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# ANALYZING EOQ MODEL FOR IMPERFECT ITEMS WITH TIME DEPENDENT DEMAND UNDER SUBSTITUTION OF TWO PRODUCTS

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**Abstract:** The inventory model is based on the quantity of order and the time at which orders for particular goods are to be placed. In this paper, models with defective items have been discussed and the EOQ approach has been shown. One-way substitution of item is used in case of scarcity of the first item. The demand for the items is based on the Heaviside function. It is noticed that the presence of defective items hampers the demand of the two items differently. To establish the virtue of our model, a numerical example and sensitivity analysis is imparted.

**Keywords:** EOQ, defective items, substitution of items.

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**MSC:** 90B05.

## 1. INTRODUCTION

Inventory is a stock of goods that have financial worth and are kept in different forms by an organization in its custody awaiting packaging, processing, transformation, or sale in the future. Inventories are fundamentally used everywhere. Because of the highly competitive market, companies and firms are continuously working on various strategies to ensure customer retention. The availability of different varieties of the same kind of items has led to the division of customers, thus sustainability in the market has become an important concern. Unavailability of inventory results in disappointment among customers which leads to financial loss or benevolence. In case of unavailability of a particular item that a customer wishes to buy, the customer might move to another store or just wait for the item to arrive or maybe switch to another similar product with the expert merchandising of the shopkeeper. The case in which the customer switches to another similar product is the case of substitution. The choice of substitutes is made keeping various factors into consideration like low cost, maximum profit provided it serves the same purpose as the product for which it is a substitute. To hold the customer, the availability of similar items with desired features should be available. Ignorance of options available for substitution may result in disinterest and dissatisfaction among customers. Cost-effective items with large order sizes should be considered as a substitute. Also, a decision maker should observe the effect of substitution to determine the order quantities.

This model shows how the stock availability of one item affects the demand for other items. The substitute item may incur the transfer cost and may also include transformation cost or may be used in its actual form. There are various benefits of substitution between items in inventory systems. The situation when a particular item is out of stock can be controlled by using substitutes. It also reduces the total holding cost as it requires holding less amount of items of different quality of the same type of items. Substitution can also be used to minimize the amount of perishable items which have an early expiry. It also induces inventory-sharing agreements between numerous systems which permits one to use another's inventory. This leads to an increase in inventory collating to combat demand variabilities and to help in limiting the safety stocks. Sometimes, substitution can also be used as a backup strategy, when an ordinary item is used as a substitute for a customary item when the latter is out of stock. In this paper, the demand considered is based on the function called the *Heaviside* function, defined by

$$H(t-\tau) = \begin{Bmatrix} 0 & \text{if } t < \tau \\ 1 & \text{if } t \geq \tau \end{Bmatrix}$$

It is a discontinuous function which is a kind of step function. EOQ-type models are used to decide the optimal inventory policy. A fraction of both the types of items in this model is defective and is inspected and removed before sale. The presence of damaged items also affects the overall optimality. Substitution of one item with another item is allowed. In many cases, it is observed that two or more batches of a product receive different keenness of purchase by customers. In this manner, we can perceive that substitution between items is very much realistic and hence it is useful to incorporate its outcomes into classical inventory models too.

## 2. LITERATURE SURVEY

Several studies and research have been developed in the context of inventory models. Various pre-existing models consider an inventory model with substitution with different demand patterns. Sharma et al. [1] (2017) have focused on developing a replenishment model for deteriorating products for ramp-type demand rate under the consideration of inflation and partial backlogging. Trade credit is introduced and the demand pattern following ramp type which is a quadratic function of time has been discussed in detail by Sharma et al. [2] (2018). Khedlekar et al. [3] (2021) considered the demand of retailers as a linear function of time and price and optimized the total profit function. They used an integrated three-layer supply chain policy for multi-channel and multi-echelon. Kumar and Kumar (2021) [4] surveys a model for declining medicinal products with price-sensitive demand under the impact of inflation. They studied the impact of order size and time interval on total average cost. Khedlekar and Singh [5](2022) used ameliorating items with time and price-varying demand over a fixed time interval to optimize the length of time during which there is no demand, the length of time during which the inventory level becomes zero and to maximize the total profit. Khedlekar et al.[6](2023) attempted to generate a stochastic inventory model with a quadratic demand function based on price. They aimed to find the optimal price, optimal replenishment, and optimal preservation technology.

