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A MARKOVIAN BATCH ARRIVAL QUEUEING SYSTEM WITH DISASTERS, WORKING BREAKDOWNS, AND IMPATIENCE: MATHEMATICAL MODELING AND ECONOMIC ANALYSIS

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Abstract: Machining systems are essential for many industrial applications, such as manufacturing, processing, and assembly. However, these systems are often exposed to various sources of uncertainty and disruption, such as disasters and customer impatience. These factors can adversely affect the performance, reliability, and profitability of the machining systems. Consequently, modeling and analyzing machining systems under these conditions becomes crucial. In this paper, we deal with a Markovian multi-server queueing system with batch arrival, Bernoulli feedback, and customers' impatience (balking and reneging). The system undergoes disastrous interruptions that force all customers–whether waiting or currently in service–to exit, leading to server failures. Moreover, the system dynamically alternates between main servers and substitute servers based on the occurrence of disasters. These substitute servers operate at reduced rates compared to the main servers. Our contributions include deriving the stability condition for the system and obtaining the probability generating function of steady-state probabilities, enabling us to derive essential performance measures. Additionally, we develop a cost model and conduct an economic analysis for the system.

Keywords: Markovian queueing models, disasters, impatience.

MSC: 60K25, 68M20, 90B22.

1. INTRODUCTION

Over the past two decades, queueing systems with disasters have garnered significant attention due to the rapid development of communication systems and networks. Disasters in these systems lead to the forced departure of all present customers, including the one being served. Such queueing models find applications in various domains. For instance, in computer networks, a virus can act as a delete operation, wiping out all stored data. Extensive research works have been conducted on the subject. Notable contributions include a $M/M/1$ queue with catastrophes by [1], a multi-server retrial queue with negative customers and disasters by [2], and a finite-source discrete-time $Geo/Geo/1$ queue with disasters by [3]. Subsequently, [4] extended the work [3] to $GI/Geo/1$ queues, while [5] investigated $M/G/1$ queues with disasters and working breakdowns. [6] studied $M/G/1$ queues with distastes in a multi-phase random environment and [7] explored $GI/M/1$ queues in multi-phase random environments with disasters and working breakdowns. Recently, [8] discussed a discrete-time $Geo/Geo/1$ queue with feedback, repair and disaster.

Customer behavior, such as balking and reneging, plays a crucial role in real-world queueing systems, where arrivals may be discouraged by long queues. Abundant literature exists on this topic, [9] studied a queue with disasters and impatient customers in which during the breakdown period, the new arrivals become impatient. Then, [10] presented optimal and equilibrium balking strategies in the single server Markovian queue with catastrophes and derived the corresponding Nash equilibrium and social optimal strategies. Later, they analyzed the effect of catastrophes on the strategic customer behavior in queueing systems [11]. Recently, [12] presented the transient analysis of impatient customers in a Markovian single server queue with disasters queue in random environment. Other contributions on impatience customers' in different queueing models can be found in [13, 14, 15, 16, 17, 18, 19, 20, 21].

Queueing systems with batch arrivals have a long history, dating back to the works of [22, 23, 24, 25, 26, 27, 28]. Subsequent researches have explored priority queues [29], server vacations [30], and heavy traffic limit theories [31, 32, 33]. Recent studies have investigated batch arrivals in conjunction with multiple working vacation [34], disasters and vacation [35], retrial queues [36], fluid queues [37], breakdowns and vacation [38, 39], vacation/working vacation queues with impatience [40, 41, 42], and group clearance [43].

This paper presents a novel contribution by considering an infinite-capacity multi-server Markovian queue with batch arrivals, Bernoulli feedback, working breakdowns, balking, and reneging. This model assumes that service continues at a reduced rate during repair periods, reflecting practical scenarios such as computer networks under virus attacks or machine replacements in manufacturing systems. The main contributions of this work can be summarized as follows:

- Establishing the stability condition for the proposed queueing system.
- Obtaining the steady-state solution for the system by using probability generating functions (PGFs), which provide a powerful approach for analyzing discrete probability distributions and stochastic processes.
- Deriving important performance measures from the steady-state probability distributions.
- Formulating a cost model for the queueing system to conduct an economic analysis.
- Performing a numerical analysis to validate the analytical results and investigate the impact of different system parameters on the performance measures, total expected cost, and total expected profit.

The body of the remainder of this paper is organized: Description of the system and a practical application of the suggested queueing model are given in Section 2. In Section 3, the analysis of the system is established. In Section 4, we formulated the performance measures of the system. In Section 5, we present a cost model. Then, in Section 6, numerical simulation results are provided and finally, in Section 7, we conclude the work done.

2. MODEL DESCRIPTION

An infinite-capacity multi-server queue with batch arrivals, Bernoulli feedback, disasters, working breakdowns, balking, and reneging is considered:

- Customers arrive in batches according to a Poisson process with rate λ. We consider our system in which the size of an arriving batch is drawn from an independent and identically distributed sequence of random variables. We assume that the times of arrivals are given by a Poisson process. The arrival batch size X is a random variable with probability mass function $\mathbb{P}(X = l) = b_l; l = 1, 2, ...$ They are served in accordance with First Come First Served 'FCFS' discipline.
- The service time during normal busy period are supposed to exponentially distributed with rate μ.
- During the busy period, the system may break down. At this time, all customers present are removed out and the system (all the servers as one station) is sent for a reparation. The inter-arrival times between successive breakdowns are assumed to be distributed exponentially with rate η. Repair times have an exponential distribution with rate ϑ.

- On arrival, if a batch of customers find the c servers busy, they may decide to enter the system with a certain probability θ, or balk wit a complementary probability $\theta' = 1 - \theta$. More precisely, we suppose that the number of customers in the batch is $n(\geq 1)$ then all customers of arrival batch join the system with probability θ, if $n < c$ and all leave the system without receiving service (balk) with probability θ', otherwise.

- During the repair period of the primary servers, new customers can be served by substitute servers. The service times during this period are assumed to be exponentially distributed with a rate ν, where $\nu < \mu$. Once the repair of the servers is completed, the service by the substitute servers is immediately stopped, and the primary servers restart operations at their regular service rate. Additionally, once the system is repaired and the queue becomes empty, all the primary servers return simultaneously to the system, remain idle, and wait for new arrivals.

- During a repair period, customers can get impatient; each customer activates an impatient timer 'T', exponentially distributed with rate χ. If the customer has not been served before its impatience time has expired he leaves the system without getting a service.

- If a customer is not happy with current service, he can retry many times as a feedback customer with some probability β' or leave the system with a complementary probability β.

- The inter-arrival times, repair times, impatience times, service times are supposed to be mutually independent.

2.1. Practical application of the proposed model

The proposed queuing model with batch arrivals, Bernoulli feedback, disasters, working breakdowns, balking, and reneging has practical applications in various manufacturing and production systems, particularly in the electronics industry. Consider a manufacturing facility that produces electronic devices such as smartphones, tablets, or laptops. The devices arrive in batches of random sizes according to a Poisson arrival process and join the queue/server for processing.

The manufacturing system comprises multiple servers, which are specialized machines or workstations responsible for quality checks, testing, and assembly operations on the devices. These servers operate in parallel, and the devices are served following the First-Come First-Served (FCFS) discipline.

However, the system is susceptible to catastrophic events like power failures, fires, or shortage of supplies, which force the machines (servers) to stop their service and evacuate the devices. In such cases, all existing devices in the system are rejected and lost, and the system undergoes a repair process of random duration.

During the repair period, the system can utilize backup generators and emergency staff to provide a substitute service to the arriving devices. However, the service rate of this substitute service is typically lower than the regular service rate. Devices arriving during a normal or breakdown period can decide whether to enter the system or balk (leave without receiving service) based on a certain probability.

Furthermore, devices already in the system during the repair process can also decide whether to stay or leave based on their impatience time. Each device activates an impatience timer with an exponentially distributed duration. If the device's service is not completed before its impatience time expires, it leaves the system without receiving service. Such devices are considered defective products and sent back to the factory.

