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Research Article

SOLVING CONSTRAINED MATRIX GAMES WITH FUZZY RANDOM LINEAR CONSTRAINTS

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Abstract: In real-world games, players may face an uncertain environment where fuzziness and randomness coexist. The main difficulty in dealing with games involving fuzziness and randomness arises when comparing the payoffs. The purpose of this paper is to introduce a new approach to deal with constrained matrix games where the entries of the constraint matrices are LR-fuzzy random variables. Our methodology is based on constructing a new matrix game using the chance constraint method adapted to the probability-possibility measures. First, a specific type of saddle point is defined as an equilibrium solution. Then, conditions for the existence of the proposed solution are established. Further, a technique based on second-order programming for computing the saddle point is presented. Finally, a numerical illustration of the approach is provided.

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1. INTRODUCTION

Game theory is employed to resolve conflict situations and was originally developed by John von Neumann and Oskar Morgenstern [1] in their book "The Theory of Games and Economic Behavior". They applied Neumann's theory of games of strategy [2] to competitive business. Game theory finds applications in various fields, including artificial intelligence, engineering, economics, computer science, psychology, and more.

One of the basic models in game theory is the matrix game. It is a zero-sum two-person game in normal form, with a finite set of pure strategies for each player. A mixed strategy is defined by enabling a player to randomly select his strategy based on a probability distribution on his pure strategy set. The particular structure and simplicity of matrix games make them interesting in many ways, and their analysis is tractable. Minimax theorem [2] is the fundamental basis for both the game structure and its resolution. Thus, in mixed strategies, any matrix game has at least one saddle point equilibrium. The computation of the saddle point is obtained by solving a primal-dual pair of linear programs [3]. However, in real life, usually games occur in uncertain (fuzzy, random, fuzzy random etc.) environments.

Fuzzy Set theory [4, 5] provides effective techniques and approaches for handling fuzzy games. Fuzzy set theory was introduced in noncooperative games in the paper of Butnariu [6]. Later, Compos [7] considered matrix games where the payoffs are triangular fuzzy numbers. Since then, games involving fuzziness have been thoroughly investigated (e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]). The work of Kumar[8] investigates a multi-objective matrix game with fuzzy goals. Under the assumption that each player has a fuzzy goal for each of the payoffs, he formulated the max-min solution. Later, Roy and Bhaumik [9] examined Triangular Type-2 Intuitionistic Fuzzy Matrix Game (TT2IFMG). A new ranking function is used to get relevant solutions of TT2IFMG and an application to water management is presented. In [10], Deli developed some models for games, where the payoffs are represented with simplified neutrosophic sets. He gives an application of simplified neutrosophic sets to two-person zero-sum matrix games. In Seikh et al [11], a new methodology is proposed to solve matrix games with payoffs of triangular hesitant fuzzy type, and an application to the market share problem is presented. Jana and Roy [12], transformed a Multiple Attribute Decision Making problem under hesitant fuzzy information into a matrix game. They developed the technique for order preference by similarity to an ideal solution based on an ordered weighted aggregation operator and hybrid hesitant fuzzy normalized Euclidean distance. The superiority of their approaches is established. Seikh et al. [13] developed a mathematical model of a matrix game triangular dense fuzzy lock sets payoffs. A new defuzzification function is used to solve the considered game. Seikh and Karmakar [14] examined matrix games with payoffs of triangular dense fuzzy lock sets (TDFLSs) type. First, they defined the possibility, necessity, and credibility measures of triangular dense fuzzy lock sets. Using the credibility expectation, the authors developed two linear programming models to find the credibility equilibrium strategies for the players and the value of the game. In Seikh et al. [15] a matrix game with rough interval pay-offs is considered. They investigated two different solution methodologies to solve such a game. Seikh et al. [17] formulated a new defuzzification model of triangular type-2 fuzzy variables (TT2FVs). They showed its superiority compared to other existing models and developed a matrix game in a type-2 fuzzy environment to show the applicability of their defuzzification method in a real-world situation. In the paper of Karmakar et al. [16], the Minkowski distance of Type-2 intuitionistic fuzzy sets based on the Hausdorff metric is proposed. Then a similarity measure of the Type-2 intuitionistic fuzzy set is formed. Next, they solved the matrix games by utilizing the proposed distance measure. Jangid and Kumar [20] proposed a novel way to deal with uncertainty in a two-person zero-sum matrix game with payoffs expressed as fuzzy rough numbers. They obtained complete and reasonable solutions to these games. More recently, Karmakar et al. [19] considered the payoffs of a bimatrix game in the form of dense fuzzy lock sets. They defined the weighted average defuzzification function. The Nash equilibrium strategies are computed using an auxiliary dense fuzzy non-linear programming problem. An application to a natural disaster management problem.

Two methods commonly used in games with probabilistic uncertainty are the chance-constrained approach and the expected value approach. The main idea of the expected value approach is that, by computing the expected values of payoffs, the original uncertain game is transformed into a deterministic game (e.g., [21, 22, 23, 24]). The chance-constrained approach, pioneered by Charnes and Cooper [25] in stochastic optimization problems, is one of the most widely used methods for handling uncertain parameters in optimization problems. It was extended to game theory by Charnes et al. (1968) [25]. The authors analyzed chance-constrained matrix games. These games, along with bimatrix chance-constrained games, have been extensively examined by Cassidy et al. [26], Singh and Lisser [27], Singh et al. [28], and Peng et al. [29]. More recently, Achemine and Larbani [30] studied the Z-equilibrium in bimatrix games with random payoffs using chance constraints.

On the other hand, fuzzy random variables [31, 32, 33] play a significant role in optimization problems where model parameters are subject to both randomness and fuzziness. In the literature, two types of finite two-person games involving fuzzy random variables have been investigated: matrix games and bimatrix games, and corresponding equilibrium solutions are introduced [34, 24, 35, 36, 37, 38]. To deal with matrix games with fuzzy random payoffs, Mondal and Roy [35], Xu and Li [24], and Xu et al.[34], used the expected value operator to define the fuzzy random expected minimax equilibrium and investigated its existence and computation. Yano [36] focused on bimatrix games with fuzzy random variables payoffs. Using possibility and necessity measures, he introduced equilibrium solution concepts and proposed two algorithms to compute the proposed solutions. Yano [38] further proposed an extension of this approach to multi-objective fuzzy random bimatrix games. Achemine and Larbani [37] considered a bimatrix game where the payoffs are fuzzy random variables. Using chance constraints, the authors proposed an approach based on probability and possibility measures. In their approach, they investigated the existence and computation of Nash equilibrium.