**Inventory models with substitution**:

Parlar and Goyal [7] (1984) used two-item inventory systems with substitution to obtain optimal order sizes. Gurnani and Drezner [8] (2000) generalized the model of Drezner et al.[9] (1995) under hierarchical substitution scenarios. Karakul and Chan [10](2008) have studied substitution in inventory control policy under newsvendor approaches. Nagarajan and Rajagopalan [11] (2008) used the newsvendor approach for substitution in inventory control policy. Pineyro and Viera [12] (2010) took into account one-way substitution and implemented a tabu search procedure. They obtained the maximum remanufacturing quantity for an economic lot-sizing problem. Liu et al. [13] (2013) considered the effect of substitution when two loss-reluctant retailers are opposing substitutable products under stochastic demand rate and deterministic substitution rate. Mukhopadhyay, A. and Goswami, A. [14] (2017) considered a random proportion of defective items and inspection policy with constant demand and used the concept of replacement of defective items with good quality items.

**Inventory models with defective items**:

Zhang and Gerchak [15] (1990) examined joint lot sizing and analyzed it with the hypothesis that a random proportion of lot size is faulty. Jaber et al. [16](2008)

made assumptions about a learning curve and showed that the percentage of defective items depends on the learning curve. A detailed survey of the recent inventory models with imperfect items is provided by Khan et al.[17] (2011). Roy et al. [18] (2011) developed an EOQ model for defective items where a portion of demand was partially backlogged. Singh and Singh[19] (2011) have considered an imperfect production process with an exponential demand rate. Yu et al.[20] (2012) expanded Salameh and Jaber [21] (2000) when a part of imperfect items can be used as perfect quality and this utilization will affect the exhaustion of the leftover perfect quality in the stock. Maddaha and Jaber [22] (2008) extended the work of Salameh and Jaber by assuming that the fraction of imperfect items per lot decreases according to a learning curve, which was experimentally authenticated. Sharma et al. (2022) [23] in their paper have focussed on the development of a flexible inventory model which requires rework on imperfect and defective items. Their model supports decision-making by considering a volume flexibility system for smooth runs.
The composition of this paper is as follows: In Sections 2 and 3 notations and assumptions considered for the model are shown. In Section 4, the EOQ model with the mathematical formulation is shown that acknowledges the consequence of faulty items and substitution. In Section 5, a numerical example is given and sensitivity analysis is accomplished to observe the consequences of change of different parameters in the optimal policy. Sections 6 and 7 provide the observations and conclusions with managerial insights.

Table 1: Contribution of this model and the pre-existing models

| **Authors** | **Defective items** | **Substitution** | **Time-Dependent Demand** |
|---|---|---|---|
| Sharma et al. (2017) | ✗ | ✗ | ✗ |
| Sharma et al. (2018) | ✗ | ✗ | ✓ |
| Khedlekar et al. (2021) | ✗ | ✗ | ✓ |
| Kuman and Kumar (2021) | ✗ | ✗ | ✗ |
| Khedlekar and Singh (2022) | ✗ | ✗ | ✓ |
| Khedlekar et al. (2023) | ✗ | ✓ | ✗ |
| Parlar and Goyal(1984) | ✗ | ✓ | ✗ |
| Gurnani and Drezner (2000) | ✓ | ✓ | ✗ |
| Drezner et al. (1995) | ✓ | ✗ | ✗ |
| Karakul and Chan(2008) | ✗ | ✓ | ✗ |
| Nagarajan and Rajagopalan (2008) | ✗ | ✓ | ✗ |
| Pineyro and Viera (2010) | ✓ | ✓ | ✗ |
| Liu et al. (2013) | ✗ | ✓ | ✗ |
| Mukhopadhyay and Goswami(2017) | ✓ | ✓ | ✗ |
| Zhang and Gherchak(1990) | ✓ | ✗ | ✗ |
| Jaber et al. (2008) | ✓ | ✗ | ✗ |
| Khan et al. (2011) | ✓ | ✗ | ✗ |
| Roy et al. (2011) | ✓ | ✗ | ✗ |
| Singh and Singh (2011) | ✓ | ✗ | ✓ |
| Yu et al. (2012) | ✓ | ✗ | ✗ |
| Salameh and Jaber (2000) | ✓ | ✗ | ✗ |
| Maddaha and Jaber (2008) | ✓ | ✗ | ✗ |
| Sharma et al. (2022) | ✓ | ✗ | ✗ |
| Present study | ✓ | ✓ | ✓ |

This paper considers a model with defective items following time-dependent demand and one-way substitution which has not been studied in the past. This model provides insights to the firms and helps in making optimal decisions under such circumstances.

