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ANALYSIS OF HETEROGENEOUS TWO SERVER QUEUEING SYSTEM WITH MULTIPLE WORKING VACATIONS AND SERVER BREAKDOWNS

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Abstract: The steady state analysis and the time-dependent behaviour of a heterogeneous two-server multiple vacation queueing model is examined in this research work. The model also takes into account the possibility of a breakdown happening when server 1 is busy while server 2 is on working vacation and both servers are busy, with the restoration procedure starting right away. The stability condition is established through the use of matrix geometric technique, and the steady-state probability vector of the system's number of customers has been evaluated as a quasi-birth and death (QBD) process. Some system performance measures are obtained. Numerical analysis for both steady-state and transient analysis are given to our proposed model.

Keywords: Matrix geometric method, transient analysis, performance measures, quasi birth death process, server breakdown, multiple working vacations.

MSC: 60K25, 60K30, 90B22.

1. INTRODUCTION

In many queueing systems now in use, servers may become unavailable for an arbitrary amount of time while the system is empty. This random period of server absence, known as a server vacation, could indicate that the server is doing a secondary activity. When the server finishes a vacation and there is no customer in the system, the server may take another vacation. This type of vacation called Multiple vacations. In some situations servers can work in vacation also, this type of vacations is called working vacation. Queuing systems with vacations have been established over the past 20 years for many different kinds of uses such as production and computer communication systems. Doshi [1] and Takagi [2] carried out several great surveys on vacation models. Laxmi [3] analysed the single working vacation with impatience customers. A working vacation queue with server breakdown is researched by Madhu Jain [4], and Agarwal[5]. Ammar [6] gives a matrix geometric solution to the single server queueing system with discouraged arrivals and reneging.Several researchers have studied multi-server queues with vacation. In the beginning, the $M/M/c$ queue with exponentially distributed vacation times was examined by Levy [7] and Vinod [8]. Tian [9] deeply discused about $M/M/c$ queueing system with vacation, and gives conditional stochastic decomposition results for waiting time and queue length. Numerous researchers, including Kao and Narayanan [10], Igaki [11], Chao and Zhao [12], Zhang and Tian [13], and Houalef [14], have studied multi-server queueing systems with vacations. Also Shanmugasundaram [15], Yang [16] are analysed about the mullti server queueing systems with working vacations.

The above-discussed research on multi-server queueing systems make the assumption that all of the servers are homogenous, meaning that they all offer the same rates of service throughout the system. This premise might only be true in particular places. In a queueing system with human servers, it is difficult to recognise the aforementioned assumption. Few studies on multi-server queueing system with server vacation and varying service rates among different servers can be found. Madan et al. [17] studied Bernoulli schedules with a single vacation policy in the $M/M/2$ queue, where two servers offer heterogeneous exponential service to customers. For different server states, they were able to derive steady-state probability generating functions for the system size. Using matrix-geometric method, a study on two heterogenous servers with multiple vacations is given by Kumar and Madheswari [18]. Singh [19] discusses the two server Markovian queues with balking and provides a comparison between heterogeneous and homogeneous servers. Mahalakshmi [20] examined the two server queueing systems in conjunction with server failure. Balasubramanian [21] also analysed about the two server Queues with server breakdown. A study on a heterogeneous servers with server breakdown is given by Reni sagayaraj [22]. Recently Liu et al. [23] studied single server queueing model with server breakdown, Karthick and Suvitha [24] analysed about multi server vacation queueing models with breakdown. Sudhesh et al. [25] has done a time dependent analysis of a two server vacation queueing model. However, no comprehensive study exists on the multiserver working vacation queueing model encompassing various breakdown possibilities. As a result, we have proposed this model.

2. THE MODEL DESCRIPTION

The system under consideration is a two-server, multiple-working vacation queueing system subject to server breakdown. It is important to note that the servers' service rates vary. The assumptions of the model are described as following:

1. Customers arrive at a rate of λ according to a Poisson. Customers arrive and join the waiting line in order of their arrival. It is assumed that both the system capacity and the total number of potential customers are infinite.
2. Based on First-Come, First-Served (FCFS) principles, the two servers provide heterogeneous exponential service to customers at service rates of μ_1 and μ_2.
3. Every server is allowed to take an independent vacation when there are no customers waiting in queue.When a vacation period expires and there is a customer in the system, service will start. If not, the server takes another vacation right away and keeps doing so until it returns and discovers that at least one customer is still waiting.
4. If there's at least one customer in the system after the vacation ends, server 1 gets busy, and server 2 goes into a working vacation mode. During the working vacation server 2 provides service to customers with slow service rate μ_{v2}. The vacation rates are θ_1 for server 1 and $p\theta_2$ for server 2. The rate for server 2 from working vacation to busy is $q\theta_2$. Vacation follows an exponential distribution.
5. In addition, there are three possibles for breakdown.
   - In a single server busy state, there are two possibilities for breakdown. One is that server 1 gets a breakdown while server 2 is still working on working vacation, second is that server 2 gets a breakdown while server 1 is still working during a regular busy period.
   - Servers may breakdown in both servers busy state.