In some real-world game situations, the players' choice of strategies is constrained by linear inequalities. We refer to these games as constrained games. Charnes [39] conducted the initial research on constrained matrix games, showing that these games can be solved by the resolution of a pair of primal-dual linear programming problems. In [40], an example of the constrained matrix game is provided. According to Owen [41], any classical constrained matrix game always has optimal strategies and game value. Li and Cheng [42] pointed out that there is no existing methodology in the literature for constrained matrix games where payoffs are represented by fuzzy numbers. They used multi-objective programming techniques to study these games. A method for solving constrained matrix games with payoffs represented by triangular fuzzy numbers is proposed by Li and Hong [43]. Later, Nan and Li [44] introduced an effective linear programming technique for constrained matrix games with interval payoffs. An α-cut linear programming approach for dealing with fuzzy constrained matrix games is developed by Li and Hong [45]. Applying the concept of recurrent neural networks, Mansoori et al. [46] investigated constrained matrix games with fuzzy payoffs and fuzzy constraints. The book by Verma and Kumar [47] provides an overview of the literature on constrained matrix games with fuzzy payoffs. The authors addressed several current approaches, identified some flaws in earlier studies on these games, and proposed a novel approach to solving them. A novel approach to finding a complete solution for constrained matrix games with fuzzy payoffs is proposed in Verma [48]. More recently, Djebara et al. examined the constrained matrix game with fuzzy payoffs and fuzzy linear constraints.[49]. The authors proposed an innovative approach based on a ranking function [50, 51] and the chance-constrained method. They introduced a kind of saddle point, provided sufficient existence conditions, and reduced the computation of this solution to the resolution of primal-dual linear programs.

In 2019, Singh and Lisser [52] explored a stochastic variant of the constrained matrix game originally studied by Charnes [39]. The authors formulated each player's stochastic linear constraints as chance constraints and demonstrated that the saddle point can be obtained by solving a primal-dual pair of programs. However, to the best of our knowledge, no research has been conducted on constrained matrix games where the constraints are fuzzy random variables.

In this paper, we focus on a constrained matrix game where strategy sets are subject to fuzzy random linear constraints. We propose a new approach to deal with this game. The main contributions of this study are summarized as follows:

- Developing a new type of constrained matrix games.
- Formulating a simplified mathematical model, more suitable for numerical computations of saddle point. Indeed, since the fuzzy random constraints do not define crisp feasible sets, we assume that the constraints will hold with probability-possibility levels.
- Giving sufficient conditions to obtain an equivalent deterministic constrained matrix game.
- Solving the derived constrained matrix game using second-order cone programming.
- Illustrating the methodology using an example of a competitive situation between two firms.

Apart from the introductory section, the present paper is organized as follows. Section 2 provides a review of notations and definitions related to constrained matrix games and fuzzy random variables. The third section presents the constrained matrix game, where the strategy sets are subject to fuzzy random linear constraints. Then, a new constrained matrix game with respect to probability-possibility constraints is proposed and a kind of saddle point is introduced. In the fourth section, sufficient existence conditions of the proposed solution are given. In section 5, a second-order cones programs method is presented to compute this solution. In Section 6, an illustrative example is discussed to show the effectiveness of our study. Section 7discusses related work. Finally, a conclusion is made in Section 8.

2. PRELIMINARY DEFINITIONS

In this section, we will review fundamental definitions related to matrix games, constrained matrix games, fuzzy sets, and fuzzy numbers.

2.1. Constrained Matrix Games (CMG)

This subsection reviews some of the fundamental terms and definitions used in our paper, particularly in the context of matrix games and constrained matrix games. The reader is referred to the work of Verma and Kumar [47] and Bector and Chandra [53] for different notations, terminology, and fundamentals related to this subject.

Let $\mathbb{R}^n$ denote the n-dimensional euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Assume that $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ be an $(m \times n)$ real matrix and $e^T = (1, 1, ..., 1)$ be a vector of 'ones' whose dimension is specified as per the specific context.

The triplet $G = (P, Q, A)$, where $P = \{p \in \mathbb{R}^m_+, e^T p = 1\}$ and $Q = \{q \in \mathbb{R}^n_+, e^T q = 1\}$, is referred to as a crisp two-person zero-sum matrix game G.

P (respectively Q) is said the strategy space for player I (respectively player II) and A is called the payoff matrix. Hence, for Player I (or Player II), the elements of P (or Q) that have the form $p = (0, 0, \cdots, 1, \cdots, 0)^T = e_i$, where 1 is at the i^{th} position (or, alternatively, $q = (0, 0, \cdots, 1, \cdots, 0)^T = e_j$, where 1 is at the j^{th} place) are referred to be pure strategies.

In a situation where player I selects i^{th} pure strategy and player II selects j^{th} pure strategy, the amount that player II pays player I is denoted by a_{ij}. In a zero-sum game, player I's payment to player II equals $-a_{ij}$; that is, one player's gain is another player's loss.

Since elements of P (respectively Q) are set of all probability distribution over $\mathbb{I} = \{1, 2, ..., m\}$ (respectively $\mathbb{J} = \{1, 2, ..., n\}$), the quantity $E(p, q) = p^T Aq$ is termed the expected payoff of player I by player II.

Furthermore, it is a common assumption to assume that player I maximizes while player II minimizes. The expected payoff for player I is equal to the expected loss for player II since player I is the maximizing player and player II is the minimizing player. The triplet $PG = (I, J, A)$ is called the pure form of the game G, whenever G is the mixed extension of the pure game PG.

The triplet $G = (P, Q, A)$ represents a two-person zero-sum game, often known as a matrix game. Where P refers to the mixed strategy space of player I, Q refers to the mixed strategy space of player II, and A refers to the payoff matrix which introduces the function $E : P \times Q \to \mathbb{R}$ given by $E(p,q) = p^T Aq = \sum_{j=1}^{n}(\sum_{i=1}^{m} p_i a_{ij})q_j$, which is known as the expected payoff function, where T represents the transpose operator.

In some real-world matrix game problems, player strategies are constrained by linear inequalities, rather than being in P or Q. These situations involve constrained matrix games, a model initialy examined by Charnes [39]. The mathematical description of this game is as follows. Players I and II must select their mixed strategies, p and q, from constrained sets defined by linear inequalities. Assuming that $B \in \mathbb{R}^{r\times m}$, $b \in \mathbb{R}^r$, and r is a positive integer, let $S_1 = \{p \in P,\ \ Bp \leq b\}$ represent the constrained set of strategies for player I. The constrained set of strategies for player II is represented as $S_2 = \{q \in Q|Dq \geq d\}$, where $d \in \mathbb{R}^s$, $D \in \mathbb{R}^{s\times n}$, and s is a positive integer. The expected payoff for player I, if he chooses any mixed strategy $p \in S_1$ and player II chooses any mixed strategy $q \in S_2$ is defined as follows

$$p^T Aq = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i a_{ij} q_j.$$

We denote this game by $CMG = (S_1, S_2, A)$.

The purpose of player I (resp. player II) in this game is to find a strategy p (resp. q) that solves the linear programming problem $(P1)$(resp. $(P2)$) given a strategy q (resp. p) of player II (resp. player I).

$$(P1)\begin{cases} \max\limits_{p} p^T Aq \\ subject \quad to \\ Bp \leq b \\ p \in P \end{cases}, \quad (P2)\begin{cases} \min\limits_{q} p^T Aq \\ subject \quad to \\ Dq \geq d \\ q \in Q \end{cases}$$

Definition 2.1. *(Owen,1982) Assume that there exist a mixed strategies $p^* \in S_1$ and $q^* \in S_2$ so that*

$$p^{*T} Aq^* = \max_{p\in S_1}\min_{q\in S_2}(p^T Aq) = \min_{q\in S_2}\max_{p\in S_1}(p^T Aq).$$

Then, (p^*, q^*) and $v = p^{*T} Aq^*$ are said a saddle point and a value of the constrained matrix game CMG, respectively.