After receiving service, if a device is not satisfied with the quality or requires additional processing, it can rejoin the queue for another service attempt with a certain feedback probability. This feedback mechanism allows devices to retry the service until they meet the desired quality standards.

3. ANALYSIS OF THE SYSTEM

3.1. Steady-state equations

Let $N(t)$ be the number of customers in the system and let $J(t)$ denote the status of the server at time t. If $J(t) = 1$, the system is functioning, serving customers, whereas if $J(t) = 2$, the system is down, undergoing a repair process.

Let $\{(N(t), J(t)); t \geq 0\}$ represent two-dimensional infinite state continuous-time Markov chain with state space $\mathcal{S} = \{(n, j) : n \geq 0, j = 1, 2\}$.

Let $\pi_{n,j} = \lim_{t\to\infty} \mathbb{P}\{N(t) = n, J(t) = j\}$, $n \geq 0, j = 1, 2$ define the system state probabilities of the process $\{(N(t), J(t)), t \geq 0\}$. Figure 1 depicts the state transition diagram of the queueing model under consideration.

Then, based on the theory of Markov process, it is easy to show that the steady-state equations of the model are:

1. If $J(t) = 1$, normal busy period:

$$\begin{aligned}
\lambda\pi_{0,1} &= \vartheta\pi_{0,2} + \beta\mu\pi_{1,1}, \quad n = 0, && (1)\\
(\lambda + \eta + \beta\mu)\pi_{1,1} &= \lambda b_1\pi_{0,1} + \vartheta\pi_{1,2} + 2\beta\mu\pi_{2,1}, \quad n = 1, && (2)\\
(\lambda + \eta + n\beta\mu)\pi_{n,1} &= \lambda\sum_{m=1}^{n} b_m\pi_{n-m,1} + \vartheta\pi_{n,2} + (n+1)\beta\mu\pi_{n+1,1}, \\
&\quad 2 \leq n \leq c-1, && (3)\\
(\theta\lambda + \eta + n\beta\mu)\pi_{n,1} &= \lambda\sum_{m=1}^{n} b_m\pi_{n-m,1} + \vartheta\pi_{n,2} + c\beta\mu\pi_{n+1,1}, \quad n = c, && (4)\\
(\theta\lambda + \eta + c\beta\mu)\pi_{n,1} &= \theta\lambda\sum_{m=1}^{n} b_m\pi_{n-m,1} + \vartheta\pi_{n,2} + c\beta\mu\pi_{n+1,1}, \quad n \geq c, && (5)
\end{aligned}$$

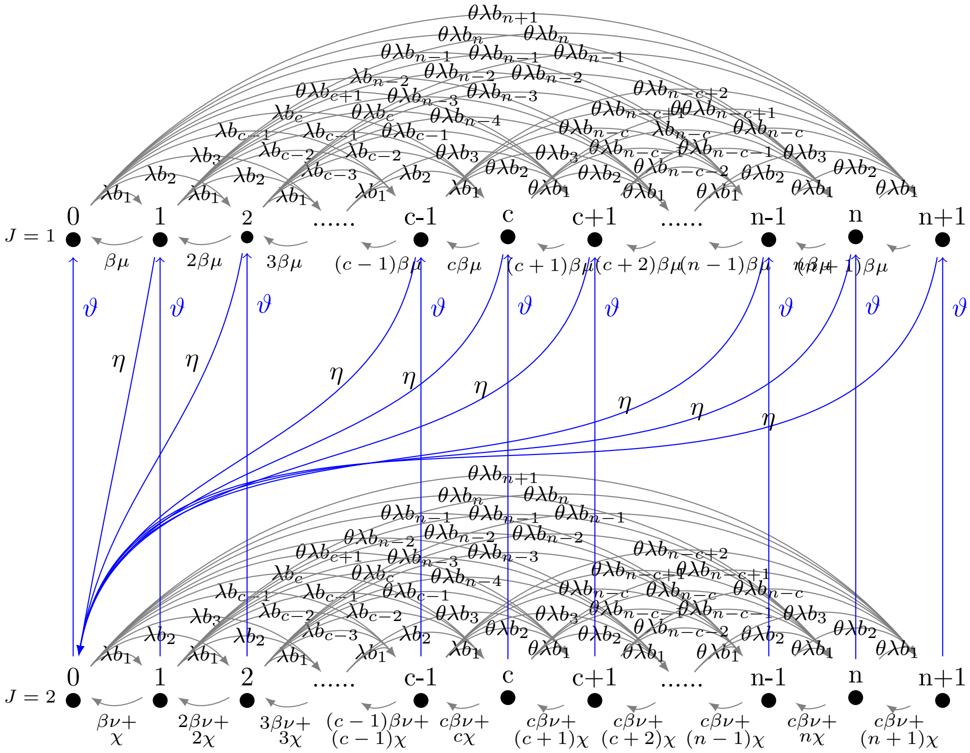

Figure 1: State-transition-rate diagram

2. If $J(t)=2$, working breakdown period:

$$(\lambda+\vartheta)\pi_{0,2} = \eta\sum_{n=1}^{\infty}\pi_{n,1}+(\beta\nu+\chi)\pi_{1,2},\ \ n=0, \tag{6}$$

$$(\lambda+\vartheta+\beta\nu+\chi)\pi_{1,2} = \lambda b_1\pi_{0,2}+2(\beta\nu+\chi)\pi_{n+1,2},\ \ n=1, \tag{7}$$

$$(\lambda+\vartheta+n(\beta\nu+\chi))\pi_{n,2} = \lambda\sum_{m=1}^{n}b_m\pi_{n-m,2}+(n+1)(\beta\nu+\chi)\pi_{n+1,2},\quad 2\le n\le c-1, \tag{8}$$

$$(\theta\lambda+\vartheta+n(\beta\nu+\chi))\pi_{n,2} = \lambda\sum_{m=1}^{n}b_m\pi_{n-m,2}+(c\beta\nu+(n+1)\chi)\pi_{n+1,2},\quad n=c, \tag{9}$$

$$(\theta\lambda+\vartheta+c\beta\nu+n\chi)\pi_{n,2} = \theta\lambda\sum_{m=1}^{n}b_m\pi_{n-m,2}+(c\beta\nu+(n+1)\chi)\pi_{n+1,2},\quad n\ge c, \tag{10}$$

3.2. Stability condition

According to Neuts [44], the infinitesimal generator $\mathbf{Q}$ for the bivariate process $\{(N(t), J(t)); t \geq 0\}$ is defined as follows:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_1^{(0)} & \mathbf{C}_2^{(0)} & \dots & \dots & \mathbf{C}_c^{(0)} & \mathbf{C}_{c+1}^{(1)} & \dots & \dots & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_1^{(0)} & \mathbf{C}_2^{(0)} & \dots & \mathbf{C}_{c-1}^{(0)} & \mathbf{C}_c^{(1)} & \dots & \dots & & \\ \mathbf{D} & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_1^{(0)} & \dots & \mathbf{C}_{c-2}^{(0)} & \mathbf{C}_{c-1}^{(1)} & \dots & \dots & & \\ \vdots & & \ddots & \ddots & \ddots & & & & & & \\ \mathbf{D} & & & \mathbf{B}_{c-1} & \mathbf{A}_{c-1} & \mathbf{C}_1^{(0)} & \mathbf{C}_2^{(1)} & \dots & \dots & & \\ \mathbf{D} & & & & \mathbf{B}_c & \mathbf{A}_c & \mathbf{C}_1^{(1)} & \dots & \dots & & \\ \mathbf{D} & & & & & \mathbf{B}_{c+1} & \mathbf{A}_{c+1} & \mathbf{C}_1^{(1)} & \dots & & \\ \vdots & & & & & & \ddots & \ddots & \ddots & & \\ \mathbf{D} & & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1^{(1)} & \\ \mathbf{D} & & & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1^{(1)} \\ \vdots & & & & & & & & & \ddots & \ddots \end{pmatrix},$$