   The breakdown follows exponential distribution with rates α_1 and α_2 for servers 1 and 2 respectively.
6. Also the repair of the servers starts immediately, repairs follow exponential distribution with rates β_1 and β_2.

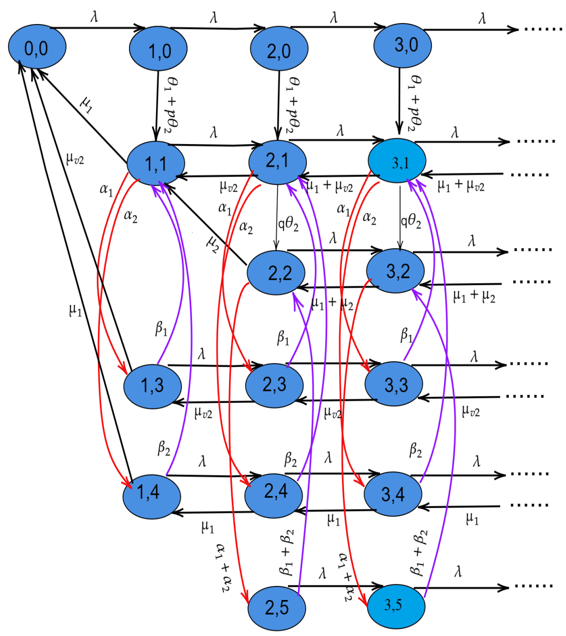

Figure 1: Transition Diagram

2.1. Practical Application

The proposed model is motivated by the following scenario in a wireless communication network: Consider a wireless communication network with two types of signal transmitters, one being a high-powered master transmitter and the other a lower-powered assistant transmitter. The master transmitter can transmit data quickly and over long distances, while the slave transmitter is slower and has a shorter range. Both transmitters decide to enter a low-power mode or standby mode if no users are waiting for service. The standby mode will continue until at least one user needs service. After completion of the standby mode, if at least one user is waiting for service, the master transmitter will transmit data with regular coverage, while the assistant transmitter serves at a slower rate. If the network detects at least two active users requiring a strong signal, it will instantly reactivate both transmitters to provide their primary, high-powered services. Additionally, the network may experience temporary outages, initiating an immediate restoration process.

2.2. The Quasi-Birth-and-Death (QBD) process

The number of customers in the system at time t is indicated by $L(t)$ and let

$$I(t) = \begin{cases} 0, & \text{Both servers are on vacation} \\ 1, & \text{Server 1 being busy and server 2 being in a working vacation} \\ 2, & \text{Both servers being busy} \\ 3, & \text{Server 1 is breakdown and server 2 being in a working vacation} \\ 4, & \text{Server 2 is breakdown and Server 1 being busy} \\ 5, & \text{Both servers are on breakdown} \end{cases}$$

Consequently, $X(t) = \{L(t); I(t)\}$ is a QBD process with the state space indicated by Ω as shown below:
$\Omega = \{(0,0)\} \cup \{(1,j), j = 0,1,3,4\} \cup \{(n,j), n \geq 2, j = 0,1,2,3,4,5\}$
Using lexicographical sequence for the states, The Markov chain's infinitesimal generator, Q is defined as follows:

$$Q = \begin{bmatrix} B_{00} & B_{01} & \dots & & & & \\ B_{10} & B_{11} & B_{12} & \dots & & & \\ \vdots & B_{21} & B_{22} & A_0 & \dots & & \\ & \vdots & A_2 & A_1 & A_0 & \dots & \\ & & \vdots & A_2 & A_1 & A_0 & \dots \\ & & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where
$B_{00} = \begin{bmatrix} -\lambda \end{bmatrix}$, $B_{01} = \begin{bmatrix} \lambda & 0 & 0 & 0 \end{bmatrix}$, $B_{10} = \begin{bmatrix} 0 & \mu_1 & \mu_{v_2} & \mu_1 \end{bmatrix}^T$,

$$B_{11} = \begin{bmatrix} V_1 & \theta_1 + p\theta_2 & 0 & 0 \\ 0 & -(\lambda + \alpha_1 + \alpha_2 + \mu_1) & \alpha_1 & \alpha_2 \\ 0 & \beta_1 & V_2 & 0 \\ 0 & \beta_2 & 0 & V_3 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_{v2} & 0 \\ 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} V_1 & \theta_1 + p\theta_2 & 0 & 0 & 0 & 0 \\ 0 & V_4 + \mu_1 & q\theta_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & V_5 + \mu_1 & 0 & 0 & -(\alpha_1 + \alpha_2) \\ 0 & \beta_1 & 0 & V_2 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & V_3 & 0 \\ 0 & 0 & \beta_1 + \beta_2 & 0 & 0 & -(\lambda + \beta_1 + \beta_2) \end{bmatrix},$$

