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HIGHER-ORDER MOND-WEIR DUALITY OF SET-VALUED FRACTIONAL MINIMAX PROGRAMMING PROBLEMS

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Abstract: In this paper, we consider a set-valued fractional minimax programming problem (abbreviated as SVFMPP) (MFP), in which both the objective and constraint maps are set-valued. We use the concept of higher-order α-cone arcwisely connectivity, introduced by Das [1], as a generalization of higher-order cone arcwisely connected set-valued maps. We explore the higher-order Mond-Weir (MWD) form of duality based on the supposition of higher-order α-cone arcwisely connectivity and prove the associated higher-order converse, strong, and weak theorems of duality between the primary (MFP) and the analogous dual problem (MWD).

Keywords: Contingent epiderivative, convex cone, arcwisely connectivity, duality, set-valued map.

MSC: 26B25, 49N15.

1. INTRODUCTION

Mathematical science, economics, and operational research are the primary fields of study for a class of problems known as fractional minimax programming problems, or FMPPs for short. In 1990, two alternative duality models were developed and the theorems of duality for FMPPs were established by Yadav and Mukherjee [2] for differentiable case of FMPPs. Later, in 1995, the theorems of duality of FMPPs were identified and two distinct forms of modified duality models were established by Chandra and Kumar [3] for differentiable case of FMPPs. The optimality criteria and the strong and weak theorems of duality of FMPPs were investigated by Weir [4] and Bector and Bhatia [5]. Zamlai [6] established the sufficient and necessary criteria of optimality and derived theorems of duality for FMPPs under the assumption of generalized invexity. Liu and Wu [7] established the theorems of duality of FMPPs and articulated the necessary criteria of optimality under the assumption of (F, α)-convexity concept. Ahmad [8] established the necessary conditions of optimality and developed the theorems of duality for FMPPs using the assumption of α-invexity hypothesis. Liang and Shi [9] established the theorems of duality for FMPPs and presented sufficient criteria of optimality via the assumption of (F, α, ρ, d)-convexity hypothesis. Lai et al. [10] investigated the parametric theorems of duality and established the necessary and sufficient criteria of optimality under the generalized convexity supposition for nondifferentiable case of FMPPs. Lai and Lee [11] developed the theorems of duality under the generalized convexity supposition for nondifferentiable case of FMPPs. Ahmad and Husain [12] developed the necessary criteria of optimality and demonstrated the theorems of duality of FMPPs under the (F, α, ρ, d)-convexity supposition. Using higher-order contingent derivatives, Li et al. [13, 14] established the sufficient and necessary criteria of optimality of set-valued optimization problems (in short, SVOPs) in 2008. Additionally, the higher-order Mond-Weir dual of SVOPs was presented, and the theorems of duality under convexity suppositions were examined. In 1976, Avriel [15] presented arcwisely connectivity, a generalized form of convexity where a continuous arc is used instead of the line segment connecting two elements. The class of convex set-valued maps (abbreviated as SVMs) is a special type of set-valued cone arcwisely maps. This concept was established by Fu and Wang [16] and Lalitha et al. [17].

The notion of contingent derivative is a fundamental generalization of Frechet differentiability from the single-valued to the set-valued case. This concept has been widely applied in set-valued optimization theory as well. The sufficient and necessary optimality criteria do not generally coincide under the assumption of contingent derivative. Therefore, contingent derivatives are not exactly the right tool for developing optimality criteria in set-valued optimization. The notion of contingent epiderivative is one potential generalization of directional derivatives in the single-valued convex case. In the contingent epiderivative, the epigraph is used in place of the graph, and the derivative is single-valued. These are the primary distinctions of two derivatives from one another. For cone-convex SVMs, contingent epiderivatives have the unique characteristic of being sublinear, if they exist at all. Therefore, higher-order contingent epiderivatives are of greater interest while studying set-valued optimization problems.

Arcwise connectedness is a generalization of convexity where the line segment connecting two places is replaced by a continuous arc. We introduce the notion of higher-order α-cone arcwise connectedness of SVMs as a generalization of cone arcwise connected SVMs. For $\alpha = 0$, we derive the conventional notion of higher-order cone arcwise connectedness of SVMs. We also develop an example of a SVM that is not higher-order cone arcwise connected, but is higher-order α-cone arcwise connected.

We employ the notion of higher-order α-cone arcwisely connectivity of SVMs, introduced by Das [1], to solve SVFMPPs. The concept of higher-order α-cone arcwisely connectivity is more widely applicable than higher-order α-cone convexity. We illustrate it by studying an example in our work. As objective functions and constraints in SVFMPPs, we primarily deal with α-cone arcwisely connected SVMs of higher order.

Das and Nahak [18] established the higher-order sufficient KKT requirements via higher-order contingent epiderivative and higher-order α-cone convexity assumptions for the set-valued optimization problem. Under α-cone convexity assumptions, the duals of higher-order Mond-Weir, Wolfe, and mixed types are formulated, and the associated higher-order duality theorems are proved. In this study, however, we have demonstrated the duality theorems and sufficient KKT conditions of SVFMPPs under the hypothesis of α-cone arcwise connectedness. Instead of α-cone convex SVMs, we essentially deal with α-cone arcwise connected SVMs as objective function and constraint. This study presents more broadly relevant cases than previous ones.

In order to establish the sufficient optimality conditions of SVFMPPs in a more generalized case, our main objective is to implement the higher-order contingent epiderivative and higher-order α-cone arcwise connectedness assumptions on the objective functions and constraints. We also investigate the weak, strong, and converse duality theorems of Mond-Weir type under higher-order contingent epiderivative and higher-order α-cone arcwise connectivity assumptions. For $\alpha = 0$, our results improve the ones currently available in the literature.

The following abbreviations have been used often in this paper. Set-valued fractional minimax programming problems are denoted by SVFMPPs, Karush-Kuhn-Tucker by KKT, set-valued optimization problems by SVOPs, fractional minimax programming problems by FMPPs, arcwise connected subsets by ACS, real normed space by RNS, set-valued maps by SVMs, and "with respect to" by "w.r.t.".

This is how the paper is organized. Section 2 covers the definitions and fundamental concepts of SVMs. The concept of higher-order α-cone arcwisely connectivity of SVMs is discussed in Section 3. A SVFMPP (MFP) is formulated in Section 4, and higher-order sufficient KKT requirements of the problem are finally demonstrated in Section 5. In Section 6, the higher-order duality theorem of Mond-Weir type are presented under generalized higher-order cone arcwisely connectivity suppositions. Section 7 ends with the concluding remarks.

2. DEFINITIONS AND OVERVIEWS

Assume that Δ is a real normed space (in short, RNS) and $\emptyset \neq \Psi \subseteq \Delta$. Then Ψ is defined to be a cone if $\tau\delta \in \Psi$, $\forall \delta \in \Psi$ and $\tau \in \mathbb{R}$ with $\tau \geq 0$. Moreover, Ψ is defined to be proper if $\Psi \neq \Delta$, nontrivial if $\Psi \neq \{\theta_\Delta\}$, solid if $\text{int}(\Psi) \neq \emptyset$, closed if $\overline{\Psi} = \Psi$, pointed if $\Psi \cap (-\Psi) = \{\theta_\Delta\}$, and convex if

$$\tau\Psi + (1-\tau)\Psi \subseteq \Psi, \forall \tau \in [0,1],$$

where $\text{int}(\Psi)$ and $\overline{\Psi}$ indicate the interior and closure of Ψ, correspondingly and θ_Δ represents the zero of Δ.

Let Ψ be a pointed cone in Δ. There are two different types of cone orders w.r.t. Ψ in Δ. For $\delta_1, \delta_2 \in \Delta$, we have

$$\delta_1 \leq \delta_2 \text{ if } \delta_2 - \delta_1 \in \Psi$$

and

$$\delta_1 < \delta_2 \text{ if } \delta_2 - \delta_1 \in \text{int}(\Psi).$$

The following minimality concepts are usually introduced w.r.t. a pointed solid convex cone Ψ in a RNS Δ, .

Definition 2.1. *Let $\emptyset \neq \widetilde{\Delta} \subseteq \Delta$. Then weakly minimal, ideal minimal, and minimal elements of $\widetilde{\Delta}$ are defined as*

(i) *$\delta' \in \widetilde{\Delta}$ is called a weakly minimal element of $\widetilde{\Delta}$ if there exists no $\delta \in \widetilde{\Delta}$, fulfilling $\delta < \delta'$.*

(ii) *$\delta' \in \widetilde{\Delta}$ is called an ideal minimal element of $\widetilde{\Delta}$ if $\delta' \leq \delta, \forall \delta \in \widetilde{\Delta}$.*

(iii) *$\delta' \in \widetilde{\Delta}$ is called a minimal element of $\widetilde{\Delta}$ if there exists no $\delta \in \widetilde{\Delta} \setminus \{\delta'\}$, fulfilling $\delta \leq \delta'$.*

We presume that w-min($\widetilde{\Delta}$), I-min($\widetilde{\Delta}$), and min($\widetilde{\Delta}$) correspondingly represent the sets of weakly minimal elements, ideal minimal elements, and minimal elements of $\widetilde{\Delta}$.

Aubin [19, 20] introduced the concept of contingent cone in a RNS.

Definition 2.2. *[19, 20] Let Δ be a RNS, $\emptyset \neq \widetilde{\Delta} \subseteq \Delta$, and $\delta' \in \overline{\widetilde{\Delta}}$. The contingent cone to $\widetilde{\Delta}$ at δ' is specified by $T(\widetilde{\Delta}, \delta')$ and is defined as follows:*

An element $\delta \in T(\widetilde{\Delta}, \delta')$ if there exist sequences $\{\tau_n\}$ in $\mathbb{R}$, with $\tau_n \to 0^+$ and $\{\delta_n\}$ in Δ, with $\delta_n \to \delta$, fulfilling

$$\delta' + \tau_n \delta_n \in \widetilde{\Delta}, \quad \forall n \in \mathbb{N},$$

or, there exist sequences $\{\tau_n\}$ in $\mathbb{R}$, with $\tau_n > 0$ and $\{\delta'_n\}$ in $\widetilde{\Delta}$, with $\delta'_n \to \delta'$, fulfilling

$$\tau_n(\delta'_n - \delta') \to \delta, \text{ as } n \to \infty.$$

Aubin [19, 20] also introduced the concept of contingent set of higher order in a RNS.

