Yugoslav Journal of Operations Research
xx (20xx), Number xx, xxx–xxx
DOI: https://doi.org/10.2298/YJOR231215009A

A NEW STANDARDIZATION-BASED RANKING METHOD FOR GENERALIZED TRAPEZOIDAL PICTURE FUZZY NUMBERS

Hande GÜNAY AKDEMIR
Department of Mathematics, Giresun University, Giresun, Türkiye
hande.akdemir@giresun.edu.tr

Semiha AYDIN
Institute of Science, Giresun University, Giresun, Türkiye
Ministry of Education, Giresun High School, Giresun, Türkiye
semiha1928@hotmail.com

Received: December 2023 / Accepted: March 2024

Abstract: A generalized fuzzy number (GFN), whose height is not necessarily 1, is used in situations when expert opinions are not completely reliable. This subnormality complicates operations based on the extension principle. Moreover, complexity is inherited in non-standard fuzzy numbers (FNs). This paper aims to present a unified approach for comparing generalized and trapezoidal types of FNs, intuitionistic FNs (IFNs), and picture FNs (PFNs). If some of the hesitation, neutrality, and refusal are assumed to be resolved, then the uncertainty is reduced while making a non-standard FN standardized. The method uses the weighted average membership function (WAMF) to standardize generalized IFNs (GIFNs) and generalized PFNs (GPFNs). WAMF employs parameters describing the behavioral patterns when decision-makers encounter situations involving risk. Then, the ranking process can be continued with the calculation of the centroid point of the resulting GFN. One of the main advantages of this approach is that the computations are straightforward due to the presence of piecewise linearity, enabling us to employ numerical integration. Furthermore, we adapt operations for generalized trapezoidal PFNs (GTPFNs) to mitigate the counter-intuitive consequences resulting from utilizing the minimum operator. The effectiveness of the method is discussed through benchmarks and its implementation in multi-attribute decision-making (MADM).

Keywords: Centroid index, generalized fuzzy numbers, generalized intuitionistic fuzzy numbers, generalized picture fuzzy numbers, multi-attribute decision-making, ranking.

MSC: 03B72, 03F55, 90Bxx, 90C70, 91B06.

1. INTRODUCTION

Deterministic decision-making occasionally ignores the underlying uncertainty of reality in practical contexts. It becomes too challenging to make decisions when the data are collected insufficiently or are uncertain in numerical quantities. Fuzzy sets only consider the grade of membership when modeling imprecise parameters; they ignore the degree of hesitation and assume that the degree of non-membership is the complement of one. In reality, the experts provide all or some of the data with uncertainty and hesitancy because of fluctuations, inaccurate measurements, or other factors. The intuitionistic fuzzy set (IFS), first proposed by Atanassov [1] in 1978, is a more comprehensive version of a fuzzy set that includes an autonomous non-membership function. Only the lower and upper memberships are known in IFSs; the actual membership function is not known. Picture fuzzy sets (PFSs) [2], which are more general forms than IFSs, introduce additional neutrality and refusal degrees alongside the acceptance and rejection degrees. When individuals are faced with decisions that require a broader range of responses, such as "accept," "abstain," "reject," and "refuse," the utilization of PFSs can offer benefits. The application of non-standard fuzzy set theory addresses the challenge of dealing with imprecise and inadequate information.

For approximate reasoning and MADM in uncertain environments, fuzzy ranking is an essential step in choosing the best alternative among several others. Establishing ranking methods has been a challenging task, and none of the existing techniques are qualified as ideal since they yield inconsistent, misleading, and counter-intuitive outcomes. Numerous studies on fuzzy ranking procedures have been published in the past few decades. For both normal and subnormal FNs, Cheng [3] developed a ranking method based on the Euclidian distance to the origin. Chu and Tsao [4] proposed a method based on the area between the centroid and origin points to avoid contradictions when ranking negative numbers. Later, Wang et al. [5] revised these methods. See also [6], [7], and [8]. Due to the extensive scope of this study, it is impractical to conduct an in-depth assessment of the literature on generalized fuzzy ranking methods, and interested readers are advised to refer to the most recent publication, [9], for further information.

Regarding the concept of generalized intuitionistic ordering, expected value, score, and accuracy functions were utilized in [10] and [11]. Li [12] introduced the concepts of the value and ambiguity associated with a triangular GIFN. Additionally, the author proposed a ranking technique that incorporates the ratio of the value index to the ambiguity index. Nishad et al. [13] presented a ranking approach that satisfies the fundamental axioms of ranking functions. This method utilizes the centroid and circumcenter of the membership and non-membership functions. Additionally, refer to [14] for an application of how the suggested ranking function can be used to address a fully intuitionistic fuzzy transportation problem. A signed distance-based defuzzification method was introduced in [15] to solve solid transportation problems with symmetrical generalized trapezoidal IFNs (GTIFNs) as unit costs. Beg et al. [16] investigated an ordering mechanism founded on the index of optimism-based expected value for the solution of the generalized triangular intuitionistic fuzzy transportation problem. In [17], the authors described a method of ranking GTIFNs based on the area of both membership and non-membership parts of the numbers. The same formula was used in [18] and [19]. In [20], formulas for the weighted possibility means of membership and non-membership functions were given for GTIFNs as well as an application for multi-attribute group decision-making. See also [21, 22, 23, 24, 25] for different variations of aggregation processes. Other group decision-making applications can be found in [26] and [27]. Su et al. [28] devised a weighted trapezoidal intuitionistic fuzzy Bonferroni harmonic mean, analyzed its characteristics, and demonstrated its applicability in multi-attribute group decision-making. See also [29] and [30]. Das and Guha [31, 32] employed a centroidal ranking technique for comparing GTIFNs. They also presented aggregation operators and related operational laws. Nyagam et al. [33] discussed the shortcomings and limitations of the existing methods and suggested improved indexes for comparing GTIFNs to overcome them. Area-based parametric methods using cut-set representations were provided in [34] and [35]. By taking into account the decision-makers' attitude when defuzzifying triangular GIFNs, Wu et al. [36] constructed a decision framework for offshore wind power site selection by combining the methods of ANP and PROMETHEE. The assignment problem in [37] involved GTIFN cost coefficients and utilized a ranking technique based on the centroid of centroids. The same formulation was used in [38]. Garai [39] developed the notions of possibilistic mean, standard deviation, and magnitude within the context of GTIFNs. Also, the author proposed a ranking technique that takes into account the decision-maker's magnitude, which is based on the mean and standard deviation. The paper [40] introduced the definitions of various generalized intuitionistic fuzzy reliability characteristics, including reliability, conditional reliability, hazard rate, and mean time to failure functions. It also discussed a specific scenario called the two-parameter Pareto generalized intuitionistic fuzzy reliability analysis. Wang [41] compared GTIFNs through the extended fuzzy preference relation, which quantifies the preference degree of two GTIFNs, not through defuzzification. The study in [42] focused on a generalized intuitionistic fuzzy flow shop scheduling problem. The approach used the centroid index, which calculates the geometric center based on the horizontal and vertical axes. MADM applications can be found in [12], [43], [44], [45], [46], [47], [48], [49], [50], and [39].

Mitchell [51] interpreted IFNs as an ensemble of standard FNs, while we similarly consider GIFNs and GPFNs to consist of two or three GFNs, respectively. The first step in our ranking process is the conversion of GIFNs and GPFNs into GFNs by means of their WAMFs [52]. This dissolves hesitancy, neutrality, and refusal with the help of risk behavior parameters, which are based on Yager's idea of incorporating indeterminacy into the membership function [53]. More precisely, GTIFNs can be transformed into linear generalized hexagonal FNs, whereas GTPFNs can be simplified into linear generalized octagonal FNs [54]. The WAMF framework utilizes criteria that characterize the behavioral tendencies of individuals when they are confronted with situations that entail risk. To further explore the topic of standardization, specifically in terms of reduction, readers are advised to consult [55], which is another study using the WAMF. A version of WAMF that is similar but not normalized was designed for triangular GIFNs in [56]. It is called the resultant membership function.

As far as we know, there is a limited amount of research in the existing literature about the sorting of trapezoidal PFNs. A formal definition of a GTPFN was given in [57], along with some associated arithmetic operations and a ranking function. See also [58] and [59] that used the same definition. This study first aims to address the issue of unreasonable outcomes resulting from the use of the minimum operator in determining the overall maximal membership value in arithmetic operations. Thereby, we propose an alternative way of identifying the maximal membership values with reduced error. The secondary objective of this study is to propose a novel integrated methodology for the comparison of arbitrary generalized trapezoidal FNs (GTFNs), GTIFNs, and/or GTPFNs. This gives us a way to compare any two FNs, GIFNs, or GPFNs that are triangular, trapezoidal, normal, or subnormal. Additionally, this paper presents the 1-9 picture fuzzy linguistic scale that experts use to express their opinions in a risk analysis application. We apply the proposed ranking method to address MADM problems in non-standard fuzzy environments.

Considering the research gap in the field, we briefly underscore our motivations and technical contributions as follows:

- Most of the previous works have primarily focused on developing ranking methods for GFNs and/or GIFNs. However, it is frequently observed that there are degrees of neutrality and refusal in expert opinions. To address this issue, we examine the processes of decision-making in situations that involve a picture fuzzy environment.
- In the existing literature, it is observed that most of the papers employed the minimum operator in all arithmetic operations to determine the overall maximal membership value, or abandoned linearity and shape-preserving. This has led to inconsistencies in rankings or complications in calculations. To assist MADM processes, we extend existing arithmetic operations with less error and information loss.
- When we have to deal with problems where people make decisions based on their attitudes and there is hesitation, neutrality, and refusal, we reduce non-standard FNs to GFNs using a parameterized membership function to dissolve indeterminacy and/or refusal. This standardization enables us to make use of the vast literature on generalized fuzzy ranking.
- We obtain a prevailing framework strategy for dealing with similar problems whose parameters satisfy or do not satisfy the normality condition and/or are a combination of several types.
- The literature already benefits from the merits of the centroidal methods initially developed for GTFNs. We universalize the centroid-based ranking

procedure. In this way, we have simplified the procedure for some fuzzy extensions.

- We use normalization to rescale the domain because sorting negative values using the centroid method produces inconsistent results.
- Considering failures and the severity of losses, the proposed method is applied to the risk analysis of production systems, where experts provide their input in linguistic terms. The suggested picture fuzzy scale provides greater flexibility and a broader range of alternatives regarding the number of linguistic terms compared to fuzzy and intuitionistic scales.

The rest of this paper is constructed as follows: Section 2 presents some preliminary definitions of fuzzy sets. Section 3 outlines the arithmetic operations. Section 4 covers some existing, straightforward, and widely used generalized intuitionistic fuzzy ranking methods. Section 5 introduces a ranking procedure founded on centroids and provides straightforward benchmark instances for illustration purposes. MADM applications are given to support the proposed approach in Section 6. The paper draws some conclusions in Section 7.

2. PRELIMINARIES

In this section, we present some of the terminology and preliminary concepts required for our discussion.