$$A_0 = \lambda I_6, \; A_1 = \begin{bmatrix} V_1 & \theta_1 + p\theta_2 & 0 & 0 & 0 & 0 \\ 0 & V_4 & q\theta_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & V_5 & 0 & 0 & -(\alpha_1 + \alpha_2) \\ 0 & \beta_1 & 0 & V_2 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & V_3 & 0 \\ 0 & 0 & \beta_1 + \beta_2 & 0 & 0 & -(\lambda + \beta_1 + \beta_2) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\mu_1 + \mu_{v_2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mu_1 + \mu_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{v2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$V_1 = -(\lambda + \theta_1 + p\theta_2)$, $V_2 = -(\lambda + \beta_1 + \mu_{v_2})$
$V_3 = -(\lambda + \beta_2 + \mu_1)$, $V_4 = -(\lambda + \alpha_1 + \alpha_2 + \mu_1 + \mu_{v2} + q\theta_2)$
$V_5 = -(\lambda + \alpha_1 + \alpha_2 + \mu_1 + \mu_2)$

3. THE STEADY STATE ANALYSIS

In order for the system to reach a steady state, we need to first find the stability condition. Then, a matrix geometric solution approach is used to calculate the system's steady-state probabilities. Furthermore provided are the rate matrix and boundary probability vector computations.

3.1. Stability Condition

We first define the matrix $A = A_0 + A_1 + A_2$ in order to derive the stationary condition. Then the matrix A can be written as

$$A = \begin{bmatrix} -(\theta_1 + p\theta_2) & (\theta_1 + p\theta_2) & 0 & 0 & 0 & 0 \\ 0 & -(\alpha_1 + \alpha_2 + q\theta_2) & q\theta_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & -(\alpha_1 + \alpha_2) & 0 & 0 & \alpha_1 + \alpha_2 \\ 0 & \beta_1 & 0 & -\beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \beta_1 + \beta_2 & 0 & 0 & -(\beta_1 + \beta_2) \end{bmatrix}$$

It is clear that A is a generator of irreducible Markov processes. Let π be the stationary probability vector for this Markov process, with $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$. The linear equations are thus satisfied by π:

$$\pi A = 0, \pi e = 1$$

Neuts (1981) asserted that the system is stable if and only if $\pi A_0 e < \pi A_2 e$. Specifically, the system is stable if and only if $\rho < 1$ where

$$\rho = \frac{\lambda(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)}{(\mu_1 + \mu_2)(\beta_1 + \beta_2)}$$

3.2. Matrix geometric Solution

The stationary random variables $L(t)$ and $I(t)$ should represent the system's customer base and the status of its servers, respectively. We use to represent the stationary probability
$P_{n,i} = \lim_{t \to \infty} P\{L(t) = n, I(t) = i\}$, $(n, i) \in \Omega$. The boundary probability vector P of the generator Q exists under the stationary condition $\rho < 1$. The partitioning for this stationary probability vector P is $P = (p_0, p_1, p_2, ...)$, where $p_0 = p_{00}$, $p_1 = (p_{10}, p_{11}, p_{13}, p_{14})$, $p_i = (p_{i0}, p_{i1}, p_{i2}, p_{i3}, p_{i4}, p_{i5})$ for $i \geq 2$. The following equations generate the sub vectors p_i,

$$p_0 B_{00} + p_1 B_{10} = 0, \tag{1}$$

$$p_0 B_{01} + p_1 B_{11} + p_2 B_{21} = 0, \tag{2}$$

$$p_1 B_{12} + p_2 B_{22} + p_3 A_2 = 0, \tag{3}$$

$$p_i A_0 + p_{i+1} A_1 + p_{i+2} A_2 = 0 \text{ for } i \geq 2,$$

$$p_i = p_2 R^{(i-2)} \text{ for } i \geq 3, \tag{4}$$

According to Neuts [26], there exists a minimal non-negative solution for the rate matrix R. Substituting the (4) in (3) we have

$$p_1 B_{12} + p_2 (B_{22} + R A_2) = 0 \tag{5}$$

and the normalizing condition is

$$p_0 e_1 + p_1 e_2 + p_2 (I - R)^{-1} e_3 = 1 \tag{6}$$

where the column vectors e_1, e_2, and e_3 each have one element in the correct sequence. The matrix R is the minimum non-negative answer to the following matrix quadratic equation:

$$R^2 A_2 + R A_1 + A_0 = 0. \tag{7}$$

According to Netus and Lucantoni [27],consider the sequence of matrices $\{R(n), k \geq 0\}$ with initial value $R(0) = 0$. From (7) we arrive at

$$R(n+1) = -[A_0 + R^2(n) A_2] A_1^{-1}, n \geq 0. \tag{8}$$

The above equation (8) monotonically converges to the minimal non negative solution to (7) with spectral radius less than 1. Using $R(0) = 0$, we compute first iteration of R matrix i.e., $R(1)$ then using first iteration of R Matrix we can compute second iteration of R matrix i.e., $R(2)$. Similarly, we can compute the further iterations of R matirx. Due to the positive values of $-A_1^{-1}$ and $A_0 + R^2 A_2$, the value of R will converge. As a result, the components of R will rise monotonically after each repetition.