Definition 2.3. *[19, 20] Assume that Δ is a RNS and $\emptyset \neq \widetilde{\Delta} \subseteq \Delta$, $\delta' \in \overline{\widetilde{\Delta}}$, $k \in \mathbb{N}$, with $k \geq 2$, and $\delta_1, ..., \delta_{k-1} \in \Delta$. Then the contingent set $T^{(k)}(\widetilde{\Delta}, \delta', \delta_1, ..., \delta_{k-1})$ of k-th order to $\widetilde{\Delta}$ at $(\delta', \delta_1, ..., \delta_{k-1})$ is defined as:*

$\delta \in T^{(k)}(\widetilde{\Delta}, \delta', \delta_1, ..., \delta_{k-1})$ if there exist some sequences $\{\tau_n\}$ in $\mathbb{R}$ and $\{\delta_n\}$ in Δ, together with $\tau_n \to 0^+$ and $\delta_n \to \delta$, so that

$$\delta' + \tau_n\delta_1 + ... + {\tau_n}^{k-1}\delta_{k-1} + {\tau_n}^k\delta_n \in \widetilde{\Delta}, \forall n \in \mathbb{N},$$

equivalently, $\delta \in T^{(k)}(\widetilde{\Delta}, \delta', \delta_1, ..., \delta_{k-1})$ *if there exist some sequences* $\{\tau_n\}$ *in* $\mathbb{R}$ *and* $\{\delta'_n\}$ *in* Δ*, together with* $\delta'_n \in \widetilde{\Delta}, \forall n \in \mathbb{N}$*,* $\tau_n \to 0^+$*, and* $\delta'_n \to \delta'$*, so that*

$$\frac{\delta'_n - \delta' - \tau_n\delta_1 - ... - {\tau_n}^{k-1}\delta_{k-1}}{{\tau_n}^k} \to \delta, \text{ as } n \to \infty.$$

Let Γ, Δ be RNSs, and Let Ψ be a pointed cone in Δ.. Let $\chi : \Gamma \to 2^\Delta$ be a SVM. The definitions given for the image, domain, epigraph, and graph χ are as follows:

$$\chi(\widetilde{\Gamma}) = \bigcup_{\gamma \in \widetilde{\Gamma}} \{\chi(\gamma)\}, \text{ for any } \emptyset \neq \widetilde{\Gamma} \subseteq \Gamma,$$

$$\mathrm{dom}(\chi) = \{\gamma \in \Gamma : \chi(\gamma) \neq \emptyset\},$$

$$\mathrm{epi}(\chi) = \{(\gamma, \delta) \in \Gamma \times \Delta : \delta \in \chi(\gamma) + \Psi\},$$

and

$$\mathrm{gr}(\chi) = \{(\gamma, \delta) \in \Gamma \times \Delta : \delta \in \chi(\gamma)\}.$$

Jahn and Rauh [21] introduced the concept of contingent epiderivative of SVMs.

Definition 2.4. *[21] Assume that* Γ*,* Δ *are RNSs,* $\chi : \Gamma \to 2^\Delta$ *is a SVM with* $\mathrm{dom}(\chi) = \Gamma$*, and* $(\gamma', \delta') \in \mathrm{gr}(\chi)$*. A function* $\overrightarrow{d}\chi(\gamma', \delta') : \Gamma \to \Delta$ *having epigraph similar to the contingent cone to the epigraph of* χ *at* (γ', δ')*, i.e.,*

$$\mathrm{epi}(\overrightarrow{d}\chi(\gamma', \delta')) = T(\mathrm{epi}(\chi), (\gamma', \delta')),$$

is stated to be contingent epiderivative of χ *at* (γ', δ')*.*

The notion of contingent derivatives of higher-order of SVMs were originally introduced by Aubin and Frankowska [20].

Definition 2.5. *[20] Suppose that* $k \in \mathbb{N}$*, with* $k \geq 2$*,* Γ*,* Δ *are RNSs,* $\chi : \Gamma \to 2^\Delta$ *is a SVM with* $\mathrm{dom}(\chi) = \Gamma$*,* $(\gamma', \delta') \in \mathrm{gr}(\chi)$*, and* $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1}) \in \Gamma \times \Delta$*. Then the contingent derivative* $d^{(k)}\chi(\gamma', \delta', \gamma_1, \delta_1, ..., \gamma_{k-1}, \delta_{k-1})$ *of k-th order of* χ *at* (γ', δ') *for* $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$ *is the SVM from* Γ *to* Δ *defined by*

$$\begin{aligned} &\mathrm{gr}(d^{(k)}\chi(\gamma', \delta', \gamma_1, \delta_1, ..., \gamma_{k-1}, \delta_{k-1})) \\ &= T^{(k)}(\mathrm{gr}(\chi), (\gamma', \delta'), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})). \end{aligned}$$

Let Γ, Δ be RNSs, Ψ be a pointed cone in Δ, and $\chi : \Gamma \to 2^\Delta$ be a SVM. Let us define a SVM $\chi + \Psi : \Gamma \to 2^\Delta$ by

$$(\chi + \Psi)(\gamma) = \chi(\gamma) + \Psi, \forall \gamma \in \mathrm{dom}(\chi).$$

Li and Chen [22] developed the concept of generalized contingent epi-derivative of higher-order of SVMs.

Definition 2.6. *[22] Suppose that $k \in \mathbb{N}$, with $k \geq 2$, Γ, Δ are RNSs, $\chi : \Gamma \to 2^{\Delta}$ is a SVM with $\mathrm{dom}(\chi) = \Gamma$, $(\gamma', \delta') \in \mathrm{gr}(\chi)$, and $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1}) \in \Gamma \times \Delta$. Then the generalized contingent epi-derivative of k-th order of χ at (γ', δ') for $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$, specified by $\overrightarrow{d}_g^{(k)} \chi(\gamma', \delta', \gamma_1, \delta_1, ..., \gamma_{k-1}, \delta_{k-1})$, is the SVM from Γ to Δ defined by*

$$\begin{aligned} &\overrightarrow{d}_g^{(k)} \chi(\gamma', \delta', \gamma_1, \delta_1, ..., \gamma_{k-1}, \delta_{k-1})(\gamma) \\ &= \min\{\delta \in \Delta : (\gamma, \delta) \in T^{(k)}(\mathrm{epi}(\chi), (\gamma', \delta'), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1}))\}, \\ &\gamma \in \mathrm{dom}(d^{(k)}(\chi + \Psi)(\gamma', \delta', \gamma_1, \delta_1, ..., \gamma_{k-1}, \delta_{k-1})). \end{aligned}$$

We now focus on the concept of cone convexity of SVMs, which was presented by Borwein [23].

Definition 2.7. *[23] Let $\emptyset \neq \widetilde{\Gamma} \subseteq \Gamma$ and $\widetilde{\Gamma}$ is a convex set. A SVM $\chi : \Gamma \to 2^{\Delta}$, with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$, is defined to be Ψ-convex on $\widetilde{\Gamma}$ if $\forall \gamma_1, \gamma_2 \in \widetilde{\Gamma}$ and $\tau \in [0, 1]$,*

$$\tau\chi(\gamma_1) + (1 - \tau)\chi(\gamma_2) \subseteq \chi(\tau\gamma_1 + (1 - \tau)\gamma_2) + \Psi.$$

The following proposition was developed for contingent derivative of higher-order of SVMs by Li et al. [14].

Proposition 2.1. *[14] Suppose that Γ, Δ are RNSs and χ is Ψ-convex on a nonempty convex subset $\widetilde{\Gamma}$ of Γ, then $\forall \gamma, \gamma' \in \widetilde{\Gamma}$ and $\forall \delta' \in \chi(\gamma')$,*

$$\chi(\gamma) - \delta' \subseteq d^{(k)}\chi(\gamma', \delta', \gamma_1 - \gamma', \delta_1 - \delta', ..., \gamma_{k-1} - \gamma', \delta_{k-1} - \delta')(\gamma - \gamma'),$$

where $\gamma_1, ..., \gamma_{k-1} \in \widetilde{\Gamma}$, $\delta_1 \in \chi(\gamma_1) + \Psi, ..., \delta_{k-1} \in \chi(\gamma_{k-1}) + \Psi$.

Avriel [15] established the concept of arcwisely connectivity as a generalization of convexity.

Definition 2.8. *[15] A subset $\widetilde{\Gamma}$ of a RNS Γ is stated to be an arcwisely connected set if $\forall \gamma_1, \gamma_2 \in \widetilde{\Gamma}$ there exists a continuous arc $\Lambda_{\gamma_1,\gamma_2} : [0, 1] \to \widetilde{\Gamma}$ fulfilling $\Lambda_{\gamma_1,\gamma_2}(0) = \gamma_1$ and $\Lambda_{\gamma_1,\gamma_2}(1) = \gamma_2$.*

Fu and Wang [16] and Lalitha et al. [17] developed the concept of cone arcwisely connected SVMs.

Definition 2.9. *[16, 17] Let $\widetilde{\Gamma}$ be an arcwisely connected subset (in short, ACS) of a RNS Γ and $\chi : \Gamma \to 2^{\Delta}$ be a SVM, with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$. Then χ is stated to be Ψ-arcwisely connected on $\widetilde{\Gamma}$ if*

$$(1 - \tau)\chi(\gamma_1) + \tau\chi(\gamma_2) \subseteq \chi(\Lambda_{\gamma_1,\gamma_2}(\tau)) + \Psi, \quad \forall \gamma_1, \gamma_2 \in \widetilde{\Gamma} \text{ and } \forall \tau \in [0, 1].$$

Khanh and Tung [24] developed the concept of η-arcwisely connectivity of SVMs.