2.2. LR-Fuzzy Numbers

We review some properties and concepts of fuzzy set theory in this section since they are crucial to understanding the proposed approach.

Definition 2.2. *[4] Let U be a universe set. A fuzzy set $\widetilde{A}$ in U is a set of ordered pairs, $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))/x \in U\}$, where the function $\mu_{\widetilde{A}} : U \mapsto [0,1]$ is called membership function, which assigns to each elements $x \in U$ a real number $\mu_{\widetilde{A}}(x)$ in the interval $[0,1]$. The value $\mu_{\widetilde{A}}(x)$ is the degree of membership of x in $\widetilde{A}$.*

A fuzzy set that is convex and normal is called a fuzzy number [4]. In the following definition, we recall an important class of fuzzy numbers.

Definition 2.3. *[5] (Dubois & Prade, 1979) A LR-fuzzy number is a fuzzy number $\widetilde{a}$ with membership function $\mu_{\widetilde{a}}$ given as*

$$\mu_{\widetilde{a}}(t) = \begin{cases} L(\frac{a-t}{\alpha}) & \text{if } t \leq a, \\ R(\frac{t-a}{\beta}) & \text{if } t \geq a, \end{cases}$$

where a is the mean value of $\widetilde{a}$, α and β (non negative) are its left and right spreads, respectively, L and R are the left and right shape functions, respectively, defined as follows $L, R : [0,1] \longrightarrow [0,1]$ with $L(1) = R(1) = 0$ and $L(0) = R(0) = 1$ are decreasing and continuous.
Using its mean value, left and right spreads, and shape functions, a LR-fuzzy number $\widetilde{a}$ is written as $\widetilde{a} = (a, \alpha, \beta)_{LR}$.

In the following, we recall an important concept for fuzzy numbers ranking.

Definition 2.4. *[54] (Dubois & Prade, 1978) The α-cut, $\alpha \in [0,1]$, of a LR-fuzzy number $\widetilde{a} = (a, \alpha_1, \beta_1)$ is a closed interval defined by*

$$a_\alpha = \{x | \mu_A(x) \geq \alpha\} = [a^L_\alpha, a^R_\alpha],$$

For a LR-fuzzy number with invertible and non-increasing functions L and R, the α-cut is given as

$$[a^L_\alpha, a^R_\alpha] = [a - \alpha_1 L^{-1}(\alpha),\ a + \beta_1 R^{-1}(\alpha)].$$

We recall Sakawa's [55] lemma for ranking fuzzy numbers.

Lemma 2.1. *(Sakawa, 1993) Let $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ be two fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$:*
$Pos\{\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2\} \geq \alpha$ *if and only if* $\lambda^R_{1,\alpha} \geq \lambda^L_{2,\alpha}$,
$Nec\{\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2\} \geq \alpha$ *if and only if* $\lambda^L_{1,1-\alpha} \geq \lambda^R_{2,\alpha}$,
where $\lambda^R_{1,\alpha}, \lambda^L_{1,\alpha}$ and $\lambda^R_{2,\alpha}, \lambda^L_{2,\alpha}$ are the left and the right side endpoints of the α-level sets (α-cuts) $[\lambda^R_{1,\alpha}, \lambda^L_{1,\alpha}]$ and $[\lambda^R_{2,\alpha}, \lambda^L_{2,\alpha}]$, of $\widetilde{\lambda_1}$ and $\widetilde{\lambda}_2$, respectively.

The concept of a fuzzy random variable was initially introduced by Feron [31]. Subsequently, many scholars have extended and applied this concept to model situations where both fuzziness and randomness coexist. In this work, we will use the definition of Liu and Liu liu [33].

Definition 2.5. *[33] Let $(\Omega, \mathcal{A}, Pr)$ be a probability space. A fuzzy random variable ξ is a mapping function $\omega \longrightarrow \xi(\omega)$, $\omega \in \Omega$, from $(\Omega, \mathcal{A}, \mathrm{P})$ to the set of fuzzy numbers $\mathcal{P}(\mathbb{R})$ in the set of real line. such that for any Borel set B of $\mathbb{R}$, and $pos\{\xi(\omega) \in B\}$ is a measurable function of ω.*

We consider the following definition of a LR-fuzzy random variable [56].

Definition 2.6. *[56] Let a be a random variable, the realization of which, for a given event $\omega \in \Omega$ is $a(\omega)$. Then, a fuzzy random variable Z is said to be a L-R fuzzy random variable, denoted by $Z = (a, \alpha, \beta)_{LR}$, if its realized values $Z(\omega) = (a(\omega), \alpha, \beta)_{LR}$ for any event $\omega \in \Omega$ are L-R fuzzy numbers defined by*

$$\mu_Z(t) = \begin{cases} L(\frac{a(\omega)-t}{\alpha}) & \text{if } t \leq a(\omega), \\ R(\frac{t-a(\omega)}{\beta}) & \text{if } a(\omega) \leq t, \end{cases}$$

where α and β are positive constants.

3. CONSTRAINED MATRIX GAMES WITH FUZZY RANDOM LINEAR CONSTRAINTS

In the constrained matrix game $CMG = (S_1, S_2, A)$, assume that the entries of the matrices B and D that define, respectively, the player I and player II constraints are fuzzy random variables and are denoted by $\tilde{B}^w$ and $\tilde{D}^w$, respectively. We take into consideration the scenario where each player wants to maximize his payoff such that each of his fuzzy random constraint is satisfied with a given probability-possibility. So, the fuzzy random constraints of each player are replaced with the individual chance constraints.

For a given strategy q (resp. p) of player II (resp. player I), player I (resp. player II) is searching for a strategy p (resp. q) that solves an optimization problem (P3) (resp. (P4)).