3.3. Boundary Probability Vectors

To obtain the boundary probability vectors p_0, p_1, and p_2, equations (1) to (6) must be solved. We can also define the matrices

$$D = B_{10} B_{00}^{-1} B_{01} - B_{11},$$

we have

$$D = \begin{bmatrix} \lambda + \theta_1 + p\theta_2 & -(\theta_1 + p\theta_2) & 0 & 0 \\ -\mu_1 & \lambda + \mu_1 + \alpha_1 + \alpha_2 & -\alpha_1 & -\alpha_2 \\ -\mu_{v2} & -\beta_1 & \lambda + \beta_1 + \mu_{v2} & 0 \\ -\mu_1 & -\beta_2 & 0 & \lambda + \mu_1 + \beta_2 \end{bmatrix}$$

Here, the D matrix that is shown above is invertible. The following theorem yields the boundary probability vectors p_0, p_1, and p_2.

Theorem 1. *The following are the boundary probability vectors:*
$p_0 = -p_2 B_{21} D^{-1} B_{10} B_{00}^{-1}$
$p_1 = p_2 B_{21} D^{-1}$
and as shown in the following equations, p_2 is calculated:

$$\begin{cases} p_2(B_{21} D^{-1} B_{12} + B_{22} + RA_2) = 0 \\ p_2[-B_{21} D^{-1} B_{10} B_{00}^{-1} e_1 + B_{21} D^{-1} e_2 + (I - R)^{-1} e_3] = 1 \end{cases} \tag{9}$$

Proof. We know B_{00} is invertable and from (1) we get

$$p_0 = -p_1 B_{10} B_{00}^{-1} \tag{10}$$

from (2) we get

$$p_1 = p_2 B_{21} D^{-1} \tag{11}$$

using (11) in equation (10) we get

$$p_0 = -p_2 B_{21} D^{-1} B_{10} B_{00}^{-1} \tag{12}$$

using (10) and (11) in (5) and (6) we arrive (9).
This gives the required proof. □

3.4. Remark

If $\alpha_i = \beta_i = 0$ for $i = 1, 2$ in the current model, the stability condition is reduced to the stability condition of the $M/M/2$ multiple vacation queueing system, which is studied by Krishnamoorthy and Sreenivasan [28].

4. TRANSIENT ANALYSIS

4.1. Govering Equations

Let $P_{n,j}(t)$ be the time-dependent probability for the system to be in state j with n customers at time t.

$$P_{0,0}^{'}(t) = -\lambda P_{0,0}(t) + \mu_1 P_{1,1}(t) + \mu_{v2} P_{1,3}(t) + \mu_1 P_{1,1}(t) \tag{13}$$

$$P_{n,0}^{'}(t) = -(\lambda + \theta_1 + p\theta_2) P_{n,0}(t) + \lambda P_{n-1,0}(t) \text{ for } n \geq 1 \tag{14}$$

$$P_{1,1}^{'}(t) = -(\lambda + \mu_1 + \alpha_1 + \alpha_2) P_{1,1}(t) + (\theta_1 + p\theta_2) P_{1,0}(t) + \mu_{v2} P_{2,1}(t) + \mu_2 P_{2,2}(t) + \beta_1 P_{1,3}(t) + \beta_2 P_{1,4}(t) \tag{15}$$

$$P_{2,1}^{'}(t) = -(\lambda + \mu_{v2} + \alpha_1 + \alpha_2 + q\theta_1) P_{2,1}(t) + (\theta_1 + p\theta_2) P_{2,0}(t) + (\mu_1 + \mu_{v2}) P_{3,1}(t) + \beta_1 P_{2,3}(t) + \beta_2 P_{2,4}(t) + \lambda P_{1,1}(t) \tag{16}$$

$$P_{n,1}^{'}(t) = -(\lambda + \mu_1 + \mu_{v2} + \alpha_1 + \alpha_2 + q\theta_2) P_{n,1}(t) + (\theta_1 + p\theta_2) P_{n,0}(t) + \beta_1 P_{n,3}(t) + \beta_2 P_{n,4}(t) + \lambda P_{n-1,1}(t) + (\mu_1 + \mu_{v2}) P_{n+1,1}(t) \text{ for } n \geq 3 \tag{17}$$

$$P_{2,2}^{'}(t) = -(\lambda + \mu_2 + \alpha_1 + \alpha_2) P_{2,2}(t) + q\theta_2 P_{2,1}(t) + (\beta_1 + \beta_2) P_{2,5}(t) + (\mu_1 + \mu_2) P_{3,2}(t) \tag{18}$$