Definition 2.10. *[24] A subset $\widetilde{\Gamma}$ of a RNS Γ is stated to be an η-arcwisely connected set, with $\eta : \widetilde{\Gamma} \times \widetilde{\Gamma} \times [0,1] \to \Gamma$, if for every $\gamma_1, \gamma_2 \in \widetilde{\Gamma}$ and $\tau \in [0,1]$,*

$$\gamma_1 + \tau\eta(\gamma_1, \gamma_2, \tau) \in \widetilde{\Gamma}.$$

For $\eta : \widetilde{\Gamma} \times \widetilde{\Gamma} \times [0,1] \to \Gamma$, a SVM $\chi : \Gamma \to 2^{\Delta}$ is stated to be Ψ-η-arcwisely connected on an η-arcwisely connected set $\widetilde{\Gamma}$ if for every $\gamma_1, \gamma_2 \in \widetilde{\Gamma}$ and $\tau \in [0,1]$,

$$\lim_{\tau \to 0^+} \tau\eta(\gamma_1, \gamma_2, \tau) = 0$$

and

$$(1-\tau)\chi(\gamma_1) + \tau\chi(\gamma_2) \subseteq \chi(\gamma_1 + \tau\eta(\gamma_1, \gamma_2, \tau)) + \Psi.$$

Suppose that $\chi : \Gamma \to 2^{\Delta}$ is a SVM, together with $\emptyset \neq \widetilde{\Gamma} \subseteq \Gamma$ and $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$.

Let us assume that $(\gamma', \delta'), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1}) \in \Gamma \times \Delta$ together with $\gamma', \gamma_1, ..., \gamma_{k-1} \in \widetilde{\Gamma}$, $\delta' \in \chi(\gamma')$, $\delta_1 \in \chi(\gamma_1) + \Psi, ..., \delta_{k-1} \in \chi(\gamma_{k-1}) + \Psi$.

The α-cone convexity of higher-order of SVMs was introduced by Das and Nahak [18].

Definition 2.11. *[18] Assume that Γ, Δ are RNSs, $\emptyset \neq \widetilde{\Gamma} \subseteq \Gamma$, Ψ is a convex cone of Δ which is both pointed and solid, $\alpha \in \mathbb{R}$, $e \in \mathrm{int}(\Psi)$, and $\chi : \Gamma \to 2^{\Delta}$ is a SVM, together with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$. Assume that χ is generalized contingent epiderivable of k-th order at (γ', δ') for $(\gamma_1 - \gamma', \delta_1 - \delta'), ..., (\gamma_{k-1} - \gamma', \delta_{k-1} - \delta')$. Then χ is stated to be α-Ψ-convex of k-th order w.r.t. e at (γ', δ') for $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$ on $\widetilde{\Gamma}$ if*

$$\chi(\gamma) - \delta' \subseteq \overrightarrow{d}_g^{(k)}\chi(\gamma', \delta', \gamma_1 - \gamma', \delta_1 - \delta', ..., \gamma_{k-1} - \gamma', \delta_{k-1} - \delta')$$
$$(\gamma - \gamma') + \alpha\|\gamma - \gamma'\|^2 e + \Psi, \forall \gamma \in \widetilde{\Gamma}.$$

Das et al. [25] presented the concept of α-cone arcwisely connectivity of SVMs by generalizing cone arcwisely connected SVMs and therefore developed the sufficient conditions of optimality for various forms of SVOPs.

Definition 2.12. *[25] Let $\widetilde{\Gamma}$ be an ACS of a RNS Γ, $e \in \mathrm{int}(\Psi)$, and $\chi : \Gamma \to 2^{\Delta}$ be a SVM, with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$. Then χ is stated to be α-Ψ-arcwisely connected w.r.t. e on $\widetilde{\Gamma}$ if there exists $\alpha \in \mathbb{R}$, fulfilling*

$$(1-\tau)\chi(\gamma_1) + \tau\chi(\gamma_2) \subseteq \chi(\Lambda_{\gamma_1,\gamma_2}(\tau)) + \alpha\tau(1-\tau)\|\gamma_1 - \gamma_2\|^2 e + \Psi,$$
$$\forall \gamma_1, \gamma_2 \in \widetilde{\Gamma} \text{ and } \forall \tau \in [0,1].$$

Theorem 2.1. *[25] Let $\widetilde{\Gamma}$ be an ACS of a RNS Γ, $e \in \mathrm{int}(\Psi)$, and $\chi : \Gamma \to 2^{\Delta}$ be α-Ψ-arcwisely connected w.r.t. e on $\widetilde{\Gamma}$. Let $\gamma' \in \widetilde{\Gamma}$ and $\delta' \in \chi(\gamma')$. Then,*

$$\chi(\gamma) - \delta' \subseteq \overrightarrow{d}\chi(\gamma', \delta')(\Lambda'_{\gamma',\gamma}(0+)) + \alpha\|\gamma - \gamma'\|^2 e + \Psi, \quad \forall \gamma \in \widetilde{\Gamma}.$$

Let $\widetilde{\Gamma}$ be an ACS of a RNS Γ. Throughout the paper, we assume that $\Lambda'_{\gamma',\gamma}(0+)$ exists $\forall \gamma, \gamma' \in \widetilde{\Gamma}$, where

$$\Lambda'_{\gamma',\gamma}(0+) = \lim_{\tau \to 0^+} \frac{\Lambda_{\gamma',\gamma}(\tau) - \Lambda_{\gamma',\gamma}(0)}{\tau}.$$

Let Γ, Δ be RNSs, $\emptyset \neq \widetilde{\Gamma} \subseteq \Gamma$, $\chi : \Gamma \to 2^{\Delta}$ be a SVM, and Ψ be a pointed convex cone in Δ.

Definition 2.13. *A SVM $\chi : \Gamma \to 2^{\Delta}$ is defined to be upper semicontinuous if $\chi^+(\widetilde{\Delta}) = \{\gamma \in \Gamma : \chi(\gamma) \subseteq \widetilde{\Delta}\}$ is open in Γ for arbitrary open subset $\widetilde{\Delta}$ of Δ.*

Definition 2.14. *Let $\emptyset \neq \widetilde{\Delta} \subseteq \Delta$. Then $\widetilde{\Delta}$ is stated to be Ψ-semicompact if all open cover of complements having the form*

$$\{(\delta_j + \Psi)^c : \delta_j \in \widetilde{\Delta}, j \in J\}$$

possesses a finite subcover, in which case J is an arbitrary index set.

Definition 2.15. *A SVM $\chi : \Gamma \to 2^{\Delta}$ is defined to be Ψ-semicompact-valued if $\chi(\gamma)$ is Ψ-semicompact, $\forall \gamma \in \mathrm{dom}(\chi)$.*

We consider the following SVOP (P):

$$\max_{\gamma \in \widetilde{\Gamma}} \quad \chi(\gamma), \tag{P}$$

The maximizer for the problem (P) is defined as follows:

Definition 2.16. *Let $\gamma' \in \widetilde{\Gamma}$ and $\delta' \in \chi(\gamma')$. Then (γ', δ') is defined to be a maximizer of* (P) *if there exist no $\gamma \in \widetilde{\Gamma}$ and $\delta \in \chi(\gamma)$ fulfilling*

$$\delta' < \delta.$$

The existence results for the solutions of SVOPs in RNSs were established by Corley [26] when the objective map is an upper semicontinuous and cone semicompact-valued.

Theorem 2.2. *[26] Let Γ, Δ be RNSs, $\emptyset \neq \widetilde{\Gamma} \subseteq \Gamma$, $\emptyset \neq \Psi \subseteq \Delta$, and $\overline{\Psi}$ is a pointed convex cone in Δ. Assume that $\chi : \Gamma \to 2^{\Delta}$ be Ψ-semicompact-valued and upper semicontinuous. In this case, there exists a maximizer for* (P)*.*

3. HIGHER-ORDER α-CONE ARCWISELY CONNECTIVITY

Das [27] and Das and Nahak [28] established the second-order sufficient optimality conditions and developed the duality results of set-valued fractional programming problems and set-valued optimization problems, respectively. Pokharna and Tripathi [29] introduced the optimality conditions and studied the duality theorems for E-minimax fractional programming problems. The concept of higher-order α-cone arcwisely connectivity of SVMs was first introduced by Das [1]. He developed the results of duality for set-valued parametric optimization problems under the contingent epiderivative and higher-order α-cone arcwisely connectivity suppositions and provided sufficient KKT conditions of optimality. For $\alpha = 0$, we have the standard notion of cone arcwisely connectivity of SVMs, introduced by Fu and Wang [16] and Lalitha et al. [17].

Assume that $\chi : \Gamma \to 2^{\Delta}$ is a SVM, together with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$. Suppose that $(\gamma', \delta'), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1}) \in \Gamma \times \Delta$ together with $\gamma', \gamma_1, ..., \gamma_{k-1} \in \widetilde{\Gamma}$, $\delta' \in \chi(\gamma')$, $\delta_1 \in \chi(\gamma_1) + \Psi, ..., \delta_{k-1} \in \chi(\gamma_{k-1}) + \Psi$.