Definition 1 (Standard Fuzzy Set). *A standard fuzzy set* $\widetilde{A}$ *on the universe of discourse* X *is defined as:*

$$\widetilde{A} = \left\{ \left\langle x, \mu_{\widetilde{A}}(x) \right\rangle \middle| \, x \in X \right\},$$

where the membership function $\mu_{\widetilde{A}} : X \to [0,1]$ *denotes the grade of belonging of the element* x *to the set* $\widetilde{A}$.

$\widetilde{A}$ is said to be normal if there exists $\exists x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$. If not, it is referred to as subnormal. The membership function μ is said to be convex if $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\right)$ for $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0,1]$. A normal fuzzy subset of the real line with a convex membership function is referred to as an FN.

Definition 2 (GFN). *Let* $p, m, n, q \in \mathbb{R}, p \leq m \leq n \leq q, \alpha \in (0,1]$. *A GFN* $\widetilde{A}$ *is a fuzzy subset of the real line* $\mathbb{R}$, *whose membership function* $\mu_{\widetilde{A}}$ *satisfies the following conditions:*

(i) $\mu_{\widetilde{A}} : \mathbb{R} \to [0,\alpha]$ *is continuous,*

(ii) $\mu_{\widetilde{A}}(x) = 0$ *for* $x < p$ *or* $x > q$,

(iii) $\mu_{\widetilde{A}}(x)$ *is strictly increasing on* $[p,m]$ *and strictly decreasing on* $[n,q]$,

(iv) $\mu_{\widetilde{A}}(x) = \alpha$, *for* $x \in [m, n]$.

If normality holds, that is, if $\alpha = 1$, *then* $\widetilde{A}$ *is referred to as an FN. Otherwise,* $\widetilde{A}$ *is called subnormal with the maximal membership value* $\alpha \in (0, 1)$. *The parameter* α, *which determines the height of* $\widetilde{A}$ *and represents the level of confidence in expert opinions, enables the adoption of a more comprehensive framework.*

Definition 3 (IFS). *A set*

$$\widetilde{A} = \left\{ \left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x) \right\rangle \middle| x \in X \right\}$$

is said to be an IFS on a universal set X, *in which the functions* $\mu_{\widetilde{A}} : X \to [0, 1]$ *and* $\nu_{\widetilde{A}} : X \to [0, 1]$ *denote the membership and non-membership functions of* $\widetilde{A}$, *respectively, such that* $\mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$ *for all* $x \in X$. $\pi_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x)$ *is called the degree of hesitancy.*

An IFS $\widetilde{A}$ is said to be normal if there exists $\exists x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$. The non-membership function ν is said to be concave if $\nu_{\widetilde{A}}(\lambda x_1 + (1 - \lambda) x_2) \leq \max\left(\nu_{\widetilde{A}}(x_1), \nu_{\widetilde{A}}(x_2)\right)$ for $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0, 1]$. $\widetilde{A}$ is said to be convex if $\mu_{\widetilde{A}}(x)$ is convex, and $\nu_{\widetilde{A}}(x)$ is concave. If an intuitionistic fuzzy subset of the real line is normal and convex, then it is referred to as an IFN. On the other hand, each GIFN $\widetilde{A}$ may be envisioned as a conjunction of two GFNs with the membership functions $\mu_{\widetilde{A}}(x)$ and $(1 - \nu)_{\widetilde{A}}(x) = 1 - \nu_{\widetilde{A}}(x)$. If $(1 - \nu_{\widetilde{A}}(x)) = \mu_{\widetilde{A}}(x)$ or the degree of hesitancy $\pi_{\widetilde{A}}(x) = 0$ for all $x \in X$, then the GIFN reduces to a GFN.

Definition 4 (WAMF for a GIFN). *[53] The WAMF for a GIFN* $\widetilde{A}$ *can be given as:*

$$\begin{aligned} \overline{\mu}_{\widetilde{A}}(x) &= \mu_{\widetilde{A}}(x) + \lambda \pi_{\widetilde{A}}(x) \\ &= (1 - \lambda) \mu_{\widetilde{A}}(x) + \lambda \left(1 - \nu_{\widetilde{A}}(x)\right), \end{aligned}$$

where $\lambda \in [0, 1]$ (*for risk-aversion* $\lambda \in [0, 1/2)$).

The value of λ denotes the amount of indeterminacy we can resolve in favor of a positive answer. The higher the value of the degree of optimism λ, the more indeterminacy is dissolved in favor of the membership degree, and the lower the value, the more indeterminacy is eliminated in favor of the non-membership degree. Here, the most sensible (and also risk-neutral) allocation is to all equally by choosing $\lambda = 1/2$, but, if necessary, this distribution can be linked to the parameter λ.

Definition 5 (PFS). *A PFS* $\widetilde{A}$ *on* X *is of the form:*

$$\widetilde{A} = \left\{ \left\langle x, \mu_{\widetilde{A}}(x), \eta_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x) \right\rangle \middle| x \in X \right\},$$

where $\mu_{\widetilde{A}}(x), \eta_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x) \in [0, 1]$ *are independent positive, neutral, and negative membership functions, respectively, and they follow the condition* $\mu_{\widetilde{A}}(x) + \eta_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$ *for all* $x \in X$. $\pi_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \eta_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x)$ *is called the degree of refusal.*

A PFS $\widetilde{A}$ is said to be normal if there exist $\exists x_1, x_2 \in X$ such that $\mu_{\widetilde{A}}(x_1) = 1$, and $\eta_{\widetilde{A}}(x_2) = 1$. In this paper, each GPFN $\widetilde{A}$ with its positive, neutral, and negative membership functions $\mu_{\widetilde{A}}(x), \eta_{\widetilde{A}}(x)$, and $\nu_{\widetilde{A}}(x)$, is assumed to be a combination of three GFNs with the membership functions $\mu_{\widetilde{A}}(x)$, $\eta_{\widetilde{A}}(x)$, and $\left(1-\nu_{\widetilde{A}}(x)\right)$, defined as fuzzy subsets of the real line. If $\eta_{\widetilde{A}}(x) = 1-\nu_{\widetilde{A}}(x)-\mu_{\widetilde{A}}(x)$ or $\eta_{\widetilde{A}}(x) = 0$ for all $x \in X$, then the GPFN reduces to a GIFN. To summarize, when the refusal or neutrality degree vanishes, a GPFN becomes a GIFN.

Definition 6 (WAMF for a GPFN). *[52] The WAMF for a GPFN $\widetilde{A}$ can be defined as:*

$$\begin{aligned}\overline{\mu}_{\widetilde{A}}(x) &= \mu_{\widetilde{A}}(x) + \lambda \pi_{\widetilde{A}}(x) + \omega \eta_{\widetilde{A}}(x) \\ &= (1-\lambda)\,\mu_{\widetilde{A}}(x) + \lambda\left(1-\nu_{\widetilde{A}}(x)\right) + (\omega-\lambda)\,\eta_{\widetilde{A}}(x),\end{aligned}$$

where $0 \leq \lambda \leq \omega \leq 1$ (for risk-aversion, $2\lambda < \omega \leq 1$).

The values of λ and ω represent the amount of refusal and neutrality that can be resolved in favor of the positive membership, respectively.

Definition 7 (GTPFN). *[57] $\widetilde{A}$, which is a GTPFN with a convex neutrality on the real line $\mathbb{R}$, is characterized by its respective positive, neutral, and negative membership functions:*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & \text{if } x < p_1 \\ \frac{\alpha(x-p_1)}{m-p_1}, & \text{if } p_1 \leq x < m \\ \alpha, & \text{if } m \leq x < n \\ \frac{\alpha(q_1-x)}{q_1-n}, & \text{if } n \leq x < q_1 \\ 0, & \text{if } x \geq q_1, \end{cases}$$

$$\eta_{\widetilde{A}}(x) = \begin{cases} 0, & \text{if } x < p_2 \\ \frac{\gamma(x-p_2)}{m-p_2}, & \text{if } p_2 \leq x < m \\ \gamma, & \text{if } m \leq x < n \\ \frac{\gamma(q_2-x)}{q_2-n}, & \text{if } n \leq x < q_2 \\ 0, & \text{if } x \geq q_2, \end{cases}$$

and

$$\nu_{\widetilde{A}}(x) = \begin{cases} 1, & \text{if } x < p_3 \\ 1-\frac{(1-\beta)(x-p_3)}{m-p_3}, & \text{if } p_3 \leq x < m \\ \beta, & \text{if } m \leq x < n \\ 1-\frac{(1-\beta)(q_3-x)}{q_3-n}, & \text{if } n \leq x < q_3 \\ 1, & \text{if } x \geq q_3, \end{cases}$$

where $p_3 \leq p_2 \leq p_1 < m \leq n < q_1 \leq q_2 \leq q_3$ and $0 < \alpha+\gamma+\beta \leq 1$. It is important to note that α and γ are referred to as the maximal positive and neutral membership, respectively. Likewise, β represents the minimal negative membership. We denote $\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \gamma, \beta \rangle$. If $m = n$, then we obtain a triangular GPFN.

Without the neutrality function, which means if $\gamma = 0$, we obtain a GTIFN for arbitrary $p_2 \in [p_3, p_1]$ and $q_2 \in [q_1, q_3]$, in addition, without the hesitancy function, i.e., if $\alpha + \beta = 1$, and $p_3 = p_2 = p_1$, and $q_1 = q_2 = q_3$, we obtain a GTFN.

Definition 8 (WAMF for a GTPFN). *The WAMF for a GTPFN*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\, ; \alpha, \gamma, \beta \rangle$$

is defined as:

$$\overline{\mu}_{\widetilde{A}}(x) = \begin{cases} 0, & \text{if } x < p_3 \\ f^L_{\widetilde{A}}(x), & \text{if } p_3 \leq x \leq m \\ (1-\lambda)\alpha + \lambda(1-\beta) + (\omega - \lambda)\gamma, & \text{if } m < x < n \\ f^R_{\widetilde{A}}(x), & \text{if } n \leq x \leq q_3 \\ 0, & \text{if } q_3 < x, \end{cases}$$

where

$$f^L_{\widetilde{A}} : [p_3, m] \to [0, (1-\lambda)\alpha + \lambda(1-\beta) + (\omega - \lambda)\gamma]$$

and

$$f^R_{\widetilde{A}} : [n, q_3] \to [0, (1-\lambda)\alpha + \lambda(1-\beta) + (\omega - \lambda)\gamma]$$

are piecewise continuous left-side (respectively right-side) functions which are strictly increasing (respectively strictly decreasing) on their domain. Also,

$$f^L_{\widetilde{A}}(x) = (1-\lambda) f^{\mu_L}_{\widetilde{A}}(x) + \lambda f^{(1-\nu)_L}_{\widetilde{A}}(x) + (\omega - \lambda) f^{\eta_L}_{\widetilde{A}}(x),$$

$$f^R_{\widetilde{A}}(x) = (1-\lambda) f^{\mu_R}_{\widetilde{A}}(x) + \lambda f^{(1-\nu)_R}_{\widetilde{A}}(x) + (\omega - \lambda) f^{\eta_R}_{\widetilde{A}}(x),$$

where

$$f^{\mu_L}_{\widetilde{A}}(x) = \begin{cases} 0, & \text{if } p_3 \leq x < p_1 \\ \frac{\alpha(x-p_1)}{m-p_1}, & \text{if } p_1 \leq x \leq m, \end{cases}$$

$$f^{\eta_L}_{\widetilde{A}}(x) = \begin{cases} 0, & \text{if } p_3 \leq x < p_2 \\ \frac{\gamma(x-p_2)}{m-p_2}, & \text{if } p_2 \leq x \leq m, \end{cases}$$

$$f^{(1-\nu)_L}_{\widetilde{A}}(x) = \frac{(1-\beta)(x-p_3)}{m-p_3},$$

$$f^{\mu_R}_{\widetilde{A}}(x) = \begin{cases} \frac{\alpha(q_1-x)}{q_1-n}, & \text{if } n \leq x < q_1, \\ 0, & \text{if } q_1 \leq x \leq q_3, \end{cases}$$

$$f^{\eta_R}_{\widetilde{A}}(x) = \begin{cases} \frac{\gamma(q_2-x)}{q_2-n}, & \text{if } n \leq x < q_2, \\ 0, & \text{if } q_2 \leq x \leq q_3, \end{cases}$$

and

$$f^{(1-\nu)_R}_{\widetilde{A}}(x) = \frac{(1-\beta)(q_3-x)}{q_3-n}.$$

Example 9. *Consider the GTPFN denoted by*

$$\widetilde{A} = \langle (1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5) \, ; 0.6, 0.2, 0.1 \rangle \, .$$