where N is a sufficiently large number such that when the number of customers $n \geq N$, we approximate the matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ by $\mathbf{A}_N$ and $\mathbf{B}_N$, respectively. In the proposed queueing model, the approximation of the matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ by $\mathbf{A}_N$ and $\mathbf{B}_N$ for $n \geq N$ is employed to facilitate numerical analysis and computation. This approximation is based on the assumption that when the queue length exceeds a certain threshold N, the transition rates within the same level and to the next level can be considered constant, as the dynamics of the system do not significantly change for large queue lengths. The value of N is judiciously chosen such that the difference between the exact and approximated transition rates becomes negligible for $n \geq N$. Specifically, a sufficiently large N is selected, and the matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ are calculated using the steady-state equations and transition rate expressions. Then, for all $n \geq N$, the matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ are approximated by the constant matrices $\mathbf{A}_N$ and $\mathbf{B}_N$, respectively. These approximated matrices are subsequently used in the matrix representation of the Markov chain and in the numerical computations. While this approximation introduces some error, it is a practical consideration that enables the analysis of large-scale queueing systems by improving computational tractability.
Each sub-matrix of the matrix $\mathbf{Q}$ is done as:

$$\mathbf{A}_0 = \begin{pmatrix} -\lambda & 0 \\ \vartheta & -(\lambda+\vartheta) \end{pmatrix}, \quad \mathbf{C}_l^{(0)} = \begin{pmatrix} \lambda b_l & \\ & \lambda b_l \end{pmatrix}, \quad 1 \leq l < c$$

$$\mathbf{B}_1 = \begin{pmatrix} \beta\mu & \eta \\ 0 & \beta\nu+\chi \end{pmatrix}, \quad \mathbf{C}_l^{(1)} = \begin{pmatrix} \theta\lambda b_l & \\ & \theta\lambda b_l \end{pmatrix}, \quad l \geq c,$$

$$\mathbf{B}_n = \begin{pmatrix} n\beta\mu & 0 \\ 0 & n(\beta\nu+\chi) \end{pmatrix}, \quad 2 \leqslant n \leqslant c-1$$

$$\mathbf{B}_n = \begin{pmatrix} c\beta\mu & 0 \\ 0 & c\beta\nu+n\chi \end{pmatrix}, \quad c \leqslant n \leqslant N-1$$

$$\mathbf{B}_n = \begin{pmatrix} c\beta\mu & 0 \\ 0 & c\beta\nu+N\chi \end{pmatrix}, \quad n \geq N, \quad \mathbf{D} = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_n = \begin{pmatrix} -(\lambda + n\beta\mu + \eta) & 0 \\ \vartheta & -(\lambda + n(\beta\nu + \chi) + \vartheta) \end{pmatrix}, \quad 1 \leqslant n \leqslant c-1$$

$$\mathbf{A}_n = \begin{pmatrix} -(\theta\lambda + c\beta\mu + \eta) & 0 \\ \vartheta & -(\theta\lambda + c\beta\nu + n\chi + \vartheta) \end{pmatrix}, \quad c \leqslant n \leqslant N-1$$

$$\mathbf{A}_n = \begin{pmatrix} -(\theta\lambda + c\beta\mu + \eta) & 0 \\ \vartheta & -(\theta\lambda + c\beta\nu + N\chi + \vartheta) \end{pmatrix}, \quad n \geq N$$

In the following Theorem, we present the stability condition of our queueing system.

Theorem 1. *The Markov process* $\{(N(t); J(t)), t \geq 0\}$ *is ergodic if and only if*

$$\theta b\lambda < [c\beta\mu\vartheta + (c\beta\nu + N\chi)\eta]\frac{1}{\vartheta + \eta}, \quad \textit{where} \ \ b = \sum_{l=1}^{\infty} l b_l. \tag{11}$$

Proof. Based on [44], the approximated system is stable and the steady-state probability vector exists if and only if

$$\mathbf{x}\sum_{l=1}^{\infty} l\mathbf{C}_1^{(l)}\mathbf{e}_n < \mathbf{x}\mathbf{B}_N\mathbf{e}_n, \tag{12}$$

where $\mathbf{x} = [x_1, x_2]$ is the invariant probability vector of the matrix:

$$\mathbf{F} = \mathbf{D} + \mathbf{B}_N + \mathbf{A}_N + \sum_{l=1}^{\infty} \mathbf{C}_1^{(l)},$$

and $\mathbf{e}_n$ denotes a column vector with size n with all elements equal to one. Further, $\mathbf{x}$ satisfies:

$$\begin{cases} \mathbf{x}\mathbf{F} = \mathbf{0}, \\ \mathbf{x}\mathbf{e}_n = 1. \end{cases}$$

Solving the above two equations, we get

$$\mathbf{x} = [x_1, x_2] = \left[\frac{\vartheta}{\vartheta + \eta}, \frac{\eta}{\vartheta + \eta}\right].$$

Then, by substituting $\mathbf{x}$, $\mathbf{e}_n$, $\mathbf{C}_1^{(l)}$, and $\mathbf{B}_N$ into Equation (12), we find the stability condition (11). □

3.3. Analysis of the steady-state probability distribution

Define the probability generating functions as:

$G_j(z)=\sum_{n=0}^{\infty}\pi_{n,j}z^n, |z|=1,\ j=1,2,\ G_j^{'}(z)=\frac{d}{dz}G_j(z)=\sum_{n=1}^{\infty}n\pi_{n,j}z^{n-1},\ j=1,2,$

and $B(z)=\sum_{n=1}^{\infty}b_nz^n$, with $B(1)=\sum_{n=1}^{\infty}b_n=1$.
Multiplying Eqs. (1)-(5) by z^n and summing all possible values of n, we get:

$$\begin{array}{l}[\theta\lambda z(B(z)-1)+c\beta\mu(1-z)-z\eta]\,G_1(z)+z\vartheta G_2(z)=\lambda\theta^{'}z\psi_1(z)\\ +\beta\mu(1-z)\psi_2(z)-\lambda\theta^{'}z\psi_3(z)-z\eta\pi_{0,1},\end{array} \tag{13}$$

where
$\psi_1(z)=\sum_{n=0}^{c-1}\pi_{n,1}z^n$, $\psi_2(z)=\sum_{n=0}^{c-1}(c-n)\pi_{n,1}z^n$, and $\psi_3(z)=\sum_{n=1}^{c}\sum_{m=1}^{n}b_m\pi_{n-m,1}z^n$.
Similarly, multiplying Eqs. (6)-(10) by z^n then summing all possible values of n, we obtain:

$$\begin{array}{l}\chi z(1-z)G_2^{'}(z)+[\theta\lambda z(B(z)-1)+c\beta\nu(1-z)-z\vartheta]G_2(z)=\lambda\theta^{'}z\varphi_1(z)\\ +\beta\nu(1-z)\varphi_2(z)-\lambda\theta^{'}z\varphi_3(z)-\eta zG_1(1)+z\eta\pi_{0,1},\end{array} \tag{14}$$

with
$\varphi_1(z)=\sum_{n=0}^{c-1}\pi_{n,2}z^n$, $\varphi_2(z)=\sum_{n=0}^{c-1}(c-n)\pi_{n,2}z^n$, and $\varphi_3(z)=\sum_{n=1}^{c}\sum_{m=1}^{n}b_m\pi_{n-m,2}z^n$.
Next, using the recursive method, we get:

$$\begin{cases}\pi_{n,1}=L_n\pi_{0,1}+R_n\pi_{0,2}+N_nG_1(1),\\ \pi_{n,2}=A_n\pi_{0,2}+B_n\pi_{0,1}+C_nG_1(1),\end{cases}$$

where

$$A_n=\begin{cases}1, & \text{n=0;}\\ \dfrac{\lambda+\vartheta}{\beta\nu+\chi}, & \text{n=1;}\\ \varpi_nA_{n-1}+\frac{\kappa}{n}\sum_{m=1}^{n-1}b_mA_{n-m-1}, & n\geqslant 2,\end{cases}$$