Definition 3.1. *[1] Suppose that Γ, Δ are RNSs, $\widetilde{\Gamma}$ is an ACS of Γ, Ψ is a solid pointed convex cone of Δ, $\alpha \in \mathbb{R}$, $e \in \mathrm{int}(\Psi)$, and $\chi : \Gamma \to 2^{\Delta}$ is a SVM, together with $\widetilde{\Gamma} \subseteq \mathrm{dom}(\chi)$. Suppose that χ is generalized contingent epiderivable of k-th order at (γ', δ') for $(\gamma_1 - \gamma', \delta_1 - \delta'), ..., (\gamma_{k-1} - \gamma', \delta_{k-1} - \delta')$. Then χ is stated to be α-Ψ-arcwisely connected of k-th order w.r.t. e at (γ', δ') for $(\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$ on $\widetilde{\Gamma}$ if*

$$\chi(\gamma) - \delta' \subseteq \overrightarrow{d}_g^{(k)} \chi(\gamma', \delta', \gamma_1 - \gamma', \delta_1 - \delta', ..., \gamma_{k-1} - \gamma', \delta_{k-1} - \delta')(\Lambda'_{\gamma', \gamma}(0+)) + \alpha \|\gamma - \gamma'\|^2 e + \Psi, \forall \gamma \in \widetilde{\Gamma}.$$

Remark 3.1. *If $\alpha > 0$, then χ is stated to be strongly α-Ψ-arcwisely connected of k-th order, if $\alpha = 0$, we have the usual concept of Ψ-arcwisely connectivity of k-th order, and if $\alpha < 0$, then χ is stated to be weakly α-Ψ-arcwisely connected of k-th order. Obviously, strongly α-Ψ-arcwisely connectivity of k-th order $\Rightarrow$ Ψ-arcwisely connectivity of k-th order $\Rightarrow$ weakly α-Ψ-arcwisely connectivity of k-th order.*

For $\alpha = 0$ and $\Lambda_{\gamma_1, \gamma_2}(\tau) = \gamma_1 + \tau \eta(\gamma_1, \gamma_2, \tau)$, with $\gamma_1 + \eta(\gamma_1, \gamma_2, 1) = \gamma_2$, we have the concept of η-arcwisely connectivity of k-th order. When $\Lambda_{\gamma_1, \gamma_2}(\tau) = \gamma_1 + \tau \eta(\gamma_1, \gamma_2, \tau)$, with $\gamma_1 + \eta(\gamma_1, \gamma_2, 1) = \gamma_2$, strongly α-Ψ-arcwisely connectivity of k-th order $\Rightarrow$ η-arcwisely connectivity of k-th order $\Rightarrow$ weakly α-Ψ-arcwisely connectivity of k-th order.

4. FORMULATION OF THE MAIN PROBLEM

Let $\emptyset \neq \widetilde{\Gamma} \subseteq \mathbb{R}^n$ and $\widetilde{\Delta}$ be a nonempty compact subset of $\mathbb{R}^m$. Let M_1 and M_2 be $n \times n$ positive semidefinite matrices. Let $\chi, \zeta : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ and $\Omega : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be SVMs, with

$$\widetilde{\Gamma} \times \widetilde{\Delta} \subseteq \mathrm{dom}(\chi) \cap \mathrm{dom}(\zeta) \text{ and } \widetilde{\Gamma} \subseteq \mathrm{dom}(\Omega).$$

Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$. Consider a SVFMPP

$$\begin{aligned} &\underset{a \in \widetilde{\Gamma}}{\text{minimize}} \quad \max \bigcup_{b \in \widetilde{\Delta}} \frac{\chi(a, b) + (a^T M_1 a)^{\frac{1}{2}}}{\zeta(a, b) - (a^T M_2 a)^{\frac{1}{2}}} \\ &\text{subject to} \quad \Omega(a) \cap (-\mathbb{R}^p_+) \neq \emptyset. \end{aligned} \tag{MFP}$$

Assume that $\emptyset + X = \emptyset$ and $\emptyset - X = \emptyset$ for every $X \subseteq \mathbb{R}$. Define a SVM $\mu : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ by

$$\mu(a,b) = \frac{\chi(a,b) + (a^T M_1 a)^{\frac{1}{2}}}{\zeta(a,b) - (a^T M_2 a)^{\frac{1}{2}}}, \forall (a,b) \in \mathbb{R}^n \times \mathbb{R}^m.$$

assuming that

$$\chi(a,b) + (a^T M_1 a)^{\frac{1}{2}} \geq 0$$

and

$$\zeta(a,b) - (a^T M_2 a)^{\frac{1}{2}} > 0, \forall (a,b) \in \widetilde{\Gamma} \times \widetilde{\Delta}.$$

We assume that the SVM $\mu(a,.) : \mathbb{R}^m \to 2^{\mathbb{R}}$ is upper semicontinuous as well as $\mathbb{R}_+$-semicompact-valued on $\widetilde{\Delta}$, $\forall a \in \widetilde{\Gamma}$. Hence, by Theorem 2.2, $\max \bigcup_{b \in \widetilde{\Delta}} \mu(a,b)$ always exists, $\forall a \in \widetilde{\Gamma}$. Since $\mu(a,b) \subseteq \mathbb{R}$, for every $a \in \widetilde{\Gamma}$ there exists only one solution of $\max \bigcup_{b \in \widetilde{\Delta}} \mu(a,b)$. The feasible set of (MFP) is

$$S' = \{a \in \widetilde{\Gamma} : \Omega(a) \cap (-\mathbb{R}^p_+) \neq \emptyset\}.$$

The following defines the minimizer of (MFP).

Definition 4.1. *Let $a' \in S'$ be a feasible element of* (MFP) *and $c' = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a',b)$. Then (a',c') is defined to be a minimizer of* (MFP) *if there exist no $a \in S'$ and $c = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a,b)$, with $a \neq a'$, fulfilling*

$$c < c'.$$

For $a \in \widetilde{\Gamma}$, define

$$I(a) = \{j : 0 \in \Omega_j(a), 1 \leq j \leq p\}$$

$$J(a) = \{1, ..., p\} \setminus I(a),$$

$$B(a) = \Big\{b' \in \widetilde{\Delta} : \max \bigcup_{b \in \widetilde{\Delta}} \mu(a,b) \in \mu(a,b')\Big\},$$

and

$$K(a) = \Big\{(r, c^*, \widetilde{b}) \in \mathbb{N} \times \mathbb{R}^r_+ \times \mathbb{R}^{mr} : 1 \leq r \leq n, c^* = (c^*_1, ..., c^*_r) \in \mathbb{R}^r_+,$$
$$\text{with} \sum_{i=1}^{r} c^*_i = 1, \widetilde{b} = (\overline{b_1}, ..., \overline{b_r}), \text{ with } \overline{b_j} \in B(a), j = 1, ..., r\Big\}.$$

As $\mu(a,.)$ is upper semicontinuous as well as $\mathbb{R}_+$-semicompact-valued on $\widetilde{\Delta}, \forall a \in \widetilde{\Gamma}$, we have

$$B(a') \neq \emptyset, \forall a' \in S'.$$

Let M be an $n \times n$ positive semidefinite matrix. Then, $\forall a, d \in \mathbb{R}^n$,

$$a^T M d \leq (a^T M a)^{\frac{1}{2}} (d^T M d)^{\frac{1}{2}}.$$

Moreover, if $(d^T M d)^{\frac{1}{2}} \leq 1$, we have

$$a^T M d \leq (a^T M a)^{\frac{1}{2}}. \tag{4.1}$$

5. HIGHER-ORDER SUFFICIENT CRITERIA OF OPTIMALITY

Assume that $\chi, \zeta : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ and $\Omega : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be SVMs, with

$$\widetilde{\Gamma} \times \widetilde{\Delta} \subseteq \mathrm{dom}(\chi) \cap \mathrm{dom}(\zeta) \text{ and } \widetilde{\Gamma} \subseteq \mathrm{dom}(\Omega).$$

Suppose that $a' \in \widetilde{\Gamma}$, $k, r \in \mathbb{N}$, with $k \geq 2$, and $\overline{b_i} \in B(a')$, $(1 \leq i \leq r)$. Let $(a', \overline{c_i'}), (a_1, c_{1i}), ..., (a_{k-1}, c_{(k-1)i}), (a', \overline{c_i''}), (a_1, c_{1i}'), ..., (a_{k-1}, c_{(k-1)i}') \in \mathbb{R}^n \times \mathbb{R}$ with $a', a_1, ..., a_{k-1} \in \widetilde{\Gamma}$, $\overline{c_i'} \in \chi(a', \overline{b_i})$, $c_{1i} \in \chi(a_1, \overline{b_i}) + \mathbb{R}_+, ..., c_{(k-1)i} \in \chi(a_{k-1}, \overline{b_i}) + \mathbb{R}_+$, $\overline{c_i''} \in \zeta(a', \overline{b_i})$, and $c_{1i}' \in \zeta(a_1, \overline{b_i}) + \mathbb{R}_+, ..., c_{(k-1)i}' \in \zeta(a_{k-1}, \overline{b_i}) + \mathbb{R}_+$.

Also, assume that $(a', d_j'), (a_1, d_{1j}), ..., (a_{k-1}, d_{(k-1)j}) \in \mathbb{R}^n \times \mathbb{R}$ together with $d_j' \in \Omega_j(a')$, $d_{1j} \in \Omega_j(a_1) + \mathbb{R}_+, ..., d_{(k-1)j} \in \Omega_j(a_{k-1}) + \mathbb{R}_+$, $(1 \leq j \leq p)$, where $\Omega = (\Omega_1, ..., \Omega_p)$.