$$(P3) \begin{cases} \max\ p^T A q \\ subject \quad to \\ Pr\{\omega | Pos\{\tilde{B}_k^w p \leq b_k\} \geq \delta_k^1\} \geq \gamma_k^1, \ \ k \in \mathcal{J}_1, \\ p \in P, \end{cases}$$

$$(P4) \begin{cases} \min\ p^T A q \\ subject \quad to \\ Pr\{\omega | Pos\{\tilde{D}_l^w q \geq d_l\} \geq \delta_l^2\} \geq \gamma_l^2, \ \ 1 \in \mathcal{J}_2, \\ q \in Q, \end{cases}$$

where Pos is a fuzzy number possibility measure and Pr is a probability measure.
The k^{th} row of the fuzzy random matrix $\tilde{B}^\omega$ is $\tilde{B}_k^\omega = (\tilde{B}_{k1}^\omega, \tilde{B}_{k2}^\omega, ..., \tilde{B}_{km}^\omega)$, $k \in \mathcal{J}_1$
The l^{th} row of the fuzzy random matrix $\tilde{D}^\omega$,$\tilde{D}_l^\omega = (\tilde{D}_{l1}^\omega, \tilde{D}_{l2}^\omega, ..., \tilde{D}_{ln}^\omega)$, $l \in \mathcal{J}_2$
$\delta_k^1 \in [0, 1]$ is the possibility level for k^{th} constraint of player I, and $\delta_l^2 \in [0, 1]$ is the possibility level for l^{th} constraint of player II;
$\gamma_k^1 \in [0, 1]$ is the probability level for k^{th} constraint of player I, and $\gamma_l^2 \in [0, 1]$ is the probability level for l^{th} constraint of player II.
Let $\delta^1 = (\delta_k^1)_{k=1}^r$, $\delta^2 = (\delta_k^2)_{l=1}^s$, $\delta = (\delta^1, \delta^2)$, $\gamma^1 = (\gamma_k^1)_{k=1}^r$, $\gamma^2 = (\gamma_k^2)_{l=1}^s$ and $\gamma = (\gamma^1, \gamma^2)$.
The above matrix game with individual chance constraints is denoted by $\Gamma(\delta, \gamma)$. Denote the player's strategies sets by
$S^m(\delta^1, \gamma^1) = \{p \in \mathbb{R}^m | p \in P, Pr\{\omega | Pos\{\tilde{B}_k^\omega p \leq b_k\} \geq \delta_k^1\} \geq \gamma_k^1, \forall k \in \mathcal{J}_1\}$,
and $S^n(\delta^2, \gamma^2) = \{q \in \mathbb{R}^n | q \in Q, Pr\{\omega | Pos\{\tilde{D}_l^\omega q \geq d_l\} \geq \delta_l^2\} \geq \gamma_l^{(2)}, \forall l \in \mathcal{J}_2\}$.

Remark 3.1. *A strategy profile $(p^*, q^*) \in S^m(\delta^1, \gamma^1) \times S^n(\delta^2, \gamma^2)$ is said to be saddle point of the matrix game $\Gamma(\delta, \gamma)$ if and only if p^* and q^* simultaneously solve the optimization problems $(P3)$ and $(P4)$.*

Definition 3.1. *A strategy profile $(p^*, q^*) \in S^m(\delta^1, \gamma^1) \times S^n(\delta^2, \gamma^2)$ is saddle point at $\delta = (\delta^1, \delta^1) \in [0,1]^r \times [0,1]^s$ possibility levels and $\gamma = (\gamma^1, \gamma^2) \in [0,1]^r \times [0,1]^s$ probability levels of the game $\Gamma(\delta, \gamma)$ if*

$$p^T A q^* \leq p^{*T} A q^* \leq p^{*T} A q, \forall p \in S^m(\delta^1, \gamma^1), q \in S^n(\delta^2, \gamma^2).$$

4. EXISTENCE CONDITIONS OF A SADDLE POINT

In Lemma 2, we reformulate the sets strategies of the players under some conditions on constraint matrices.

Lemma 4.1. *Assume that $\widetilde{B}_{ki}(\omega)$ and $\widetilde{D}_{lj}(\omega)$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, $k \in \mathcal{J}_1$ and $l \in \mathcal{J}_2$, are characterized by the following membership functions :*

$$\mu_{\widetilde{B}_{ki}(\omega)}(t) = \begin{cases} L(\frac{B_{ki}(\omega)-t}{\alpha^{(1)}_{ki}}) & \text{if } t \leq B_{ki}(\omega), \ \omega \in \Omega, \\ R(\frac{t-B_{ki}(\omega)}{\beta^{(1)}_{ki}}) & \text{if } B_{ki}(\omega) \geq t, \end{cases}$$

$$\mu_{\widetilde{D}_{lj}(\omega)}(t) = \begin{cases} L(\frac{D_{lj}(\omega)-t}{\alpha^{(2)}_{lj}}) & \text{if } t \leq D_{lj}(\omega), \ \omega \in \Omega, \\ R(\frac{t-D_{lj}(\omega)}{\beta^{(2)}_{lj}}) & \text{if } D_{lj}(\omega) \geq t, \end{cases}$$

where the row vector $B^\omega_k = (B^\omega_{k1}, B^\omega_{k2}, ..., B^\omega_{km}) \sim N(\mu^1_k, \Sigma^1_k)$, $k \in \mathcal{J}_1$ follows a multivariate normal distribution with location parameter $\mu^1_k \in \mathbb{R}^m$ and positive definite scale matrix Σ^1_k. And $D^\omega_l = (D^\omega_{l1}, D^\omega_{l2}, ..., D^\omega_{ln}) \sim N(\mu^2_l, \Sigma^2_l)$, $l \in \mathcal{J}_2$ follow a multivariate normal distribution with location parameter $\mu^2_l \in \mathbb{R}^n$ and positive definite scale matrix Σ^2_l.

Then, for all $\delta = (\delta^1, \delta^2) \in [0,1]^r \times [0,1]^s$ and $\gamma = (\gamma^1, \gamma^2) \in [0,1]^r \times [0,1]^s$, we have

$$S^m(\delta^1, \gamma^1) = \{p \in \mathbb{R}^m | p \in P, \ p^T(\mu^1_k - L^{-1}(\delta^1_k)\alpha^{(1)}_k) + \phi^{-1}(\gamma^1_k)||(\Sigma^1_k)^{\frac{1}{2}} p|| \leq b_k, \forall k \in \mathcal{J}_1\},$$

$$S^n(\delta^2, \gamma^2) = \{q \in \mathbb{R}^n | q \in Q, q^T(-\mu^2_l - R^{-1}(\delta^2_l)\beta^{(2)}_l) + \phi^{-1}(\gamma^2_l)||(\Sigma^2_l)^{\frac{1}{2}} q|| \leq -d_l, \forall l \in \mathcal{J}_2\},$$

where $\beta^{(1)}_k = (\beta^{(1)}_{k1}, \beta^{(1)}_{k2}, ..., \beta^{(1)}_{km})^T, k \in \mathcal{J}_1$, $\beta^{(2)}_l = (\beta^{(2)}_{l1}, \beta^{(2)}_{l2}, ..., \beta^{(2)}_{ln})^T, l \in \mathcal{J}_2$, $\alpha^{(1)}_k = (\alpha^{(1)}_{k1}, \alpha^{(1)}_{k2}, ..., \alpha^{(1)}_{km})^T, k \in \mathcal{J}_1$ and $\alpha^{(2)}_l = (\alpha^{(2)}_{l1}, \alpha^{(2)}_{l2}, ..., \alpha^{(2)}_{ln})^T, l \in \mathcal{J}_2$.