Please refer to Figure 1 for the graphical representation of the membership functions. The values of λ *and* ω *are assumed to be* 0.5 *and* 1*, respectively. The figure illustrates that WAMF is composed of three distinct segments. The left side consists of a non-decreasing piecewise linear function, while the right side has a non-increasing piecewise linear function. Both of those segments are characterized by three breaking points. Located at the core, there exists a function that remains constant.*

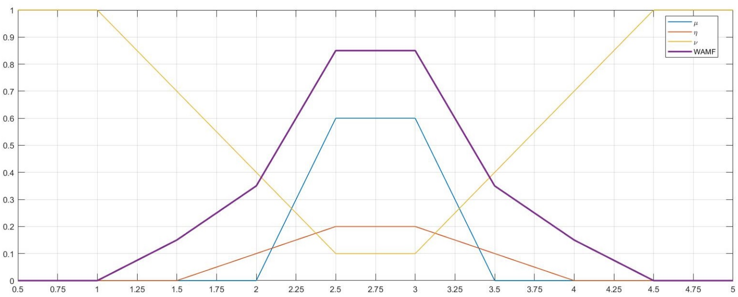

Figure 1: Membership plots of Example 9

3. ARITHMETIC OPERATIONS

The minimum operator is commonly employed to determine the maximal membership values in all arithmetic operations defined for GTFNs. However, in this case, representations with narrower most likely intervals of a GTFN not having the lowest maximal membership value are not taken into account. Moreover, rankings exhibit inconsistency in this manner. To offer a unified method, the subsequent arithmetic operations are designed to function globally over GTFNs, GTIFNs, GTPFNs, and the corresponding reductions between them. We do not desire to give up shape preservation and use nonlinear extended fuzzy arithmetic [60] due to its computational complexity. In order to address the limitations associated with minimal operator usage, the following arithmetic operations are adapted from [61] for GTPFNs. See also [62] and [63] for details. Also note that an outcome of moderate reliability is obtained by adding a highly reliable non-negative value to a somewhat less reliable non-negative value.

Definition 10. *Let*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3) \, ; \alpha_1, \gamma_1, \beta_1 \rangle$$

and

$$\widetilde{B} = \langle (p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3)\,; \alpha_2, \gamma_2, \beta_2 \rangle$$

be two GTPFNs, and $c \in \mathbb{R}$, then we have the following shape-preserving and error-reducing operations:

(i) *Addition of two GTPFNs*

$$\widetilde{A} + \widetilde{B} = \langle (p_3 + p'_3, p_2 + p'_2, p_1 + p'_1, m + m', n + n', q_1 + q'_1, q_2 + q'_2, q_3 + q'_3)\,; \alpha, \gamma, \beta \rangle,$$

where

$$\alpha = \frac{\left\|\widetilde{A}_\mu\right\| \alpha_1 + \left\|\widetilde{B}_\mu\right\| \alpha_2}{\left\|\widetilde{A}_\mu\right\| + \left\|\widetilde{B}_\mu\right\|}, \gamma = \frac{\left\|\widetilde{A}_\eta\right\| \gamma_1 + \left\|\widetilde{B}_\eta\right\| \gamma_2}{\left\|\widetilde{A}_\eta\right\| + \left\|\widetilde{B}_\eta\right\|}, \beta = \frac{\left\|\widetilde{A}_{1-\nu}\right\| \beta_1 + \left\|\widetilde{B}_{1-\nu}\right\| \beta_2}{\left\|\widetilde{A}_{1-\nu}\right\| + \left\|\widetilde{B}_{1-\nu}\right\|},$$

$$\begin{aligned}
\left\|\widetilde{A}_\mu\right\| &= \frac{|p_1| + |m| + |n| + |q_1|}{4}, & \left\|\widetilde{B}_\mu\right\| &= \frac{|p'_1| + |m| + |n| + |q'_1|}{4}, \\
\left\|\widetilde{A}_\eta\right\| &= \frac{|p_2| + |m| + |n| + |q_2|}{4}, & \left\|\widetilde{B}_\eta\right\| &= \frac{|p'_2| + |m| + |n| + |q'_2|}{4}, \\
\left\|\widetilde{A}_{1-\nu}\right\| &= \frac{|p_3| + |m| + |n| + |q_3|}{4}, & \left\|\widetilde{B}_{1-\nu}\right\| &= \frac{|p'_3| + |m| + |n| + |q'_3|}{4}.
\end{aligned}$$

(ii) *Scalar Multiplication*

$$c\widetilde{A} = \begin{cases} \left\langle \; (cp_3, cp_2, cp_1, cm, cn, cq_1, cq_2, cq_3)\,; \alpha_1, \gamma_1, \beta_1 \; \right\rangle, & \text{if } c \geq 0 \\ \left\langle \; (cq_3, cq_2, cq_1, cn, cm, cp_1, cp_2, cp_3)\,; \alpha_1, \gamma_1, \beta_1 \; \right\rangle, & \text{if } c < 0. \end{cases}$$

(iii) *Multiplication of two GTPFNs*

$$\widetilde{A} \times \widetilde{B} = \langle (P_3, P_2, P_1, M, N, Q_1, Q_2, Q_3)\,; \alpha, \gamma, \beta \rangle,$$

where

$$\begin{aligned}
P_3 &= \min\{p_3p'_3, p_3q'_3, q_3p'_3, q_3q'_3\}, \\
P_2 &= \min\{p_2p'_2, p_2q'_2, q_2p'_2, q_2q'_2\}, \\
P_1 &= \min\{p_1p'_1, p_1q'_1, q_1p'_1, q_1q'_1\}, \\
M &= \min\{mm', mn', nm', nn'\}, \\
N &= \max\{mm', mn', nm', nn'\}, \\
Q_1 &= \max\{p_1p'_1, p_1q'_1, q_1p'_1, q_1q'_1\}, \\
Q_2 &= \max\{p_2p'_2, p_2q'_2, q_2p'_2, q_2q'_2\}, \\
Q_3 &= \max\{p_3p'_3, p_3q'_3, q_3p'_3, q_3q'_3\}, \\
\alpha &= \alpha_1\alpha_2, \\
\gamma &= (\alpha_1 + \gamma_1)(\alpha_2 + \gamma_2) - \alpha_1\alpha_2, \\
\beta &= \beta_1 + \beta_2 - \beta_1\beta_2.
\end{aligned}$$

Interval arithmetic is used in division, similar to multiplication, and maximal and minimal memberships are calculated in the same way.

Example 11. *[60] Consider the following two GTFNs,*

$$\widetilde{A} = \langle (1,1,1,2,3,5,5,5)\,; 0.3, 0, 0.7\rangle$$

and

$$\widetilde{B} = \langle (1,1,1,2,3,5,5,5)\,; 0.9, 0, 0.1\rangle\,.$$

Intuitively, we have $\widetilde{A} \prec \widetilde{B}$*. If* $\widetilde{A} \prec \widetilde{B}$*, then* $2\widetilde{A} \prec \widetilde{A} + \widetilde{B}$ *by adding* $\widetilde{A}$ *on both sides. By using the addition and scalar multiplication defined in Definition 10, we obtain as follows:*

$$\widetilde{A} + \widetilde{B} = \langle (2,2,2,4,6,10,10,10)\,; 0.6, 0, 0.4\rangle\,,$$
$$2\widetilde{A} = \langle (2,2,2,4,6,10,10,10)\,; 0.3, 0, 0.7\rangle$$

which gives a consistent ordering.

Here, since $\left\|\widetilde{A}_{\mu}\right\| = \left\|\widetilde{B}_{\mu}\right\|$ *and* $\left\|\widetilde{A}_{1-\nu}\right\| = \left\|\widetilde{B}_{1-\nu}\right\|$*, we determine the overall maximal membership (and minimal non-membership) of* $\widetilde{A} + \widetilde{B}$ *as 0.6 (and 0.4, respectively), which is the average of the maximal memberships of 0.3 and 0.9 (and the average of the minimal non-memberships of 0.7 and 0.1).*

Example 12. *[64] Consider the following three GTIFNs,*

$$\widetilde{A} = \langle (10,11,12,15,15,18,19,20)\,; 0.6, 0, 0.4\rangle\,,$$

$$\widetilde{B} = \langle (3,3,4,5,5,6,7,8)\,; 0.7, 0, 0.3\rangle\,,$$

and

$$\widetilde{C} = \langle (3,3,4,5,5,6,7,8)\,; 0.9, 0, 0.1\rangle\,.$$

Intuitively, since $\widetilde{B} \neq \widetilde{C}$*, then* $\widetilde{A} + \widetilde{B} \neq \widetilde{A} + \widetilde{C}$*. By using the addition defined in Definition 10, we obtain as follows:*

$$\widetilde{A} + \widetilde{B} = \langle (13,14,16,20,20,24,26,28)\,; 0.6250, 0, 0.3741\rangle\,,$$
$$\widetilde{A} + \widetilde{C} = \langle (13,14,16,20,20,24,26,28)\,; 0.6750, 0, 0.3222\rangle\,.$$

The utilization of the minimum operator in the addition operation yields the aforementioned counter-intuitive outcome.

4. SOME EXISTING INTUITIONISTIC RANKING INDEXES

This section covers some straightforward and widely used ranking methods for GTIFNs from the literature. They will be utilized in the benchmark examples for comparison. Our notation is implemented here.