## 3. NOTATIONS

The following notations are used in this model:

Table 2: Notations

| Symbols | Description |
|---|---|
| **Parameters** | |
| $d_i$ | annual demand for products $i = 1, 2$ |
| $q_i$ | order quantity for products $i = 1, 2$ |
| $h_i$ | holding cost for products $i = 1, 2$ |
| $c_t$ | transfer cost |
| $c_o$ | ordering cost |
| $\mu$ | time interval during which no substitution is required |
| $T$ | total cycle time |
| $z_i$ | fraction of imperfect items for products $i = 1, 2$ |
| $t_i$ | time epoch at which screening occurs for products $i = 1, 2$ |
| $s_i$ | screening rate for products $i = 1, 2$ |
| $E[.]$ | probabilistic expectation operator |
| **Decision Variables** | |
| $\mu$ | time interval during which no substitution is required |
| $T$ | total cycle time |

## 4. ASSUMPTIONS

The following assumptions have been made in the model to obtain results:

1. The demand rate of each type of items is given by $D(t) = a + bt[1 - H(t - \mu)]$ This type of demand is considered to show that the demand increases upon the arrival of new item and after a certain time it becomes constant.
2. Only one-way substitution is allowed in this model. Substitution takes place only when the minor item is exhausted.
3. Substitute is ample whose shortage never happens. The major item used as substitute is available in large amounts.
4. A fraction of both the items is defective. The presence of defective items affects the optimality of the total average cost function.
5. Screening rate of both types of items is greater than their demand rates i.e. $s_i > d_i, i = 1, 2$. Inspection of defective items is important to ensure that only good quality of items are received by the customers in order to sustain in the market.

6. Replenishment is spontaneous i.e. lead time is negligible.The stock gets refilled without any delay in time.
7. $E[z_i] < 1 - \frac{d_i}{s_i}, i = 1, 2$. The expectation operator of defective items helps in assuming the loss occured due to the production of defective items.
8. At the end of each cycle, inventory reaches zero. The inventory is refilled only after one cycle is completed.
9. The following relation holds: $h_2 > h_1$ , i.e. the choice of substitute is made in such a way that the holding cost of substitute is less than that of the minor item.
10. $t_i = \frac{q_i}{s_i}$ for products $i = 1, 2$, i.e. the screening time depends on the quantity of items available and the screening rate.

## 5. EOQ MODEL FOR DEFECTIVE ITEMS WITH SHORTAGE AND SUBSTITUTION

We consolidate the existence of defective items in this paper. The existence of defective items proposes for bigger lot size whereas a substitute will propose a smaller size. The purpose is to examine the consequence of defective items. In this inventory model, both major and minor items are available initially. The stock availability of minor item is limited and as a result, the customers tend to consider the substitution of items. The shortage of minor item thus affects the demand and the substitution behaviour for the major items.

### 5.1. Mathematical formulation

We consider the following model under two cases represented by figure 2:

1. $\mu \leq t_1$
2. $\mu \geq t_1$

In the two cases considered, the number of perfect items of major items remains constant. Shortage of minor items is fulfilled by perfect quality major-type items. The demand function for the two items is given by:

$$\begin{aligned} d_1 &= a + bt \\ d_2 &= a \end{aligned} \tag{1}$$

a. **Costs associated with minor item:** The graph representing the inventory for the minor item is given by figure (1):

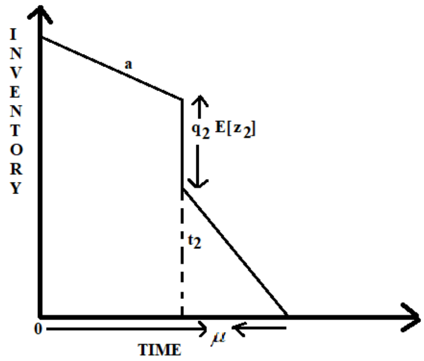


Figure 1: Inventory vs time graph of minor item

The differential equation administrating the inventory level at time $t \in [0, t_2]$ is given by

$$\frac{dI_2}{dt} = -a, I_2(0) = q_2 \tag{2}$$

Integrating both sides and applying the initial condition, we obtain

$$I_2(t) = q_2 - at, t \in [0, t_2] \tag{3}$$

The differential equation administrating the inventory level at time $t \in [t_2, \mu]$ is given by

$$\frac{dI_2}{dt} = -a, I_2(\mu) = 0 \tag{4}$$

Integrating both sides and applying the initial condition, we obtain

$$I_2(t) = q_2(1 - E[z_2]) - at \tag{5}$$

At time $t = \mu, I_2(\mu) = 0$, so from equation (5) we get

$$q_2 = \frac{a\mu}{1 - E[z_2]} \tag{6}$$

The holding cost of the substitute item is given by

$$\begin{aligned} H.C_2 &= h_2 \int_0^{\mu} I_2(t)dt \\ &= h_2 \left[ \int_0^{t_2} (q_2 - at)dt + \int_{t_2}^{\mu} (q_2[1 - E[z_2]] - at)dt \right] \\ &= h_2 \left[ \frac{q_2^2 E[z_2]}{s_2} + q_2[1 - E[z_2]]\mu - \frac{a\mu^2}{2} \right] \left( \because t_2 = \frac{q_2}{s_2} \right) \end{aligned} \tag{7}$$

