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Research Article

EFFICIENCY OF MANAGING A STOCHASTIC INVENTORY SYSTEM IN A DECLINING MARKET FOR NON-INSTANTANEOUS DETERIORATING ITEMS UNDER PARTIAL BACKLOGS

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Abstract: In a declining market, companies must adapt their strategies during the mature stage to maintain competitiveness. One of the effective approach to sustaining demand is through targeted promotional efforts, which can extend the products demand in the market. Additionally, implementing an appropriate pricing strategy is essential to ensure products are sold before its start deteriorate or the season ends. This paper explores the role of promotional efforts in shaping pricing policies for non-instantaneous deteriorating items in the declining demand scenario. In the model, we consider the demand as a function of both price and time, incorporating assumptions of partial backlogging for shortages, losts sales, fixed holding costs, and deterioration rate. Using these assumptions, we develop an inventory model that starts with a shortage and ends with zero inventory. The objective is to maximize expected profit per unit time by optimizing both the selling price and replenishment schedules. The innovation of this study lies in integrating promotional efforts with pricing strategies, specifically designed for deteriorating items in a declining market. We present a numerical example to demonstrate the model's effectiveness and conduct a sensitivity analysis to examine its performance under varying conditions. Finally, we provide managerial insights and outline potential directions for future research.

Keywords: Stochastic demand, price and time declining demand, non-instantaneous deterioration, promotional efforts, partial backlogging, lost sales.

MSC: 90B05, 90B30, 90B50.

1. INTRODUCTION

In recent times, managing inventory decisions has become increasingly complex for manufacturers and retailers, particularly when dealing with deteriorating products. Specifically, for items categorized as *non-instantaneous deteriorating products*—goods whose quality declines gradually over time—inventory management presents significant challenges. This phenomenon, explored by Keswani & Khedlekar [1], Shah et al. [2] are crucial for developing effective inventory models. Such items include electronic components, medicines, foodstuffs, fruits, and vegetables, which maintain their quality for a period before beginning to deteriorate. Given that many businesses face this issue, incorporating this phenomenon into inventory models is essential for accurately reflecting real-world scenarios. The selling price is a key factor influencing product demand, with lower prices generally leading to increased demand. Keswani & Khedlekar [3] have tackled this issue by developing an inventory model that determines optimal pricing and lot sizes for resellers, accounting for price-dependent demand, time-dependent deterioration, and partial backlogging. In today's competitive global markets, where rapid advancements in information technology and internet connectivity drive the landscape, effective pricing policies are crucial for sellers to craft profitable strategies.

Despite the wealth of research, as highlighted in the literature review and Table 1, a comprehensive model focusing specifically on non-instantaneous deteriorating items—one that incorporates price, promotional efforts, time-dependent demand, and backlogging rates—remains lacking. This article addresses this gap by proposing a model tailored for managing non-instantaneous deteriorating inventory. Our model integrates pricing and replenishment policies with the goal of maximizing profits. We define criteria for identifying optimal decision variables and present a detailed algorithm to efficiently determine the best pricing and inventory strategies.

Table 1: Gaps and Contributions of the Past Studies Related to Present Study

Authors	Det.	SD	PE	PDD	TDD	Dec.D
Keswani & Khedlekar [3]	✓		✓	✓	✓	
Hollier & Mak [4]	✓				✓	✓
Jalan & Chaudhuri [5]	✓				✓	✓
Wee [6]	✓			✓	✓	✓
Zhang *et al.* [7]		✓		✓		
He *et al.* [8]		✓	✓	✓		
Maihami & Karimi [9]	✓	✓	✓			
Singh *et al.* [10]	✓	✓		✓		
Chen *et al.* [11]		✓	✓			
Shah *et al.* [12]	✓			✓	✓	
Soni & Chauhan [13]	✓	✓	✓		✓	
Rastogi & Singh [14]	✓			✓		
Rapolu & Kandpal [15]	✓	✓	✓	✓		
This model (present study)	✓	✓	✓	✓	✓	✓

Note: Det. = Deterioration, SD = Stochastic Demand, PE = Promotional Efforts, PDD = Price Dependent Demand, TDD = Time Dependent Demand, Dec.D = Declining Demand

2. LITERATURE SURVEY

Inventory management is a technique of maintaining stock at the desired level, encompassing various aspects such as product demand, storage capacity, maintenance, inventory decisions, strategies to stay competitive, promotional policies, quality maintenance, business reputation, and financial growth. Inventory managers strive to increase demand and maintain profitability even in declining markets. To address these challenges, mathematical models are needed to determine how much and when to order, considering restrictions on production, storage, time, and money. Inventory modeling involves developing optimal policies based on realistic parameters and results. Since the initial development of the classical Economic Order Quantity (EOQ) model, numerous researchers have extended it to represent practical scenarios more accurately. In real markets, decay or deterioration is a natural phenomenon affecting products such as vegetables, fruits, foods, perfumes, chemicals, pharmaceuticals, radioactive substances, and electronic equipment. This deterioration is crucial in inventory management. Subsequent researchers, including Keswani [16] incorporated non-instantaneous deterioration into their models. Recently, Rapolu [15] proposed an inventory model with different non-instantaneous deterioration rates, price, advertisement-dependent demand under trade credit.

Declining markets introduce substantial challenges for inventory management, particularly as markets transition from periods of maturity to declining sales. This decline is driven by both internal and external factors that contribute to reduced demand. Mak and Hollier [4] were pioneers in addressing this issue, proposing replenishment policies for deteriorating items in markets where demand diminishes exponentially and deterioration occurs at a constant rate. Building on their work, researchers like Wee & Hui-Ming [6], Jalan & Chaudhuri [5], Mathur & Dwivedi [17], have further explored how declining demand interacts with other influencing factors. This study aims to achieve optimal results by considering the realities of declining demand in the context of modern challenges such as pandemics, global conflicts, and climate change. Demand plays a pivotal role in driving production and ordering decisions, making it crucial for the success of businesses Rastogi [14]. Whether demand is constant, deterministic, or stochastic, it can be shaped by a range of factors, including pricing, inventory levels, timing, promotions, advertising, and seasonal variations. Recent studies by Jose et al. [18], Saha et al. [19], and Bhavani & Mahapatra [20] have delved into the effects of these variables on demand. Traditional inventory models often simplify demand as predictable and constant, but real-world demand is far more complex and uncertain, necessitating its treatment as a random variable. The classic "newsvendor"problem, which deals with random demand for perishable goods, is a well-known example of this challenge. Researchers such as Soni & Chauhan [13], Soni & Suthar [21], and Shah et al. [12] have made significant strides in enhancing these models, integrating both price-sensitive and time-dependent factors to create more realistic and practical approaches for managing inventory in today's fluctuating markets.

