\overline{F^-} = (\overline{f_1^-}, \overline{f_2^-})$, where:

$$\overline{f_1^-} = \min\{\min_{\mathbf{x}\in M_L} f_1(\mathbf{x}), \min_{C\in ZO(M_{BN})} \max_{\mathbf{x}\in ZI(C)} f_1(\mathbf{x})\} = \min\{-12, 4^-\} = -12$$

$$\overline{f_2^-} = \max_{\mathbf{x}\in M^U} f_2(\mathbf{x}) = 22$$

Remark 1. *If $r \in \mathbb{R}$, then $r^- = r - \epsilon$ and $r^+ = r + \epsilon$ where $\epsilon > 0, \epsilon \simeq 0$. The value of ϵ is chosen by the DM.*

Step (5): Let us assume that $w_1 = w_2 = 0.5$ and $p = 1$. The least probable distance to the PIS is:

$$\begin{aligned}(d_1^{PIS}(\mathbf{x}))_L &= 0.5\left(\frac{6 - 2x_1 + 3x_2}{6 - (-12)}\right) + 0.5\left(\frac{x_1 + 5x_2 - (-10)}{20 - (-10)}\right)\\ &= \frac{1}{3} - \frac{7}{180}x_1 + \frac{1}{6}x_2\end{aligned}$$

The most probable distance to the PIS is:

$$\begin{aligned}(d_1^{PIS}(\mathbf{x}))^U &= 0.5\left(\frac{10 - 2x_1 + 3x_2}{10 - (-12)}\right) + 0.5\left(\frac{x_1 + 5x_2 - (-10)}{22 - (-10)}\right)\\ &= \frac{135}{352} - \frac{21}{704}x_1 + \frac{103}{704}x_2\end{aligned}$$

Step (6): The least probable distance to the NIS is:

$$\begin{aligned}(d_1^{NIS}(\mathbf{x}))_L &= 0.5\left(\frac{2x_1 - 3x_2 - (-12)}{6 - (-12)}\right) + 0.5\left(\frac{20 - x_1 - 5x_2}{20 - (-10)}\right)\\ &= \frac{2}{3} + \frac{7}{180}x_1 - \frac{1}{6}x_2\end{aligned}$$

The most probable distance to the NIS is:

$$\begin{aligned}(d_1^{NIS}(\mathbf{x}))^U &= 0.5\left(\frac{2x_1 - 3x_2 - (-12)}{10 - (-12)}\right) + 0.5\left(\frac{22 - x_1 - 5x_2}{22 - (-10)}\right)\\ &= \frac{217}{352} + \frac{21}{704}x_1 - \frac{103}{704}x_2\end{aligned}$$

Step (7): The lower bound of the least probable distance to the PIS is:

$$\begin{aligned}\underline{(d_1^{PIS})_L} = \min \quad & (d_1^{PIS}(\mathbf{x}))_L\\ \text{s.t.} \quad & \mathbf{x} \in M^U\\ & -12 \leq f_1(\mathbf{x}) \leq 6\\ & -10 \leq f_2(\mathbf{x}) \leq 20\end{aligned}$$

$$\underline{(d_1^{PIS})_L} = 0 \quad \text{at} \quad \mathbf{x}_{PL} = (0, -2)$$

The lower bound of the least probable distance to the NIS is:

$$\underline{(d_1^{NIS})_L} = (d_1^{NIS}(0, -2))_L = 1$$

The upper bound of the least probable distance to the NIS is:

$$\begin{aligned}\overline{(d_1^{NIS})_L} = \max \quad & (d_1^{NIS}(\mathbf{x}))_L\\ \text{s.t.} \quad & \mathbf{x} \in M^U\\ & -12 \leq f_1(\mathbf{x}) \leq 6\\ & -10 \leq f_2(\mathbf{x}) \leq 20\end{aligned}$$

$$\overline{(d_1^{NIS})_L} = 1 \quad \text{at} \quad \mathbf{x}_{NL} = (0, -2)$$

The upper bound of the least probable distance to the PIS is:

$$\overline{(d_1^{PIS})_L} = (d_1^{PIS}(0, -2))_L = 0$$

The lower bound of the most probable distance to the PIS is:

$$\begin{aligned} \underline{(d_1^{PIS})^U} = \min \quad & (d_1^{PIS}(\mathbf{x}))^U \\ \text{s.t.} \quad & \mathbf{x} \in M^U \end{aligned}$$

$$\underline{(d_1^{PIS})^U} = \frac{1}{32} \quad \text{at} \quad \mathbf{x}_{PU} = (2, -2)$$