$$P_{n,2}^{'}(t) = -(\lambda + \mu_1 + \mu_2 + \alpha_1 + \alpha_2) P_{n,2}(t) + q\theta_2 P_{n,1}(t) + (\beta_1 + \beta_2) P_{n,5}(t) + \lambda P_{n-1,2}(t) + (\mu_1 + \mu_2) P_{n+1,2}(t) \text{ for } n \geq 3 \tag{19}$$

$$P_{1,3}^{'}(t) = -(\lambda + \mu_{v2} + \beta_1) P_{1,3}(t) + \alpha_1 P_{1,1}(t) + \mu_{v_2} P_{2,3}(t) \tag{20}$$

$$P_{n,3}^{'}(t) = -(\lambda + \mu_{v2} + \beta_1) P_{n,3}(t) + \alpha_1 P_{n,1}(t) + \mu_{v_2} P_{n+1,3}(t) + \lambda P_{n-1,3}(t) \text{ for } n \geq 2 \tag{21}$$

$$P_{1,4}^{'}(t) = -(\lambda + \mu_1 + \beta_2) P_{1,4}(t) + \alpha_2 P_{1,1}(t) + \mu_1 P_{2,4}(t) \tag{22}$$

$$P_{n,4}^{'}(t) = -(\lambda + \mu_1 + \beta_2) P_{n,4}(t) + \alpha_2 P_{n,1}(t) + \mu_1 P_{n+1,4}(t) + \lambda P_{n-1,4}(t) \text{ for } n \geq 2 \tag{23}$$

$$P_{2,5}^{'}(t) = -(\lambda + \beta_1 + \beta_2) P_{2,5}(t) + (\alpha_1 + \alpha_2) P_{2,2}(t) \tag{24}$$

$$P_{n,5}^{'}(t) = -(\lambda + \beta_1 + \beta_2) P_{n,5}(t) + (\alpha_1 + \alpha_2) P_{n,2}(t) + \lambda P_{n-1,5}(t) \text{ for } n \geq 3 \tag{25}$$

with the initial state probabilities given by, all probabilities should be zero apart from $P_{0,0}(0)$.

4.2. The Solution Approach

Most queuing systems utilise a set of differential equations as their governing equations. There are several techniques to analyse the model involving steady state governing equations. Techniques like the matrix analytical approach and probability generating methods are used frequently. However, often it is difficult to discover the analytical solution because of the transient and complex character of the differential equations using the queuing models. In this study, we take into account a numerical method based on the Runge-Kutta method for finding the answers to the set of differential equations. The transient numerical results corresponding to the differential-difference equation of the model can be found using the MATLAB *ode45* function.

5. PERFORMANCE MEASURES

We have computed the following performance metrics for the proposed model.

1. Mean number of customers in the system
   $E(L) = p_1 e_2 + p_2 (I-R)^{-2} R^{-1} e_3 - p_2 R^{-1} e_3.$
2. The probability that
   - both servers are on vacation $P_v{=}P_0 + P_1 e_4^1 + P_2(I-R)^{-1} e_6^1$
   - server 1 being busy and server 2 being in a working vacation $P_{1b}^{2WV} = P_1 e_4^2 + P_2(I-R)^{-1} e_6^2$
   - both servers are on busy $P_b = P_2(I-R)^{-}1 e_6^3$
   - server 1 is breakdown and server 2 being in a working vacation $P_{1br} = P_1 e_4^3 + P_2(I-R)^{-1} e_6^4$
   - server 2 is breakdown and Server 1 being busy $P_{2br} = P_1 e_4^4 + P_2(I-R)^{-1} e_6^5$
   - the both servers are breakdown $P_b = P_2(I-R)^{-1} e_6^6$
     Here e_l^m is a column vector of order $l \times 1$ with m^{th} element equal to one and other elements are zero. Where $l = 4, 6, m = 1, 2, 3, 4, 5, 6$.
3. The following cost components that are incurred per unit of time are considered while formulating the cost function:
   - C_1 : Cost associated with holding each customer in the system
   - C_2 : Costs associated with the system when both servers are on vacation
   - C_3 : Costs associated with the system server 1 being busy and server 2 being working vacation on the system
   - C_4 : Costs associated with the system when both servers are busy
   - C_5 : Costs associated with the system when it is in a server 1 breakdown
   - C_6 : Costs associated with the system when it is in a server 2 breakdown
   - C_7 : Costs associated with maintaining the system when both servers are breakdown
   - C_{R1} : Costs associated with repairing server 1
   - C_{R2} : Costs associated with repairing server 2

- C_s : Costs associated with providing customer service

The whole cost at time t is then calculated as

$$TC(t) = C_1E(L(t)) + C_2P_{n,0}(t) + C_3P_{n,1}(t) + C_4P_{n,2}(t) + C_5P_{n,3}(t) \\ + C_6P_{n,4}(t) + C_7P_{n,5}(t) + C_{R1}\beta_1 + C_{R2}\beta_2 + C_s(\mu_1 + \mu_2 + \mu_{v2}).$$