$$B_n=\begin{cases}0, & \text{n=0;}\\ \dfrac{\eta}{\beta\nu+\chi}, & \text{n=1;}\\ \varpi_nB_{n-1}+\frac{\kappa}{n}\sum_{m=1}^{n-1}b_mB_{n-m-1}, & n\geqslant 2,\end{cases}$$

$$C_n=\begin{cases}0, & \text{n=0;}\\ \dfrac{-\eta}{\beta\nu+\chi}, & \text{n=1;}\\ \varpi_nC_{n-1}+\frac{\kappa}{n}\sum_{m=1}^{n-1}b_mC_{n-m-1}, & n\geqslant 2,\end{cases}$$

$$L_n = \begin{cases} 1, & \text{n=0;} \\ \dfrac{\lambda}{\beta\mu}, & \text{n=1;} \\ \zeta_n L_{n-1} + \frac{\kappa'}{n}\displaystyle\sum_{m=1}^{n-1} b_m L_{n-m-1} + \frac{R_1}{n} B_{n-1}, & n \geqslant 2, \end{cases}$$

$$R_n = \begin{cases} 0, & \text{n=0;} \\ \dfrac{-\vartheta}{\beta\mu}, & \text{n=1;} \\ \zeta_n R_{n-1} + \frac{\kappa'}{n}\displaystyle\sum_{m=1}^{n-1} b_m R_{n-m-1} + \frac{R_1}{n} A_{n-1}, & n \geqslant 2, \end{cases}$$

$$N_n = \begin{cases} 0, & \text{n=0;} \\ 0, & \text{n=1;} \\ \zeta_n N_{n-1} + \frac{\kappa'}{n}\displaystyle\sum_{m=1}^{n-1} b_m N_{n-m-1} + \frac{R_1}{n} C_{n-1}, & n \geqslant 2, \end{cases}$$

such that
$\kappa = \dfrac{-\lambda}{\beta\nu + \chi}$, $\varpi_n = \dfrac{\lambda + \vartheta}{n(\beta\nu + \chi)} + \dfrac{n-1}{n}$, $\kappa' = \dfrac{-\lambda}{\beta\mu}$, and $\zeta_n = \dfrac{\lambda + \eta}{n\beta\mu} + \dfrac{n-1}{n}$.
Further, for $z \neq 1$ *and* $z \neq 0$, Eqs. (13) and (14) can be respectively written as:

$$\begin{aligned} G_1(z) = &-\frac{z\vartheta}{\xi(z)} G_2(z) + \left[\frac{\lambda\theta' z(L_1(z) - L_0(z)) + \beta\mu(1-z)L_2(z) - z\eta}{\xi(z)}\right]\pi_{0,1} \\ &+ \left[\frac{\lambda\theta' z(R_1(z) - R_0(z)) + \beta\mu(1-z)R_2(z)}{\xi(z)}\right]\pi_{0,2} \\ &+ \left[\frac{\lambda\theta' z(N_1(z) - N_0(z)) + \beta\mu(1-z)N_2(z)}{\xi(z)}\right] G_1(1), \end{aligned} \tag{15}$$

where $\xi(z) = \theta\lambda z(B(z) - 1) + c\beta\mu(1-z) - z\eta$,

$$L_0(z) = \sum_{n=1}^{c}\sum_{m=1}^{n} b_m L_{n-m} z^n, \quad L_1(z) = \sum_{n=0}^{c-1} L_n z^n, \quad L_2(z) = \sum_{n=0}^{c-1}(c-n)L_n z^n,$$
$$R_0(z) = \sum_{n=1}^{c}\sum_{m=1}^{n} b_m R_{n-m} z^n, \quad R_1(z) = \sum_{n=0}^{c-1} R_n z^n, \quad R_2(z) = \sum_{n=0}^{c-1}(c-n)R_n z^n,$$
$$N_0(z) = \sum_{n=1}^{c}\sum_{m=1}^{n} b_m N_{n-m} z^n, \quad N_1(z) = \sum_{n=0}^{c-1} N_n z^n, \quad N_2(z) = \sum_{n=0}^{c-1}(c-n)N_n z^n,$$

and

$$\begin{aligned} G_2'(z) &+ \left[\frac{\theta\lambda}{\chi}H'(z) + \frac{c\beta\nu}{z\chi} - \frac{\vartheta}{\chi(1-z)}\right]G_2(z) \\ &= \left[\frac{\theta'\lambda}{\chi(1-z)}\left[A_1(z) - A_0(z)\right] + \frac{\beta\nu}{z\chi}A_2(z)\right]\pi_{0,2} \\ &+ \left[\frac{\theta'\lambda}{\chi(1-z)}\left[B_1(z) - B_0(z)\right] + \frac{\beta\nu}{z\chi}B_2(z) + \frac{\eta}{\chi(1-z)}\right]\pi_{0,1} \\ &\quad + \left[\frac{\theta'\lambda}{\chi(1-z)}\left[C_1(z) - C_0(z)\right] + \frac{\beta\nu}{z\chi}C_2(z) - \frac{\eta}{\chi(1-z)}\right]G_1(1), \end{aligned} \tag{16}$$

with

$$\begin{aligned} A_0(z) &= \sum_{n=1}^{c}\sum_{m=1}^{n} b_m A_{n-m} z^n, \;\; A_1(z) = \sum_{n=0}^{c-1} A_n z^n, \;\; A_2(z) = \sum_{n=0}^{c-1}(c-n)A_n z^n, \\ B_0(z) &= \sum_{n=1}^{c}\sum_{m=1}^{n} b_m B_{n-m} z^n, \;\; B_1(z) = \sum_{n=0}^{c-1} B_n z^n, \;\; B_2(z) = \sum_{n=0}^{c-1}(c-n)B_n z^n, \\ C_0(z) &= \sum_{n=1}^{c}\sum_{m=1}^{n} b_m C_{n-m} z^n, \;\; C_1(z) = \sum_{n=0}^{c-1} C_n z^n, \;\; C_2(z) = \sum_{n=0}^{c-1}(c-n)C_n z^n, \end{aligned}$$

and $H(z) = \int_0^z \frac{B(x)-1}{1-x}dx$, $H'(z) = \frac{B(z)-1}{1-z}$.

To solve Eq. (14), we multiply both sides of Eq. (16) by $e^{\frac{\lambda\theta}{\chi}H(z)}(1-z)^{\frac{\vartheta}{\chi}}z^{\frac{c\beta\nu}{\chi}}$, then we get:

$$\begin{aligned} G_2(z) = \frac{e^{-\frac{\lambda\theta}{\chi}H(z)}}{(1-z)^{\frac{\vartheta}{\chi}}z^{\frac{c\beta\nu}{\chi}}} &\Bigg\{ \left(\frac{\theta'\lambda}{\chi}K_0(z) + \frac{\beta\nu}{\chi}K_1(z)\right)\pi_{0,2} \\ &+ \left(\frac{\theta'\lambda}{\chi}K_2(z) + \frac{\beta\nu}{\chi}K_3(z) + \frac{\eta}{\chi}K_4(z)\right)\pi_{0,1} + \\ &\left[\frac{\theta'\lambda}{\chi}K_5(z) + \frac{\beta\nu}{\chi}K_6(z) - \frac{\eta}{\chi}K_4(z)\right]G_1(1)\Bigg\}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} K_0(z) &= \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}-1}x^{\frac{c\beta\nu}{\chi}}\left(A_1(x) - A_0(x)\right)dz, \\ K_1(z) &= \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}}x^{\frac{c\beta\nu}{\chi}-1}A_2(x)dz, \\ K_2(z) &= \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}-1}x^{\frac{c\beta\nu}{\chi}}\left(B_1(x) - B_0(x)\right)dz, \\ K_3(z) &= \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}}x^{\frac{c\beta\nu}{\chi}-1}B_2(x)dz, \end{aligned}$$