Proof. 1) For all $k \in \mathcal{J}_1$ and $l \in \mathcal{J}_2$, we have the equivalence $Pos\{\tilde{B}^\omega_k p \leq b_k\} \geq \delta^1_k \Longleftrightarrow Pos\{\sum_{i=1}^{m} \tilde{B}^\omega_{ki} p_i \leq b_k\} \geq \delta^1_k$. Due to Lemma 2.1, we write

$$Pos\{\sum_{i=1}^{m} \tilde{B}_{ki}^{\omega} p_i \leq b_k\} \geq \delta_k^1$$

$$\Longleftrightarrow \sum_{i=1}^{m} p_i B_{ki}^{\omega} - L^{-1}(\delta_k^1) \sum_{i=1}^{n} p_i \alpha_{ki}^{(1)} \leq b_k.$$

Consequently,

$$Pr\{\omega \in \Omega | B_k^{\omega} p - L^{-1}(\delta_k^1)\alpha_k^{(1)} p \leq b_k\} \geq \gamma_k^1$$

$$\Longleftrightarrow Pr\left(\frac{B_k^{\omega} p - p^T \mu_k^1}{\sqrt{p^T \Sigma_k^1 p}} \leq \frac{b_k + L^{-1}(\delta_k^1) p^T \alpha_k^{(1)} - p^T \mu_k^1}{\sqrt{p^T \Sigma_k^1 p}}\right) \geq \gamma_k^1.$$

Since $\frac{B_k^{\omega} p - p^T \mu_k^1}{\sqrt{p^T \Sigma_k^1 p}}$ follows a univariate normal distribution with parameters 0 and 1 and $\phi(.)$ is the distribution function of the normal random variable $N(0,1)$, we obtain

$$Pr\left(B_k^{\omega} p - L^{-1}(\delta_k^1) p^T \alpha_k^{(1)} \leq b_k\right) \geq \gamma_k^1 \Longleftrightarrow \phi(\tfrac{b_k + L^{-1}(\delta_k^1)\alpha_k^{(1)} p - p^T \mu_k^1}{\sqrt{p^T \Sigma_k^1 p}}) \geq \gamma_k^1.$$

Since $\Sigma_k^1 \succ 0$, then $p^T \Sigma_k^1 p = \parallel (\Sigma_k^1)^{\frac{1}{2}} p \parallel$, hence

$$Pr\left(B_k^{\omega} p - L^{-1}(\delta_k^1) p^T \alpha_k^{(1)} \leq b_k\right) \geq \gamma_k^1 \Longleftrightarrow$$
$$p^T \mu_k^1 - L^{-1}(\delta_k^1) p^T \alpha_k^{(1)} + \phi^{-1}(\gamma_k^1) ||(\Sigma_k^1)^{\frac{1}{2}} p|| \leq b_k,$$

where $\phi^{-1}(.)$ is a quantile function of a normal random variable $N(0,1)$,
$S^m(\delta^1, \gamma^1) = \{p \in \mathbb{R}^m | p \in P,\ p^T(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(1)}) + \phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}} p|| \leq b_k, \forall k \in \mathcal{J}_1\}$,
2) For the second player, for all $l \in \mathcal{J}_2$, similarly as in 1), $\frac{-D_l^{\omega} q + q^T \mu_l^2}{\sqrt{q^T \Sigma_l^2 q}}$ is the univariate normal distribution with parameters 0 and 1 and we show that

$$S^m(\delta^2, \gamma^2) = \{q \in \mathbb{R}^n | q \in Q, q^T(-\mu_l^2 - R^{-1}(\delta_l^2)\beta_l^{(2)}) + \phi^{-1}(\gamma_l^2)||(\Sigma_l^2)^{\frac{1}{2}} q|| \leq -d_l, \forall l \in \mathcal{J}_2\}.$$

□

In the sequel, we assume that the following assumption holds. It guarantees a strong duality result.

Assumption 1. The sets $S^m(\delta^1, \gamma^1)$ and $S^n(\delta^2, \gamma^2)$ are strictly feasible.

Theorem 4.1. *Assume that $\widetilde{B}_{ki}(\omega)$ and $\widetilde{D}_{lj}(\omega)$, $i \in I$, $j \in J$, $k \in \mathcal{J}_1$ and $l \in \mathcal{J}_2$, are LR-fuzzy random variables characterized by the following membership functions :*

$$\mu_{\widetilde{B}_{ki}(\omega)}(t) = \begin{cases} L(\frac{B_{ki}(\omega) - t}{\alpha_{ki}^{(1)}}) & \text{if } t \leq B_{ki}(\omega),\ \omega \in \Omega, \\ R(\frac{t - B_{ki}(\omega)}{\beta_{ki}^{(1)}}) & \text{if } B_{ki}(\omega) \geq t, \end{cases}$$

$$\mu_{\widetilde{D}_{lj}(\omega)}(t) = \begin{cases} L(\frac{D_{lj}(\omega) - t}{\alpha_{lj}^{(2)}}) & \text{if } t \leq D_{lj}(\omega),\ \omega \in \Omega, \\ R(\frac{t - D_{lj}(\omega)}{\beta_{lj}^{(2)}}) & \text{if } D_{lj}(\omega) \geq t, \end{cases}$$

where
$B_k^\omega = (B_{k1}^\omega, B_{k2}^\omega, \cdots, B_{km}^\omega) \sim N(\mu_k^1, \Sigma_k^1)$, $k \in \mathcal{J}_1$ *follows a multivariate normal distribution with location parameter* $\mu_k^1 \in \mathbb{R}^m$ *and scale matrix* $\Sigma_k^1 \in \mathbb{R}^{m\times m}$ *(*$\sigma_k^1 \succ 0$*);*
$D_l^\omega = (D_{l1}^\omega, D_{l2}^\omega, \cdots, D_{ln}^\omega) \sim Ellip(\mu_l^2, \Sigma_l^2)$, $l \in \mathcal{J}_2$ *follows a multivariate normal distribution with location parameter* $\mu_l^2 \in \mathbb{R}^n$ *and scale matrix* $\Sigma_l^2 \in \mathbb{R}^{n\times n}$ *(*$\Sigma_l^2 \succ 0$*).*

Then, there exists a saddle point at δ *possibility levels and* γ *probability levels for the game* $\Gamma(\delta,\gamma)$ *for all* $\delta \in [0,1]^r \times [0,1]^s$ *and* $\gamma \in (0.5,1]^r \times (0.5,1]^s$.

Proof. For any $\delta \in [0,1]^r \times [0,1]^s$ and $\gamma \in (0.5,1]^r \times (0.5,1]^s$. For all $k \in \mathcal{J}_1$, we have
• $S^m(\delta^1,\gamma^1)$ and $S^n(\delta^2,\gamma^2)$ non empty and are compact sets;
• $\phi^{-1}(\gamma_k^1) \geqslant 0$ and using the property of norm $||.||_2$,
we deduce that the function

$$F_k^1(p) = +p^T\mu_k^1 + L^{-1}(\delta_k^{(1)})p^T\alpha_k^1 + \phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}}p||$$

is a convex function of p. Then $S^m(\delta^1,\gamma^1)$ is convex.

We follow the same reasoning to show the convexity of $S^n(\delta^2,\gamma^2)$. Using the minimax theorem [2], we can deduce the existence of at least a saddle point since the function p^TAq is continuous. □

5. COMPUTATION OF THE SADDLE POINT

By solving two primal-dual programs, as will be detailed below, one may obtain a saddle point of the game $\Gamma(\delta,\gamma)$.