Definition 13. *[20] Let*

$$\widetilde{A} = \langle (p, p, p, m, n, q, q, q)\, ; \alpha, 0, \beta \rangle \,,$$

be a GTIFN. The weighted possibility means for the membership and non-membership functions are:

$$\begin{aligned} m_\mu(\widetilde{A}) &= \frac{1}{6}\,(p + 2m + 2n + q)\,\alpha, \\ m_\nu(\widetilde{A}) &= \frac{1}{6}\,(p + 2m + 2n + q)\,(1-\beta), \end{aligned} \tag{1}$$

respectively. The lexicographic method for the ranking of $\widetilde{A}$ and $\widetilde{B}$ is defined as follows:

(i) *if $m_\mu(\widetilde{A}) < m_\mu(\widetilde{B})$, then $\widetilde{A} \prec \widetilde{B}$;*

(ii) *else if $m_\mu(\widetilde{A}) > m_\mu(\widetilde{B})$, then $\widetilde{A} \succ \widetilde{B}$;*

(iii) *else*

- **(a)** *if $m_\nu(\widetilde{A}) < m_\nu(\widetilde{B})$, then $\widetilde{A} \prec \widetilde{B}$;*
- **(b)** *else if $m_\nu(\widetilde{A}) > m_\nu(\widetilde{B})$, then $\widetilde{A} \succ \widetilde{B}$;*
- **(c)** *else $\widetilde{A} \sim \widetilde{B}$.*

Definition 14. *[21, 26] Let*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\, ; \alpha, 0, \beta \rangle \,,$$

be a GTIFN. The risk-based defuzzification function can be given as follows:

$$h(\widetilde{A}) = \begin{cases} \frac{1}{6}\,(p_3 + 2m + 2n + q_3)\,(1-\beta)\,, & \text{if } r \to 0 \\ \frac{1}{18}\left[(p_1 + 2m + 2n + q_1)\,\alpha + (p_3 + 2m + 2n + q_3)\,2\,(1-\beta)\right], & \text{if } r = \frac{1}{2} \\ \frac{1}{12}\left[(p_1 + 2m + 2n + q_1)\,\alpha + (p_3 + 2m + 2n + q_3)\,(1-\beta)\right], & \text{if } r = 1 \\ \frac{1}{18}\left[(p_1 + 2m + 2n + q_1)\,2\alpha + (p_3 + 2m + 2n + q_3)\,(1-\beta)\right], & \text{if } r = 2 \\ \frac{1}{6}\,(p_1 + 2m + 2n + q_1)\,\alpha, & \text{if } r \to \infty, \end{cases} \tag{2}$$

where $r > 0$ is the risk parameter. For values of r less than 1, greater than 1, or equal to 1, rankings can be classified as optimistic, pessimistic, or neutral, respectively.

Definition 15. *[17] Consider a GTIFN:*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\,; \alpha, 0, \beta \rangle\,.$$

A defuzzification function can be given as follows:

$$R(\widetilde{A}) = \frac{\alpha S(\mu_{\widetilde{A}}) + \beta S(\nu_{\widetilde{A}})}{\alpha + \beta}, \tag{3}$$

where

$$S(\mu_{\widetilde{A}}) = \left(\frac{2p_1 + 7m + 7n + 2q_1}{18}\right)\left(\frac{7\alpha}{18}\right)$$

and

$$S(\nu_{\widetilde{A}}) = \left(\frac{2p_3 + 7m + 7n + 2q_3}{18}\right)\left(\frac{11 + 7\beta}{18}\right).$$

Definition 16. *[38] Consider a GTIFN:*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\,; \alpha, 0, \beta \rangle\,.$$

A defuzzification function can be given as follows:

$$R(\widetilde{A}) = \left(\frac{p_3 + p_1 + 7m + 7n + q_1 + q_3}{18}\right)\left(\frac{4\alpha + 5\beta}{18}\right). \tag{4}$$

5. PROPOSED RANKING APPROACH

This present study introduces a novel methodology for ranking GTPFNs, which builds upon the existing strategy based on ordering GTFNs by calculating the Euclidean distance between the centroid point and the origin. We use Wang et al.'s revised centroid calculation method [5] as a basis. Initially, we convert the given GTPFN to a GTFN. Then, we determine the centroid point using numerical integration. As a byproduct of the code used, we can also calculate the expected value when we want to express GTPFNs with a singular numerical representation.

In order to rank negative values, the min-max normalization approach is employed, wherein the domain is rescaled using the formula $x \leftarrow \frac{x - min}{max - min}$. In this context, min and max denote the minimal and maximum elements of the domain, respectively. This prevents the outcomes from becoming misleading. The maximal and minimal membership values remain unchanged during the process of normalization.

The computations are conducted on a PC running MS Windows 10 Pro, equipped with an Intel Core i5-7400 CPU (3.00 GHz) and 4 GB of RAM, using MATLAB R2019a. We particularly use the built-in function “trapz” for numerical integration.

Definition 17 (Centroid distance method based on [5]). *Let*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\,; \alpha, \gamma, \beta \rangle$$

be a GTPFN. The centroid points and distance index are provided as follows:

$$x_0(\widetilde{A}) = \frac{\int_{p_3}^{m} x f_{\widetilde{A}}^{L}(x)\,dx + \int_{m}^{n} x\theta dx + \int_{n}^{q_3} x f_{\widetilde{A}}^{R}(x)\,dx}{\int_{p_3}^{m} f_{\widetilde{A}}^{L}(x)\,dx + \int_{m}^{n} \theta dx + \int_{n}^{q_3} f_{\widetilde{A}}^{R}(x)\,dx},$$

$$y_0(\widetilde{A}) = \frac{\int_{0}^{\theta} y g_{\widetilde{A}}^{R}(y)\,dy - \int_{0}^{\theta} y g_{\widetilde{A}}^{L}(y)\,dy}{\int_{0}^{\theta} g_{\widetilde{A}}^{R}(y)\,dy - \int_{0}^{\theta} g_{\widetilde{A}}^{L}(y)\,dy},$$

$$R(\widetilde{A}, \lambda, \omega) = \sqrt{(x_0(\widetilde{A}))^2 + (y_0(\widetilde{A}))^2},$$

where $\theta = (1-\lambda)\alpha + \lambda(1-\beta) + (\omega - \lambda)\gamma$, $g_{\widetilde{A}}^{L}$ *and* $g_{\widetilde{A}}^{R}$ *are the inverse functions of* $f_{\widetilde{A}}^{L}$ *and* $f_{\widetilde{A}}^{R}$ *given in Definition 8, respectively. The ranking of* $\widetilde{A}$ *and* $\widetilde{B}$ *is defined as follows:*

(i) *If* $R(\widetilde{A}, \lambda, \omega) < R(\widetilde{B}, \lambda, \omega)$, *then* $\widetilde{A} \prec \widetilde{B}$,

(ii) *Else if* $R(\widetilde{A}, \lambda, \omega) > R(\widetilde{B}, \lambda, \omega)$, *then* $\widetilde{A} \succ \widetilde{B}$,

(iii) *Else* $\widetilde{A} \sim \widetilde{B}$.

Note that, according to the additive property of definite integrals, the integral:

$$\int_{p_3}^{m} x f_{\widetilde{A}}^{L}(x)\,dx$$

is the sum of three parts of integrals over $[p_3, p_2]$, $[p_2, p_1]$, and $[p_1, m]$. The same situation exists for the integrals:

$$\int_{n}^{q_3} x f_{\widetilde{A}}^{R}(x)\,dx, \int_{p_3}^{m} f_{\widetilde{A}}^{L}(x)\,dx, \int_{n}^{q_3} f_{\widetilde{A}}^{R}(x)\,dx,$$

$$\int_{0}^{\theta} y g_{\widetilde{A}}^{R}(y)\,dy, \int_{0}^{\theta} y g_{\widetilde{A}}^{L}(y)\,dy, \int_{0}^{\theta} g_{\widetilde{A}}^{R}(y)\,dy, and \int_{0}^{\theta} g_{\widetilde{A}}^{L}(y)\,dy.$$

Regarding GTIFNs, due to the presence of two distinct breaking points, the integrals are composed of two parts.

Definition 18 (Expected value of a GTPFN). *[65, 66] Let*

$$\widetilde{A} = \langle (p_3, p_2, p_1, m, n, q_1, q_2, q_3)\,; \alpha, \gamma, \beta \rangle$$

be a GTPFN. The expected value, which is the midpoint of the expected interval, has the following definition:

$$EV(\widetilde{A}) = \frac{1}{2}\left[\int_0^{\theta} g_{\widetilde{A}}^{L}(y)\,dy + \int_0^{\theta} g_{\widetilde{A}}^{R}(y)\,dy\right],$$

where $\theta = (1-\lambda)\alpha + \lambda(1-\beta) + (\omega - \lambda)\gamma$, $g_{\widetilde{A}}^{L}$ *and* $g_{\widetilde{A}}^{R}$ *are the inverse functions of* $f_{\widetilde{A}}^{L}$ *and* $f_{\widetilde{A}}^{R}$ *given in Definition 8, respectively.*

Akram et al. [57] derived this value as:

$$EV(\widetilde{A}) = \frac{1}{4}\left(\alpha\left(p_1 + m + n + q_1\right) + \gamma\left(p_2 + m + n + q_2\right) + (1-\beta)\left(p_3 + m + n + q_3\right)\right) \tag{5}$$

and used it as a ranking function. Note that our expected value involves extra risk parameters λ *and* ω*, therefore, it is different from theirs.*

Example 19. *[67] Let*

$$\widetilde{A_1} = \langle (3,3,3,5,6,8,8,8)\,; 1, 0, 0 \rangle,$$

$$\widetilde{A_2} = \langle (3,3,3,5,6,8,8,8)\,; 0.8, 0, 0.2 \rangle,$$

$$\widetilde{A_3} = \langle (3,3,3,5,6,9,9,9)\,; 1, 0, 0 \rangle,$$

and

$$\widetilde{A_4} = \langle (3,3,3,5,6,9,9,9)\,; 0.7, 0, 0.3 \rangle$$

be GTFNs. According to the ranking index values, which are given in Table 1, $\widetilde{A_2} \prec \widetilde{A_1} \prec \widetilde{A_4} \prec \widetilde{A_3}$ *with arbitrary* $\lambda = \omega = 0.5$*. The risk parameters* λ *and* ω *are not applicable for GTFNs. The ordering coincides with the one found in [67].*

Let us now interpret the results obtained using the indexes given in Section 4. According to Table 2, indexes (3) *and* (4) *give irrational results. Intuitively, we have* $\widetilde{A_2} \prec \widetilde{A_1}$ *and* $\widetilde{A_4} \prec \widetilde{A_3}$*. The result with indexes* (1) *and* (2) *is different from ours, but it is acceptable.*

Example 20. *[68, 69] Let*

$$\widetilde{A} = \langle (0.2, 0.2, 0.2, 0.4, 0.6, 0.8, 0.8, 0.8)\,; 0.35, 0, 0.65 \rangle,$$

Table 1: Ranking indexes of Example 19

	EV	x_0	y_0	R
$\widetilde{A}_1$	5.5000	5.5000	0.3889	5.5137
$\widetilde{A}_2$	4.4000	5.5000	0.3111	5.5088
$\widetilde{A}_3$	5.7500	5.8095	0.3810	5.8220
$\widetilde{A}_4$	4.0250	5.8095	0.2667	5.8156
Result		$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_4 \prec \widetilde{A}_3$		