In today's volatile market, shortages have become a critical concern for researchers and businesses alike. Shortages occur when demand cannot be fulfilled immediately, leading to unmet customer needs. Impatient customers may choose to shop elsewhere, resulting in order cancellations and significant financial losses. Traditionally, many inventory models have operated under the assumption that "shortages are permitted and completely backlogged." Two primary inventory policies have been developed to address shortages: Inventory Followed by Shortages (IFS) and Shortages Followed by Inventory (SFI). Initially, most models relied on the IFS policy, but the SFI policy, which starts each cycle with a shortage and ends with zero inventory, has gained prominence. Goyal *et al.* [22] were the first to demonstrate that the SFI policy could reduce inventory costs compared to the IFS policy. Since then, inventory models by researchers like Shaikh *et al.* [23] have increasingly adopted the SFI replenishment approach.

Another essential aspect of inventory management is promotional efforts. Promotions such as gifts, price discounts, special displays, extended payment terms, and advertising can significantly influence customer purchasing behavior. For instance, Maihami & Kamalabadi [24] developed an inventory model for non-instantaneous deteriorating items that incorporates promotional efforts, price-sensitive stochastic demand, and partially backlogged shortages. Similarly, Singh & Rathore [10] presented a model for deteriorating items that accounts for production reliability and stochastic demand, both with and without shortages. Chen *et al.* [11] tackled optimization challenges related to inventory replenishment, production, and promotion under the risks of production disruptions and stochastic demand. Furthermore, Soni & Suthar [21] formulated a profit-maximization inventory model for deteriorating items by incorporating both stochastic price-sensitive demand and promotional activities. Shah *et al.* [12] explored the impact of supplier discounts for bulk orders. Other researchers, such as Dash [25], Jaggi *et al.* [26] and Palanivel *et al.* [27] have developed models that integrate pricing, promotion, and inventory control to enable more effective decision-making.

Recent studies have yet to fully integrate contemporary challenges such as economic volatility, global supply chain disruptions, and the rise of advanced predictive analytics. There is also a pressing need to incorporate multi-factor demand influences and sophisticated promotional strategies into inventory models, which could lead to more resilient and adaptive solutions. The literature review underscores the necessity for more comprehensive and up-to-date research, particularly in areas like declining markets, stochastic demand, shortages, and promotional efforts. Addressing these aspects is essential for developing robust inventory management strategies that can withstand unexpected market fluctuations. Table 1 outlines the specific research gaps identified in the current study, reinforcing the call for more holistic approaches in future investigations.

3. RESEARCH GAP AND RELEVANCE OF THE PRESENT STUDY

Table 1 highlights the distinctive features of this study in contrast to previous research. The existing literature identifies significant gaps in the analysis of stochastic inventory models for non-instantaneous deteriorating items. This study seeks to bridge these gaps by proposing a robust inventory model that integrates joint promotional cost-sharing policies. The following research gaps have been identified and addressed in this work:

1) Previous models often assume constant or deterministic demand rates, which fail to capture the unpredictable nature of real-world demand. Stochastic demand, which introduces variability and uncertainty, has been infrequently studied. Our model addresses this by incorporating stochastic demand, making it more applicable to practical scenarios where demand fluctuates.
2) Many decision-making models assume that unmet demand is fully backlogged, an idealized scenario not reflective of real business environments. In practice, some unmet demand results in lost sales, while other customers may be willing to wait, leading to partial backlogging. Our model provides a more realistic representation by incorporating both partial backlogging and lost sales.
3) While numerous studies have examined inventory management for deteriorating items, none have integrated critical factors such as price and time sensitive demand, promotional efforts, partial backlogs (shortages), and lost sales in a comprehensive manner. Additionally, the combined impact of these factors on maximizing profit through promotional efforts in a declining market has been rarely explored. Our model uniquely integrates all these elements, offering a holistic approach to inventory management.

3.1. Novel Contributions

Addressing the challenges of a declining market and minimizing losses from decreased demand are crucial for sustaining products. Our proposed model tackles these issues by developing an inventory model for non-instantaneous deteriorating items with price and time dependent demand which is stochastic in nature, incorporating partial backlogging and lost sales. The novelty of our approach is rooted in its comprehensive and integrated framework, which includes the following innovative features:

a) The model incorporates demand that varies with both price and time, offering a more realistic reflection of dynamic market conditions.
b) Unlike traditional models that assume fixed demand rates, our model employs stochastic (random) demand rates to better capture market volatility and unpredictability.
c) By accounting for partial backlogging based on customer waiting time and considering lost sales, our model presents a more accurate representation of real-world scenarios and customer behavior.

d) The impact of promotional efforts on demand is integrated into the model, aiding in the optimization of marketing strategies and enhancing profitability. This aspect is especially crucial for maintaining product relevance and stimulating demand in a declining market.

Additionally, we explore the influence of promotional efforts on stochastic demand in a declining market and perform a sensitivity analysis to examine how these efforts vary with different parameters. Our model provides efficient solutions for determining optimal selling prices and replenishment strategies. We utilize graphical analysis to demonstrate the concavity of the profit function, ensuring the reliability of our results.

The proposed model aids retailers and manufacturers by addressing key questions such as:

1. How much inventory should be ordered, and for how long should it be held to minimize costs and maximize profits?
2. How can demand be accelerated in a declining market through strategic decisions and promotional efforts?
3. What pricing policy can enhance profits even in a declining market environment?
4. What is the impact of with and without promotional efforts on the total profit of retailers and manufacturers, and how can these efforts be optimized?