6. NUMERICAL ILLUSTRATIONS

In this section, numerical results are provided in order to demonstrate the proposed model's practical functionality. Here tables and graphs are formulated using the MATLAB software. We splitted this section into two subsections for steady state numerical analaysis and transient state numerical analysis. The cost elements and default parameters for both subsections are taken as $C_1 = 90$, $C_2 = 20$, $C_3 = 30$, $C_4 = 40$, $C_5 = 25$, $C_6 = 35$, $C_7 = 45$, $C_{R1} = 60$, $C_{R2} = 50$, $C_s = 55$, $\lambda = 0.5$, $\mu_1 = 2.0$, $\mu_2 = 1.5$, $\mu_{v1} = 1.2$, $\theta_1 = 0.9$, $\theta_2 = 0.8$, $p = 0.4$, $q = 0.6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\beta_1 = 0.7$, $\beta_2 = 0.5$.

6.1. For Steady State

Table 1 displays the steady state probabilities for various states corresponding to the number of customers in the system. In table 1 its observed that if the number of customers increases, steady state probability values decrease. Figures 2 and 3 demonstrate that the variation of expected system size depends on the arrival rate for various breakdown and repair rates for servers 1 and 2. From the Figures 2 and 3 we observed that if the breakdown rates for server 1 and server 2 increase, the expected system size increases, and if the repair rates for server 1 and 2 increase, the expected system size decreases.

Table 1: Steady state probabilities

n	p_{n0}	p_{n1}	p_{n2}	p_{n3}	p_{n4}	p_{n5}
0	00.4896614	-	-	-	-	-
1	0.1423434	0.0982361	-	0.0213276	0.0113827	-
2	0.0442454	0.0646550	0.0129519	0.0180962	0.0089578	0.0045275
3	0.0128620	0.0141960	0.0051370	0.0066343	0.0026745	0.0024161
4	0.0037390	0.0036358	0.0017720	0.0021795	0.0007539	0.0010470
5	0.0010869	0.0010086	0.0005971	0.0006857	0.0002118	0.0004110
6	0.0003160	0.0002896	0.0002001	0.0002112	0.0000601	0.0001528
7	0.0000918	0.0000843	0.0000667	0.0000642	0.0000172	0.0000549
8	0.0000267	0.0000246	0.0000222	0.0000194	0.0000050	0.0000193
9	0.0000078	0.0000072	0.0000073	0.0000058	0.0000014	0.0000066
10	0.0000023	0.0000021	0.0000024	0.0000017	0.0000004	0.0000023
11	0.0000007	0.0000006	0.0000008	0.0000005	0.0000001	0.0000008
12	0.0000002	0.0000002	0.0000003	0.0000002	0.0000000	0.0000003
13	0.0000001	0.0000001	0.0000001	0.0000000	0.0000000	0.0000001

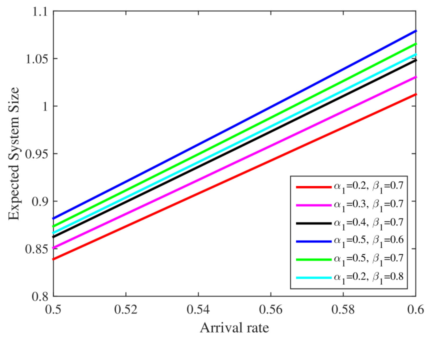

Figure 2: Arival rate Vs Expected system size

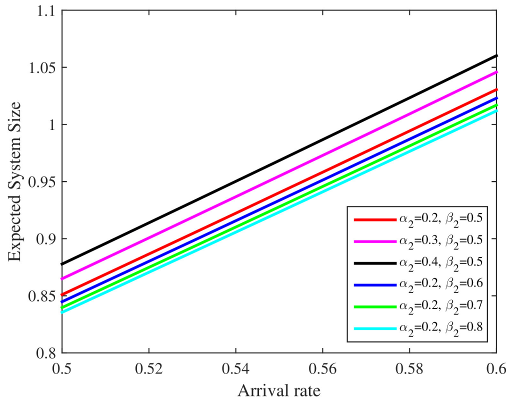

Figure 3: Arival rate Vs Expected system size

6.2. For Transient Analysis

Figures 4-9 depicts the time dependent behaviour of $P_{n,0}(t)$, $P_{n,1}(t)$, $P_{n,2}(t)$, $P_{n,3}(t)$, $P_{n,4}(t)$, $P_{n,5}(t)$. Its evident that all the probability curves starts with 0 and reach the steady state as t increases. From the Figures 4 - 9 we observed that if the number of customers increases, then the probability values decrease. Tables 2 and 3 displays the variation of expected system size by breakdown and repair rate of server 1 and server 2. From these two tables, we notice that if the breakdown rate increases, the expected system size increases, and if the repair rate increases, the expected system size decreases. Table 4 establishes the total cost of the system for different values of breakdown and repair rates. Tables 4 and 5 displays the variation of total cost by breakdown and repair rate for server 1 and server 2. From these tables, we observed that if the breakdown rate increases, the total cost increases, and if the repair rate increases, the total cost decreases. Figures 10 - 13 dipicts the effect of time on total cost for differet values of λ, μ_1, μ_2, μ_{v2}. From these figures, we notice that the total cost reaches a steady state after some time.