Table 2: Defuzzifications of Example 19

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	Result
Index (1)	5.5000	4.4000	5.6667	3.9667	$\widetilde{A}_4 \prec \widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$
Index (2)	5.5000	4.4000	5.6667	3.9667	$\widetilde{A}_4 \prec \widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$
Index (3)	2.1389	2.1267	2.1821	2.2943	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_4$
Index (4)	1.2222	1.2833	1.2469	1.3404	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2 \prec \widetilde{A}_4$

and

$$\widetilde{B} = \langle (0.1, 0.1, 0.1, 0.2, 0.3, 0.4, 0.4, 0.4)\,; 0.7, 0, 0.3 \rangle$$

be GTFNs with the same expected value of 0.1750. According to the ranking index values of $R(\widetilde{A}, 1/2, 1/2) = 0.5208$ *and* $R(\widetilde{B}, 1/2, 1/2) = 0.3841$*, so* $\widetilde{A} \succ \widetilde{B}$*. The ordering corresponds to the ones documented in references [68] and [69]. Moreover,*

$$-\widetilde{A} = \langle (0, 0, 0, 2/7, 4/7, 6/7, 6/7, 6/7)\,; 0.35, 0, 0.65 \rangle\,,$$

and

$$-\widetilde{B} = \langle (4/7, 4/7, 4/7, 5/7, 6/7, 1, 1, 1)\,; 0.7, 0, 0.3 \rangle$$

after normalization. The ranking indexes are 0.4527 and 0.8381, respectively, which implies $-\widetilde{A} \prec -\widetilde{B}$ *consistently.*

Example 21. *[70] Let*

$$\widetilde{A} = \langle (1, 1, 1, 3, 3, 5, 5, 5)\,; 0.3, 0, 0.2 \rangle$$

and

$$\widetilde{B} = \langle (4, 4, 4, 8, 8, 9, 9, 9)\,; 0.4, 0, 0.1 \rangle$$

Table 3: Ranking indexes of Example 21

		EV	x_0	y_0	R
$\lambda = 0.25$	$\widetilde{A}$	1.2750	3.0000	0.1417	3.0033
$\omega = 0.5$	$\widetilde{B}$	3.8063	7.0000	0.1750	7.0022
Risk-neutral ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.2$	$\widetilde{A}$	1.2000	3.0000	0.1333	3.0030
$\omega = 0.5$	$\widetilde{B}$	3.6250	7.0000	0.1667	7.0020
Pessimistic ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.3$	$\widetilde{A}$	1.3500	3.0000	0.1500	3.0037
$\omega = 0.5$	$\widetilde{B}$	3.9875	7.0000	0.1833	7.0024
Optimistic ranking					$\widetilde{A} \prec \widetilde{B}$

be two triangular GIFNs. According to the risk-attitude ranking index values, which are given in Table 3, $\widetilde{A} \prec \widetilde{B}$.

Using the arithmetic operations in Definition 10, we obtain:

$$\widetilde{A} - \widetilde{B} = \langle (-8, -8, -8, -5, -5, 1, 1, 1)\, ; 0.3707, 0, 0.1293 \rangle$$

and

$$\widetilde{B} - \widetilde{B} = \langle (-5, -5, -5, 0, 0, 5, 5, 5)\, ; 0.4, 0, 0.1 \rangle\,,$$

and after normalization, we get:

$$\widetilde{A} - \widetilde{B} = \langle (0, 0, 0, 3/13, 3/13, 9/13, 9/13, 9/13)\, ; 0.3707, 0, 0.1293 \rangle$$

and

$$\widetilde{B} - \widetilde{B} = \langle (3/13, 3/13, 3/13, 8/13, 8/13, 1, 1, 1)\, ; 0.4, 0, 0.1 \rangle\,.$$

In conclusion, we obtain a consistent ranking as $\widetilde{A} - \widetilde{B} \prec \widetilde{B} - \widetilde{B}$. *The obtained outcomes align with the findings presented in reference [70]. See also Table 4.*

Example 22. *[70] Let*

$$\widetilde{A} = \langle (1, 1, 1, 4, 5, 8, 8, 8)\, ; 0.6, 0, 0.3 \rangle$$

and

$$\widetilde{B} = \langle (1, 1, 1, 3, 6, 8, 8, 8)\, ; 0.7, 0, 0.2 \rangle$$

Table 4: Ranking indexes of Example 21

		EV	x_0	y_0	R
$\lambda = 0.25$	$\widetilde{A} - \widetilde{B}$	0.1430	0.3077	0.1652	0.3493
$\omega = 0.5$	$\widetilde{B} - \widetilde{B}$	0.3231	0.6154	0.1750	0.6398
Risk-neutral ranking					$\widetilde{A} - \widetilde{B} \prec \widetilde{B} - \widetilde{B}$
$\lambda = 0.2$	$\widetilde{A} - \widetilde{B}$	0.1358	0.3077	0.1569	0.3454
$\omega = 0.5$	$\widetilde{B} - \widetilde{B}$	0.3077	0.6154	0.1667	0.6376
Pessimistic ranking					$\widetilde{A} - \widetilde{B} \prec \widetilde{B} - \widetilde{B}$
$\lambda = 0.3$	$\widetilde{A} - \widetilde{B}$	0.1502	0.3077	0.1736	0.3533
$\omega = 0.5$	$\widetilde{B} - \widetilde{B}$	0.3385	0.6154	0.1833	0.6421
Optimistic ranking					$\widetilde{A} - \widetilde{B} \prec \widetilde{B} - \widetilde{B}$

be two GTIFNs. According to the risk-attitude ranking index values, which are given in Table 5, $\widetilde{A} \prec \widetilde{B}$. But, Singh et al. [70] presented the ranking of these GIFNs, which is denoted as $\widetilde{A} \sim \widetilde{B}$. By using the weighted possibility means for the membership function (1)*:*

$$m_\mu(\widetilde{A}) = 2.7, m_\mu(\widetilde{B}) = 3.15 \Rightarrow \ \widetilde{A} \prec \widetilde{B}.$$

See Table 6 for the results found with the defuzzification function (2)*. These results coincide with ours, but they contradict the results obtained with indexes* (3) *and* (4)*.*

Example 23. *Let us now perform a sensitivity analysis on a synthetic example. Suppose that*

$$\widetilde{A_i} = \langle (0.1, 0.1, 0.1, 0.3, 0.5, 0.8, 0.8, 0.8)\,; \alpha_i, 0, \beta_i \rangle\,, i = 1, 2, 3$$

are GTIFNs such that $(\alpha_1, \beta_1) = (0.5, 0.2)$*,* $(\alpha_2, \beta_2) = (0.6, 0.4)$*, and* $(\alpha_3, \beta_3) = (0.4, 0.3)$*. Additionally, we consider pessimistic, risk-neutral, and optimistic orderings, with a fixed* $\omega = 0.8$ *and the corresponding parameter values being* $\lambda = 0.1, 0.4, 0.7$*. See Table 7 for results observed to be sensitive to the changes in the* λ *parameter.*

See also Table 8 for the results found with the defuzzification function (2)*. The pessimistic (or respectively optimistic) results are similar for* $r > 1$ *(or respectively* $r < 1$*). The neutral ranking is the same. However, the defuzzification function* (2) *fails to compare different numbers for* $r = 1/2$ *and* $r = 2$*.*

Example 24. *[71] Let*

$$\widetilde{A} = \langle (-2, -2, -1, 0, 0, 1, 1, 2)\,; 1, 0, 0 \rangle$$

Table 5: Ranking indexes of Example 22

		EV	x_0	y_0	R
$\lambda = 0.25$	$\widetilde{A}$	2.8125	4.5000	0.2344	4.5061
$\omega = 0.5$	$\widetilde{B}$	3.2625	4.5000	0.3142	4.5110
Risk-neutral ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.2$	$\widetilde{A}$	2.7900	4.5000	0.2325	4.5060
$\omega = 0.5$	$\widetilde{B}$	3.2400	4.5000	0.3120	4.5108
Pessimistic ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.3$	$\widetilde{A}$	2.8350	4.5000	0.2362	4.5062
$\omega = 0.5$	$\widetilde{B}$	3.2850	4.5000	0.3163	4.5111
Optimistic ranking					$\widetilde{A} \prec \widetilde{B}$

Table 6: Defuzzifications of Example 22

	$r \to 0$	$r = \frac{1}{2}$	$r = 1$	$r = 2$	$r \to \infty$
$\widetilde{A}$	3.150	3.000	2.925	2.850	2.700
$\widetilde{B}$	3.600	3.450	3.375	3.300	3.150
Result	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \prec \widetilde{B}$

Table 7: Ranking indexes of Example 23

	$\lambda = 0.1$	$\lambda = 0.4$	$\lambda = 0.7$
$\widetilde{A_1}$	0.4808	0.4984	0.5179
$\widetilde{A_2}$	0.4943	0.4943	0.4943
$\widetilde{A_3}$	0.4640	0.4790	0.4963
Result	$\widetilde{A_2} \succ \widetilde{A_1} \succ \widetilde{A_3}$	$\widetilde{A_1} \succ \widetilde{A_2} \succ \widetilde{A_3}$	$\widetilde{A_1} \succ \widetilde{A_3} \succ \widetilde{A_2}$

and

$$\widetilde{B} = \langle (-3, -2, -2, 0, 0, 2, 3, 3)\,; 1, 0, 0 \rangle$$

be two triangular IFNs. After normalization, we obtain:

$$\widetilde{A} = \langle (1/6, 1/6, 1/3, 1/2, 1/2, 2/3, 2/3, 5/6)\,; 1, 0, 0 \rangle$$

Table 8: Defuzzifications of Example 23

	$r \to 0$	$r = \frac{1}{2}$	$r = 1$	$r = 2$	$r \to \infty$
$\widetilde{A_1}$	0.3333	0.2917	0.2708	0.2500	0.2083
$\widetilde{A_2}$	0.2500	0.2500	0.2500	0.2500	0.2500
$\widetilde{A_3}$	0.2917	0.2500	0.2292	0.2083	0.1667

and

$$\widetilde{B} = \langle (0, 1/6, 1/6, 1/2, 1/2, 5/6, 1, 1)\,; 1, 0, 0 \rangle\,.$$

According to the risk-attitude ranking index values, which are given in Table 9, $\widetilde{A} \prec \widetilde{B}$. *A distinct comparison is derived from the one presented in [71].*

By using the weighted possibility means for the membership functions (1)*:*

$$m_\mu(\widetilde{A}) = m_\mu(\widetilde{B}) = m_\nu(\widetilde{A}) = m_\nu(\widetilde{B}) = 0 \Rightarrow \widetilde{A} \sim \widetilde{B}.$$

The process of representing a GTIFN as a GTPFN is not unique or singular in nature. By varying the value of p_2 *within the range of* $[p_3, p_1]$ *and* q_2 *within the range of* $[q_1, q_3]$*, we can observe that various yet equivalent representations are achieved. Here, as an IFN,* $\widetilde{A}$ *is the same as its negation,* $-\widetilde{A}$*. This means that* $\widetilde{A}$ *is symmetrical with respect to the* $y-$*axis. A similar situation exists for* $\widetilde{B}$*. In this particular scenario, the proposition asserting that if* $\widetilde{A} \prec \widetilde{B}$*, then* $-\widetilde{A} \succ -\widetilde{B}$ *is deemed invalid. See [71] for more detail. In this case, the most reasonable result is obtained with the lexicographic method* (1)*.*