We establish the higher-order sufficient KKT conditions of SVFMPP (MFP) using the assumption of the higher-order α-cone arcwisely connectivity

Theorem 5.1. ***(Higher-order sufficient criteria of optimality)*** *Let $\widetilde{\Gamma}$ be an ACS of $\mathbb{R}^n$, a' be a feasible element of* (MFP)*, and $c' = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b)$. Assume that there exist $r \in \mathbb{N}$, where $1 \leq r \leq n$, $c^* = (c_1^*, ..., c_r^*) \in \mathbb{R}_+^r$, with $\sum_{i=1}^{r} c_i^* = 1$, $\overline{b_i} \in B(a')$, $(1 \leq i \leq r)$, $d, \delta \in \mathbb{R}^n$, $d^* = (d_1^*, ..., d_p^*) \in \mathbb{R}_+^p$, and $d_j' \in \Omega_j(a') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$, fulfilling*

$$\begin{aligned}
&\sum_{i=1}^{r} c_i^* \Big(\overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) \\
&+ M_1 d - c' (\overrightarrow{d}_g^{(k)} (-\zeta) \\
&(., \overline{b_i})(a', -\overline{c_i''}, a_1 - a', -c_{1i}' + \overline{c_i''}, ..., a_{k-1} - a', -c_{(k-1)i}' + \overline{c_i''}) - M_2 \delta) \Big) \\
&(\Lambda_{a',a}'(0+)) \\
&+ \sum_{j=1}^{p} d_j^* \overrightarrow{d}_g^{(k)} \Omega_j(a', d_j', a_1 - a', d_{1j} - d_j', ..., a_{k-1} - a', d_{(k-1)j} - d_j') \\
&(\Lambda_{a',a}'(0+)) \geq 0, \forall a \in \widetilde{\Gamma},
\end{aligned} \tag{5.2}$$

$$\sum_{j=1}^{p} d_j^* d_j' = 0, \tag{5.3}$$

$$d^T M_1 d \leq 1, \delta^T M_2 \delta \leq 1, \tag{5.4}$$

$$(a'^T M_1 a')^{\frac{1}{2}} = a'^T M_1 d, \tag{5.5}$$

and

$$(a'^T M_2 a')^{\frac{1}{2}} = a'^T M_2 \delta. \tag{5.6}$$

Suppose that $\chi(.,\overline{b_i})$ *is* α_i-$\mathbb{R}_+$*-arcwisely connected of* k*-th order w.r.t.* 1 *at* $(a', \overline{c_i'})$ *for the elements* $(a_1, c_{1i}), ..., (a_{k-1}, c_{(k-1)i})$*,* $(.)^T M_1 d$ *is* $\overline{\alpha}_i$-$\mathbb{R}_+$*-arcwisely connected w.r.t.* 1*,* $-\zeta(.,\overline{b_i})$ *is* α_i'-$\mathbb{R}_+$*-arcwisely connected of* k*-th order w.r.t.* 1 *at* $(a', -\overline{c_i''})$ *for the elements* $(a_1, -c_{1i}'), ..., (a_{k-1}, -c_{(k-1)i}')$*,* $(.)^T M_2 \delta$ *is* $\overline{\alpha'}_i$-$\mathbb{R}_+$*-arcwisely connected w.r.t.* 1 *and* Ω_j*,* $(1 \leq j \leq p)$*, is* ν_j-$\mathbb{R}_+$*-arcwisely connected of* k*-th order w.r.t.* 1 *at* $(a', \overline{d_j'})$ *for* $(a_1, d_{1j}), ..., (a_{k-1}, d_{(k-1)j})$*, on* $\widetilde{\Gamma}$*, satisfying*

$$\sum_{i=1}^{r} c_i^* \Big(\alpha_i + \overline{\alpha_i} - c'(\alpha_i' + \overline{\alpha_i'})\Big) + \sum_{j=1}^{p} d_j^* \nu_j \geq 0. \tag{5.7}$$

Then (a', c') *is a minimizer of* (MFP)*.*

Proof. Assume that (a', c') is not a minimizer of (MFP). Then there exist $a \in S'$ and $c = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a, b)$, with $a \neq a'$, fulfilling

$$c < c'.$$

Since $\overline{b_i} \in B(a')$, $i = 1, ..., r$, we have

$$\max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b) \in \mu(a', \overline{b_i}).$$

As $c' = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b)$, we have

$$c' \in \mu(a', \overline{b_i}), i = 1, ..., r.$$

Let $c_i \in \mu(a, \overline{b_i})$. Again, as $c = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a, b)$ and $\overline{b_i} \in B(a') \subseteq \widetilde{\Delta}$, we have

$$c_i \leq c.$$

Hence,

$$c_i < c'.$$

As $c' \in \mu(a', \overline{b_i})$, there exist $\overline{c_i'} \in \chi(a', \overline{b_i})$ and $\overline{c_i''} \in \zeta(a', \overline{b_i})$ fulfilling

$$c' = \frac{\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}}}{\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}}.$$

So,

$$\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) = 0, \forall i = 1, ..., r. \quad (5.8)$$

Since $c_i \in \mu(a, \overline{b_i})$, there exist $c_i' \in \chi(a, \overline{b_i})$ and $c_i'' \in \zeta(a, \overline{b_i})$ fulfilling

$$c_i = \frac{c_i' + (a^T M_1 a)^{\frac{1}{2}}}{c_i'' - (a^T M_2 a)^{\frac{1}{2}}}.$$

Hence,

$$\frac{c_i' + (a^T M_1 a)^{\frac{1}{2}}}{c_i'' - (a^T M_2 a)^{\frac{1}{2}}} < c'.$$

So,

$$c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) < 0, \forall i = 1, ..., r. \quad (5.9)$$

From (4.1) and (5.4), we have

$$\begin{aligned} &\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta)) \Big) \\ &\leq \sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) \Big). \end{aligned}$$

Again, from (5.9), we have

$$\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) \Big) < 0.$$

From (5.8),

$$\sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) \Big) = 0.$$

Again, from (5.5) and (5.6) we have

$$\begin{aligned} &\sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) \Big) \\ &= \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta)) \Big). \end{aligned}$$

Hence, we have

$$\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta))\Big)$$
$$< \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta))\Big).$$

As $a \in S'$, there exists

$$d_j \in \Omega_j(a) \cap (-\mathbb{R}_+).$$

Since $d_j^* \geq 0$ $(1 \leq j \leq p)$, we have

$$d_j^* d_j \leq 0, \forall j, \text{ with } 1 \leq j \leq p.$$

So,

$$\sum_{j=1}^{p} d_j^* d_j \leq 0.$$

From (5.3), we have

$$\sum_{j=1}^{p} d_j^* d_j' = 0.$$

Hence,

$$\sum_{j=1}^{p} d_j^* d_j \leq \sum_{j=1}^{p} d_j^* d_j'.$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d_j \\ &< \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d_j'.\end{aligned} \tag{5.10}$$

As it is presumed that $\chi(., \overline{b_i})$ is α_i-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', \overline{c_i'})$ for $(a_1, c_{1i}), ..., (a_{k-1}, c_{(k-1)i})$, on $\widetilde{\Gamma}$ and $\overline{c_i'} \in \chi(a', \overline{b_i})$, we have

$$\begin{aligned}&\chi(a, \overline{b_i}) - \overline{c_i'} \\ &\subseteq \overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) \\ &\qquad (\Lambda'_{a',a}(0+)) + \alpha_i \|a - a'\|^2 + \mathbb{R}_+.\end{aligned}$$

Again, as $c_i' \in \chi(a, \overline{b_i})$, we have

$$\begin{aligned}&c_i' - \overline{c_i'} \\ &\in \overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) \\ &\qquad (\Lambda'_{a',a}(0+)) + \alpha_i \|a - a'\|^2 + \mathbb{R}_+.\end{aligned} \tag{5.11}$$

Suppose that $(.)^T M_1 d$ is $\overline{\alpha}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$, we have

$$a^T M_1 d - a'^T M_1 d \geq M_1 d(\Lambda'_{a',a}(0+)) + \overline{\alpha_i}\|a - a'\|^2 + \mathbb{R}_+. \tag{5.12}$$

As it is presumed that $-\zeta(.,\overline{b_i})$ is α'_i-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', -\overline{c''_i})$ for $(a_1, -c'_{1i}), ..., (a_{k-1}, -c'_{(k-1)i})$, on $\widetilde{\Gamma}$ and $\overline{c''_i} \in \zeta(a', \overline{b_i})$, we have

$$\begin{aligned} & -\zeta(a, \overline{b_i}) + \overline{c''_i} \\ \subseteq\ & \overrightarrow{d}_g^{(k)}(-\zeta)(.,\overline{b_i})(a', -\overline{c''_i}, a_1 - a', -c'_{1i} + \overline{c''_i}, ..., a_{k-1} - a', -c'_{(k-1)i} + \overline{c''_i}) \\ & (\Lambda'_{a',a}(0+)) + \alpha'_i\|a - a'\|^2 + \mathbb{R}_+. \end{aligned}$$

Again, as $c''_i \in \zeta(a, \overline{b_i})$, we have

$$\begin{aligned} & -c''_i + \overline{c''_i} \in \overrightarrow{d}_g^{(k)}(-\zeta)(.,\overline{b_i}) \\ & (a', -\overline{c''_i}, a_1 - a', -c'_{1i} + \overline{c''_i}, ..., a_{k-1} - a', -c'_{(k-1)i} + \overline{c''_i}) \\ & (\Lambda'_{a',a}(0+)) + \alpha'_i\|a - a'\|^2 + \mathbb{R}_+. \end{aligned} \tag{5.13}$$

Since $(.)^T M_2 \delta$ is $\overline{\alpha'}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$, we have

$$a^T M_2 \delta - a'^T M_2 \delta \geq M_2 \delta(\Lambda'_{a',a}(0+)) + \overline{\alpha'_i}\|a - a'\|^2 + \mathbb{R}_+. \tag{5.14}$$

As Ω_j, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', \overline{d'_j})$ for the elements $(a_1, d_{1j}), ..., (a_{k-1}, d_{(k-1)j})$, on $\widetilde{\Gamma}$ and $d'_j \in \Omega_j(a') \cap (-\mathbb{R}_+)$, we have

$$\begin{aligned} \Omega_j(a) - d'_j \subseteq\ & \overrightarrow{d}_g^{(k)}\Omega_j(a', d'_j, a_1 - a', d_{1j} - d'_j, ..., a_{k-1} - a', d_{(k-1)j} - d'_j) \\ & (\Lambda'_{a',a}(0+)) + \nu_j\|a - a'\|^2 + \mathbb{R}_+. \end{aligned}$$