Using equation (6), we obtain

$$H.C_2 = h_2\left[\frac{a\mu^2}{2} + \frac{E[z_2]a^2\mu^2}{(1-E[z_2])^2 s_2}\right] \tag{8}$$

b. **Costs associated with major item:** There are two possible cases for the inventory of the major item. The graphs (2) represent the two cases:

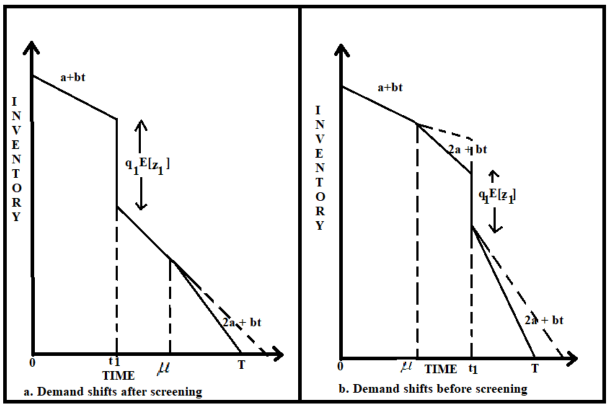


Figure 2: Inventory vs time graph of major item

The differential equation administrating the inventory level at time $t \in [0, \mu]$ is given by

$$\frac{dI_1}{dt} = -a - bt, I_1(0) = q_1 \tag{9}$$

Integrating both sides and applying the initial condition, we obtain

$$I_1(t) = q_1 - at - \frac{bt^2}{2} \tag{10}$$

At $t = \mu, I_1(\mu) = q_1 - a\mu - \frac{b\mu^2}{2}$
The differential equation administrating the inventory level at time $t \in [\mu, t_1]$ is given by

$$\frac{dI_1(t)}{dt} = -2a - bt \tag{11}$$

Solving differential equation and applying the initial condition, we get

$$I_1(t) = q_1 - a(2t - \mu) - \frac{bt^2}{2} \tag{12}$$

At $t = t_1, I_1(t_1) = q_1 - a(2t_1 - \mu) - \frac{bt_1^2}{2}$
The differential equation administrating the inventory level at time $t \in [t_1, T]$ is given by

$$\frac{dI_1}{d1} = -2a - bt \tag{13}$$

Solving the above equation, we obtain

$$I_1(t) = q_1(1 - E[z_1]) - a(2t - \mu) - \frac{bt^2}{2} \tag{14}$$

The holding cost of the major item is given by

$$H.C_1 = h_1 \int_0^T I_1(t)dt$$

$$= h_1 \left[ \int_0^\mu I_1(t)dt + \int_\mu^{t_1} I_1(t)dt + \int_{t_1}^T I_1(t)dt \right]$$

$$= h_1 \left[ -\frac{a\mu^2}{2} + q_1(1 - E[z_1])T - aT^2 + a\mu T - \frac{bT^3}{3} - q_1 E[z_1]t_1 \right] \tag{15}$$

$$= h_1 \left[ -\frac{a\mu^2}{2} + q_1(1 - E[z_1])T - aT^2 + a\mu T - \frac{bT^3}{3} - q_1 E[z_1]\frac{q_1}{s_1} \right]$$

$$\left( \because t_1 = \frac{q_1}{s_1} \right)$$

Using $I_1(T) = 0$, we obtain $q_1 = \frac{bT^2}{2(1-E[z_1])} + \frac{a(2T-\mu)}{1-E[z_1]}$ Thus, the holding cost is given by

$$H.C_1 = h_1 \left[ -\frac{a\mu^2}{2} + \frac{bT^3}{6} + aT^2 - \frac{E[z_1]}{s_1(1 - E[z_1])^2}(bT^2 + a(2T - \mu))^2 \right] \tag{16}$$

The other associated costs for the items are given as:

$$\text{Ordering Cost, OC} = c_o \tag{17}$$

$$\text{Transfer cost, CT} = c_t a(T - \mu) \tag{18}$$

Thus, the total cost is given by

$$T.C = H.C_1 + H.C_2 + OC + CT$$

$$= h_1 \left[ -\frac{a\mu^2}{2} + \frac{bT^3}{6} + aT^2 - \frac{E[z_1]}{s_1(1 - E[z_1])^2}(bT^2 + a(2T - \mu))^2 \right] \tag{19}$$

$$+ h_2 \left[ \frac{a\mu^2}{2} + \frac{E[z_2]a^2\mu^2}{(1 - E[z_2])^2 s_2} \right] + c_o + c_t a(T - \mu)$$