This study stands out for its comprehensive approach, integrating stochastic demand, partial backlogging, lost sales, and promotional efforts into a single inventory model for deteriorating items. By addressing these factors, our model offers a robust tool for improving profitability and decision-making in challenging market conditions.

4. NOTATIONS AND HYPOTHESES OF THE MODEL

4.1. Notations

Following notations are used throughout this paper.

Parameters

O_a : Ordering cost per order,
c_p : Purchase cost per unit,
c_h : Holding cost per unit per time period,
c_s : Backorder cost per unit per time period,
c_1 : Cost of lost sales per unit,
c_d : Deterioration cost per unit over time,
$I(t)$: Inventory level at time t,
ε : Random variable representing uncertainty, with $E(\varepsilon) = \mu$,
ρ : Promotional efforts (e.g., advertising), where $\rho \geq 1$,
$*$: Denotes optimal value,
a_m : Market potential, where $a > b, c$,
b : Demand sensitivity to price,
c : Demand sensitivity to time,
θ : Deterioration rate, $0 \leq \theta < 1$,
δ_b : Backlogging rate.

Decision Variables

t_b : Time period allowing shortages, where $0 \leq t \leq t_b$,
t_r : Time when inventory is depleted post-replenishment,
s_p : Selling price per unit.

Additional Variables

π : Total profit per time period for the inventory system,
π_{avg} : Optimal average profit per time period,
$I_{short}(t)$: Inventory level during shortages at time t, where $0 \leq t \leq t_b$,
$I_{hold}(t)$: On-hand inventory level at time t, where $t_b \leq t \leq t_b + t_r$,
$D(s_p,t)$: Demand function dependent on price and time, which decreases abruptly,
q : Total order quantity per cycle, with q_b as backordered quantity and q_r as replenished quantity.

4.2. Hypotheses

To mathematically formulate the model, we propose the following hypotheses:

a) The inventory model considers a single item that experiences non-instantaneous deterioration.
b) The product is abundantly available in the competitive market, implying an infinite replenishment rate with zero lead time.
c) The stochastic demand function, $D(s_p,t) + E(\varepsilon)$, is continuous and depends on both price and time. Here, $D(s_p,t)$ is expressed as $D(s_p,t) = a_m - bs_p - ct$, where $a_m > 0$, $b \neq 0$, $c \neq 0$, $s_p > 0$, and ε is a non-negative continuous random variable with $E(\varepsilon) = \mu$. The parameters a_m, b, and c represent potential market demand, price sensitivity, and time sensitivity, respectively.
d) The distribution of the random variable ε is fixed and independent of time, with the parameters of the demand function remaining constant over the time horizon.
e) The cost of promotional effort, denoted as $PC(\rho, D(s_p,t))$, increases with both the effort and the basic demand. This cost is given by:

$$PC = K(\rho - 1)^2 \left[\int_0^1 (D(s_p,t) + \varepsilon)\, dt \right]^\eta$$

where $K > 0$ and η is a constant. This formulation, adopted from Soni & Chauhan [13] and Maihami & Karimi [9] captures the relationship between promotional effort, market demand, and promotional costs. Notably, $\rho = 1$ represents a scenario with no promotional policy, as indicated in previous studies.

f) The demand increases with the introduction of a promotional factor, expressed as $\rho(D(s_p,t) + \varepsilon)$, where $\rho \geq 1$.
g) The item deteriorates at a constant rate θ, where $0 \leq \theta < 1$.

h) During periods of shortage, a portion of the unmet demand is back-ordered, while the rest is lost or partially backlogged. The backlogging rate is given by:

$$\beta(t) = \begin{cases} e^{-\delta_b t} & \text{if } \delta_b > 0 \\ 1 & \text{if } \delta_b = 0 \end{cases} \tag{1}$$

where $0 < \delta_b < 1$ is the backlogging parameter, and t represents the waiting time until the next replenishment.

5. MATHEMATICAL MODELLING AND ANALYSIS

Consider a scenario where the business experiences an initial shortage during the interval $[0, t_b]$, which is partially backlogged. At time t_b, the backlogged demand q_b is immediately fulfilled through a replenishment of quantity q, while the remaining demand q_r is satisfied over the interval $[t_b, t_r]$, leading to a complete depletion of inventory by time t_r. Following this, the inventory decreases due to price- and time-dependent stochastic demand as well as item deterioration. To address the declining demand affected by price fluctuations and uncertainty, a promotional effort parameter ρ is introduced to influence demand over time. Figure 1 illustrates the inventory depletion process, comparing the effect of promotional efforts during the interval (t_b, t_r) with the scenario without promotional efforts during the interval (t_b, t_r'). Both intervals demonstrate the interaction between stochastic demand and reduced deterioration.

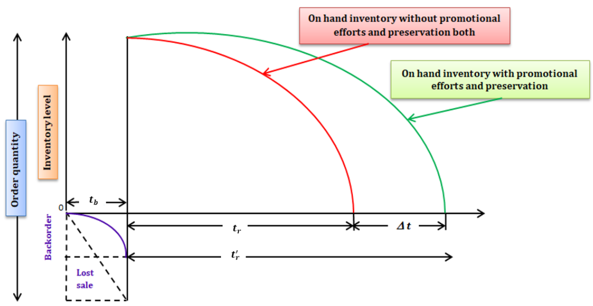

Figure 1: Graphical representation of inventory system

Based on the scenario described, the state of the shortage inventory at time t within the interval $[0, t_b]$ can be represented by the following differential equation:

$$\frac{dI_{short}(t)}{dt} = -\rho\,(D(s_p, t) + \varepsilon)\,\beta(t_b - t), \quad 0 \le t \le t_b \tag{2}$$

with the initial condition $I_{short}(0) = 0$.

The solution to the above differential equation is given by:

$$I_{short}(t) = -\rho \int_0^t (D(s_p, t) + \varepsilon)\,\beta(t_b - t)\,dt \tag{3}$$

Next, the status of the on-hand inventory at any time t within the interval $[0,t_r]$ is governed by the following differential equation:

$$\frac{dI_{hold}(t)}{dt} = -\theta I_{hold}(t) - \rho\left(D(s_p,t)+\varepsilon\right), \quad 0 \le t \le t_r \tag{4}$$

with the boundary condition $I_{hold}(t_r) = 0$.