Since $d_j \in \Omega_j(a) \cap (-\mathbb{R}_+)$, we have

$$\begin{aligned} & d_j - d'_j \\ \in\ & \overrightarrow{d}_g^{(k)}\Omega_j(a', d'_j, a_1 - a', d_{1j} - d'_j, ..., a_{k-1} - a', d_{(k-1)j} - d'_j) \\ & (\Lambda'_{a',a}(0+)) + \nu_j\|a - a'\|^2 + \mathbb{R}_+. \end{aligned} \tag{5.15}$$

From (5.2), (5.7), (5.11), (5.12), (5.13), (5.14), and (5.15), we have

$$\begin{aligned} & \sum_{i=1}^{r} c_i^*\Big(c'_i + (a^T M_1 d) - c'(c''_i - (a^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d_j \\ & \geq \sum_{i=1}^{r} c_i^*\Big(\overline{c'_i} + (a'^T M_1 d) - c'(\overline{c''_i} - (a'^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d'_j, \end{aligned}$$

which contradicts (5.10). Hence, (a', c') is a minimizer of (MFP). $\square$

6. HIGHER-ORDER MOND-WEIR TYPE DUAL

We consider a higher-order dual (MWD) of Mond-Weir form in accordance with the primal problem (MFP), where the SVMs $\chi(.,\overline{b_i})$, $-\zeta(.,\overline{b_i})$; $\overline{b_i} \in B(a'), a' \in \widetilde{\Gamma}$, and Ω_j are higher-order contingent epiderivable SVMs.

$$\text{maximize} \quad c' \qquad \text{(MWD)}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{r} c_i^* \Big(\overrightarrow{d}_g^{(k)} \chi(.,\overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', \\
&c_{(k-1)i} - \overline{c_i'}) + M_1 d \\
&- c'(\overrightarrow{d}_g^{(k)}(-\zeta)(.,\overline{b_i})(a', -\overline{c_i''}, a_1 - a', -c_{1i}' + \overline{c_i''}, ..., a_{k-1} - a', \\
&- c_{(k-1)i}' + \overline{c_i''}) - M_2 \delta)\Big) \\
&(\Lambda_{a',a}'(0+)) \\
&+ \sum_{j=1}^{p} d_j^* \overrightarrow{d}_g^{(k)} \Omega_j(a', d_j', a_1 - a', d_{1j} - d_j', ..., a_{k-1} - a', \\
&d_{(k-1)j} - d_j')(\Lambda_{a',a}'(0+)) \geq 0, \forall a \in \widetilde{\Gamma}, \\
&\text{for some } r \in \mathbb{N}, (1 \leq r \leq n) \text{ and } \overline{b_i} \in B(a'), \\
&\sum_{j=1}^{p} d_j^* d_j' \geq 0, \\
&d^T M_1 d \leq 1, \delta^T M_2 \delta \leq 1, (a'^T M_1 a')^{\frac{1}{2}} = a'^T M_1 d, \\
&(a'^T M_2 a')^{\frac{1}{2}} = a'^T M_2 \delta, \text{ for some } d, \delta \in \mathbb{R}^n, \\
&a' \in \widetilde{\Gamma}, c' = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b), d' = (d_1', ..., d_p'), d_j' \in \Omega_j(a'), \\
&c^* = (c_1^*, ..., c_r^*), d^* = (d_1^*, ..., d_p^*), c_i^* \geq 0, d_j^* \geq 0, \sum_{i=1}^{r} c_i^* = 1, \\
&\text{where } 1 \leq i \leq r \text{ and } 1 \leq j \leq p.
\end{aligned}$$

A element (a', c', d', c^*, d^*) meeting every constraints of (MWD) is defined to be a feasible element of (MWD).

Definition 6.1. *A feasible element* (a', c', d', c^*, d^*) *of the problem* (MWD) *is defined to be a maximizer of* (MWD) *if there exists no feasible element* (a, c, d, c_1^*, d_1^*) *of* (MWD) *fulfilling*

$$c' < c.$$

Theorem 6.1. ***(Higher-order weak duality)*** *Let* $\widetilde{\Gamma}$ *be an ACS of* $\mathbb{R}^n$, a_0 *be feasible to* (MFP) *and* (a', c', d', c^*, d^*) *be a feasible element of* (MWD).

Suppose that $\chi(.,\overline{b_i})$ *is* α_i-$\mathbb{R}_+$*-arcwisely connected of k-th order w.r.t.* 1 *at* $(a',\overline{c_i'})$ *for the elements* $(a_1,c_{1i}),...,(a_{k-1},c_{(k-1)i})$*,* $(.)^T M_1 d$ *is* $\overline{\alpha}_i$-$\mathbb{R}_+$*-arcwisely connected w.r.t.* 1*,* $-\zeta(.,\overline{b_i})$ *is* α_i'-$\mathbb{R}_+$*-arcwisely connected of k-th order w.r.t.* 1 *at* $(a',-\overline{c_i''})$ *for* $(a_1,-c_{1i}'),...,(a_{k-1},-c_{(k-1)i}')$*,* $(.)^T M_2\delta$ *is* $\overline{\alpha'}_i$-$\mathbb{R}_+$*-arcwisely connected w.r.t.* 1 *and* Ω_j*,* $(1\leq j\leq p)$*, is* ν_j-$\mathbb{R}_+$*-arcwisely connected of k-th order w.r.t.* 1 *at* $(a',\overline{d_j'})$ *for* $(a_1,d_{1j}),...,(a_{k-1},d_{(k-1)j})$*, on* $\widetilde{\Gamma}$*, satisfying*

$$\Big(\alpha_i+\overline{\alpha_i}-c'(\alpha_i'+\overline{\alpha_i'})\Big)+\sum_{j=1}^{p}d_j^*\nu_j\geq 0. \tag{6.16}$$

Then,

$$\max\bigcup_{b\in\widetilde{\Delta}}\mu(a_0,b)\nleq c'.$$

Proof. Using the method of contradiction, we demonstrate the proof. Assume that for $c_0=\max\bigcup_{b\in\widetilde{\Delta}}\mu(a_0,b)$, $c_0<c'$. Since $\overline{b_i}\in B(a')$, $i=1,...,r$, we have

$$\max\bigcup_{b\in\widetilde{\Delta}}\mu(a',b)\in\mu(a',\overline{b_i}).$$

As $c'=\max\bigcup_{b\in\widetilde{\Delta}}\mu(a',b)$, we have

$$c'\in\mu(a',\overline{b_i}), i=1,...,r.$$

Let $c_i\in\mu(a_0,\overline{b_i})$. Again, as $c_0=\max\bigcup_{b\in\widetilde{\Delta}}\mu(a_0,b)$ and $\overline{b_i}\in B(a')\subseteq\widetilde{\Delta}$, we have

$$c_i\leq c_0.$$

Hence,

$$c_i<c'.$$

As $c'\in\mu(a',\overline{b_i})$, there exist $\overline{c_i'}\in\chi(a',\overline{b_i})$ and $\overline{c_i''}\in\zeta(a',\overline{b_i})$ fulfilling

$$c'=\frac{\overline{c_i'}+(a'^TM_1a')^{\frac{1}{2}}}{\overline{c_i''}-(a'^TM_2a')^{\frac{1}{2}}}.$$

So,

$$\overline{c_i'}+(a'^TM_1a')^{\frac{1}{2}}-c'(\overline{c_i''}-(a'^TM_2a')^{\frac{1}{2}})=0,\forall i=1,...,r. \tag{6.17}$$

Since $c_i\in\mu(a_0,\overline{b_i})$, there exist $c_i'\in\chi(a_0,\overline{b_i})$ and $c_i''\in\zeta(a_0,\overline{b_i})$ fulfilling

$$c_i=\frac{c_i'+(a_0^TM_1a_0)^{\frac{1}{2}}}{c_i''-(a_0^TM_2a_0)^{\frac{1}{2}}}.$$

Hence,

$$\frac{c'_i + (a_0^T M_1 a_0)^{\frac{1}{2}}}{c''_i - (a_0^T M_2 a_0)^{\frac{1}{2}}} < c'.$$

So,

$$c'_i + (a_0^T M_1 a_0)^{\frac{1}{2}} - c'(c''_i - (a_0^T M_2 a_0)^{\frac{1}{2}}) < 0, \forall i = 1, ..., r. \tag{6.18}$$

From (4.1) and the constraints of the problem (MWD), we have

$$\sum_{i=1}^{r} c_i^* \Big(c'_i + (a_0^T M_1 d) - c'(c''_i - (a_0^T M_2 \delta)) \Big)$$
$$\leq \sum_{i=1}^{r} c_i^* \Big(c'_i + (a_0^T M_1 a_0)^{\frac{1}{2}} - c'(c''_i - (a_0^T M_2 a_0)^{\frac{1}{2}}) \Big).$$

Again, from (6.18), we have

$$\sum_{i=1}^{r} c_i^* \Big(c'_i + (a_0^T M_1 a_0)^{\frac{1}{2}} - c'(c''_i - (a_0^T M_2 a_0)^{\frac{1}{2}}) \Big) < 0.$$

From (6.17),

$$\sum_{i=1}^{r} c_i^* \Big(\overline{c'_i} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c''_i} - (a'^T M_2 a')^{\frac{1}{2}}) \Big) = 0.$$

Again, from the constraints of the problem (MWD), we have

$$\sum_{i=1}^{r} c_i^* \Big(\overline{c'_i} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c''_i} - (a'^T M_2 a')^{\frac{1}{2}}) \Big)$$
$$= \sum_{i=1}^{r} c_i^* \Big(\overline{c'_i} + (a'^T M_1 d) - c'(\overline{c''_i} - (a'^T M_2 \delta)) \Big).$$