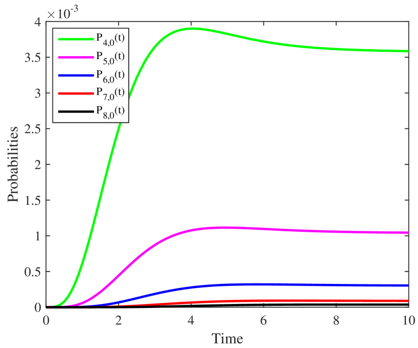

Figure 4: Probabilities $P_{n,0}(t)$ Vs Time

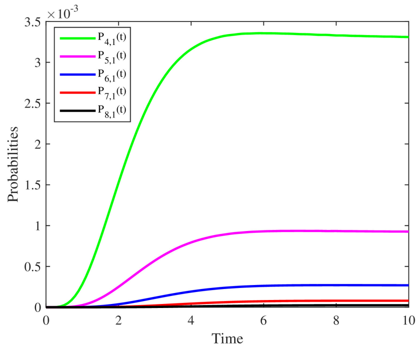

Figure 5: Probabilities $P_{n,1}(t)$ Vs Time

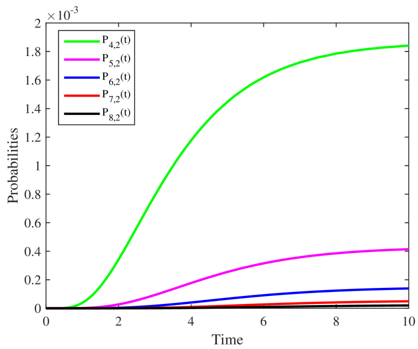

Figure 6: Probabilities $P_{n,2}(t)$ Vs Time

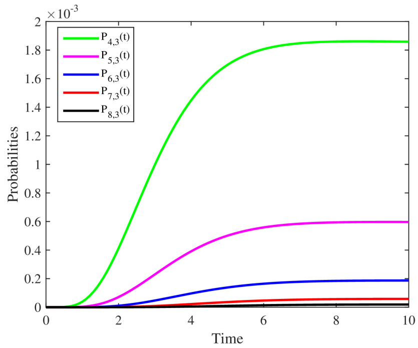

Figure 7: Probabilities $P_{n,3}(t)$ Vs Time

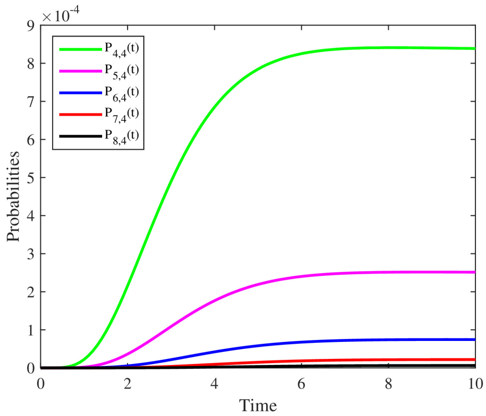

Figure 8: Probabilities $P_{n,4}(t)$ Vs Time

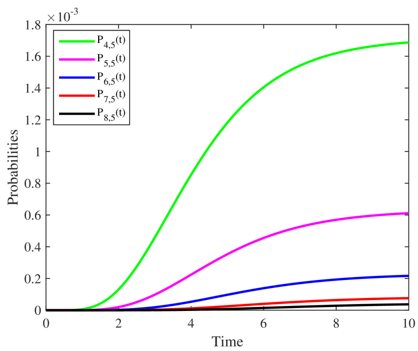

Figure 9: Probabilities $P_{n,5}(t)$ Vs Time

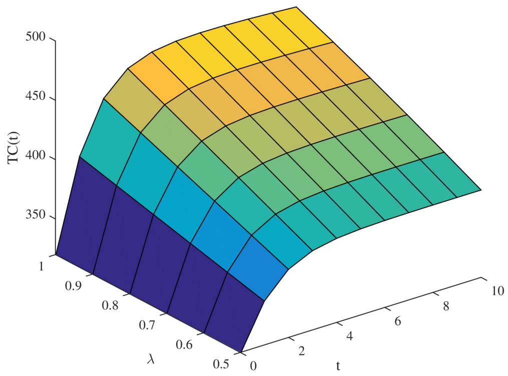

Figure 10: Time versus Total cost for different values of λ

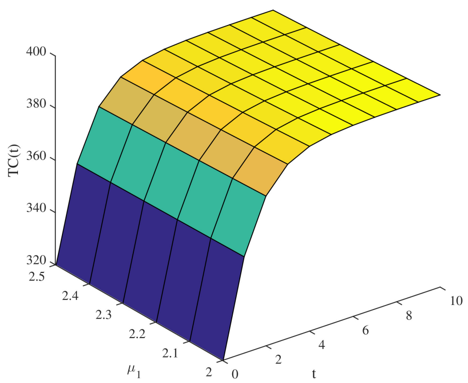

Figure 11: Time versus Total cost for different values of μ_1

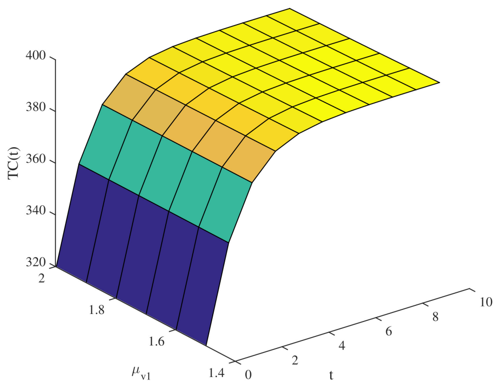

Figure 12: Time versus Total cost for different values of μ_2

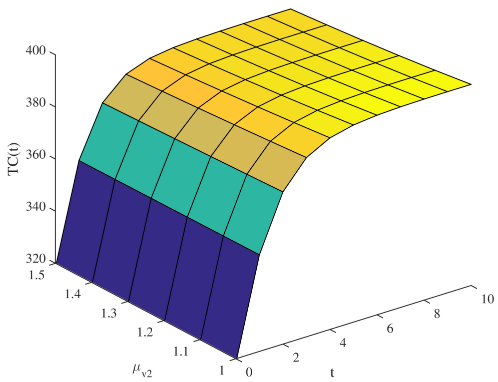

Figure 13: Time versus Total cost for different values of μ_{v2}

Table 2: Expected system size for different values of α_1, β_1

(α_1, β_1)	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$
(0.2, 0.7)	0.62809	0.76143	0.78218	0.78492	0.78460	0.78355
(0.3, 0.7)	0.63039	0.76965	0.79440	0.79877	0.79901	0.79811
(0.4, 0.7)	0.63256	0.77724	0.80572	0.81175	0.81263	0.81197
(0.5, 0.6)	0.63501	0.78698	0.82170	0.83126	0.83386	0.83397
(0.5, 0.7)	0.63461	0.78427	0.81623	0.82394	0.82555	0.82517
(0.5, 0.8)	0.63423	0.78181	0.81140	0.81762	0.81849	0.81777

Table 3: Expected system size for different values of α_2, β_2

(α_2, β_2)	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$
(0.2, 0.5)	0.63461	0.78427	0.81623	0.82394	0.825	0.82517
(0.3, 0.5)	0.63399	0.78355	0.81660	0.82530	0.82753	0.82750
(0.4, 0.5)	0.63338	0.78283	0.81688	0.82656	0.82945	0.82980