Total average cost, T.A.C. $= \frac{T.C}{T}$ Hence,

$$\text{T.A.C.} = h_1 \left[ -\frac{a\mu^2}{2T} + \frac{bT^2}{6} + aT - \frac{E[z_1]}{s_1(1 - E[z_1])^2}(bT + \frac{a}{T}(2T - \mu))^2 \right] + h_2 \left[ \frac{a\mu^2}{2T} + \frac{E[z_2]a^2\mu^2}{T(1 - E[z_2])^2 s_2} \right] + \frac{c_o}{T} + ac_t \left( 1 - \frac{\mu}{T} \right) \tag{20}$$

### 5.2. Solution Procedure

To obtain the optimal solution, the procedure followed is: Differentiate the total cost function $T.A.C.$ with respect to decision variables $T$ and $\mu$ partially. Solve the equations $\frac{\partial(T.A.C)}{\partial T} = 0$ and $\frac{\partial(T.A.C.)}{\partial\mu} = 0$ for $T$ and $\mu$. Evaluate $\frac{\partial^2(T.A.C.)}{\partial\mu^2}, \frac{\partial^2(T.A.C.)}{\partial T^2}$ and $\frac{\partial^2 T.A.C.}{\partial\mu\partial T}$ and calculate the Hessian matrix. The convexity of total cost function is established when $\frac{\partial^2 TAC}{\partial\mu^2} > 0$ and $H(\mu,T) = \frac{\partial^2(T.A.C.)}{\partial\mu^2}\frac{\partial^2(T.A.C.)}{\partial T^2} - \frac{\partial^2 T.A.C.}{\partial\mu\partial T} > 0$. Allocate the values to all inventory parameters except decision variables. Substitute these values in the total cost function to demonstrate the effects of different parameters on the decision variables and the total average cost function TAC.

**Theorem 1.** *If the parameter of the cost function given by the equation (21) satisfy* $c_o > \frac{T^2}{2}\left[2ac_t\mu - bT - \frac{a\mu^2(h_2-h_1)}{T^2} + \frac{c_t}{h_2-h_1} - \frac{\mu h_2 T^2}{h_2-h_1}\right]$ *then the optimal cost can be obtained uniquely with optimally unique parameters* $T^*$ *and* $\mu^*$.
*(For Proof See Appendix)*

## 6. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

### 6.1. Numerical Example

To demonstrate and authenticate the evolved inventory model, we have chosen an arbitrary data and validate our model using software Wolfram Mathematica 9. The values of various used parameters can be taken as follows:
$a = 2000, b = 1200, s_1 = 10000 units, s_2 = 35 units, h_1 = \$25/unit/year, h_2 = \$40/unit/year, E[z_1] = 0.3, E[z_2] = 0.3, c_o = \$2000/cycle, c_t = \$16/unit.$
After applying the solution procedure and substituting the above values for the parameters, the optimal total average cost $T.A.C.$ obtained is \$24459, the demand shift time is 0.0010 week and the total cycle time is 0.2334 week. The convexity of the total cost function is shown in the graph (3) and it ensures the existence of unique optimal solution.

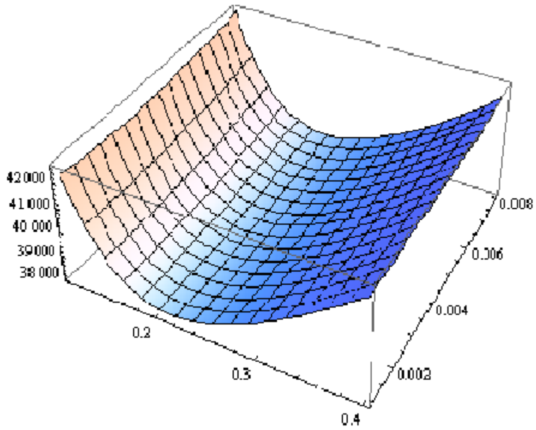


Figure 3: Convexity of the Total Average Cost function

The graph (3) indicates the existence of a minimum cost in correspondance with the assigned values of various parameters and shows the effects on the total average cost in response to change in the values of parameters.

### 6.2. Sensitivity Analysis

The model considers the case of substitution of one type of items due to shortages. To understand the effect of various parameters on the optimal solution, sensitivity analysis has been conducted. The numerical results obtained using Wolfram Msthematica 9 software are shown in the following table 3. Table 3 provides the valuable information aobut how different parameters, presence of defective items and substitution affect the overall total average cost.