$K_4(z) = \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}-1}x^{\frac{c\beta\nu}{\chi}}dz,$
$K_5(z) = \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}-1}x^{\frac{c\beta\nu}{\chi}}\left(C_1(x)-C_0(x)\right)dz,$
$K_6(z) = \int_0^z e^{\frac{\lambda\theta}{\chi}H(x)}(1-x)^{\frac{\vartheta}{\chi}}x^{\frac{c\beta\nu}{\chi}-1}C_2(x)dz.$
Taking limit as $z \to 1$ in Eq. (17) we get:

$$\begin{aligned} G_2(1) = e^{-\frac{\lambda\theta}{\chi}H(1)}[&\left(\frac{\theta'\lambda}{\chi}K_0(1) + \frac{\beta\nu}{\chi}K_1(1)\right)\pi_{0,2} \\ &+\left(\frac{\theta'\lambda}{\chi}K_2(1) + \frac{\beta\nu}{\chi}K_3(1) + \frac{\eta}{\chi}K_4(1)\right)\pi_{0,1} \\ &+\left(\frac{\theta'\lambda}{\chi}K_5(1) + \frac{\beta\nu}{\chi}K_6(1) - \frac{\eta}{\chi}K_4(1)\right)G_1(1)]\lim_{z\to 1}(1-z)^{\frac{-\vartheta}{\chi}}z^{\frac{-c\beta\nu}{\chi}}. \end{aligned} \tag{18}$$

Since $G_2(1) = \sum_{n=0}^{\infty}\pi_{n,2} > 0$ and $\lim_{z\to 1}(1-z)^{\frac{-\vartheta}{\chi}}z^{\frac{-c\beta\nu}{\chi}} = \infty$ we must have that :

$$\begin{aligned} &\left(\frac{\theta'\lambda}{\chi}K_0(1) + \frac{\beta\nu}{\chi}K_1(1)\right)\pi_{0,2} + \left(\frac{\theta'\lambda}{\chi}K_2(1) + \frac{\beta\nu}{\chi}K_3(1) + \frac{\eta}{\chi}K_4(1)\right)\pi_{0,1} \\ &+\left(\frac{\theta'\lambda}{\chi}K_5(1) + \frac{\beta\nu}{\chi}K_6(1) - \frac{\eta}{\chi}K_4(1)\right)G_1(1) = 0. \end{aligned} \tag{19}$$

Therefore

$$G_1(1) = \Theta_1\pi_{0,2} + \Theta_2\pi_{0,1}, \tag{20}$$

where

$$\Theta_1 = \frac{-\theta'\lambda K_0(1) - \beta\nu K_1(1)}{\theta'\lambda K_5(1) + \beta\nu K_6(1) - \eta K_4(1)},$$

and

$$\Theta_2 = \frac{-\eta K_4(1) - \theta'\lambda K_2(1) - \beta\nu K_3(1)}{\theta'\lambda K_5(1) + \beta\nu K_6(1) - \eta K_4(1)}.$$

Further, by taking $z = 1$ in Eq. (15), we find

$$\begin{aligned} G_2(1) = &\left[\frac{\eta + \lambda\theta'(N_1(1) - N_0(1))}{\vartheta}\right]G_1(1) + \left[\frac{\lambda\theta'(L_1(1) - L_0(1)) - \eta}{\vartheta}\right]\pi_{0,1} \\ &+\left[\frac{\lambda\theta'(R_1(1) - R_0(1))}{\vartheta}\right]\pi_{0,2}. \end{aligned}$$

(21)

By substituting Eq. (20) into Eq. (21) we obtain:

$$G_2(1) = \Psi_1 \pi_{0,2} + \Psi_2 \pi_{0,1}, \tag{22}$$

where

$$\begin{aligned} \Psi_1 &= \frac{(\eta + \lambda\theta^{'}(N_1(1) - N_0(1)))\Theta_1}{\vartheta} + \frac{\lambda\theta^{'}(R_1(1) - R_0(1))}{\vartheta}, \\ \Psi_2 &= \frac{(\eta + \lambda\theta^{'}(N_1(1) - N_0(1)))\Theta_2}{\vartheta} + \frac{\lambda\theta^{'}(L_1(1) - L_0(1)) - \eta}{\vartheta}. \end{aligned}$$

Next, by taking $z = 1$ in Eqs. (13)-(14), we respectively have:

$$-\eta G_1(1) + \vartheta G_2(1) = \lambda\theta^{'}(\psi_1(1) - \psi_3(1)) - \eta\pi_{0,1}, \tag{23}$$

$$\eta G_1(1) - \vartheta G_2(1) = \lambda\theta^{'}(\varphi_1(1) - \varphi_3(1)) + \eta\pi_{0,1}. \tag{24}$$

Summing both (23) and (24) we obtain:

$$\psi_1(1) - \psi_3(1) = \varphi_3(1) - \varphi_1(1), \tag{25}$$

where
$\varphi_3(1) = A_0(1)\pi_{0,2} + B_0(1)\pi_{0,1} + C_0(1)G_1(1),$
$\varphi_1(1) = A_1(1)\pi_{0,2} + B_1(1)\pi_{0,1} + C_1(1)G_1(1),$
$\psi_3(1) = L_0(1)\pi_{0,1} + R_0(1)\pi_{0,2} + N_0(1)G_1(1),$
$\psi_1(1) = L_1(1)\pi_{0,1} + R_1(1)\pi_{0,2} + N_1(1)G_1(1).$

Further, by substituting Eq. (25) into (20) we find:

$$\pi_{0,2} = \Gamma(1)\pi_{0,1}, \tag{26}$$

where,

$$\Gamma(1) = \frac{L_0(1) - L_1(1) + B_0(1) - B_1(1) - \Theta_2\left[C_1(1) - C_0(1) + N_1(1) - N_0(1)\right]}{A_1(1) - A_0(1) + R_1(1) - R_0(1) + \Theta_1\left[C_1(1) - C_0(1) + N_1(1) - N_0(1)\right]}.$$

The following Theorem presents the steady-state probabilities of the considered queueing system.

Theorem 2. *Under the stability condition, the steady-state probabilities are given by*

$$\pi_{.,1} = G_1(1) = \left[\Theta_1\Gamma(1) + \Theta_2\right]\pi_{0,1}, \tag{27}$$

and

$$\pi_{.,2} = G_2(1) = \left[\Psi_1\Gamma(1) + \Psi_2\right]\pi_{0,1}, \tag{28}$$

where

$$\pi_{0,1} = \left\{\Theta_1\Gamma(1) + \Theta_2 + \Psi_1\Gamma(1) + \Psi_2\right\}^{-1}.$$

Proof. By substituting Eq. (26) into Eqs. (20) and (22) we get $\pi_{.,1}$ and $\pi_{.,2}$, respectively.
Then, using the normalization condition:

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2}\pi_{n,j}=1\Leftrightarrow\pi_{.,1}+\pi_{.,2}=1,\ \text{ we obtain }\ \pi_{0,1}.$$

□

Let L denote the number of customers in the system. Then we have $E(L) = E(L_1) + E(L_2)$, where $E(L_1)$ is the mean system size when the system is on busy period and $E(L_2)$ represents the mean system size when the system is on working repair.