Theorem 5.1. *Assume that the fuzzy random variables* $\widetilde{B}_{ki}(\omega)$ *and* $\widetilde{D}_{lj}(\omega)$, $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$,$k \in \mathcal{J}_1$ *and* $l \in \mathcal{J}_2$, *are characterized by the following membership functions :*

$$\mu_{\widetilde{B}_{ki}(\omega)}(t) = \begin{cases} L(\frac{B_{ki}(\omega)-t}{\gamma_{ki}^{(1)}}) & \text{if } t \le B_{ki}(\omega),\ \ \omega \in \Omega, \\ R(\frac{t-B_{ki}(\omega)}{\beta_{ki}^{(1)}}) & \text{if } B_{ki}(\omega) \ge t, \end{cases}$$

$$\mu_{\widetilde{D}_{lj}(\omega)}(t) = \begin{cases} L(\frac{D_{lj}(\omega)-t}{\gamma_{lj}^{(2)}}) & \text{if } t \le D_{lj}(\omega),\ \ \omega \in \Omega, \\ R(\frac{t-D_{lj}(\omega)}{\beta_{lj}^{(2)}}) & \text{if } D_{lj}(\omega) \ge t, \end{cases}$$

where the row vector $B_k^\omega = (B_{k1}^\omega, B_{k2}^\omega, \cdots, B_{km}^\omega) \sim N(\mu_k^1, \Sigma_k^1)$, $k \in \mathcal{J}_1$ *follows a multivariate normal distribution with location parameter* $\mu_k^1 \in \mathbb{R}^m$ *and positive definite scale matrix* $\Sigma_k^1 \in \mathbb{R}^{m\times m}$ *(*$\sigma_k^1 \succ 0$*). And* $D_l^\omega = (D_{l1}^\omega, D_{l2}^\omega, ..., D_{ln}^\omega) \sim N(\mu_l^2, \Sigma_l^2)$, $l \in \mathcal{J}_2$ *follows a multivariate normal distribution with location parameter* $\mu_l^2 \in \mathbb{R}^n$ *and positive definite scale matrix* $\Sigma_l^2 \in \mathbb{R}^{n\times n}$ *(*$\Sigma_l^2 \succ 0$*).*
Then, for a given $\delta \in [0,1]^r \times [0,1]^s$, $\gamma \in (0.5,1]^r \times (0.5,1]^s$, (p^*,q^*) *is a saddle point of the game* $\Gamma(\delta,\gamma)$ *if and only if there exists* $(\upsilon^{1*}, (\rho_k^{1*})_{k=1}^p, \lambda^{1*})$ *and* $(\upsilon^{2*}, (\rho_l^{2*})_{l=1}^q, \lambda^{2*})$ *such that* $(q^*, \upsilon^{1*}, (\rho_k^{1*})_{k=1}^p, \lambda^{1*}))$ *and* $(p^*, \upsilon^{2*}, (\rho_l^{2*})_{l=1}^q, \lambda^{2*})$ *are optimal solutions of primal-dual pair of SOCPs (P) and (D) respectively.*

$$
(P)\begin{cases}
\min\limits_{\upsilon^1,(\rho_k^1)_{k=1}^r} \upsilon^1 - \sum\limits_{k=1}^r \lambda_k^1 b_k \\
s.t.\ (i)\ \ Aq + \sum\limits_{k=1}^r \lambda_k^1(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(2)}) - \sum\limits_{k=1}^r \left(\Sigma_k^1\right)^{\frac{1}{2}} (\rho_k^1) \geq \upsilon^1 \mathbf{1}_m, \\
(ii)\ \ -q^T(\mu_l^2 - R^{-1}(\delta_l^{(2)})\beta_l^{(2)}) + \phi^{-1}(\gamma_l^2)||(\Sigma_l^2)^{\frac{1}{2}}q|| \leq -d_l, \forall l \in \mathcal{J}_2, \\
(iii)\ \ ||\rho_k^1|| \geq \lambda_k^1 \phi^{-1}(\gamma_k^1), \forall k \in \mathcal{J}_1, \\
(iv)\ \ \ \sum\limits_{j\in\mathcal{J}} q_j = 1, \\
(v)\ q_j \geq 0, \forall j \in \mathcal{J}, \\
(vi)\ \lambda_k^1 \geq 0, \forall k \in \mathcal{J}_1,
\end{cases}
$$

where $1_m = (1, \cdots, 1) \in \mathbb{R}^m$.

$$
(D)\begin{cases}
\max\limits_{\upsilon^2,(\rho_l^2)_{l=1}^s} \upsilon^2 + \sum\limits_{l=1}^r \lambda_l^2 d_l \\
s.t.\ (i)\ \ A^T p - \sum\limits_{l=1}^s \lambda_l^2(\mu_l^2 - R^{-1}(\delta_l^2)\beta_l^{(2)}) - \sum\limits_{l=1}^s \left(\Sigma_l^2\right)^{\frac{1}{2}} (\rho_l^2) \leq \upsilon^2 \mathbf{1}_n, \\
(ii)\ \ p^T(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(1)}) + \phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}}p|| \leq b_k, \forall k \in \mathcal{J}_1, \\
(iii)\ \ ||\rho_l^2|| \leq \lambda_l^2 \phi^{-1}(\gamma_l^2), \forall l \in \mathcal{J}_2, \\
(iv)\ \ \sum\limits_{i\in\mathcal{I}} p_i = 1, \\
(v)\ \ p_i \geq 0, \forall i \in \mathcal{I}, \\
(vi)\ \ \lambda_l^2 \geq 0, \forall l \in \mathcal{J}_2,
\end{cases}
$$

where $1_n = (1, \cdots, 1) \in \mathbb{R}^n$.

Proof. Assume that $\gamma \in (0.5, 1]^r \times (0.5, 1]^s$ and $\delta \in [0, 1]^r \times [0, 1]^s$. According to Definition 2.1, a strategy profile (p^*, q^*) is a saddle point for game $\Gamma(\delta, \gamma)$ if and only if

$$p^* A q^* = \max_{p\in S^m(\delta^1,\gamma^1)} \min_{q\in S^n(\delta^2,\gamma^2)} p^T A q = \min_{q\in S^n(\delta^2,\gamma^2)} \max_{p\in S^m(\delta^1,\gamma^1)} p^T A q,$$

and

$$p^* \in \arg \max_{p\in S^m(\delta^1,\gamma^1)} p^T A q^*,$$

$$p^* \in \arg \min_{q\in S^n(\delta^2,\gamma^2)} {p^*}^T A q.$$

Considering $\delta \in [0, 1]^p \times [0, 1]^q$ and $\gamma \in (0.5, 1]^r \times (0.5, 1]^s$, the constraints $p^T(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(1)}) + \phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}}p|| \leq b_k, k \in \mathcal{J}_1$, and $q^T(-\mu_l^2 - R^{-1}(\delta_l^2)\beta_l^{(2)}) + \phi^{-1}(\gamma_l^2)||(\Sigma_l^2)^{\frac{1}{2}}q|| \leq -d_l, l \in \mathcal{J}_2$ are, as noted in [57], second-order cone constraints.