The solution to this differential equation is:

$$I_{hold}(t) = \rho e^{-g(t)} \left(\int_0^{t_r} (D(s_p,t)+\varepsilon)\, e^{g(x)}\, dx\right) \tag{5}$$

where $g(t) = \int_0^t \theta\, dt$.

Consequently, the loss in sales at any time t is calculated as:

$$I_{lost}(t) = \rho\left(D(s_p,t)+\varepsilon\right)(1-\beta(t_b-t)), \quad 0 \le t \le t_b \tag{6}$$

Based on these inventory levels, we can now calculate the various inventory-related costs and revenue per cycle. These include the following:

- **Total Quantity:** The total replenishment size (including backlog) is given by:

$$\begin{aligned} q =& I_{hold}(t) - I_{short}(1) \\ =& \rho\left(D(s_p,t)+\varepsilon\right) \\ & \times \left\{ e^{g(0)} \int_0^{t_r} e^{g(x)}\, dx + \int_0^{t_b} \beta(t_b-t)\, dt \right\} \end{aligned} \tag{7}$$

- **Lost Sale Cost:** The expected cost of lost sales during the interval $[0,t_b]$ is:

$$\begin{aligned} TLC =& E\left(c_l \int_0^{t_b} I_{lost}(t)\, dt\right) \\ =& c_1 \rho \int_0^{t_b} (D(s_p,x)+\mu) \\ & \times [1-\beta(t_b-x)]\, dx \end{aligned} \tag{8}$$

- **Shortage Cost:** The expected cost of stock-outs and backlogs over the interval $[0,t_b]$ is:

$$\begin{aligned} TSC =& E\left(c_s \int_0^{t_b} [-I_{short}(t)]\, dt\right) \\ =& c_s \rho \left[\int_0^{t_b} \left\{ \int_0^{t} (D(s_p,x)+\mu)\, \beta(t_b-x)\, dx \right\} dt\right] \end{aligned} \tag{9}$$

- **Holding Cost:** The expected holding cost during the interval $[0,t_r]$ is:

$$\begin{aligned} TIC =& E\left(c_h \int_0^{t_r} I_{hold}(t)\, dt\right) \\ =& c_h \rho \left[\int_0^{t_r} e^{g(x)} \left\{ \int_t^{t_r} (D(s_p,x)+\mu)\, e^{g(x)}\, dx \right\} dt\right] \end{aligned} \tag{10}$$

- **Purchasing Cost:** The expected purchasing cost is:

$$
\begin{aligned}
TPC = E(c_0 q) &= c_0 \rho e^{g(0)} \left\{ \int_0^{t_r} (D(s_p, x) + \mu)\, e^{g(x)}\, dx \right. \\
&\quad \left. + \int_0^{t_b} \beta (t_b - t)\, (D(s_p, t) + \mu)\, dt \right\}
\end{aligned} \tag{11}
$$

- **Sales Revenue:** The expected revenue from sales is:

$$
\begin{aligned}
TRV &= E\left(s_p \rho \int_0^{t_r} (D(s_p, x) + \varepsilon) dx - I_{short}(t_b) \right) \\
&= s_p \rho \left[\int_0^{t_r} (D(s_p, x) + \mu) dx \right. \\
&\quad \left. + \int_0^{t_b} (D(s_p, x) + \mu) \beta_b (t_b - x) dx \right]
\end{aligned} \tag{12}
$$

- **Promotional Investments:** The expected cost of promotional efforts is:

$$
\begin{aligned}
PC &= E\left(K(\rho - 1)^2 \left[\int_0^{t_b + t_r} (D(s_p, t) + \varepsilon)\, dt \right]^{\eta} \right) \\
&= K \left[(t_b + t_r)(D(s_p, t) + \mu) \right]^{\eta} (-1 + \rho)^2
\end{aligned} \tag{13}
$$

- **Deterioration Cost:** The expected cost due to deterioration during the shortage period is:

$$
\begin{aligned}
TDC &= E\left(c_d \int_0^{t_r} \theta I_{hold}(t)\, dt \right) \\
&= \rho c_d \theta \int_0^{t_r} e^{g(t)} \left\{ \int_t^{t_r} (D(s_p, x) + \mu)\, e^{(1 - m(\xi)) g(x)}\, dx \right\} dt
\end{aligned} \tag{14}
$$

6. SOLUTION PROCEDURE

6.1. Total Profit Function

When all of the cost and profit elements are added together, the total integrated profit

$$
\begin{aligned}
\pi(t_b, t_r, s_p) &= TRV \\
&\quad - (O_a + TSC + TLC + TIC + TPC + PC + TDC)
\end{aligned} \tag{15}
$$

$$\begin{aligned}
\pi(t_b,t_r,s_p) \quad = \quad & s_p\rho\left\{\int_0^{t_r}(D(s_p,x)+\mu)dx\right. \\
& \left.+\int_0^{t_b}(D(s_p,x)+\mu)\beta(t_b-x)dx\right\} \\
& -\left(O_a+c_s\rho\left[\int_0^{t_b}\left\{\int_0^{t}(D(s_p,x)+\mu)\beta(t_b-x)dx\right\}dt\right]\right. \\
& +c_1\rho\int_0^{t_b}(D(s_p,x)+\mu)\left(1-\beta_b(t_b-x)\right)dx \\
& +C_h\rho\left[\int_0^{t_r}e^{g(x)}\left\{\int_t^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right\}dt\right] \\
& +C_0\rho e^{g(0)}\left\{\int_0^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right. \\
& \left.\left.+\int_0^{t_b}\beta(t_b-x)(D(s_p,x)+\mu)dx\right\}\right) \\
& +K(\rho-1)^2\left[\int_0^{(t_b+t_r)}(D(s_p,t)+\mu)dt\right]^{\eta} \\
& +\rho C_d\theta\int_0^{t_r}e^{g(t)}\left(\int_t^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right)dt \qquad (16)
\end{aligned}$$

Therefore, the total average profit is

$$\pi_{avg}(t_b,t_r,s_p)=\frac{\pi(t_b,t_r,s_p)}{t_b+t_r} \qquad (17)$$

6.2. Optimization of the proposed inventory model

To optimize the proposed inventory model, we utilize a classical optimization approach to analytically solve the mathematical formulation. The decision variables namely s_p, t_b, and t_r are optimized through a continuous optimization technique. Since the problem involves multiple decision variables, the Hessian matrix is employed to verify the optimality of the objective function.