Theorem 3. *The mean system sizes during busy and working breakdown periods can be expressed as*

$$\begin{aligned}E(L_1)=&\frac{1}{\eta}\left[(\theta\lambda B^{'}(1)-c\beta\mu-\eta)(\Theta_1\Gamma(1)+\Theta_2)+\vartheta(\Psi_1\Gamma(1)+\Psi_2)+\eta\right]\pi_{0,1}\\&+\frac{1}{\eta}\left[\lambda\theta^{'}(\psi_3(1)+\psi_3^{'}(1)-\psi_1(1)-\psi_1^{'}(1))+\beta\mu\psi_2(1)+\vartheta E(L_2)\right],\end{aligned}$$

and

$$\begin{aligned}E(L_2)=&\frac{1}{\chi+\vartheta}[\left[\left(\theta\lambda B^{'}(1)-c\beta\nu-\vartheta\right)[\Psi_1\Gamma(1)+\Psi_2]-\eta+\eta(\Theta_1\Gamma(1)+\Theta_2)\right]\pi_{0,1}\\&+\lambda\theta^{'}(\varphi_3(1)+\varphi_3^{'}(1)-\varphi_1(1)-\varphi_1^{'}(1))+\beta\nu\varphi_2(1)].\end{aligned}$$

Proof. Setting $z\longrightarrow 1$ and using L'Hospital rule in Eq. (13), we get:

$$\begin{aligned}&\left[\theta\lambda B^{'}(1)-c\beta\mu-\eta\right]G_1(1)+\vartheta G_2(1)+\vartheta G_2^{'}(1)-\eta G_1^{'}(1)\\&\qquad=\lambda\theta^{'}\left[-\psi_3(1)-\psi_3^{'}(1)+\psi_1(1)+\psi_1^{'}(1)\right]-\beta\mu\psi_2(1)-\eta\pi_{0,1},\end{aligned}$$

where $G_1(1)$ and $G_2(1)$ are given in Eqs. (27) and (28), respectively. Therefore

$$\begin{aligned}E(L_1)&=\lim_{z\to 1}G_1^{'}(z)\\&=\frac{1}{\eta}\left[(\theta\lambda B^{'}(1)-c\beta\mu-\eta)(\Theta_1\Gamma(1)+\Theta_2)+\vartheta(\Psi_1\Gamma(1)+\Psi_2)+\eta\right]\pi_{0,1}\\&+\frac{1}{\eta}\left[\lambda\theta^{'}(\psi_3(1)+\psi_3^{'}(1)-\psi_1(1)-\psi_1^{'}(1))+\beta\mu\psi_2(1)+\vartheta E(L_2)\right].\end{aligned}$$

Next, differentiating Eq. (14) and taking $z = 1$, we find:

$$\begin{aligned}(\chi+\vartheta)G_2^{'}(1)=&\left[\theta\lambda B^{'}(1)-c\beta\nu-\vartheta\right]G_2(1)+\lambda\theta^{'}(\varphi_3(1)+\varphi_3^{'}(1)-\varphi_1(1)\\&-\varphi_1^{'}(1))+\beta\nu\varphi_2(1)-\eta\left[\pi_{0,1}-G_1(1)\right].\end{aligned}$$

Thus

$$
\begin{aligned}
E(L_2) &= \lim_{z\to 1} G_2^{'}(z) \\
&= \frac{1}{\chi+\vartheta}\Big[\Big[\Big(\theta\lambda B^{'}(1) - c\beta\nu - \vartheta\Big)\left[\Psi_1\Gamma(1)+\Psi_2\right] - \eta + \eta(\Theta_1\Gamma(1)+\Theta_2)\Big]\pi_{0,1} \\
&\quad +\lambda\theta^{'}(\varphi_3(1)+\varphi_3^{'}(1)-\varphi_1(1)-\varphi_1^{'}(1)) + \beta\nu\varphi_2(1)\Big].
\end{aligned}
$$

□

4. PERFORMANCE MEASURES AND COST MODEL

4.1. Performance measures

In this subpart of paper, useful performance measures are presented.

Corollary 4. *The mean number of customers in the queue is given as:*

$$E(L_q) = \sum_{n=c+1}^{\infty}(n-c)\left(\pi_{n,1}+\pi_{n,2}\right) = E(L) - c + \psi_2(1) + \varphi_2(1).$$

Corollary 5. 1. *The probability that the servers are in working repair period is presented as:*

$$P_{wr} = G_2(1) = \sum_{n=0}^{\infty}\pi_{n,2}.$$

2. *The probability that the servers are in a normal busy period is presented as:*

$$P_b = 1 - P_{wr}.$$

3. *The probability that the servers are working either during busy or repair period is presented as:*

$$P_w = \sum_{n=1}^{\infty}(\pi_{n,2}+\pi_{n,1}).$$

Corollary 6.

1. The mean number of customers served per unit time is given as:

$$
\begin{aligned}
N_s &= \beta\mu\sum_{n=1}^{c-1} n\pi_{n,1} + c\beta\mu\sum_{n=c}^{\infty}\pi_{n,1} + \beta\nu\sum_{n=1}^{c-1} n\pi_{n,2} + c\beta\nu\sum_{n=c}^{\infty}\pi_{n,2} \\
&= \beta\mu\left[cP_{busy} - \psi_2(1)\right] + \beta\nu\left[cP_{wr} - \varphi_2(1)\right].
\end{aligned}
$$

2. The average rate of abandonment of customers due to impatience is given as:

$$R_a = \chi E(L_2).$$

3. The average rate of balking is given as:

$$R_{balk} = \lambda(1-\theta)\left[\sum_{n=c}^{\infty}\pi_{n,1} + \sum_{n=c}^{\infty}\pi_{n,2}\right] = \lambda(1-\theta)\left[1 - \psi_1(1) - \varphi_1(1)\right].$$

4.2. Cost model

Cost-profit analysis is very beneficial in the application of real-life situations arising from industrial and technical situations.

The total expected cost (T_{cost}) is defined as:

$$\begin{aligned} T_{\text{cost}} &= C_b P_b + C_{rp} P_{wr} + (C_l E(L)) + (C_r R_{ren}) + c(\mu + \nu) \\ &\times (C_s + (1-\beta) C_f) + c C_p, \end{aligned}$$

where C_b is the cost per unit time during normal busy period, C_{rp}; the cost per unit time during working repair period, C_l; the holding cost per unit time, C_r; the cost per unit time when a customer is lost due to impatience, C_s; the cost per service per unit time, C_f; the cost per unit time when a customer returns to the system as a feedback, and C_p; the fixed server purchase cost per unit.

Let $\mathcal{R}$ be the revenue earned for providing service to a customer, then the total expected revenue per unit time (T_{revenue}) of the system is as:

$$T_{\text{revenue}} = R \times N_s.$$

The total expected profit (T_{profit}) per unit time of the system is as:

$$T_{\text{profit}} = T_{\text{revenue}} - T_{\text{cost}}.$$

5. NUMERICAL ANALYSIS

To validate the analytical results obtained through mathematical modeling and analysis, we employ computational techniques. These methods allow us to compute and approximate the relevant quantities, including steady-state probabilities and performance measures for the manufacturing system. By comparing the numerically obtained results with the analytical expressions, we validate the accuracy of our derived solutions.

While numerical analysis introduces approximations and potential numerical errors, it complements analytical methods by providing a practical means of verifying results and acquiring a deeper understanding of the behavior of complex queueing systems.

In this section, important numerical results are presented in the form of Tables and Graphs in order to illustrate the effect of various system parameters on different system characteristics, (T_{cost}) and (T_{profit}), using R program. The arrival batch size X follows a geometric distribution with parameter σ ,that is $P(X = l) = (1-\sigma)^{l-1}\sigma$, with $0 < \sigma < 1$, and $l = 1, 2, \ldots.$ Therefore, $B(z) = \dfrac{\sigma z}{1-(1-\sigma)z}$.

For our analysis, we consider the manufacturing system discussed above. Unless their values are indicated in the appropriate places, the model parameters are assumed to be as follows: the devices arrive in groups of random size according to a Poisson arrival process with rate $\lambda = 1.0$ devices per minute, and join the queue/server for processing. The system has $c = 3$ primary servers. The service time of each machine is exponentially distributed with rate $\mu = 1.9$ devices per minute. The time between successive breakdowns is exponentially distributed with rate $\eta = 5.0$ breakdowns per minute, and the repair time is exponentially distributed with rate $\vartheta = 2.0$ repairs per minute. During a breakdown, the substitute service time is exponentially distributed with rate $\nu = 0.5$ devices per minute, where $\nu < \mu$. The devices can decide whether to enter the system or not, based on the probability $\theta = 0.8$. The devices that are already in the system can also decide whether to stay or leave, based on their impatience time, which is exponentially distributed with rate $\chi = 0.2$ devices per minute. If a device gets a service but is not satisfied, it can retry the service with probability $\beta' = 0.5$ or leave the system with probability $\beta = 0.5$, and $\sigma = 0.7$.