Table 9: Ranking indexes of Example 24

		EV	x_0	y_0	R
$\lambda = 0.25$	$\widetilde{A}$	0.5000	0.5000	0.3083	0.5874
$\omega = 0.5$	$\widetilde{B}$	0.5000	0.5000	0.3241	0.5958
Risk-neutral ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.2$	$\widetilde{A}$	0.5000	0.5000	0.3111	0.5889
$\omega = 0.5$	$\widetilde{B}$	0.5000	0.5000	0.3253	0.5965
Pessimistic ranking					$\widetilde{A} \prec \widetilde{B}$
$\lambda = 0.3$	$\widetilde{A}$	0.5000	0.5000	0.3064	0.5864
$\omega = 0.5$	$\widetilde{B}$	0.5000	0.5000	0.3232	0.5954
Optimistic ranking					$\widetilde{A} \prec \widetilde{B}$

Example 25. *[72] Let us consider the following three GTPFNs:*

$$\widetilde{A_1} = \langle (0.2, 0.2, 0.2, 0.4, 0.4, 0.5, 0.5, 0.5)\, ; 0.4, 0.2, 0.3 \rangle\,,$$

$$\widetilde{A_2} = \langle (0.6, 0.6, 0.6, 0.6, 0.6, 0.7, 0.7, 0.7)\, ; 0.5, 0.1, 0.4 \rangle\,,$$

and

$$\widetilde{A_3} = \langle (0.3, 0.3, 0.3, 0.6, 0.7, 0.8, 0.8, 0.8)\, ; 0.3, 0.2, 0.4 \rangle\,.$$

It should be emphasized that the equivalent reduced GFN for $\widetilde{A_3}$ is trapezoidal because $p_1 = p_2 = p_3$ and $q_1 = q_2 = q_3$. Similarly, it is triangular for $\widetilde{A_1}$. According to the risk-attitude ranking index values, which are given in Table 10, $\widetilde{A_1} \prec \widetilde{A_3} \prec \widetilde{A_2}$. The ordering is identical to that which is given in [72]. The same ordering is obtained with the ranking function (5)*; expected values are 0.4875, 0.75, and 0.66, respectively.*

Table 10: Ranking indexes of Example 25

		EV	x_0	y_0	R
$\lambda = 0.4$	$\widetilde{A_1}$	0.2250	0.3667	0.2000	0.4177
$\omega = 0.8$	$\widetilde{A_2}$	0.1885	0.6333	0.2826	0.6935
	$\widetilde{A_3}$	0.3000	0.5889	0.1944	0.6202
Risk-neutral ranking				$\widetilde{A_1} \prec \widetilde{A_3} \prec \widetilde{A_2}$	
$\lambda = 0.1$	$\widetilde{A_1}$	0.2138	0.3667	0.1900	0.4130
$\omega = 0.8$	$\widetilde{A_2}$	0.1885	0.6333	0.2826	0.6935
	$\widetilde{A_3}$	0.2820	0.5889	0.1828	0.6166
Pessimistic ranking				$\widetilde{A_1} \prec \widetilde{A_3} \prec \widetilde{A_2}$	
$\lambda = 0.7$	$\widetilde{A_1}$	0.2363	0.3667	0.2100	0.4225
$\omega = 0.8$	$\widetilde{A_2}$	0.1885	0.6333	0.2826	0.6935
	$\widetilde{A_3}$	0.3180	0.5889	0.2061	0.6239
Optimistic ranking				$\widetilde{A_1} \prec \widetilde{A_3} \prec \widetilde{A_2}$	

Example 26. *[73] Let us consider two different reductions for triangular GIFNs $\widetilde{A}$ and $\widetilde{B}$, one involving the vanishing refusal and the other involving the vanishing*

neutrality. Suppose that

$$\begin{aligned}
\textit{Case 1: } \widetilde{A} &= \langle (0.3, 0.3, 0.3, 0.5, 0.5, 0.7, 0.7, 0.7)\,; 0.6, 0.1, 0.3 \rangle\,, \\
\widetilde{B} &= \langle (0.1, 0.1, 0.1, 0.5, 0.5, 0.9, 0.9, 0.9)\,; 0.5, 0.3, 0.2 \rangle\,, \\
\textit{Case 2: } \widetilde{A} &= \langle (0.3, 0.3, 0.3, 0.5, 0.5, 0.7, 0.7, 0.7)\,; 0.6, 0, 0.3 \rangle\,, \\
\widetilde{B} &= \langle (0.1, 0.1, 0.1, 0.5, 0.5, 0.9, 0.9, 0.9)\,; 0.5, 0, 0.2 \rangle\,.
\end{aligned}$$

$x_0(\widetilde{A}) = x_0(\widetilde{B}) = 0.5$ *for both cases. Therefore, the ranking index is determined only by* y_0*; that is, it is sensitive to the maximal membership value. In the first case, the ranking index is sensitive only to parameter* ω*. Likewise, in the second case, the ranking index is sensitive only to parameter* λ*. For Case 1, the maximal membership values are as follows:*

$$\begin{aligned}
\theta &= (1-\lambda)\alpha + \lambda(1-\beta) + (\omega-\lambda)\gamma \\
&= (1-\lambda)\alpha + \lambda(1-\beta) + (\omega-\lambda)(1-\alpha-\beta) \\
&= (1-\omega)\alpha + \omega(1-\beta), \\
\theta_{\widetilde{A}} &= 0.6 + 0.1\omega, \\
\theta_{\widetilde{B}} &= 0.5 + 0.3\omega.
\end{aligned}$$

Similarly, the maximal membership values for Case 2 are as follows:

$$\begin{aligned}
\theta &= (1-\lambda)\alpha + \lambda(1-\beta), \\
\theta_{\widetilde{A}} &= 0.6 + 0.1\lambda, \\
\theta_{\widetilde{B}} &= 0.5 + 0.3\lambda.
\end{aligned}$$

The optimistic, pessimistic, and neutral rankings of $\widetilde{A}$ *and* $\widetilde{B}$ *are respectively given as:*

(i) *If* $\omega > 0.5$ *(or* $\lambda > 0.5$*, respectively), then* $\theta_{\widetilde{A}} < \theta_{\widetilde{B}}$*, so* $\widetilde{A} \prec \widetilde{B}$*,*

(ii) *Else if* $\omega < 0.5$ *(or* $\lambda < 0.5$*, respectively), then* $\theta_{\widetilde{A}} > \theta_{\widetilde{B}}$*, so* $\widetilde{A} \succ \widetilde{B}$*,*

(iii) *Else (if* $\omega = 0.5$ *(or* $\lambda = 0.5$*, respectively), then)* $\theta_{\widetilde{A}} = \theta_{\widetilde{B}}$*, so* $\widetilde{A} \sim \widetilde{B}$*.*

See Table 11 for the results found with the defuzzification function (2)*. The outcomes we achieve are identical to the ones acquired here.*

6. MADM APPLICATIONS

This section contains two applications. The first one is characterized by the GTIFN parameters, and the second one is characterized by the linguistic GTPFN parameters.

Table 11: Defuzzifications of Example 26

	$r \to 0$	$r = \frac{1}{2}$	$r = 1$	$r = 2$	$r \to \infty$
$\widetilde{A}$	0.350	0.333	0.325	0.317	0.300
$\widetilde{B}$	0.400	0.350	0.325	0.300	0.250
Result	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \prec \widetilde{B}$	$\widetilde{A} \sim \widetilde{B}$	$\widetilde{A} \succ \widetilde{B}$	$\widetilde{A} \succ \widetilde{B}$

6.1. Video monitoring system selection

This application is adapted from [47]. To improve campus security, it is important to establish a video monitoring system (VMS) in a school that intends to procure surveillance systems from existing VMS providers (S_1, S_2, S_3, or S_4). The decision-maker assesses the four prospective VMS providers based on comprehensive provider evaluations, considering five factors: product mobility (a_1), false positive rate (a_2), real-time (a_3), picture quality (a_4), and security (a_5). One can find the normalized evaluations, represented as GTIFNs, of the possible VMS providers with these characteristics in [47]. The decision-maker assigns weights to the attributes based on their experience, expertise, and judgment. Let the attribute weight vector be $(0.36, 0.18, 0.05, 0.11, 0.30)^T$.

Table 12 displays the weighted average scores for each provider, which are obtained using the arithmetic operations outlined in Definition 10. It is important to remember that we do not employ the minimum operator to ascertain the overall maximal membership value. The overall α and β values are rounded to four decimal places in Table 12.

Table 12: Weighted average scores of VMS providers

	$p_3 = p_2 = p_1$	$m = n$	$q_1 = q_2 = q_3$	α	β
S_1	0.373	0.507	0.769	0.5650	0.2436
S_2	0.410	0.510	0.904	0.5217	0.2497
S_3	0.364	0.534	0.690	0.5691	0.2196
S_4	0.404	0.520	0.744	0.5689	0.2090

We consider pessimistic, risk-neutral, and optimistic orderings, with a fixed $\omega = 1$ and the corresponding parameter values being $\lambda = 0.01, 0.2, 0.5, 0.7, 0.99$. See Table 13 for results observed to be insensitive to the changes in the λ risk parameter, even for its extreme values. Previous studies offered S_4 as an option in certain instances for varying risk attitudes. Refer to [47] for further information.

6.2. Risk analysis application

This risk analysis application is adapted from [71]. Here, the parameters of the problem are the probability of failure and severity of loss, which are expressed

Table 13: Ranking indexes of VMS providers

λ	0.01	0.2	0.5	0.7	0.99
S_1	0.5812	0.5853	0.5921	0.5970	0.6045
S_2	0.6326	0.6367	0.6439	0.6491	0.6571
S_3	0.5625	0.5672	0.5751	0.5808	0.5895
S_4	0.5877	0.5924	0.6004	0.6062	0.6150
Best Provider	S_2	S_2	S_2	S_2	S_2

as linguistic GTPFNs. It is aimed at identifying the highest risk of failure in the production system for the decision-making process.

Consider three production systems in poultry farming: C_i, $i = 1, 2, 3$. Each production system consists of eight sub-components, denoted as A_{ik}, where $k = 1, \ldots, 8$. These are availability of land, financial support, availability of expert laborers, availability of clean water, transportation, availability of electricity, food supply, and good poultry. The sub-component A_{ik} is assessed in terms of two factors: the probability of failure $\widetilde{R_{ik}}$ and the severity of loss $\widetilde{W_{ik}}$, which are represented as linguistic terms given in Table 14. Assessments are constructed in Table 15 by the experts in a picture fuzzy environment.