Thus, the optimization problem tackled in this study is formulated as:

$$\max_{t_b,t_r,s_p}\pi_{avg}(t_b,t_r,s_p) \qquad (18)$$

subject to $c_p\leq s_p$, and $t_b,t_r\geq 0$

The problem can be solved in two phases,

First Phase: To maximize the objective function with respect to (t_b,t_r),

$$\max_{t_b,t_r}\pi_{avg}(t_b,t_r,s_p) \qquad (19)$$

Second Phase: To maximize with respect to s_p, that is

$$\max_{s_p}\left\{\max_{t_b,t_r}\pi_{avg}(t_b,t_r,s_p)\right\}. \tag{20}$$

subject to $c_p \leq s_p$, and $t_b,t_r \geq 0$

That is, first with regard to (t_b,t_r), then with respect to s_p.

Now, fix s_p, we have the first partial derivatives of π_{avg} with respect to t_b and t_r are

$$\frac{\partial \pi_{avg}(t_b,t_r)}{\partial t_b} = -\frac{\pi(t_b,t_r)}{(t_b+t_r)^2} + \frac{1}{t_b+t_r}\left(\frac{\partial \pi(t_b,t_r)}{\partial t_b}\right) \tag{21}$$

$$\frac{\partial \pi_{avg}(t_b,t_r)}{\partial t_r} = -\frac{\pi(t_b,t_r)}{(t_b+t_r)^2} + \frac{1}{t_b+t_r}\left(\frac{\partial \pi(t_b,t_r)}{\partial t_r}\right) \tag{22}$$

And, the second partial derivatives of π_{avg} with respect to t_b and t_r are

$$\frac{\partial^2 \pi_{avg}(t_b,t_r)}{\partial t_b^2} = \frac{2(t_b,t_r)}{(t_b+t_r)^3} - \frac{2}{(t_b+t_r)^2}\frac{\partial \pi(t_b,t_r)}{\partial t_b} + \frac{1}{t_b+t_r}\left(\frac{\partial^2 \pi(t_b,t_r)}{\partial t_b^2}\right) \tag{23}$$

$$\frac{\partial^2 \pi_{avg}(t_b,t_r)}{\partial t_r^2} = \frac{2\pi(t_b,t_r)}{(t_b+t_r)^3} - \frac{2}{(t_b+t_r)^2}\frac{\partial \pi(t_b,t_r)}{\partial t_r} + \frac{1}{t_b+t_r}\left(\frac{\partial^2 \pi(t_b,t_r)}{\partial t_r^2}\right) \tag{24}$$

For the optimum value of $\pi_{avg}(t_b,t_r)$, we have the necessary condition

$$\frac{\partial \pi_{avg}(t_b,t_r)}{\partial t_b} = 0 \tag{25}$$

$$\frac{\partial \pi_{avg}(t_b,t_r)}{\partial t_r} = 0 \tag{26}$$

and we obtain,

$$\pi(t_b,t_r) = (t_b+t_r)\frac{\partial \pi(t_b,t_r)}{\partial t_b} \tag{27}$$

$$\pi(t_b,t_r) = (t_b+t_r)\frac{\partial \pi(t_b,t_r)}{\partial t_r} \tag{28}$$

From above two equations, we have

$$\frac{\partial \pi(t_b,t_r)}{\partial t_b} = \frac{\partial \pi(t_b,t_r)}{\partial t_r} \tag{29}$$

Next, we find the first-order partial derivative of $\pi(t_b,t_r)$ with respect to t_b is

$$\begin{aligned}\frac{\partial \pi (t_b,t_r)}{\partial t_b} &= \rho(s_p+c_l-c_0)\{(a_m-bp-ct_b+\mu)\\&+ (a_m-bp+\mu)(1-e^{-t_b\delta_b})\\&-c\left(t_b+\frac{1-e^{-t_b\delta_b}}{\delta_b^2}\right)\Bigg\}\\&-c_s\rho\left\{\frac{c(1-e^{-t_b\delta})}{\delta_b}+t_be^{-t_b\delta_b}\left(a_m-bs_p+\mu+\frac{c}{\delta}\right)\right\}\\&-C_l(a_m-bs_p-ct_b+\mu)\\&+\eta cK(\rho-1)^2(t_b+t_r)(a_m-bs_p+\mu)\\&\times\left[\frac{-1}{2}c(t_b+t_r)^2+(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta-1}\end{aligned}\tag{30}$$

The first-order partial derivative of $\pi(t_b,t_r)$ with respect to t_r is

$$\begin{aligned}\frac{\partial \pi (t_b,t_r)}{\partial t_r} &= \rho s_p(a_m-bs_p-ct_r+\mu)\\&-(c_h+\theta_bc_d)\int_0^{t_r}(a_m-bs_p-ct_r+\mu)e^{g(t_r)-g(t)}dt\\&-c_0\rho(a_m-bs_p-ct_r+\mu)e^{g(t_r)}\\&+c\eta K(t_b+t_r)(\rho-1)^2(a_m-bs_p+\mu)\\&\times\left[\frac{-1}{2}c(t_b+t_r)^2+(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta-1}\end{aligned}\tag{31}$$

putting the values of $\frac{\partial\pi(t_b,t_r)}{\partial t_b}$ *and* $\frac{\partial\pi(t_b,t_r)}{\partial t_r}$ in equation $\frac{\partial\pi(t_b,t_r)}{\partial t_b}=\frac{\partial\pi(t_b,t_r)}{\partial t_r}$, we obtained