The lower bound of the most probable distance to the NIS is:

$$\underline{(d_1^{NIS})^U} = (d_1^{NIS}(2, -2))^U = \frac{31}{32}$$

The upper bound of the most probable distance to the NIS is:

$$\begin{aligned} \overline{(d_1^{NIS})^U} = \max \quad & (d_1^{NIS}(\mathbf{x}))^U \\ \text{s.t.} \quad & \mathbf{x} \in M^U \end{aligned}$$

$$\overline{(d_1^{NIS})^U} = \frac{31}{32} \quad \text{at} \quad \mathbf{x}_{NU} = (2, -2)$$

The upper bound of the most probable distance to the PIS is:

$$\overline{(d_1^{PIS})^U} = (d_1^{PIS}(2, -2))^U = \frac{1}{32}$$

Step (8):

$$\overline{(d_1)^U} = (\overline{(d_1^{PIS})^U}, \overline{(d_1^{NIS})^U}) = (\frac{1}{32}, \frac{31}{32})$$
$$\underline{(d_1)^U} = (\underline{(d_1^{PIS})^U}, \underline{(d_1^{NIS})^U}) = (\frac{1}{32}, \frac{31}{32})$$

It is not necessary to create the fuzzy membership functions μ_{PU} and μ_{NU} because $\underline{(d_1)^U} = \overline{(d_1)^U}$.

Step (9): Consider the following problem:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{PU}(\mathbf{x}) \geq \alpha \\ & \mu_{NU}(\mathbf{x}) \geq \alpha \\ & \mathbf{x} \in M^U \\ & \alpha \in [0, 1] \end{aligned} \tag{6}$$

Since $(d_1^{PIS}(\mathbf{x}))^U$ and $(d_1^{NIS}(\mathbf{x}))^U$ are not incompatible, the solution to problem (6) is $\mathbf{x}_{PU} = \mathbf{x}_{NU} = (2, -2)$ with the highest level of satisfaction (*i.e.*, $\alpha = 1$).

The set of surely-feasible, surely-compromise solutions is:

$$F_sC_s = \{\mathbf{x} \in M_L \mid \mathbf{x}\,\text{solves problem (6)}\} = \phi$$

The set of probably-feasible, surely-compromise solutions is:

$$F_pC_s = \{\mathbf{x} \in M^U \mid \mathbf{x}\,\text{solves problem (6)}\} = \{(2, -2)\}$$

Step (10):

$$\overline{(d_1)_L} = (\overline{(d_1^{PIS})_L}, \overline{(d_1^{NIS})_L}) = (\frac{1}{32}, \frac{31}{32})$$
$$\underline{(d_1)_L} = (\underline{(d_1^{PIS})_L}, \underline{(d_1^{NIS})_L}) = (\frac{1}{32}, \frac{31}{32})$$

It is not necessary to create the fuzzy membership functions μ_{PL} and μ_{NL} because $\underline{(d_1)_L} = \overline{(d_1)_L}$.

Step (11): Consider the following problem:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{PL}(\mathbf{x}) \geq \alpha \\ & \mu_{NL}(\mathbf{x}) \geq \alpha \\ & \mathbf{x} \in M^U \\ & -12 \leq f_1(\mathbf{x}) \leq 6 \\ & -10 \leq f_2(\mathbf{x}) \leq 20 \\ & \alpha \in [0, 1] \end{aligned} \tag{7}$$

Since $(d_1^{PIS}(\mathbf{x}))_L$ and $(d_1^{NIS}(\mathbf{x}))_L$ are not incompatible, the solution to problem (7) is $\mathbf{x}_{PL} = \mathbf{x}_{NL} = (0, -2)$ with the highest level of satisfaction (*i.e.*, $\alpha = 1$).

The set of surely-feasible, probably-compromise solutions is:

$$F_sC_p = \{\mathbf{x} \in M_L \,|\mathbf{x}\,\text{solves problem (7)}\} = \{(0, -2)\}$$

The set of probably-feasible, probably-compromise solutions is:

$$F_pC_p = \{\mathbf{x} \in M^U \mid \mathbf{x}\,\text{solves problem (7)}\} = \{(0, -2)\}$$

It should be noted that when $p = 1$, the same outcome is obtained whether the weights are equal or unequal. Considering the objective representation space, the slopes of the lines $(d_1^{PIS}(\mathbf{x}))^U$ and $(d_1^{NIS}(\mathbf{x}))^U$ are equal. Moreover, the slopes of the lines $(d_1^{PIS}(\mathbf{x}))_L$ and $(d_1^{NIS}(\mathbf{x}))_L$ are equal. Therefore, the solution to min $(d_1^{PIS}(\mathbf{x}))^U$ is the same as the solution to max $(d_1^{NIS}(\mathbf{x}))^U$, and the solution to min $(d_1^{PIS}(\mathbf{x}))_L$ is the same as the solution to max $(d_1^{NIS}(\mathbf{x}))_L$. The surely-compromise solution and the probably-compromise solution are always achieved with a satisfactory degree equal to 1 when $p = 1$.