To evaluate the cost and revenue of the system, we can use the following parameters: we take the cost parameters as $C_b = \$3.5$, $C_{rp} = \$2$, $C_l = \$2.5$, $C_r = \$2$, $C_s = \$0.11$, $C_f = \$0.11$, $C_p = \$1$, and $R = \$70$. These values can be adjusted according to the market conditions and the quality of the devices. Numerical results are presented in Table 1 and Figs. 2-7:

		$\pi_{0,1}$	P_{wr}	P_b	$E(L_1)$	$E(L_2)$	$E(L)$	R_{balk}	R_{ren}	N_s
	1.0	0.5156	0.3445	0.6555	0.1448	0.1783	0.3231	0.1238	0.0357	0.4922
λ	1.5	0.3815	0.4387	0.5613	0.2163	0.3288	0.5451	0.2434	0.0658	0.6480
	2.0	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	1.1	0.2873	0.5041	0.4959	0.3465	0.4884	0.8349	0.3831	0.0977	0.4797
μ	1.5	0.2889	0.5029	0.4971	0.3160	0.4872	0.8033	0.3886	0.0974	0.6244
	1.9	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	0.3	0.2874	0.5040	0.4960	0.2903	0.5051	0.7953	0.3990	0.1010	0.7349
ν	0.4	0.2884	0.5033	0.4967	0.2872	0.4959	0.7831	0.3966	0.0992	0.7553
	0.5	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	0.5	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
θ	0.7	0.2874	0.5060	0.4940	0.3270	0.5334	0.8604	0.2391	0.1067	0.7740
	0.9	0.2855	0.5094	0.4906	0.3705	0.5830	0.9535	0.0805	0.1166	0.7731
	5	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
η	7	0.3050	0.5364	0.4636	0.2251	0.5197	0.7448	0.3628	0.1039	0.6594
	9	0.3143	0.5575	0.4425	0.1853	0.5402	0.7254	0.3443	0.1080	0.5901
	1	0.1362	0.7108	0.2892	0.2064	1.0894	1.2958	0.5284	0.2179	0.8523
ϑ	1.5	0.2210	0.5926	0.4074	0.2547	0.7019	0.9566	0.4450	0.1404	0.8075
	2	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	0.1	0.2874	0.5040	0.4960	0.2914	0.5078	0.7991	0.3990	0.0508	0.7796
χ	0.15	0.2884	0.5033	0.4967	0.2877	0.4971	0.7848	0.3966	0.0746	0.7773
	0.20	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	3	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
c	5	0.3032	0.4959	0.5041	0.3723	0.5333	0.9056	0.1447	0.1067	0.6822
	7	0.3026	0.4975	0.5025	0.4112	0.5616	0.9728	0.0501	0.1123	0.6175
	0.5	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
β	0.7	0.2898	0.5022	0.4978	0.2154	0.4709	0.6862	0.4006	0.0942	1.1139
	0.9	0.2871	0.5041	0.4959	0.1425	0.4577	0.6002	0.4075	0.0915	1.4710
	0.5	0.2795	0.5030	0.4970	0.3668	0.5924	0.9592	0.5439	0.1185	0.9595
σ	0.7	0.2894	0.5025	0.4975	0.2843	0.4869	0.7712	0.3942	0.0974	0.7749
	0.9	0.3011	0.4981	0.5019	0.2474	0.4180	0.6654	0.2628	0.0836	0.6098