$$\begin{aligned}&(s_p-c_0+c_l)\Big\{(a_m-bs_p-ct_b+\mu)+(a_m-bs_p+\mu)(1-e^{-t_b\delta_b})\\&-c\left(t_b+\frac{1-e^{-t_b\delta_b}}{\delta^2}\right)\Bigg\}\\&-c_s\rho\left\{\frac{c}{\delta}(1-e^{-t_b\delta})+t_be^{-t_b\delta}\left(a_m-bs_p+\mu+\frac{c}{\delta}\right)\right\}\\&-c_l(a_m-bs_p-ct_b+\mu)\\&=s_p(a_m-bs_p-ct_2+\mu)\\&-(c_h+\theta c_d)\int_0^{t_r}(a_m-bs_p-ct_2+\mu)e^{g(t_r)-g(t)}dt\\&-c_0\rho e^{g(t_r)}(a_m-bs_p-ct_2+\mu)\end{aligned}\tag{32}$$

we assumed that the left hand side of above equation is $W(s_p,t_b)$ and right hand side is $\varphi(t_r)$.

Putting values in equations (20) to (22) and we get,

$e^{-t_b\delta}\ W(s_p,t_b) = \varphi(t_r)$

From above expression we obtain the value of t_b

$t_b = \frac{1}{\delta} ln \frac{W(s_p,t_b)}{\varphi(t_r)}$

For convenience, we take

$$\frac{O_a}{\rho} - \varphi(t_r)(t_b+t_r) - K(\rho-1)^2\left[\frac{1}{2}c(t_b+t_r)^2-(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta-1}$$
$$+c(t_b+t_r)(a_m-bs_p+\mu)+(c_h+\theta c_d)\left[\int_0^{t_r} e^{g(t)}\left\{\int_t^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right\}dt\right]$$
$$+c_0\rho\left\{\int_0^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx+\int_0^{t_b}\beta(t_b-x)(D(s_p,x)+\mu)dx\right\}+\frac{W(s_p,t_b)}{\delta_b}=0$$

We suppose an auxiliary function above equation, say $V(t_r)$, $t_r \in [0,\infty)$, where

$$V(t_r) = \frac{O_a}{\rho} - \varphi(t_r)(t_b+t_r) + \frac{K(\rho-1)^2\left[c(t_b+t_r)^2-2(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta}}{2\rho}$$
$$\left(1-\frac{2\eta\left[c(t_b+t_r)^2-(a_m-bs_p+\mu)(t_b+t_r)\right]}{c(t_b+t_r)^2-2(t_b+t_r)(a_m-bs_p+\mu)}\right)+U(s_p,t_b)+$$
$$(c_h+\theta c_d)\left[\int_0^{t_r} e^{g(t)}\left\{\int_t^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right\}dt\right]$$
$$+c_0\rho\left\{\int_0^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx+\int_0^{t_b}\beta(t_b-x)(D(s_p,x)+\mu)dx\right\} \quad (33)$$

Differentiating $V(t_r)$ with respect to t_r, we get

$$\frac{dV(t_r)}{dt_r} = -\varphi(t_r)-\varphi'(t_r)(t_b+t_r)K\left[\frac{1}{2}c(t_b+t_r)^2-(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta}$$
$$\left(\frac{4\eta(t_b+t_r)(a_m-bs_p-c(t_b+t_r)+\mu)^2}{\left[c(t_b+t_r)^2-(t_b+t_r)(a_m-bs_p+\mu)\right]^2}+\frac{2c\eta(t_b+t_r)}{c(t_b+t_r)^2-(t_b+t_r)(a_m-bs_p+\mu)}\right)$$
$$-K\left[\frac{1}{2}c(t_b+t_r)^2+(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta}\left[\frac{2\eta(a_m-bs_p-c(t_b+t_r)+\mu)}{-c(t_b+t_r)^2+(t_b+t_r)(a_m-bs_p+\mu)}\right]^{\eta-1}$$
$$-K(\rho-1)^2\eta(a_m-bs_p-c(t_b+t_r)+\mu)\left[\frac{-1}{2}c(t_b+t_r)^2+(t_b+t_r)(a_m-bs_p+\mu)\right]^{\eta-1}$$
$$\left(1+\frac{(t_b+t_r)\eta(a_m-bs_p-c(t_b+t_r)+\mu)}{\frac{1}{2}c(t_b+t_r)^2-(t_b+t_r)(a_m-bs_p+\mu)}\right)-C_0\rho(a_m-bs_p-ct_r+\mu)e^{g(t_r)}$$
$$-(c_h+\theta C_d)\int_0^{t_r}(a_m-bs_p-ct_r+\mu)e^{g(t_r)-g(t)}dt<0$$

Thus, $V(t_r)$ is strictly decreasing function of $t_r \in [0,\infty)$.

$$\delta_b s_p = \frac{O_a}{\rho} - (a_m - bs_p + \mu)(\rho - c_0)\left(\frac{1}{\delta_b}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right) - \frac{K(\rho-1)^2}{\rho} \left[\frac{-1}{2}c\left(\frac{1}{\delta_b}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right)^2 + \left(\frac{1}{\delta_b}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right)(a_m - bs_p + \mu)\right]^{\eta-1} \left[1 - \frac{\eta(-c\left(\frac{1}{\delta_b}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right) + (a_m - bs_p + \mu)}{\frac{-1}{2}c\left(\frac{1}{\delta}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right)^2 + \left(\frac{1}{\delta}\ln\frac{W(s_p,t_b)}{\varphi(t_r)}\right)(a_m - bs_p + \mu)}\right] t_b + \frac{W(s_p,t_b)}{\delta_b} \tag{34}$$

Lemma 1. *For, fix value of s_p the following inequality holds:*

(i) If $\delta_b s_p > 0$, then the unique pair of (t_b^o, t_r^o) exists which satisfy condition $\frac{\partial \pi_{avg}}{\partial t_b} = \frac{\partial \pi_{avg}}{\partial t_r}$

(ii) If $\delta_b s_p \leq 0$, then the optimal value of (t_b^*, t_r^*) obtain at fix point

$$(t_b^*, t_r^*) = \left(\frac{1}{\delta_b}\ln\frac{W(s_p,t_b)}{(s_p - c_0)(a_m - bs_p + \mu)}, 0\right) \quad \text{if } \delta_b s_p \leq 0.$$