Based on the second-order cone programming (SOCP) dual formulation provided in [57], the Lagrangian dual of the SOCP $\max\limits_{p\in S^m(\delta^1,\gamma^1)} p^T A q$ is also SOCP [58].

According to Assumption 1, the duality gap is zero. Therefore, the $\max\limits_{p\in S^m(\delta^1,\gamma^1)} \min\limits_{q\in S^n(\delta^2,\gamma^2)} p^TAq$ problem is equivalent to the following SOCP :

$$(P)\begin{cases} \min\limits_{\upsilon^1,(\rho_k^1)_{k=1}^r} \upsilon^1 - \sum\limits_{k=1}^r \lambda_k^1 b_k \\ s.t.\ (i)\ Aq + \sum\limits_{k=1}^r \lambda_k^1(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(2)}) - \sum\limits_{k=1}^r \left(\Sigma_k^1\right)^{\frac{1}{2}}(\rho_k^1) \geq \upsilon^1\mathbf{1}_m, \\ (ii)\ \ -q^T(\mu_l^2 - R^{-1}(\delta_l^{(2)})\beta_l^{(2)}) + \phi^{-1}(\gamma_l^2)||(\Sigma_l^2)^{\frac{1}{2}}q|| \leq -d_l, \forall l \in \mathcal{J}_2, \\ (iii)\ ||\rho_k^1|| \geq \lambda_k^1\phi^{-1}(\gamma_k^1), \forall k \in \mathcal{J}_1, \\ (iv)\ \ \sum_{j\in\mathcal{J}} q_j = 1, \\ (v)\ q_j \geq 0, \forall j \in \mathcal{J}, \\ (vi)\ \lambda_k^1 \geq 0, \forall k \in \mathcal{J}_1, \end{cases}$$

where $1_m = (1, \cdots, 1) \in \mathbb{R}^m$.

Applying the same argument to the player II model, the $\max\limits_{q\in S^m(\delta^1,\gamma^1)} \min\limits_{q\in S^n(\delta^2,\gamma^2)} p^TAq$ problem is equivalent to the following SOCP :

$$(D)\begin{cases} \max\limits_{\upsilon^2,(\rho_l^2)_{l=1}^q} \upsilon^2 + \sum\limits_{l=1}^s \lambda_l^2 d_l \\ s.t.\ (i)\ A^Tp - \sum\limits_{l=1}^s \lambda_l^2(\mu_l^2 - R^{-1}(\delta_l^2)\beta_l^{(2)}) - \sum\limits_{l=1}^s \left(\Sigma_l^2\right)^{\frac{1}{2}}(\rho_l^2) \leq \upsilon^2\mathbf{1}_n, \\ (ii)\ p^T(\mu_k^1 - L^{-1}(\delta_k^1)\alpha_k^{(1)}) + \phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}}p|| \leq b_k, \forall k \in \mathcal{J}_1, \\ (iii)\ ||\rho_l^2|| \leq \lambda_l^2\phi^{-1}(\gamma_l^2), \forall l \in \mathcal{J}_2, \\ (iv)\ \sum\limits_{i\in\mathcal{I}} p_i = 1, \\ (v)\ p_i \geq 0, \forall i \in \mathcal{I}, \\ (vi)\ \lambda_l^2 \geq 0, \forall l \in \mathcal{J}_2, \end{cases}$$

where $1_n = (1, \cdots, 1) \in \mathbb{R}^n$.

The SOCPs (P) and (D) are primal-dual pair of optimization problems.

Let (p^*, q^*) be a saddle point of game $\Gamma(\delta, \gamma)$.

With respect to Assumption 1, there is $(\upsilon^{1*}, (\rho_k^{1*})_{k=1}^r, \lambda^{1*})$ and $(\upsilon^{2*}, (\rho_l^{2*})_{l=1}^s, \lambda^{2*})$ such as $(q^*, (\upsilon^{1*}, (\rho_k^{1*})_{k=1}^r, \lambda^{1*}))$ and $(p^*, \upsilon^{2*}, (\rho_l^{2*})_{l=1}^s, \lambda^{2*})$ are optimal solutions of primal-dual pair (P) and (D) respectively.

Consider $(q^*, \upsilon^{1*}, (\rho_k^{1*})_{k=1}^r, \lambda^{1*})$ and $(p^*, \upsilon^{2*}, (\rho_l^{2*})_{l=1}^s, \lambda^{2*})$ be optimal solutions of the primal-dual pair (P) and (D), respectively.

Assumption 1 states that (P) and (D) are strictly feasible. Thus, for the primal-dual pair (P)-(D), strong duality holds [57]. Next up, we have

$$\upsilon^{1*} - \sum_{k=1}^r \lambda_k^{1*} b_k = \upsilon^{2*} - \sum_{l=1}^s \lambda_l^{2*} d_l.$$

Multiplying the constraint (i) from (P) by the vector p^{*T} from the left, we obtain

$$p^{*T}Aq^* + \sum_{k=1}^{r} \lambda_k^1 p^{*T}\mu_k^1 - \sum_{k=1}^{r} \lambda_k^1 L^{-1}(\delta_{k^1}) p^{*T}\alpha_k^{(1)} - \sum_{k=1}^{r} \left(\Sigma_k^1\right)^{\frac{1}{2}} p^{*T}(\rho_k^{1*}) \leq \upsilon^{1*}$$

$$\Longleftrightarrow p^{*T}Aq^* \leq \upsilon^{1*} - \sum_{k=1}^{r} \lambda_k^{1*} p^{*T}\mu_k^1 + \sum_{k=1}^{r} \lambda_k^1 L^{-1}(\delta_k^1) p^{*T}\alpha_k^{(1)} + \sum_{k=1}^{r} \left(\Sigma_k^1\right)^{\frac{1}{2}} p^{*T}(\rho_k^{1*})$$

$$\Longleftrightarrow p^{*T}Aq^* \leq \upsilon^{1*} - \sum_{k=1}^{r} \lambda_k^{1*} p^{*T}\mu_k^1 + \sum_{k=1}^{r} \lambda_k^1 L^{-1}(\delta_k^1) p^{*T}\alpha_k^{(1)} + \sum_{k=1}^{r} \lambda^{1*}\phi^{-1}(\gamma_k^1)||(\Sigma_k^1)^{\frac{1}{2}} p^*||$$

$$\Longleftrightarrow p^{*T}Aq^* \leq \upsilon^{1*} - \sum_{k=1}^{r} \lambda_k^{1*} b_k.$$

Let us consider the constraint (i) of (D)

$$A^T p^* - \sum_{l=1}^{s} \lambda_l^{2*}\mu_l^2 - \sum_{l=1}^{s} \lambda_l^1 R^{-1}(\delta_l^2)\beta_l^{(2)} - \sum_{l=1}^{s} \left(\Sigma_l^2\right)^{\frac{1}{2}} (\rho_l^{2*})^T \geq \upsilon^{*2} 1_n.$$

By using the argumentation shown above, we arrive at

$$p^{*T}Aq^* \geq \upsilon^{2*} - \sum_{l=1}^{s} \lambda_l^{2*} d_l.$$

Thus

$$p^{*T}Aq^* = \upsilon^{1*} - \sum_{k=1}^{r} \lambda_k^{1*} b_k = \upsilon^{2*} - \sum_{l=1}^{s} \lambda_l^{2*} d_l.$$

It is clear that, $p^TAq^* \leq p^{T*}Aq^*, \forall p \in S^m(\delta^1,\gamma^1))$ and, $p^{T*}Aq^* \leq p^{T*}Aq, \forall q \in S^n(\delta^2,\gamma^2))$. (p^*,q^*) is a saddle point of the game $\Gamma(\delta,\gamma)$. $\square$

6. NUMERICAL ILLUSTRATION

In this section, inspired by the idea in [43], we provide an example to illustrate the feasibility of the proposed approach.