Table 14: Linguistic scale and corresponding GTPFNs

Linguistic term	GTPFNs
Absolutely-low	$\langle(0, 0.01, 0.03, 0.06, 0.1, 0.13, 0.15, 0.16)\,; 0.2, 0.4, 0.35\rangle$
Very-low	$\langle(0.1, 0.13, 0.15, 0.16, 0.2, 0.21, 0.23, 0.26)\,; 0.3, 0.35, 0.3\rangle$
Low	$\langle(0.2, 0.21, 0.23, 0.26, 0.3, 0.33, 0.35, 0.36)\,; 0.4, 0.3, 0.25\rangle$
Fairly-low	$\langle(0.3, 0.32, 0.34, 0.36, 0.4, 0.42, 0.44, 0.46)\,; 0.5, 0.25, 0.2\rangle$
Medium	$\langle(0.4, 0.42, 0.44, 0.46, 0.5, 0.52, 0.54, 0.56)\,; 0.6, 0.2, 0.15\rangle$
Fairly-high	$\langle(0.5, 0.52, 0.54, 0.56, 0.6, 0.62, 0.64, 0.66)\,; 0.7, 0.15, 0.1\rangle$
High	$\langle(0.6, 0.62, 0.64, 0.66, 0.7, 0.72, 0.74, 0.76)\,; 0.8, 0.1, 0.05\rangle$
Very-high	$\langle(0.7, 0.72, 0.74, 0.76, 0.8, 0.82, 0.84, 0.86)\,; 0.9, 0.05, 0\rangle$
Absolutely-high	$\langle(0.8, 0.82, 0.84, 0.86, 0.9, 0.92, 0.94, 1)\,; 1, 0, 0\rangle$

We search for the answer to which farmer will be at the greatest risk of failure under such conditions. The weighted average risk $\widetilde{R_i}$ is calculated using the following formula:

$$\widetilde{R_i} = \frac{\sum\limits_{k=1}^{8} \widetilde{W_{ik}} \times \widetilde{R_{ik}}}{\sum\limits_{k=1}^{8} \widetilde{W_{ik}}}, i = 1, 2, 3. \quad (6)$$

By using Equation (6) and the arithmetic operations in Definition 10, the weighted average risks are GTPFNs calculated as follows:

$$\widetilde{R_1} = \langle (0.04, 0.06, 0.09, 0.13, 0.24, 0.34, 0.46, 0.59)\,; 0.05, 0.27, 0.60\rangle\,,$$
$$\widetilde{R_2} = \langle (0.21, 0.24, 0.28, 0.33, 0.43, 0.50, 0.57, 0.65)\,; 0.29, 0.28, 0.32\rangle\,,$$
$$\widetilde{R_3} = \langle (0.15, 0.18, 0.22, 0.26, 0.37, 0.44, 0.52, 0.61)\,; 0.21, 0.29, 0.40\rangle\,,$$

where numbers are rounded to two decimal places. Given $\omega = 1$ and $\lambda = 0.5$, the ranking indexes are 0.2879, 0.4702, and 0.4069, respectively. This indicates that C_2 has the greatest weighted average risk, while C_1 has the lowest. The results coincide with the author's findings in [71].

7. CONCLUSION

In real applications, it is common to have a limited amount of historical data. Therefore, we depend on imprecise data that is derived from expert opinions and estimations. On top of that, experts may exhibit hesitancy and lack of confidence in their assessment or even decline to provide one. In such cases, the use of non-standard FNs is required. Existing indeterminacy and/or reluctance have to be somehow overcome to determine true membership value. When there is hesitancy regarding belonging to an element in a fuzzy set, the actual membership lies between the given membership value and 1 minus the non-membership value. Hence, we calculate the true membership value as a convex combination of these values or add some proportion of this indeterminacy to the given membership value. This parametric calculation allows us to adjust risk aversion in some sense while reducing uncertainty through standardization at the same time. This process is similar in a picture fuzzy environment.

This study develops a new reduction-based ranking principle for a particular class of generalized and/or non-standard FNs by employing their centroid point and ranking index. Due to its general structure, the applicability of the suggested ranking method for reductions and analogous structures is advantageous. We observe that our method is consistent with the risk-attitudinal methods described in the literature. Two modifications are implemented, considering the critiques presented in the literature concerning the coherence of the rankings. These are ones regarding arithmetic operations and the idea of applying normalization. Important advantages include the elimination of inconsistencies in the rankings, the simplification of calculations, and the generation of the expected value as a defuzzification tool.

Table 15: Linguistic terms of $\widetilde{R_{ik}}$ and $\widetilde{W_{ik}}$ for the sub-components A_{ik}

C_i	Sub-component	$\widetilde{R_{ik}}$	$\widetilde{W_{ik}}$
C_1	A_{11}	Very-low	Absolutely-low
	A_{12}	Absolutely-low	Very-low
	A_{13}	Very-low	Absolutely-low
	A_{14}	Absolutely-low	Fairly-low
	A_{15}	Low	Fairly-low
	A_{16}	Low	Very-low
	A_{17}	Very-low	Absolutely-low
	A_{18}	Very-low	Low
C_2	A_{21}	Very-low	Absolutely-low
	A_{22}	High	Fairly-high
	A_{23}	Fairly-high	High
	A_{24}	Very-low	Fairly-low
	A_{25}	Low	Fairly-low
	A_{26}	Low	High
	A_{27}	Very-low	Absolutely-low
	A_{28}	Very-low	Fairly-high
C_3	A_{31}	Very-low	Absolutely-low
	A_{32}	Absolutely-low	Very-low
	A_{33}	High	Fairly-high
	A_{34}	Very-low	Fairly-low
	A_{35}	Low	Fairly-low
	A_{36}	Low	Medium
	A_{37}	Very-low	Absolutely-low
	A_{38}	Very-low	Fairly-high

Furthermore, the picture fuzzy scaling proposed in this article allows us to use a greater quantity of linguistic terms in MADM applications, as opposed to fuzzy and intuitionistic scaling.

Unfortunately, the method has some limitations. Although the proposed ranking technique claims it addresses the different risk perspectives of the decision-maker, it has been observed that, in some cases, it lacks sensitivity to alterations in risk parameters. Moreover, even though it finds the ranking index values very close when comparing two numbers corresponding to the fuzzy zero, which are symmetrical with respect to the y-axis, it does not successfully make the logical comparison of equality.

Based on the results of this study, the suggested approach not only enhances existing literature but also indicates other potential avenues for further investigation. Future research involves the examination of mathematical programming problems with hybrid generalized and/or non-standard parameters using newly defined arithmetic operations.

Acknowledgements: The authors express gratitude to the anonymous reviewers for their constructive comments.

Funding: This research received no external funding.

REFERENCES

[1] K. T. Atanassov and S. Stoeva, "Intuitionistic fuzzy sets," *Fuzzy sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[2] B. C. Cuong and V. H. Pham, "Some fuzzy logic operators for picture fuzzy sets," in *2015 seventh international conference on knowledge and systems engineering (KSE)*. IEEE, 2015, pp. 132–137.

[3] C.-H. Cheng, "A new approach for ranking fuzzy numbers by distance method," *Fuzzy sets and systems*, vol. 95, no. 3, pp. 307–317, 1998.

[4] T.-C. Chu and C.-T. Tsao, "Ranking fuzzy numbers with an area between the centroid point and original point," *Computers & mathematics with applications*, vol. 43, no. 1-2, pp. 111–117, 2002.

[5] Y.-M. Wang, J.-B. Yang, D.-L. Xu, and K.-S. Chin, "On the centroids of fuzzy numbers," *Fuzzy sets and systems*, vol. 157, no. 7, pp. 919–926, 2006.

[6] S.-M. Chen and K. Sanguansat, "Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers," *Expert Systems with Applications*, vol. 38, no. 3, pp. 2163–2171, 2011.

[7] P. Xu, X. Su, J. Wu, X. Sun, Y. Zhang, and Y. Deng, "A note on ranking generalized fuzzy numbers," *Expert Systems with Applications*, vol. 39, no. 7, pp. 6454–6457, 2012.

[8] H. T. X. Chi and F. Y. Vincent, "Ranking generalized fuzzy numbers based on centroid and rank index," *Applied Soft Computing*, vol. 68, pp. 283–292, 2018.

[9] R. Bihari, S. Jeevaraj, and A. Kumar, "A new ranking principle for ordering generalized trapezoidal fuzzy numbers based on diagonal distance, mean and its applications to supplier selection," *Soft Computing*, pp. 1–26, 2023.

[10] W. Jianqiang and Z. Zhong, "Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems," *Journal of Systems Engineering and Electronics*, vol. 20, no. 2, pp. 321–326, 2009.

[11] X. Li, F. Wang, X. Chen *et al.*, "Trapezoidal intuitionistic fuzzy multiattribute decision making method based on cumulative prospect theory and Dempster-Shafer theory," *Journal of Applied Mathematics*, vol. 2014, 2014.

[12] D.-F. Li, "A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems," *Computers & Mathematics with Applications*, vol. 60, no. 6, pp. 1557–1570, 2010.

[13] A. K. Nishad, S. K. Bharati, and S. R. Singh, "A new centroid method of ranking for intuitionistic fuzzy numbers," in *Proceedings of the Second International Conference on Soft Computing for Problem Solving (SocProS 2012), December 28-30, 2012*. Springer, 2014, pp. 151–159.

[14] A. K. Nishad and Abhishekh, "A new ranking approach for solving fully fuzzy transportation problem in intuitionistic fuzzy environment," *Journal of Control, Automation and Electrical Systems*, vol. 31, pp. 900–911, 2020.

[15] S. Aggarwal and C. Gupta, "Solving intuitionistic fuzzy solid transportation problem via new ranking method based on signed distance," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 24, no. 04, pp. 483–501, 2016.

[16] I. Beg, M. Bisht, and S. Rawat, "An approach for solving fully generalized intuitionistic fuzzy transportation problems," *Computational and Applied Mathematics*, vol. 42, no. 8, p. 329, 2023.

[17] A. N. Gani and V. Mohamed, "A method of ranking generalized trapezoidal intuitionistic fuzzy numbers," *International Journal of Applied Engineering Research*, vol. 10, no. 10, pp. 25 465–25 473, 2015.

[18] D. S. Dinagar and E. F. Helena, "Similarity measure on generalized trapezoidal intuitionistic fuzzy numbers with centroid ranking," *J Composition Theor*, vol. 12, no. 10, pp. 265–271, 2019.

[19] B. Abirami, V. Vamitha, and S. Rajaram, "A new approach for solving trapezoidal intuitionistic fuzzy transportation problem," *Advances in Mathematics: Scientific Journal*, vol. 9, no. 11, pp. 9149–9159, 2020.

[20] S.-P. Wan and J.-Y. Dong, "Power geometric operators of trapezoidal intuitionistic fuzzy numbers and application to multi-attribute group decision making," *Applied Soft Computing*, vol. 29, pp. 153–168, 2015.

[21] S.-P. Wan, Q.-Y. Wang, and J.-Y. Dong, "The extended VIKOR method for multi-attribute group decision making with triangular intuitionistic fuzzy numbers," *Knowledge-Based Systems*, vol. 52, pp. 65–77, 2013.

[22] S.-P. Wan, "Power average operators of trapezoidal intuitionistic fuzzy numbers and application to multi-attribute group decision making," *Applied mathematical modelling*, vol. 37, no. 6, pp. 4112–4126, 2013.

[23] S.-P. Wan and Z.-H. Yi, "Power average of trapezoidal intuitionistic fuzzy numbers using strict t-norms and t-conorms," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 5, pp. 1035–1047, 2015.

[24] S. Wan, J. Dong, and D. Yang, "Trapezoidal intuitionistic fuzzy prioritized aggregation operators and application to multi-attribute decision making," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 4, pp. 1–32, 2015.

[25] S.-P. Wan, F. Wang, L.-L. Lin, and J.-Y. Dong, "Some new generalized aggregation operators for triangular intuitionistic fuzzy numbers and application to multi-attribute group decision making," *Computers & Industrial Engineering*, vol. 93, pp. 286–301, 2016.