Table 3: Sensitivity analysis showing the effect of various parameters

| Parameter | Value | $\mu$ | $T$ | $T.A.C$ |
|---|---|---|---|---|
| | 0.1 | 0.0048 | 0.1990 | 46332.2 |
| | 0.2 | 0.0033 | 0.2111 | 37970.5 |
| $E[z_1]$ | 0.3 | 0.0010 | 0.2334 | 24459 |
| | 0.31 | 0.0007 | 0.2368 | 22683.1 |
| | 0.32 | 0.0004 | 0.2404 | 20808 |
| | 0.1 | 0.0052 | 0.2323 | 24406.7 |
| | 0.2 | 0.0020 | 0.2331 | 24446.6 |
| $E[z_2]$ | 0.3 | 0.0010 | 0.2334 | 24459 |
| | 0.4 | 0.0006 | 0.2336 | 24464.8 |
| | 0.5 | 0.0003 | 0.2337 | 24468 |
| | 30 | 0.0014 | 0.2333 | 24454.6 |
| | 35 | 0.0012 | 0.2334 | 24457.2 |
| $h_2$ | 40 | 0.0010 | 0.2334 | 24459 |
| | 45 | 0.0009 | 0.2335 | 24460.5 |
| | 50 | 0.0008 | 0.2335 | 24461.7 |
| | 14 | 0.0003 | 0.2337 | 20470.9 |
| | 15 | 0.0005 | 0.2336 | 22467.5 |
| $c_t$ | 16 | 0.0010 | 0.2333 | 24459 |
| | 17 | 0.0011 | 0.0.2333 | 26453.3 |
| | 18 | 0.0014 | 0.2332 | 28442.5 |

## 7. OBSERVATIONS

Table(3) illustrates the impact of colorful parameters on the total average cost. The following conclusion can be drawn from the table

- ➢ The anticipation parameter of a major item affects the total average cost and the demand shift time negatively whereas it shows a analogous effect on total cycle time. With the increase in the anticipation, the total average cost and the demand shift time decreases whereas the total cycle time increases.
- ➢ The anticipation parameter of minor particulars affects the total average cost and the total cycle time in a analogous manner. Both of them increase nearly with an increase in the anticipation. But the demand shift time decreases with an increase in anticipation of minor particulars.
- ➢ Change in the holding cost of minor particulars manifests a analogous effect on cycle time and total average cost but affects the demand shift time negatively.
- ➢ Transfer cost has reversible goods on the total average cost and the total cycle time. With an increase in transfer cost, the total average cost and the demand shift time show a drastic increase whereas the total cycle time decreases.

## 8. CONCLUSION AND MANAGERIAL INSIGHTS

In this paper, a two-product ineluctable inventory system with imperfect quality particulars under failure and one-way substitution is studied. Demand for minor particulars is constant whereas for major particulars it's time-dependent.

Cycle time and demand-shift time are considered to be the decision variables to decide the total average cost. Transfer cost affects the two decision variables equally whereas the holding cost shows analogous goods on both of them. The protuberance of the total average cost function ensures a unique and optimal result. From the perceptivity analysis, it's observed that the total cost is more sensitive towards the parameters $c_t$ and $h_2$. A numerical illustration is vindicated to test the validity of our model. It helps in decision-making to gain optimal results. Our work differs from former exploration works in the sense that we consider the demand to be a specific function dependent on time. Using this point of view, we arrive at an expression for the total average cost which is comparable with the model of Drezner et al. (1995) [9]. Thus, the contribution of our model is that we alter the optimality condition of the total average cost function in the basic model of Drezner et al. [9] incorporating specific demand. Also, we include the effect of defective items in both types of items when substitution is allowed.

The data are acquired from the suggested model and economically can be enforced by the enterprises to achieve their targets. The assumptions taken into account in this model like the time-dependent demand and one-way substitution handles the real circumstances of the market and give precise insights into present and future sale. A firm director can effectively use the model to minimize the total cost of the item by understanding the effect of the one-way substitution of particulars. This model can be applied in the areas of clothes, cement, and electronics industry. This recommended model also helps to finalize the acceptable cover volume to gain optimal results. It can further be extended to a multi-product inventory system. Another useful future alteration of this model can be to use positive or stochastic lead time. In addition, it can also be modified with multi-step substitution rather than single-step substitution as in this model.

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## REFERENCES

[1] A. Sharma, U. Sharma, and C. Singh, "A robust replenishment model for deteriorating items considering ramp-type demand and inflation under fuzzy environment," *International Journal of Logistics Systems and Management*, vol. 28, no. 3, pp. 287–307, 2017. https://doi.org/10.1504/IJLSM.2017.086944.