Hence, we have

$$\sum_{i=1}^{r} c_i^* \Big(c'_i + (a_0^T M_1 d) - c'(c''_i - (a_0^T M_2 \delta)) \Big)$$
$$< \sum_{i=1}^{r} c_i^* \Big(\overline{c'_i} + (a'^T M_1 d) - c'(\overline{c''_i} - (a'^T M_2 \delta)) \Big).$$

As $a_0 \in S'$, there exists

$$d_j \in \Omega_j(a_0) \cap (-\mathbb{R}_+).$$

As $d_j^* \geq 0$, where $(1 \leq j \leq p)$,

$$d_j^* d_j \leq 0, \forall j, (1 \leq j \leq p).$$

So,

$$\sum_{j=1}^{p} d_j^* d_j \leq 0.$$

From (MWD),

$$\sum_{j=1}^{p} d_j^* d_j' \geq 0.$$

Hence,

$$\sum_{j=1}^{p} d_j^* d_j \leq \sum_{j=1}^{p} d_j^* d_j'.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^{r} c_i^* \Big(c_i' + (a_0^T M_1 d) - c'(c_i'' - (a_0^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d_j \\ &< \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta))\Big) + \sum_{j=1}^{p} d_j^* d_j'. \end{aligned} \tag{6.19}$$

As it is presumed that $\chi(.,\overline{b_i})$ is α_i-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', \overline{c_i'})$ for $(a_1, c_{1i}), ..., (a_{k-1}, c_{(k-1)i})$, on $\widetilde{\Gamma}$ and $\overline{c_i'} \in \chi(a', \overline{b_i})$,

$$\begin{aligned} \chi(a_0, \overline{b_i}) - \overline{c_i'} \subseteq \overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) \\ (\Lambda'_{a', a_0}(0+)) + \alpha_i \|a_0 - a'\|^2 + \mathbb{R}_+. \end{aligned}$$

As $c_i' \in \chi(a_0, \overline{b_i})$,

$$\begin{aligned} &c_i' - \overline{c_i'} \\ &\in \overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a', c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) \\ &(\Lambda'_{a', a_0}(0+)) + \alpha_i \|a_0 - a'\|^2 + \mathbb{R}_+. \end{aligned} \tag{6.20}$$

Suppose that $(.)^T M_1 d$ is $\overline{\alpha}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$,

$$a_0^T M_1 d - a'^T M_1 d \geq M_1 d(\Lambda'_{a', a_0}(0+)) + \overline{\alpha_i} \|a_0 - a'\|^2 + \mathbb{R}_+. \tag{6.21}$$

As it is presumed that $-\zeta(., \overline{b_i})$ is α_i'-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', -\overline{c_i''})$ for $(a_1, -c_{1i}'), ..., (a_{k-1}, -c_{(k-1)i}')$, on $\widetilde{\Gamma}$ and $\overline{c_i''} \in \zeta(a', \overline{b_i})$, we have

$$\begin{aligned} &-\zeta(a_0, \overline{b_i}) + \overline{c_i''} \\ &\subseteq \overrightarrow{d}_g^{(k)} (-\zeta)(., \overline{b_i})(a', -\overline{c_i''}, a_1 - a', -c_{1i}' + \overline{c_i''}, ..., a_{k-1} - a', -c_{(k-1)i}' + \overline{c_i''}) \\ &(\Lambda'_{a', a_0}(0+)) + \alpha_i' \|a_0 - a'\|^2 + \mathbb{R}_+. \end{aligned}$$

Again, as $c''_i \in \zeta(a_0, \overline{b_i})$, we have

$$-c''_i + \overline{c''_i} \in \overrightarrow{d}_g^{(k)}(-\zeta)(., \overline{b_i})$$
$$(a', -\overline{c''_i}, a_1 - a', -c'_{1i} + \overline{c''_i}, ..., a_{k-1} - a', -c'_{(k-1)i} + \overline{c''_i}) \tag{6.22}$$
$$(\Lambda'_{a',a_0}(0+)) + \alpha'_i \|a_0 - a'\|^2 + \mathbb{R}_+.$$

Since $(.)^T M_2 \delta$ is $\overline{\alpha'}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$,

$$a_0^T M_2 \delta - a'^T M_2 \delta \geq M_2 \delta (\Lambda'_{a',a_0}(0+)) + \overline{\alpha'_i} \|a_0 - a'\|^2 + \mathbb{R}_+. \tag{6.23}$$

As Ω_j, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a', \overline{d'_j})$ for the elements $(a_1, d_{1j}), ..., (a_{k-1}, d_{(k-1)j})$, on $\widetilde{\Gamma}$ and $d'_j \in \Omega_j(a') \cap (-\mathbb{R}_+)$, we have

$$\Omega_j(a_0) - d'_j \subseteq \overrightarrow{d}_g^{(k)} \Omega_j (a', d'_j, a_1 - a', d_{1j} - d'_j, ..., a_{k-1} - a', d_{(k-1)j} - d'_j)$$
$$(\Lambda'_{a',a_0}(0+)) + \nu_j \|a_0 - a'\|^2 + \mathbb{R}_+.$$

Since $d_j \in \Omega_j(a_0) \cap (-\mathbb{R}_+)$, we have

$$d_j - d'_j$$
$$\in \overrightarrow{d}_g^{(k)} \Omega_j (a', d'_j, a_1 - a', d_{1j} - d'_j, ..., a_{k-1} - a', d_{(k-1)j} - d'_j) \tag{6.24}$$
$$(\Lambda'_{a',a_0}(0+)) + \nu_j \|a_0 - a'\|^2 + \mathbb{R}_+.$$

From (6.16), (6.20), (6.21), (6.22), (6.23), (6.24), and (MWD),

$$\sum_{i=1}^{r} c_i^* \Big(c'_i + (a_0^T M_1 d) - c'(c''_i - (a_0^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d_j$$
$$\geq \sum_{i=1}^{r} c_i^* \Big(\overline{c'_i} + (a'^T M_1 d) - c'(\overline{c''_i} - (a'^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d'_j,$$

which contradicts (6.19). Hence,

$$\max \bigcup_{b \in \widetilde{\Delta}} \mu(a_0, b) \not\leq c'.$$

It concludes the proof of the theorem. □

Theorem 6.2. ***(Higher-order strong duality)*** *Let* (a', c') *be a minimizer of* (MFP) *and* $d'_j \in \Omega_j(a') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. *Suppose that for arbitrary* $k \in \mathbb{Z}$, $(1 \leq r \leq n)$, $c_i^* \geq 0$, $\overline{b_i} \in B(a')$, $(1 \leq i \leq r)$ *with* $\sum_{i=1}^{r} c_i^* = 1$ *and* $d_j^* \geq 0$, $(1 \leq j \leq p)$, *Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6) are fulfilled at* (a', c', d', c^*, d^*). *Then* (a', c', d', c^*, d^*) *is feasible to* (MWD). *If Theorem 6.1 is satisfied, then* (a', c', d', c^*, d^*) *is a maximizer of* (MWD).

Proof. As Eqs. Eqs. (5.2), (5.3), (5.4), (5.5), and (5.6) are fulfilled at (a', c', d', c^*, d^*),

$$\sum_{i=1}^{r} c_i^* \Big(\overrightarrow{d}_g^{(k)} \chi(., \overline{b_i})(a', \overline{c_i'}, a_1 - a',$$
$$c_{1i} - \overline{c_i'}, ..., a_{k-1} - a', c_{(k-1)i} - \overline{c_i'}) + M_1 d$$
$$-c'(\overrightarrow{d}_g^{(k)} (-\zeta)(., \overline{b_i})(a', -\overline{c_i''}, a_1 - a', -c_{1i}' + \overline{c_i''}, ..., a_{k-1} - a',$$
$$-c_{(k-1)i}' + \overline{c_i''}) - M_2 \delta) \Big) (\Lambda_{a',a}'(0+))$$
$$+ \sum_{j=1}^{p} d_j^* \overrightarrow{d}_g^{(k)} \Omega_j (a', d_j', a_1 - a', d_{1j} - d_j', ..., a_{k-1} - a',$$
$$d_{(k-1)j} - d_j')(\Lambda_{a',a}'(0+)) \geq 0, \forall a \in \widetilde{\Gamma},$$

$$\sum_{j=1}^{p} d_j^* d_j' = 0,$$

$$d^T M_1 d \leq 1, \delta^T M_2 \delta \leq 1,$$
$$(a'^T M_1 a')^{\frac{1}{2}} = a'^T M_1 d,$$

and

$$(a'^T M_2 a')^{\frac{1}{2}} = a'^T M_2 \delta.$$

Hence, (a', c', d', c^*, d^*) is feasible to (MWD). Assume that Theorem 6.1 is satisfied and (a', c', d', c^*, d^*) is not a maximizer of (MWD). Let (a, c, d, c_1^*, d_1^*) be a feasible element for (MWD) fulfilling

$$c' < c.$$

It contradicts Theorem 6.1. Hence, (a', c', d', c^*, d^*) is a maximizer for (MWD). □

Theorem 6.3. ***(Higher-order converse duality)*** *Let $\widetilde{\Gamma}$ be an ACS of $\mathbb{R}^n$ and (a', c', d', c^*, d^*) be a feasible element of* (MWD).

Suppose that $\chi(., \overline{b_i})$ is α_i-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 *at $(a', \overline{c_i'})$ for $(a_1, c_{1i}), ..., (a_{k-1}, c_{(k-1)i})$, $(.)^T M_1 d$ is $\overline{\alpha}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t.* 1*, $-\zeta(., \overline{b_i})$ is α_i'-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t.* 1 *at $(a', -\overline{c_i''})$ for the elements $(a_1, -c_{1i}'), ..., (a_{k-1}, -c_{(k-1)i}')$, $(.)^T M_2 \delta$ is $\overline{\alpha'}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t.* 1 *and Ω_j, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t.* 1 *at $(a', \overline{d_j'})$ for $(a_1, d_{1j}), ..., (a_{k-1}, d_{(k-1)j})$, on $\widetilde{\Gamma}$, satisfying* (6.16)*. If a' is feasible to* (MFP)*, then (a', c') is a minimizer of* (MFP).