6.3. Proof of Optimality of Profit function

Theorem 1. *For a known value of s_p, the total profit function $\pi_{avg}(t_b, t_r, s_p)$ is concave and its global maxima attains at the point (t_b^*, t_r^*)*

Proof. Refer by Lemma 1

$$(t_b^*, t_r^*) = \begin{cases} (t_b^o, t_r^o) & \text{if } \delta_b s_p > 0 \\ \left(\frac{1}{\delta} ln \frac{W(s_p,t_b)}{((s_p - C_0)(a_m - bs_p + \mu)}, 0\right) & \text{if } \delta_b s_p \leq 0 \end{cases} \tag{35}$$

Taking optimal value of $(t_b, t_r) = (t_b^*, t_r^*)$ for differentiation of $\frac{\partial \pi}{\partial t_b}$ with respect to t_b and t_r, to get $\frac{\partial^2 \pi}{\partial t_b^2}$, $\frac{\partial^2 \pi}{\partial t_b t_r}$ and $\frac{\partial \pi}{\partial t_r}$, with respect to t_r, to get $\frac{\partial^2 \pi}{\partial t_r^2}$. Taking optimal value of $(t_b, t_r) = (t_b^*, t_r^*)$ for differentiation of $\frac{\partial \pi}{\partial t_b}$ with respect to t_b and t_r, to get $\frac{\partial^2 \pi}{\partial t_b^2}$, $\frac{\partial^2 \pi}{\partial t_b t_r}$ and $\frac{\partial \pi}{\partial t_r}$, with respect to t_r, to get $\frac{\partial^2 \pi}{\partial t_r^2}$.

$$\frac{\partial^2 \pi}{\partial t_b^2} = -\Bigg[\rho(s_p - C_0 + C_l)\left(-2c - \frac{(a - bs_p + \mu)e^{-t_b^*\delta_b}}{\delta_b} + \frac{e^{-t_b^*\delta_b}}{\delta^3}\right)$$
$$+ C_s\rho\left(\frac{e^{-t_b^*\delta_b}(c - t_b^*\delta_b + 1)}{\delta^2}\right) - c_l c$$
$$-cK(-1+\rho)^2\eta\left[\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a_m - bs_p + \mu)\right]^{\eta-1} \tag{36}$$
$$+K(\eta-1)(a_m - bs_p - c(t_b^* + t_r^*) + \mu)^2(\rho-1)^2\eta$$
$$\left[\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a_m - bs_p + \mu)\right]^{\eta-2}\Bigg]$$

$$\frac{\partial^2 \pi}{\partial t_b \partial t_r} = -\big[3cK(\eta-1)(a_m - bs_p - c(t_b^* + t_r^*) + \mu)(-1+\rho)^2\eta$$
$$\times\left(\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a - bs_p + \mu)\right)^{\eta-2}$$
$$+K(\eta-2)(\eta-1)(a_m - bs_p - c(t_b^* + t_r^*) + \mu)^3$$
$$\times\left(\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a_m - bs_p + \mu)\right)^{\eta-3} \tag{37}$$
$$+K(\eta-1)(\rho-1)^2\eta\left(\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a_m - bs_p + \mu)\right)^2$$
$$+(a_m - bs_p - c(t_b^* + t_r^*) + \mu)^3(\rho-1)^2\eta$$
$$\times\left(\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a_m - bs_p + \mu)\right)^{\eta-2}\Big]$$

$$\frac{\partial^2 \pi}{\partial t_r^2} = -[s_p\rho\,(c + (c_h + c_d)\,[(a_m - bs_p - ct_r^* + \mu)$$
$$+\int_0^{t_r^*} e^{g(t_r^*) - g(t)}\frac{\partial g(t_r^*)}{\partial t_r}\,dt\Bigg] + c_0\rho\,[(a_m - bs_p - ct_r^* + \mu)$$
$$\cdot e^{g(t_r^*)}\frac{\partial g(t_r^*)}{\partial t_r^*} - ce^{g(t_r^*)}\Bigg]\Bigg)$$
$$-K(\rho-1)^2\eta\left[\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a - bs_p + \mu)\right]^{\eta-1} \tag{38}$$
$$+K(\eta-1)(a_m - bs_p - c(t_b^* + t_r^*) + \mu)^2(-1+\rho)^2\eta$$
$$\left[\frac{-1}{2}c(t_b^* + t_r^*)^2 + (t_b^* + t_r^*)(a - bs_p + \mu)\right]^{\eta-2}$$

Next, we assess the optimality at the point (t_b^*, t_r^*) by evaluating the Hessian matrix, denoted as H. If the determinant of the Hessian matrix $|H| \geq 0$, and is positive at the point (t_b^*, t_r^*), this indicates that the solution provides a global maximum for the profit function. The optimality of the profit function with respect to (t_b^*, t_r^*) is illustrated in Figure 2.

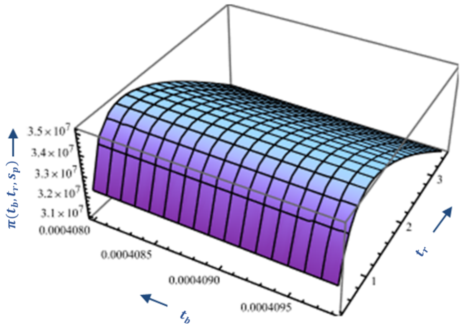

Figure 2: Variation in Profit Function with Respect to Shortage Time (t_b) and Replenishment Time (t_r)

□

Theorem 2. *For fixed values t_b^* and t_r^*, the total profit function $\pi(t_b^*, t_r^*, s_p)$ is strictly concave with respect to s_p. This concavity ensures that the profit function $\pi(t_b^*, t_r^*, s_p)$ reaches its optimal value at the corresponding s_p.*

$$\begin{aligned}
s_p^* &= \rho\,(t_b^*\delta_b - 1) \\
&[2a(t_b^* + t_r^* + t_b^* t_r^* \delta_b) - c\left(2t_b^{*2} + t_r^{*2} + t_b^* t_r^{*2}\delta\right) + 2\mu(t_b^* + t_r^* + t_b^* t_r^* \delta_b)] \\
&- \left[\frac{c_p(-t_b^* + t_r^* + t_b^{*2}\delta_b) + K(t_b^* + t_r^*)(\rho - 1)^2 + \rho c_l t_b^{*2}\delta_b(1 - t_b^*\delta_b)}{2\rho(t_b^*\delta_b - 1)(t_b^* + t_r^* + t_b^* t_r^* \delta_b)}\right]
\end{aligned} \tag{39}$$