[2] A. Sharma, U. Sharma, and C. Singh, "An analysis of replenishment model of deteriorating items with ramp-type demand and trade credit under the learning effect," *International Journal of Procurement Management*, vol. 11, no. 3, pp. 313–342, 2018.

[3] U. K. Khedlekar, A. Nigwal, N. Khedlekar, and H. Patel, "Integrated three layer supply chain inventory model for price sensitive and time dependent demand with suggested retail price by manufacturer," *International Journal of Nonlinear Analysis and Applications*, vol. 12, no. 1, pp. 1135–1152, 2021. https://doi.org/10.22075/ijnaa.2018.14334.1749.

[4] K. Kumar, N. Kumar, *et al.*, "An inventory system for varying decaying medicinal products in healthcare trade," *Yugoslav Journal of Operations Research*, vol. 31, no. 2), pp. 273–283, 2021. http://yujor.fon.bg.ac.rs/index.php/yujor/article/view/899/772.

[5] U. Khedlekar and P. Singh, "Optimal pricing policy for ameliorating items considering time- and price sensitive demand," *International Journal of Modelling and Simulation*, vol. 42, no. 3, pp. 426–440, 2022. 10.1080/02286203.2021.1924042.

[6] U. Khedlekar, L. Kumar, and M. Keswani, "A stochastic inventory model with price-sensitive demand, restricted shortage and promotional efforts," *Yugoslav Journal of Operations Research*, vol. 33, no. 4, pp. 613–642, 2023. http://dx.doi.org/10.2298/YJOR220915010K.

[7] M. Parlar and S. Goyal, "Optimal ordering decisions for two substitutable products with stochastic demands," *Opsearch*, vol. 21, no. 1, pp. 1–15, 1984.

[8] H. Gurnani and Z. Drezner, "Deterministic hierarchical substitution inventory models," *Journal of the Operational Research Society*, vol. 51, no. 1, pp. 129–133, 2000. 10.1057/palgrave.jors.2600843.

[9] Z. Drezner, H. Gurnani, and B. Pasternack, "An eoq model with substitutions between products," *Journal of the Operational Research Society*, vol. 46, no. 7, pp. 887–891, 1995. 10.1057/jors.1995.120.

[10] M. Karakul and L. Chan, "Analytical and managerial implications of integrating product substitutability in the joint pricing and procurement problem," *European Journal of Operational Research*, vol. 190, no. 1, pp. 179–204, 2008. 10.1016/j.ejor.2007.06.026.

[11] M. Nagarajan and S. Rajagopalan, "Inventory models for substitutable products: Optimal policies and heuristics," *Management Science*, vol. 54, no. 8, pp. 1453–1466, 2008. 10.1287/mnsc.1080.0871.

[12] P. Pineyro and O. Viera, "The economic lot-sizing problem with remanufacturing and one-way substitution," *International Journal of Production Economics*, vol. 124, no. 2, pp. 482–488, 2010. 10.1016/j.ijpe.2010.01.007.

[13] W. Liu, S. Song, and C. Wu, "Impact of loss aversion on the newsvendor game with product substitution," *International Journal of Production Economics*, vol. 141, no. 1, pp. 352–359, 2013. 10.1016/j.ijpe.2012.08.017.

[14] A. Mukhopadhyay and A. Goswami, "An inventory model with shortages for imperfect items using substitution of two products," *International Journal of Operational Research*, vol. 30, no. 2, pp. 193–219, 2017. 10.1504/IJOR.2017.086525.

[15] X. Zhang and Y. Gerchak, "Joint lot sizing and inspection policy in an eoq model with random yield," *IIE transactions*, vol. 22, no. 1, pp. 41–47, 1990. 10.1080/07408179008964156.

[16] M. Jaber, S. Goyal, and M. Imran, "Economic production quantity model for items with imperfect quality subject to learning effects," *International Journal of Production Economics*, vol. 115, no. 1, pp. 143–150, 2008. 10.1016/j.ijpe.2008.05.007.

[17] M. Khan, M. Jaber, A. Guiffrida, and S. Zolfaghari, "A review of the extensions of a modified eoq model for imperfect quality items," *International Journal of Production Economics*, vol. 132, no. 1, pp. 1–12, 2011. https://doi.org/10.1016/j.ijpe.2011.03.009.

[18] M. D. Roy, S. S. Sana, and K. Chaudhuri, "An economic order quantity model of imperfect quality items with partial backlogging," *International journal of systems Science*, vol. 42, no. 8, pp. 1409–1419, 2011. https://doi.org/10.1080/00207720903576498.

[19] C. Singh and S. Singh, "Imperfect production process with exponential demand rate, weibull deterioration under inflation," *International Journal of Operational Research*, vol. 12, no. 4, pp. 430–445, 2011. https://doi.org/10.1504/IJOR.2011.043551.