Proof. Assume that (a', c') is not a minimizer of (MFP). Hence there exist $a \in S'$ and $c = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a, b)$, with $a \neq a'$, fulfilling

$$c < c'.$$

Since $\overline{b_i} \in B(a')$, $i = 1, ..., r$,

$$\max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b) \in \mu(a', \overline{b_i}).$$

As $c' = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a', b)$,

$$c' \in \mu(a', \overline{b_i}), i = 1, ..., r.$$

Let $c_i \in \mu(a, \overline{b_i})$. Again, as $c = \max \bigcup_{b \in \widetilde{\Delta}} \mu(a, b)$ and $\overline{b_i} \in B(a') \subseteq \widetilde{\Delta}$,

$$c_i \leq c.$$

Hence,

$$c_i < c'.$$

As $c' \in \mu(a', \overline{b_i})$, there exist $\overline{c_i'} \in \chi(a', \overline{b_i})$ and $\overline{c_i''} \in \zeta(a', \overline{b_i})$ fulfilling

$$c' = \frac{\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}}}{\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}}.$$

So,

$$\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) = 0, \forall i = 1, ..., r. \tag{6.25}$$

Since $c_i \in \mu(a, \overline{b_i})$, there exist $c_i' \in \chi(a, \overline{b_i})$ and $c_i'' \in \zeta(a, \overline{b_i})$ fulfilling

$$c_i = \frac{c_i' + (a^T M_1 a)^{\frac{1}{2}}}{c_i'' - (a^T M_2 a)^{\frac{1}{2}}}.$$

Hence,

$$\frac{c_i' + (a^T M_1 a)^{\frac{1}{2}}}{c_i'' - (a^T M_2 a)^{\frac{1}{2}}} < c'.$$

So,

$$c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) < 0, \forall i = 1, ..., r. \tag{6.26}$$

From (4.1) and the constraints of the problem (MWD), we have

$$\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta)) \Big)$$
$$\leq \sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) \Big).$$

Again, from (6.26), we have

$$\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 a)^{\frac{1}{2}} - c'(c_i'' - (a^T M_2 a)^{\frac{1}{2}}) \Big) < 0.$$

From (6.25),

$$\sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) \Big) = 0.$$

Again, from the constraints of the problem (MWD), we have

$$\begin{aligned}&\sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 a')^{\frac{1}{2}} - c'(\overline{c_i''} - (a'^T M_2 a')^{\frac{1}{2}}) \Big) \\ &= \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta)) \Big).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta)) \Big) \\ &= \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta)) \Big).\end{aligned}$$

As $a \in S'$, there exists $d_j \in \Omega_j(a) \cap (-\mathbb{R}_+)$. As $d_j^* \geq 0$ $(1 \leq j \leq p)$,

$$d_j^* d_j \leq 0, \forall j, \text{ with } 1 \leq j \leq p.$$

So,

$$\sum_{j=1}^{p} d_j^* d_j \leq 0.$$

From (MWD),

$$\sum_{j=1}^{p} d_j^* d_j' \geq 0.$$

Hence,

$$\sum_{j=1}^{p} d_j^* d_j \leq \sum_{j=1}^{p} d_j^* d_j'.$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d_j \\ &< \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d_j'.\end{aligned} \tag{6.27}$$

As it is presumed that $\chi(.,\overline{b_i})$ is α_i-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a',\overline{c_i'})$ for $(a_1,c_{1i}),...,(a_{k-1},c_{(k-1)i})$, on $\widetilde{\Gamma}$ and $\overline{c_i'}\in\chi(a',\overline{b_i})$,

$$\chi(a,\overline{b_i})-\overline{c_i'}\subseteq\overrightarrow{d}_g^{(k)}\chi(.,\overline{b_i})(a',\overline{c_i'},a_1-a',c_{1i}-\overline{c_i'},...,a_{k-1}-a',c_{(k-1)i}-\overline{c_i'})(\Lambda'_{a',a}(0+))+\alpha_i\|a-a'\|^2+\mathbb{R}_+.$$

As $c_i'\in\chi(a,\overline{b_i})$,

$$c_i'-\overline{c_i'}\in\overrightarrow{d}_g^{(k)}\chi(.,\overline{b_i})(a',\overline{c_i'},a_1-a',c_{1i}-\overline{c_i'},...,a_{k-1}-a',c_{(k-1)i}-\overline{c_i'})(\Lambda'_{a',a}(0+))+\alpha_i\|a-a'\|^2+\mathbb{R}_+. \tag{6.28}$$

Suppose that $(.)^TM_1d$ is $\overline{\alpha}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$,

$$a^TM_1d-a'^TM_1d\geq M_1d(\Lambda'_{a',a}(0+))+\overline{\alpha_i}\|a-a'\|^2+\mathbb{R}_+. \tag{6.29}$$

As it is presumed that $-\zeta(.,\overline{b_i})$ is α_i'-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a',-\overline{c_i''})$ for $(a_1,-c_{1i}'),...,(a_{k-1},-c_{(k-1)i}')$, on $\widetilde{\Gamma}$ and $\overline{c_i''}\in\zeta(a',\overline{b_i})$, we have

$$-\zeta(a,\overline{b_i})+\overline{c_i''}\subseteq\overrightarrow{d}_g^{(k)}(-\zeta)(.,\overline{b_i})(a',-\overline{c_i''},a_1-a',-c_{1i}'+\overline{c_i''},...,a_{k-1}-a',-c_{(k-1)i}'+\overline{c_i''})(\Lambda'_{a',a}(0+))+\alpha_i'\|a-a'\|^2+\mathbb{R}_+.$$

Again, as $c_i''\in\zeta(a,\overline{b_i})$, we have

$$-c_i''+\overline{c_i''}\in\overrightarrow{d}_g^{(k)}(-\zeta)(.,\overline{b_i})(a',-\overline{c_i''},a_1-a',-c_{1i}'+\overline{c_i''},...,a_{k-1}-a',-c_{(k-1)i}'+\overline{c_i''})(\Lambda'_{a',a}(0+))+\alpha_i'\|a-a'\|^2+\mathbb{R}_+. \tag{6.30}$$

Since $(.)^TM_2\delta$ is $\overline{\alpha'}_i$-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on $\widetilde{\Gamma}$,

$$a^TM_2\delta-a'^TM_2\delta\geq M_2\delta(\Lambda'_{a',a}(0+))+\overline{\alpha_i'}\|a-a'\|^2+\mathbb{R}_+. \tag{6.31}$$

As Ω_j, $(1\leq j\leq p)$, is ν_j-$\mathbb{R}_+$-arcwisely connected of k-th order w.r.t. 1 at $(a',\overline{d_j'})$ for the elements $(a_1,d_{1j}),...,(a_{k-1},d_{(k-1)j})$, on $\widetilde{\Gamma}$ and $d_j'\in\Omega_j(a')\cap(-\mathbb{R}_+)$, we have

$$\Omega_j(a)-d_j'\subseteq\overrightarrow{d}_g^{(k)}\Omega_j(a',d_j',a_1-a',d_{1j}-d_j',...,a_{k-1}-a',d_{(k-1)j}-d_j')(\Lambda'_{a',a}(0+))+\nu_j\|a-a'\|^2+\mathbb{R}_+.$$

Since $d_j\in\Omega_j(a)\cap(-\mathbb{R}_+)$, we have

$$d_j-d_j'\in\overrightarrow{d}_g^{(k)}\Omega_j(a',d_j',a_1-a',d_{1j}-d_j',...,a_{k-1}-a',d_{(k-1)j}-d_j')(\Lambda'_{a',a}(0+))+\nu_j\|a-a'\|^2+\mathbb{R}_+. \tag{6.32}$$

From (6.16), (6.28), (6.29), (6.30), (6.31), (6.32), and (MWD),

$$\sum_{i=1}^{r} c_i^* \Big(c_i' + (a^T M_1 d) - c'(c_i'' - (a^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d_j$$
$$\geq \sum_{i=1}^{r} c_i^* \Big(\overline{c_i'} + (a'^T M_1 d) - c'(\overline{c_i''} - (a'^T M_2 \delta)) \Big) + \sum_{j=1}^{p} d_j^* d_j',$$

which contradicts (6.27). Hence, (a', c') is a minimizer of (MFP). □

7. CONCLUSIONS

In light of the broader concept of higher-order cone arcwisely connected SVMs, we use the concept of higher-order α-cone arcwisely connectivity of SVMs, introduced by Das [1]. We have the standard notion of higher-order cone arcwisely connected SVMs for $\alpha = 0$. We investigate the higher-order converse, strong, and weak theorems of duality for the Mond-Weir (MWD) form under the higher-order generalized contingent epiderivative and α-cone arcwisely connectivity suppositions.

For $k = 1$ and $\Lambda_{\gamma_1,\gamma_2}(\tau) = (1-\tau)\gamma_1 + \tau\gamma_2$, our results correspond with the sufficient optimality conditions and duality results of the problem (MFP) under the contingent epiderivative and α-cone convexity suppositions as explored in [30]. For $k = 2$ and $\Lambda_{\gamma_1,\gamma_2}(\tau) = (1-\tau)\gamma_1 + \tau\gamma_2$, our findings coincides with the second-order sufficient optimality conditions and duality results of the problem (MFP) under the second-order contingent epiderivative and second-order α-cone convexity suppositions as discussed in [31]. For $k = 1$, our findings correspond with the sufficient optimality conditions and duality results of the problem (MFP) under the contingent epiderivative and α-cone arcwisely connectivity suppositions as presented in [32].

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