Table 1: Effect of λ, μ, ν, ϑ, η, σ, χ, and β on performance measures

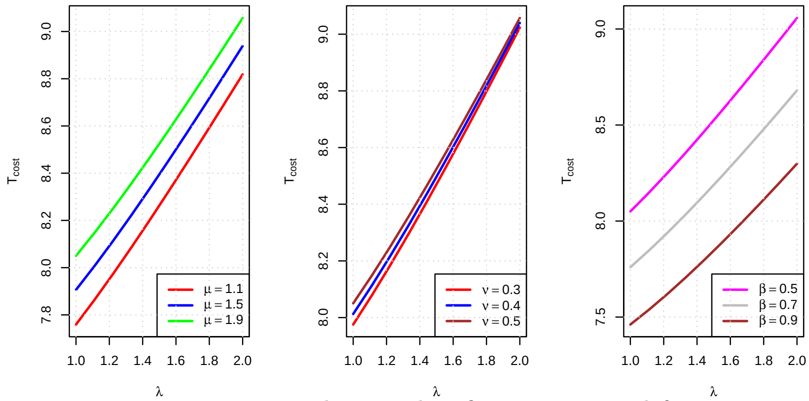

Figure 2: Total expected profit versus μ, ν, and β

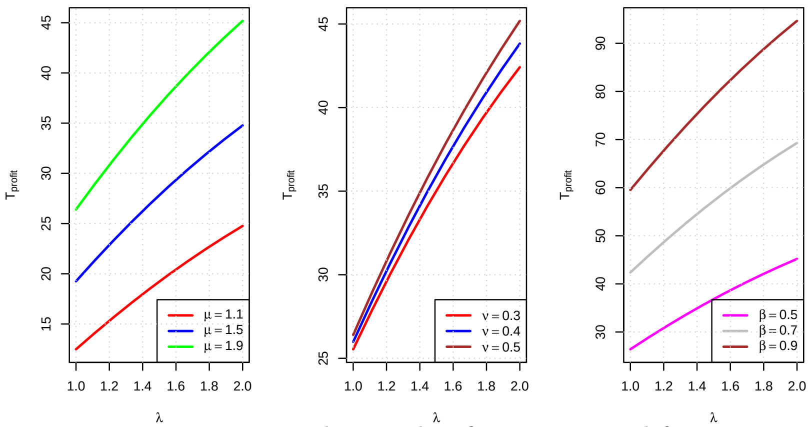

Figure 3: Total expected profit versus μ, ν, and β

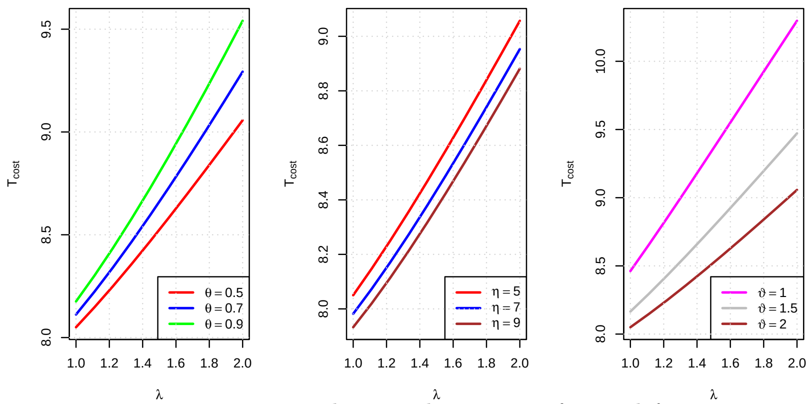

Figure 4: Total expected cost versus θ, η, and ϑ

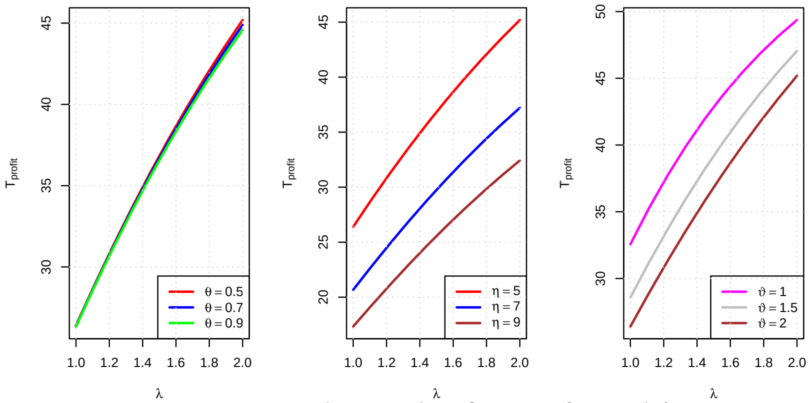

Figure 5: Total expected profit versus θ, η, and ϑ

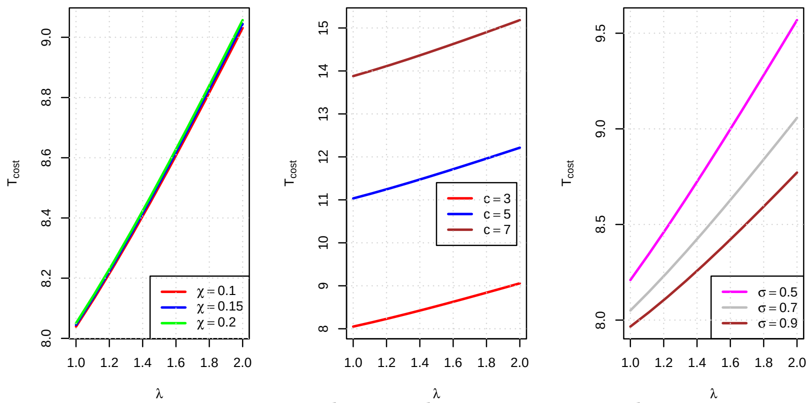

Figure 6: Total expected cost versus χ, c, and σ

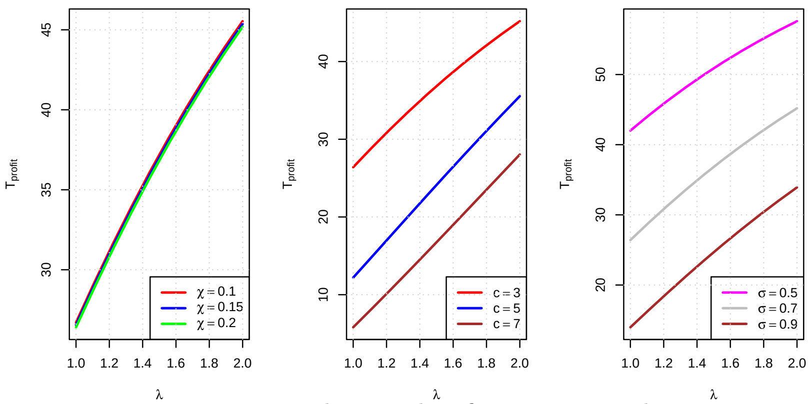

Figure 7: Total expected profit versus χ, c, and σ

5.1. Results discussion and managerial insights

The numerical experiments explored the sensitivity of various system parameters on performance measures and the cost-profit model. The key observations and potential managerial recommendations are as follows:

1. Arrival rate (λ): A higher arrival rate leads to lower idle probability ($\pi_{0,1}$) and higher expected system lengths ($E(L_1), E(L_2), E(L)$), average balking and reneging rates (R_{balk}, R_{ren}), and mean number of customers served (N_s). Consequently, both the total expected cost and profit ($T_{\text{cost}}, T_{\text{profit}}$) increase. While higher demand can boost revenue, managers must judiciously balance it against the potential impact on congestion, service quality degradation, and increased system breakdowns.
2. Service rates (μ, ν): Higher service rates during busy and breakdown periods reduce average system lengths and reneging rates, improving customer satisfaction and loyalty. Furthermore, it increases idle probability ($\pi_{0,1}$), mean number of customers served (N_s), and lowers breakdown probability (P_{wr}), enhancing profitability. However, the associated costs and feasibility constraints of increasing service rates should be carefully evaluated.
3. Non-balking probability (θ): A higher non-balking probability attracts more customers, increasing the busy period's average system length ($E(L_1)$) and breakdown probability (P_{wr}). This leads to higher customer losses (R_{ren}), reducing the mean number of customers served and the total expected profit (T_{profit}) while increasing the total expected cost (T_{cost}). Managers must strike a balance between attracting customers and managing congestion, impatience, and system breakdowns.
4. Impatience rate (χ): A higher impatience rate adversely impacts the profit by reducing the mean number of customers served due to reneging. To mitigate this, the manufacturing system should aim to decrease the impatience rate of the devices as much as possible, by providing substitute services, priority queues, or compensation schemes. This can improve customer retention and overall satisfaction.
5. Failure rate (η): A higher failure rate decreases the average system length during the busy period ($E(L_1)$), leading to more customer losses due to impatience (R_{ren}), significantly reducing the total expected profit (T_{profit}). This means that the manufacturing system can reduce the impact of catastrophic events by decreasing the failure rate and increasing the repair rate.
6. Repair rate (ϑ): Even when considering the server's ability to work during breakdowns, a higher repair rate leads to a reduction in the number of customers served and profitability. This can be explained by the fact that the reduced service rate can, sometimes, act as a bottleneck, leading to longer queues and waiting times. Consequently, fewer customers may be served overall, resulting in lower profitability despite the reduction in complete downtime. Therefore, managers must carefully evaluate the cost-benefit trade-off of investing in higher repair rates against the potential rev-

enue gains from reduced complete downtime and the potential revenue losses due to the server's reduced service capacity during breakdowns.

7. Non-feedback probability (β): Similar to service rates, a higher non-feedback probability leads to an increase in the total expected profit. This is likely due to a reduction in congestion and a more efficient utilization of system resources when fewer customers return to the system. However, a high non-feedback probability may also indicate underlying issues with service quality and customer dissatisfaction. While increasing the feedback probability can boost revenue in the short term by accommodating more retries, it may also exacerbate congestion and waiting times. To strike a balance and maintain long-term profitability, managers should focus on improving service quality through better service rates or providing compensations to enhance overall customer satisfaction. This can help retain customers and mitigate the potential negative impact of a high non-feedback probability on future business.
8. Batch size probability (σ): A higher probability of smaller batch sizes (σ closer to 1) decreases the mean number of customers in the system, customers served, the average rates of balking and reneging, probability of breakdowns, total expected cost, and total expected profit. Attracting more devices by favoring smaller batches can be a viable strategy. However, this approach also increases congestion and customer waiting times. Managers must carefully balance the trade-off between batch size probability and waiting times, as a high probability of smaller batches can exacerbate impatience and dissatisfaction among customers.
9. The number of servers (c): A higher number of servers in the system leads to higher total expected costs and lower total expected profits. This counterintuitive result can be attributed to potential inefficiencies and coordination challenges associated with managing a larger number of servers. With more servers, there is an increased likelihood of underutilization or imbalanced workload distribution, leading to inefficient resource utilization and longer waiting times for customers. Additionally, a higher number of servers may also increase the system's complexity, potentially leading to more frequent breakdowns or maintenance requirements, further contributing to reduced throughput and profitability. Therefore, managers should carefully evaluate the trade-off between the number of servers, the associated costs, and the potential impact on system efficiency and customer throughput.

Remark 7. *The choice of parameters in Table 1, while arbitrary, was carefully done to ensure the stability of the system and to observe clear behaviors that could be interpreted meaningfully. Based on the observations above, it is important to recognize that most of the results align with our intuition. However, there are some examples that are less straightforward to interpret, possibly due to the specific costs and parameters chosen.*

6. CONCLUSION

In this paper, we presented a queueing system applicable to manufacturing systems producing electronic devices like smartphones, tablets, or laptops. Our model incorporated batch arrivals, multiple servers, catastrophic events, substitute service during breakdowns, customer balking and reneging behavior, and feedback. We derived the stability condition and employed probability generating functions to obtain closed-form expressions for the steady-state probabilities and performance measures. Furthermore, we conducted a numerical analysis to evaluate the impact of different parameters on key performance metrics, total expected cost, and total expected profit.

Potential future research directions include extending the proposed model to batch service queues. It would also be interesting to explore more complex scenarios, such as repairable queueing systems with non-Markovian arrival processes for customers and non-Markovian service processes for normal and breakdown services. Such extensions would enhance the model's applicability to a wider range of real-world manufacturing scenarios.

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