[26] P. Gupta, M. K. Mehlawat, and N. Grover, "Intuitionistic fuzzy multi-attribute group decision-making with an application to plant location selection based on a new extended VIKOR method," *Information Sciences*, vol. 370, pp. 184–203, 2016.

[27] T. Wu, "Heterogeneous opinion dynamics considering consensus evolution in social network group decision-making," *Group Decision and Negotiation*, pp. 1–36, 2023.

[28] W. Su Mail, J. Zhou, S. Zeng, C. Zhang, and K. Yu, "A novel method for intuitionistic fuzzy MAGDM with Bonferroni weighted harmonic means," *Recent Patents on Computer Science*, vol. 10, no. 2, pp. 178–189, 2017.

[29] B. Dutta and D. Guha, "Trapezoidal intuitionistic fuzzy bonferroni means and its application in multi-attribute decision making," in *2013 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*. IEEE, 2013, pp. 1–8.

[30] J. Zhou, T. Baležentis, and D. Streimikiene, "Normalized weighted bonferroni harmonic mean-based intuitionistic fuzzy operators and their application to the sustainable selection of search and rescue robots," *Symmetry*, vol. 11, no. 2, p. 218, 2019.

[31] S. Das and D. Guha, "A centroid-based ranking method of trapezoidal intuitionistic fuzzy numbers and its application to MCDM problems," *Fuzzy Information and Engineering*, vol. 8, no. 1, pp. 41–74, 2016.

[32] ——, "Family of harmonic aggregation operators under inuitionistic fuzzy environment," *Scientia Iranica*, vol. 24, no. 6, pp. 3308–3323, 2017.

[33] V. L. G. Nayagam, S. Jeevaraj, and P. Dhanasekaran, "An improved ranking method for comparing trapezoidal intuitionistic fuzzy numbers and its applications to multicriteria decision making," *Neural Computing and Applications*, vol. 30, pp. 671–682, 2018.

[34] M. Darehmiraki, "A novel parametric ranking method for intuitionistic fuzzy numbers," *Iranian journal of fuzzy systems*, vol. 16, no. 1, pp. 129–143, 2019.

[35] B. Shakouri, R. Abbasi Shureshjani, B. Daneshian, and F. Hosseinzadeh Lotfi, "A parametric method for ranking intuitionistic fuzzy numbers and its application to solve intuitionistic fuzzy network data envelopment analysis models," *Complexity*, vol. 2020, pp. 1–25, 2020.

[36] Y. Wu, Y. Tao, B. Zhang, S. Wang, C. Xu, and J. Zhou, "A decision framework of offshore wind power station site selection using a promethee method under intuitionistic fuzzy environment: A case in china," *Ocean & Coastal Management*, vol. 184, p. 105016, 2020.

[37] K. Dhilipkumar and C. Praveenkumar, "A method for solving assignment problem using ranking of centroid of centroids in trapezoidal intuitionistic fuzzy number." *South East*

Asian Journal of Mathematics & Mathematical Sciences, vol. 17, no. 3, 2021.

[38] B. Pardha Saradhi, M. Madhuri, and N. Ravi Shankar, "Ordering of intuitionistic fuzzy numbers using centroid of centroids of intuitionistic fuzzy number," *International journal of mathematics trends and technology (IJMTT)*, vol. 52, no. 5, pp. 276–285, 2017.

[39] T. Garai, "Ranking method of the generalized intuitionistic fuzzy numbers founded on possibility measures and its application to MADM problem," *Advances in Computational Intelligence*, vol. 3, no. 4, p. 14, 2023.

[40] Z. Roohanizadeh, E. Baloui Jamkhaneh, and E. Deiri, "The reliability analysis based on the generalized intuitionistic fuzzy two-parameter pareto distribution," *Soft Computing*, vol. 27, no. 6, pp. 3095–3113, 2023.

[41] Y. Wang, "Ranking intuitionistic fuzzy numbers by relative preference relation," *Iranian Journal of Fuzzy Systems*, vol. 20, no. 5, pp. 121–133, 2023.

[42] T. Yogashanthi, S. Sathish, and K. Ganesan, "Generalized intuitionistic fuzzy flow shop scheduling problem with setup time and single transport facility," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 23, no. 1, pp. 34–43, 2023.

[43] M. Verma, A. Kumar, P. Singh, and Y. Singh, "Risk analysis of combustion system using vague ranking method," *Journal of Intelligent & Fuzzy Systems*, vol. 24, no. 4, pp. 765–773, 2013.

[44] J.-q. Wang, R.-r. Nie, H.-y. Zhang, and X.-h. Chen, "Intuitionistic fuzzy multi-criteria decision-making method based on evidential reasoning," *Applied Soft Computing*, vol. 13, no. 4, pp. 1823–1831, 2013.

[45] C. Luo, X. Tian, and S. Wan, "Heavy owa operator of trapezoidal intuitionistic fuzzy numbers and its application to multi-attribute decision making," *Journal of Systems Science and Information*, vol. 3, no. 1, pp. 86–96, 2015.

[46] Q. Qin, F. Liang, L. Li, Y.-W. Chen, and G.-F. Yu, "A todim-based multi-criteria group decision making with triangular intuitionistic fuzzy numbers," *Applied soft computing*, vol. 55, pp. 93–107, 2017.

[47] J. Xu, J. Dong, S. Wan, and J. Gao, "Multiple attribute decision making with triangular intuitionistic fuzzy numbers based on zero-sum game approach," *Iranian Journal of Fuzzy Systems*, vol. 16, no. 3, pp. 97–112, 2019.

[48] Z. Ashraf, M. S. Khan, A. Tiwari, and Q. Danish Lohani, "Difference sequence-based distance measure for intuitionistic fuzzy sets and its application in decision making process," *Soft Computing*, vol. 25, no. 14, pp. 9139–9161, 2021.

[49] A. Keikha, H. Garg, and H. Mishmast Nehi, "An approach based on combining choquet integral and topsis methods to uncertain magdm problems," *Soft Computing*, vol. 25, no. 10, pp. 7181–7195, 2021.

[50] D. Marimuthu and G. Mahapatra, "Multi-criteria decision-making using a complete ranking of generalized trapezoidal fuzzy numbers," *Soft Computing*, vol. 25, no. 15, pp. 9859–9871, 2021.

[51] H. Mitchell, "Ranking-intuitionistic fuzzy numbers," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 12, no. 03, pp. 377–386, 2004.

[52] H. G. Akdemir and H. G. Kocken, "New unified score functions and similarity measures for non-standard fuzzy numbers: an extended TOPSIS method addressing risk attitudes," *Neural Computing and Applications*, vol. 35, no. 19, pp. 14 029–14 046, 2023.

[53] R. R. Yager, "Some aspects of intuitionistic fuzzy sets," *Fuzzy Optimization and Decision Making*, vol. 8, pp. 67–90, 2009.

[54] L. G. N. Velu and B. Ramalingam, "Total ordering on generalized 'n'gonal linear fuzzy numbers," *International Journal of Computational Intelligence Systems*, vol. 16, no. 1, p. 23, 2023.

[55] H. G. Akdemir, H. G. Kocken, and K. Nurdan, "On value-at-risk and conditional value-at-risk measures for intuitionistic and picture fuzzy losses," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 41, no. 6, pp. 583–617, 2023.

[56] N. Veerraju and V. Prasannam, "Solving intuitionistic fuzzy linear programming problem using new GM ranking method," *Advances and Applications in Mathematical Sciences*, vol. 22, no. 2, pp. 537–555, 2022.

[57] M. Akram, I. Ullah, and T. Allahviranloo, "A new method to solve linear programming

problems in the environment of picture fuzzy sets," *Iranian Journal of Fuzzy Systems*, vol. 19, no. 6, pp. 29–49, 2022.

[58] M. A. Mehmood and S. Bashir, "Extended transportation models based on picture fuzzy sets," *Mathematical Problems in Engineering*, vol. 2022, 2022.

[59] M. K. Hasan, A. Sultana, and N. K. Mitra, "Arithmetic operations of generalized trapezoidal picture fuzzy numbers by vertex method," *American Journal of Computational Mathematics*, vol. 13, no. 01, pp. 99–121, 2023.

[60] M. Sotoudeh-Anvari and A. Sotoudeh-Anvari, "Setback in ranking fuzzy numbers: a study in fuzzy risk analysis in diabetes prediction," *Artificial Intelligence Review*, vol. 56, no. 5, pp. 4591–4639, 2023.

[61] J.-Q. Wang, R. Nie, H.-Y. Zhang, and X.-H. Chen, "New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis," *Information Sciences*, vol. 251, pp. 79–95, 2013.

[62] D. Chakraborty, D. K. Jana, and T. K. Roy, "Arithmetic operations on generalized intuitionistic fuzzy number and its applications to transportation problem," *Opsearch*, vol. 52, pp. 431–471, 2015.

[63] A. Keikha and H. M. Nehi, "Operations and ranking methods for intuitionistic fuzzy numbers, a review and new methods," *International Journal of Intelligent Systems Technologies and Applications*, vol. 1, no. 1, pp. 35–48, 2016.

[64] P. Dutta, "Multi-criteria decision making under uncertainty via the operations of generalized intuitionistic fuzzy numbers," *Granular Computing*, vol. 6, no. 2, pp. 321–337, 2021.

[65] T. Garai, D. Chakraborty, and T. K. Roy, "Possibility–necessity–credibility measures on generalized intuitionistic fuzzy number and their applications to multi-product manufacturing system," *Granular Computing*, vol. 3, pp. 285–299, 2018.

[66] P. Dutta and G. Borah, "Multicriteria decision making approach using an efficient novel similarity measure for generalized trapezoidal fuzzy numbers," *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 3, pp. 1507–1529, 2023.

[67] Q. Gu and Z. Xuan, "A new approach for ranking fuzzy numbers based on possibility theory," *Journal of Computational and Applied Mathematics*, vol. 309, pp. 674–682, 2017.

[68] A. Kumar, P. Singh, P. Kaur, and A. Kaur, "A new approach for ranking of L–R type generalized fuzzy numbers," *Expert Systems with Applications*, vol. 38, no. 9, pp. 10 906–10 910, 2011.

[69] H. Rouhparvar and A. Panahi, "A new definition for defuzzification of generalized fuzzy numbers and its application," *Applied Soft Computing*, vol. 30, pp. 577–584, 2015.

[70] P. Singh, M. Verma, and A. Kumar, "A novel method for ranking of vague sets for handling the risk analysis of compressor system," *Applied Soft Computing*, vol. 26, pp. 202–212, 2015.

[71] R. Chutia, "A novel method of ranking intuitionistic fuzzy numbers using value and θ multiple of ambiguity at flexibility parameters," *Soft Computing*, vol. 25, no. 21, pp. 13 297–13 314, 2021.

[72] C. Shit, G. Ghorai, Q. Xin, and M. Gulzar, "Harmonic aggregation operator with trapezoidal picture fuzzy numbers and its application in a multiple-attribute decision-making problem," *Symmetry*, vol. 14, no. 1, p. 135, 2022.

[73] M. Zhang and J. Nan, "A compromise ratio ranking method of triangular intuitionistic fuzzy numbers and its application to madm problems," *Iranian Journal of Fuzzy Systems*, vol. 10, no. 6, pp. 21–37, 2013.