To validate our approach, we solve the same problem using the method described in [4]. Both methods produce the same solution.

Assume that the DM values both objectives equally and prefers the Euclidean distance. Then, the least and the most probable distances to the PIS and the NIS are:

$$\left(d_2^{PIS}(\mathbf{x})\right)_L = \sqrt{0.5^2\left(\frac{6-2x_1+3x_2}{6-(-12)}\right)^2 + 0.5^2\left(\frac{x_1+5x_2-(-10)}{20-(-10)}\right)^2}$$

$$\left(d_2^{PIS}(\mathbf{x})\right)^U = \sqrt{0.5^2\left(\frac{10-2x_1+3x_2}{10-(-12)}\right)^2 + 0.5^2\left(\frac{x_1+5x_2-(-10)}{22-(-10)}\right)^2}$$

$$\left(d_2^{NIS}(\mathbf{x})\right)_L = \sqrt{0.5^2\left(\frac{2x_1-3x_2-(-12)}{6-(-12)}\right)^2 + 0.5^2\left(\frac{20-x_1-5x_2}{20-(-10)}\right)^2}$$

$$\left(d_2^{NIS}(\mathbf{x})\right)^U = \sqrt{0.5^2\left(\frac{2x_1-3x_2-(-12)}{10-(-12)}\right)^2 + 0.5^2\left(\frac{22-x_1-5x_2}{22-(-10)}\right)^2}$$

The lower bound of the least probable distance to the PIS is:

$$\begin{aligned} \underline{(d_2^{PIS})_L} = \min \quad & (d_2^{PIS}(\mathbf{x}))_L \\ \text{s.t.} \quad & \mathbf{x} \in M^U \\ & -12 \le f_1(\mathbf{x}) \le 6 \\ & -10 \le f_2(\mathbf{x}) \le 20 \end{aligned}$$

$$\underline{(d_2^{PIS})_L} = \frac{1}{12} \quad \text{at} \quad \mathbf{x}_{PL} = (0, -2)$$

The lower bound of the least probable distance to the NIS is:

$$\underline{(d_2^{NIS})_L} = (d_2^{NIS}(0,-2))_L = \frac{1}{\sqrt{2}}$$

The upper bound of the least probable distance to the NIS is:

$$\begin{aligned} \overline{(d_2^{NIS})_L} = \max \quad & (d_2^{NIS}(\mathbf{x}))_L \\ \text{s.t.} \quad & \mathbf{x} \in M^U \\ & -12 \le f_1(\mathbf{x}) \le 6 \\ & -10 \le f_2(\mathbf{x}) \le 20 \end{aligned}$$

$$\overline{(d_2^{NIS})_L} = \frac{1}{\sqrt{2}} \quad \text{at} \quad \mathbf{x}_{NL} = (0, -2)$$

The upper bound of the least probable distance to the PIS is:

$$\overline{(d_2^{PIS})_L} = (d_2^{PIS}(0,-2))_L = \frac{1}{12}$$

The lower bound of the most probable distance to the PIS is:

$$\underline{(d_2^{PIS})^U} = \min \quad (d_2^{PIS}(\mathbf{x}))^U \\ \text{s.t.} \quad \mathbf{x} \in M^U$$

$$\underline{(d_2^{PIS})^U} = 0.029553 \quad \text{at} \quad \mathbf{x}_{PU} = (1.79, -2)$$

The lower bound of the most probable distance to the NIS is:

$$\underline{(d_2^{NIS})^U} = (d_2^{NIS}(1.79, -2))^U = 0.680705$$

The upper bound of the most probable distance to the NIS is:

$$\overline{(d_2^{NIS})^U} = \max \quad (d_2^{NIS}(\mathbf{x}))^U \\ \text{s.t.} \quad \mathbf{x} \in M^U$$

$$\overline{(d_2^{NIS})^U} = \frac{\sqrt{481}}{32} \quad \text{at} \quad \mathbf{x}_{NU} = (2, -2)$$

The upper bound of the most probable distance to the PIS is:

$$\overline{(d_2^{PIS})^U} = (d_2^{PIS}(2, -2))^U = \frac{1}{32}$$

$$\overline{(d_2)^U} = (\overline{(d_2^{PIS})^U}, \overline{(d_2^{NIS})^U}) = (\frac{1}{32}, \frac{\sqrt{481}}{32})$$
$$\underline{(d_2)^U} = (\underline{(d_2^{PIS})^U}, \underline{(d_2^{NIS})^U}) = (0.029553, 0.680705)$$

It is necessary to create the fuzzy membership functions μ_{PU} and μ_{NU} because $\underline{(d_2)^U} \neq \overline{(d_2)^U}$.