Consider a situation where two competing companies are about to launch a new product with similar functionalities. In the target market, the demand for this innovative product remains essentially constant. This implies that any increase in one company's market share directly corresponds to an equivalent decrease in the market share of the other company. Each company aims to attract as many customers as possible by choosing a strategy from its set of marketing strategies, such as advertising, offering free samples, special promotions, etc.

For the sake of simplicity, let's assume that each company has two marketing strategies. Specifically, $\mathbb{I} = \{1, 2\}$ represents the pure strategy set for player I (firm f_1), and $\mathbb{J} = \{1, 2\}$ represents the pure strategy set for player II (firm f_2).

The allocation of funds among marketing strategies within a company is determined by its mixed strategy. Due to the imprecision of the available information, the management of firm f_1 (resp. f_2) cannot provide the exact amount of money required by the implementation of strategies 1 and 2. Therefore, LR-fuzzy random variables are appropriate to represent these funds.

Let $\tilde{B}^{\omega}_{11} = (N(62, 2), 26, 41)_{LR}$ and $\tilde{B}_{12} = (N(43, 3), 28, 22)_{LR}$ be the funds which firm f_1 needs when it selects strategy 1 and 2, respectively. And the firm f_2 needs the funds $\tilde{D}_{11} = (N(28, 3), 31, 30)_{LR}$ and $\tilde{D}_{12} = (N(2, 0.5), 15, 40)_{LR}$ when it takes strategy 1 and 2, respectively.

Furthermore, because of financial constraints, firm f_1 has a limited budget of $40,000$ (thousand dollars). As a result, the mixed strategies used by firm f_1 must satisfy the constraint condition $\tilde{B}^{\omega}11p_1 + \tilde{B}^{\omega}12p_2 \leq 40$. The firm f_2 only provides 17 (thousand dollars), the mixed strategies of the firm f_2 may satisfy the constraint conditions $\tilde{D}^{\omega}_{11}q_1 + \tilde{D}^{\omega}_{12}q_2 \leq 17$ or $-\tilde{D}^{\omega}_{11}q_1 - \tilde{D}^{\omega}_{12}q_2 \geq -17$.

Assuming that the payoff matrix and the data for this game are provided as follows:

$$\text{A}=\begin{pmatrix} 18 & 23 \\ 16 & 14 \end{pmatrix}$$

$$\textstyle\sum_1^1 = \begin{pmatrix} 14 & 2 \\ 2 & 14 \end{pmatrix}, \sum_1^2 = \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix},$$

$$\mu_1^1 = \begin{pmatrix} 16 \\ 18 \end{pmatrix}, \mu_1^2 = \begin{pmatrix} 10 \\ 12 \end{pmatrix}, \alpha_1^1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \beta_1^1 = \begin{pmatrix} 12 \\ 2 \end{pmatrix}, \alpha_1^2 = \begin{pmatrix} 13 \\ 5 \end{pmatrix}, \beta_1^2 = \begin{pmatrix} 7 \\ 12 \end{pmatrix},$$

$$b = \begin{pmatrix} 16 \end{pmatrix}, d = \begin{pmatrix} 14 \end{pmatrix}.$$

The saddle point $(p^*, q^*) = (0.4989, 0.5011), (0.3943, 0.6057)$ is obtained by selecting $(\delta_1^1, \delta_1^2) = (0.8, 0.8)$, $(\gamma_1^1, \gamma_1^2) = (0.6, 0.6)$, and solving the primal-dual programs in equations (P) and (D). Additionally, $v^*(\delta, \gamma) = 17.9015$ is the game's value.

7. RELATED WORK

Finite two-person games involving fuzziness and randomness are analysed in [34, 24, 35, 36, 38, 37]. In the present work, following the methodology proposed by Achemine and Larbani [37], it is assumed that the realizations of the fuzzy random variables are LR-fuzzy numbers whereas the works [34, 24, 35, 36, 38] deal with the uncertainty of basic type, the triangular fuzzy numbers. Further, the works in [34, 24, 35, 36, 38, 37] analyze matrix games and bimatrix games where the payoffs are fuzzy random variables, while in the present approach, we focus on constrained matrix games, where the payoffs are deterministic, and the entries of the constraint matrices are fuzzy random variables. To deal with matrix games with fuzzy random payoffs, Mondal and Roy [35], Xu and Li [24], and Xu et al. [34], used the expected value operator to define the fuzzy random expected minimax equilibrium and examine its existence and computation whereas our proposed methodology, as the approach outlined in [37], relies on chance constraints utilizing both probability and possibility measures. A noteworthy advantage of our suggested method is its flexibility, enabling the players to select both probability and possibility confidence levels at which they desire the constraints to be satisfied.

8. CONCLUDING REMARKS AND FUTURE WORK

In the theory of constrained matrix games introduced by Charnes [39], the linear constraints on the sets of strategies are assumed to be known with certainty. However, in many real-life situations, this assumption is not realistic. In this paper, we focus on matrix games where strategy sets are subject to fuzzy random linear constraints. Each player aims to maximize his payoff. Given that fuzzy random constraints do not define a crisp feasible set, we assume that the constraints hold at probability-possibility levels.The main results of this paper can be summarized in the following four aspects: (i) A new deterministic game is formulated considering probability-possibility constraints, and a specific type of saddle point equilibrium is introduced (ii) Sufficient existence conditions for this concept are derived; (iii) a second order cone programming method is provided to compute the proposed solution; (iv) an example of an application is presented to illustrate our approach. Remarkably little attention has been given to fuzzy random matrix games, and to the best of our knowledge, our approach is the first in the literature to address constrained matrix games with fuzzy random constraints. The present study theoretically presents a novel method for addressing fuzzy random constrained matrix games. However, the proposed methodology is highly reliant on the selection of confidence levels, potentially leading to a loss of valuable fuzzy random information. Moreover, within this approach, we deal with a normal distribution where the realizations are LR-fuzzy numbers. Consequently, further research and development are necessary to elaborate models and approaches that better reflect fuzziness and randomness. Moreover, as matrix games with fuzzy random constraints are a relatively new class of games, their further exploration is a worthy direction of research.

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