[20] H.-F. Yu, W.-K. Hsu, and W.-J. Chang, "Eoq model where a portion of the defectives can be used as perfect quality," *International Journal of Systems Science*, vol. 43, no. 9, pp. 1689–1698, 2012. https://doi.org/10.1080/00207721.2010.549593.

[21] M. K. Salameh and M. Y. Jaber, "Economic production quantity model for items with imperfect quality," *International journal of production economics*, vol. 64, no. 1-3, pp. 59–64, 2000. https://doi.org/10.1016/S0925-5273(99)00044-4.

[22] B. Maddah and M. Y. Jaber, "Economic order quantity for items with imperfect quality: revisited," *International Journal of Production Economics*, vol. 112, no. 2, pp. 808–815, 2008. https://doi.org/10.1016/j.ijpe.2007.07.003.

[23] A. Sharma, C. Singh, P. Verma, and A. Malik, "Flexible inventory system of imperfect production under deterioration and inflation," *Yugoslav Journal of Operations Research*, vol. 32, no. 4, pp. 515–528, 2022. http://dx.doi.org/10.2298/YJOR220318025S.

## 9. APPENDIX

The expression for total average cost function is given as:

$$\begin{aligned} \text{T.A.C.} = {} & h_1\left[-\frac{a\mu^2}{2T}+\frac{bT^2}{6}+aT-\frac{E[z_1]}{s_1(1-E[z_1])^2}(bT+\frac{a}{T}(2T-\mu))^2\right] \\ & + h_2\left[\frac{a\mu^2}{2T}+\frac{E[z_2]a^2\mu^2}{T(1-E[z_2])^2 s_2}\right]+\frac{c_o}{T}+ac_t\left(1-\frac{\mu}{T}\right) \end{aligned} \tag{21}$$

The convexity of $T.A.C.(\mu, T)$ is proved as following:
$H(\mu,T) = \frac{\partial^2(T.A.C.)}{\partial\mu^2}\frac{\partial^2(T.A.C.)}{\partial T^2} - \frac{\partial^2 T.A.C.}{\partial\mu\partial T} > 0$

$$\begin{aligned} & \frac{\partial^2(T.A.C.)}{\partial T^2} \\ & = \frac{2c_o}{T^3}-\frac{2ac_t\mu}{T}+h_1\left(\frac{b}{3}-\frac{a\mu^2}{T^3}-\frac{2(bT^2+2aT-a(2T-\mu))^2E[z_1]}{^4s_1(1-E[z_1])^2}\right) \\ & -\frac{2(-4aT+2a(2T-\mu))(bT^2+a(2T-\mu))E[z_1]}{s_1(1-E[z_1])^2} \\ & + h_2\left(\frac{a\mu}{T^3}+\frac{2a^2\mu^2E[z_2]}{T^3s_2(1-E[z_2])^2}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} & \frac{\partial^2(T.A.C)}{\partial\mu^2} = h_1\left(-\frac{a}{T}-\frac{2a^2E[z_1]}{T^2s_1(1-E[z_1])^2}\right)+ \\ & h_2\left(\frac{a}{T}+\frac{2a^2E[z_2]}{Ts_2(1-E[z_2])^2}\right) \end{aligned} \tag{23}$$

$$\begin{aligned} & \frac{\partial^2(T.A.C.)}{\partial\mu\partial T} = \frac{ac_t}{T^2} \\ & + h_1\left(\frac{2a(bT^2+2aT-a(2T-\mu))E[z_1]}{T^3s_1(1-E[z_1])^2}-\frac{2a(bT^2+a(2T-\mu))E[z_1]}{T^3s_1(1-E[z_1])^2}\right) \\ & + h_2\left(-\frac{a\mu}{T^2}-\frac{2a^2\mu E[z_2]}{T^2s_2(1-E[z_2])^2}\right) \end{aligned} \tag{24}$$

After calculation and substituting $E[z_1] = E[z_2] = 0$, we obtain:
$H(\mu,T) = a(h_2-h_1)\left(\frac{2c_o}{T^4}-\frac{2ac_t\mu}{T^2}+\frac{b}{3T}+\frac{a\mu^2(h_2-h_1)}{T^4}\right)-\frac{ac_t}{T^2}+\frac{a\mu h_2}{T^2}$
$H(\mu,T) > 0$ is possible if the following condition hold:

$$c_o > \frac{T^2}{2}\left[2ac_t\mu - bT - \frac{a\mu^2(h_2-h_1)}{T^2}+\frac{c_t}{h_2-h_1}-\frac{\mu h_2 T^2}{h_2-h_1}\right] \tag{25}$$

Hence, the convexity of the total average cost function follows.