$$\mu_{PU}(\mathbf{x}) = \begin{cases} 1 & (d_2^{PIS}(\mathbf{x}))^U < 0.029553 \\ \frac{(d_2^{PIS}(\mathbf{x}))^U - \frac{1}{32}}{0.029553 - \frac{1}{32}} & 0.029553 \leq (d_2^{PIS}(\mathbf{x}))^U \leq \frac{1}{32} \\ 0 & (d_2^{PIS}(\mathbf{x}))^U > \frac{1}{32} \end{cases}$$

$$\mu_{NU}(\mathbf{x}) = \begin{cases} 1 & (d_2^{NIS}(\mathbf{x}))^U > \frac{\sqrt{481}}{32} \\ \frac{(d_2^{NIS}(\mathbf{x}))^U - 0.680705}{\frac{\sqrt{481}}{32} - 0.680705} & 0.680705 \leq (d_2^{NIS}(\mathbf{x}))^U \leq \frac{\sqrt{481}}{32} \\ 0 & (d_2^{NIS}(\mathbf{x}))^U < 0.680705 \end{cases}$$

Now, we will solve the following problem:

$$\begin{aligned} \max\quad & \alpha \\ \text{s.t.}\quad & \mu_{PU}(\mathbf{x}) \geq \alpha \\ & \mu_{NU}(\mathbf{x}) \geq \alpha \\ & -5x_1 + 4x_2 \leq 16 \\ & x_1 + 4x_2 \geq -8 \\ & x_1 \leq 2 \\ & -2 \leq x_2 \leq 4 \\ & \alpha \in [0,1] \end{aligned}$$

The solution to the above problem is $(1.888, -2)$ with the level of satisfaction $\alpha = 46.4\%$.
Thus, the set of surely-feasible, surely-compromise solutions is $F_sC_s = \phi$, and the set of probably-feasible, surely-compromise solutions is $F_pC_s = \{(1.888, -2)\}$.

Now, let us consider the least probable distances to the PIS and NIS.

$$\overline{(d_2)_L} = (\overline{(d_2^{PIS})_L}, \overline{(d_2^{NIS})_L}) = (\frac{1}{12}, \frac{1}{\sqrt{2}})$$

$$\underline{(d_2)_L} = (\underline{(d_2^{PIS})_L}, \underline{(d_2^{NIS})_L}) = (\frac{1}{12}, \frac{1}{\sqrt{2}})$$

It is not necessary to create the fuzzy membership functions μ_{PL} and μ_{NL} because $\underline{(d_2)_L} = \overline{(d_2)_L}$.

Consider the following problem:

$$\begin{aligned} \max\quad & \alpha \\ \text{s.t.}\quad & \mu_{PL}(\mathbf{x}) \geq \alpha \\ & \mu_{NL}(\mathbf{x}) \geq \alpha \\ & -5x_1 + 4x_2 \leq 16 \\ & x_1 + 4x_2 \geq -8 \\ & x_1 \leq 2 \\ & -2 \leq x_2 \leq 4 \\ & -12 \leq f_1(\mathbf{x}) \leq 6 \\ & -10 \leq f_2(\mathbf{x}) \leq 20 \\ & \alpha \in [0,1] \end{aligned}$$

Since $(d_2^{PIS}(\mathbf{x}))_L$ and $(d_2^{NIS}(\mathbf{x}))_L$ are not incompatible, the solution to the above problem is $\mathbf{x}_{PL} = \mathbf{x}_{NL} = (0, -2)$ with the highest level of satisfaction (*i.e.*, $\alpha = 1$). Thus, the set of surely-feasible, probably-compromise solutions is $F_sC_p = \{(0, -2)\}$, and the set of probably-feasible, probably-compromise solutions is $F_pC_p = \{(0, -2)\}$.

5. CONCLUSION

In this paper, TOPSIS is modified to solve RMOPPs in which all the objectives are crisp functions, the decision set is a rough set, and the search space is composed of simple points from the fine universe. The basic model, the required definitions, and the flowchart depicting the proposed algorithm were introduced. Through a numerical example, we indicated that the presence of roughness in the decision set results in four optimal sets covering all possible levels of feasibility and optimality of the solutions.

A number of issues will need to be addressed in future research. These issues are summarized as follows:

1. Future research should consider RMOPPs with a rough decision set and crisp objective functions whose domains are a set of equivalence classes from the coarse universe.
2. It will be fruitful to expand our study to include RMOPPs with rough objective functions.
3. Future work should apply methods rather than TOPSIS to solve RMOPPs.
4. More research is needed on topics such as stability and parametric analysis in RMOPPs.
5. Testing the proposed method with large real-world problems and developing computer code for this method will yield significant benefits.

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