(0.2, 0.6)	0.63460	0.78363	0.81439	0.82112	0.82215	0.82148
(0.2, 0.7)	0.63459	0.78305	0.81276	0.81868	0.81927	0.81839
(0.2, 0.8)	0.63458	0.78251	0.81130	0.81655	0.81679	0.81578

Table 4: Total cost for different values of α_1, β_1

(α_1, β_1)	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$
(0.2, 0.7)	377.1833	390.0673	392.1094	392.3633	392.3054	392.1737
(0.3, 0.7)	377.3656	390.7888	393.2108	393.6234	393.6198	393.5027
(0.4, 0.7)	377.5378	391.4540	394.2305	394.8041	394.8634	394.7678
(0.5, 0.6)	377.7317	392.3047	395.6700	396.5806	396.8057	396.7851
(0.5, 0.7)	377.7006	392.0688	395.1769	395.9122	396.0421	395.9744
(0.5, 0.8)	377.6711	391.8539	394.7407	395.3344	395.3927	395.2926

Table 5: Total cost for different values of α_2, β_2

(α_2, β_2)	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$
(0.2, 0.5)	377.4100	391.1630	394.0213	394.7261	394.8709	394.8201
(0.3, 0.5)	377.3936	391.2199	394.2484	395.0826	395.3041	395.2941
(0.4, 0.5)	377.3766	391.2632	394.4463	395.4070	395.7096	395.7468
(0.2, 0.6)	377.4065	391.0931	393.8225	394.4202	394.5011	394.4177
(0.2, 0.7)	377.4032	391.0295	393.6483	394.1606	394.1946	394.0898
(0.2, 0.8)	377.4001	390.9715	393.4950	393.9384	393.9377	393.8190

7. CONCLUSION

This paper examines a two-server heterogeneous multiple vacation queueing model that accounts for server breakdown. We have given the stationary condition and boundary probability vectors for our model using the matrix geometric technique. This study also discusses the proposed queueing model's time-dependent behavior and cost analysis of the system. In the future, this work will be extended to a multi-server heterogeneous queueing system with multiple vacations and different kinds of breakdowns.

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