Proof. First, we take partial derivative of profit function $\pi(t_b^*, t_r^*, s_p)$ with respect to s_p for fixed t_b^*, t_r^*, we get

$$\begin{aligned}
\frac{\partial \pi_{avg}}{\partial s_p} = \quad & bK\eta\,(t_b^* + t_r^*)(\rho - 1)^2\,[(t_b^* + t_r^*)(a - bs_p - cx + 2\mu)]^{\eta - 1} \\
& + C_l\rho \int_0^{t_b^*} b\,(\beta\,(t_b^* - x) - 1)\,dx
\end{aligned}$$

$$\begin{aligned}
& + c_p\left(\frac{b(e^{t_r^*\theta} - 1)}{\theta} + \int_0^{t_b^*} b\beta\,(t_b^* - x)dx\right) \\
& + s_p\rho\left(-bt_r^* + \int_0^{t_b^*} -b\beta\,(t_b^* - x)dx\right)
\end{aligned}$$

$$bC_b\rho\int_0^{t_b^*}\left(\int_0^t\beta(t_b^*-x)\,dx\right)dt+\frac{b(c_h+c_d\theta)(-1+e^{t_r^*\theta}-t_r^*\theta)\rho}{\theta^2}$$

$$+\rho\,(t_r^*(a_m-bs_p-cx+\mu)+\int_0^{t_b^*}(a_m-cx-bs_p+\mu)\,\beta(t_b^*-x)\,dx\Big) \quad (40)$$

Taking $\frac{\partial\pi_{avg}}{\partial s_p}=0$, we get the analytical value of s_p

$$\begin{aligned} s_p = \; & \rho\,(t_b^*\delta_b-1)\,[2a_m\,(t_b^*+t_r^*+t_b^*t_r^*\delta_b) \\ & -c\left(2t_b^{*2}+t_r^{*2}+t_b^*t_r^{*2}\delta_b\right) \\ & +2\mu\,(t_b^*+t_r^*+t_b^*t_r^*\delta_b)\,] \\ & -\left[\frac{c_p\,(-t_b^*+t_r^*+t_b^{*2}\delta_b)}{2\rho\,(t_b^*\delta_b-1)\,(t_b^*+t_r^*+t_b^*t_r^*\delta_b)}\right. \\ & +\frac{K\,(t_b^*+t_r^*)\,(\rho-1)^2}{2\rho\,(t_b^*\delta_b-1)\,(t_b^*+t_r^*+t_b^*t_r^*\delta_b)} \\ & \left.+\frac{\rho c_l t_b^{*2}\delta_b(1-t_b^*\delta_b)}{2\rho\,(t_b^*\delta_b-1)\,(t_b^*+t_r^*+t_b^*t_r^*\delta_b)}\right] \end{aligned}$$

Next, we take the partial derivative of the profit function $\pi(t_b^*,t_r^*,s_p)$ with respect to s_p to verify the optimality of s_p and ensure that the profit is maximized.

$$\begin{aligned} \frac{\partial^2\pi_{avg}}{\partial s_p^2} = -\Bigg[& \frac{b^2\eta(\eta-1)(\rho-1)^2K(t_b+t_r)^2}{t_b+t_r} \\ & \times[(t_b+t_r)(a_m-cx-bs_p+\mu)]^{(\eta-2)} \\ & +\frac{2\rho b}{t_b+t_r}\left(t_r+\int_0^{t_b}\beta(t_b-x)\,dx\right)\Bigg] \end{aligned} \quad (41)$$

Hence, we observe that $\frac{\partial^2\pi_{avg}}{\partial s_p^2}<0$, and conclude that $\pi(t_b^*,t_r^*,s_p)$ is indeed a strictly concave function of s_p. Consequently, s_p^* represents the optimal selling price that maximizes the profit function for the fixed values of t_b^* and t_r^*.

$$\begin{aligned} s_p^* = \; & \rho\left(t_b^*\delta_b-1\right)\left[2a\left(t_b^*+t_r^*+t_b^*t_r^*\delta_b\right)\right. \\ & -c\left(2t_b^{*2}+t_r^{*2}+t_b^*t_r^{*2}\delta_b\right) \\ & \left.+2\mu\left(t_b^*+t_r^*+t_b^*t_r^*\delta_b\right)\right] \end{aligned}$$

$$-\left[\frac{c_p\left(-t_b^*+t_r^*+t_b^{*2}\delta_b\right)+K\left(t_b^*+t_r^*\right)(\rho-1)^2+\rho c_l t_b^{*2}\delta_b(1-t_b^*\delta_b)}{2\rho\left(t_b^*\delta_b-1\right)\left(t_b^*+t_r^*+t_b^*t_r^*\delta_b\right)}\right] \quad (42)$$

□

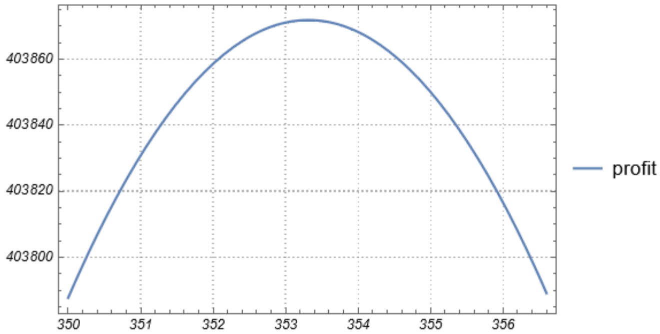

Figure 3: Expected average profit function (π_{avg}) versus the selling price (s_p^*).

Figure 3 represents the expected average profit function (π_{avg}) versus the selling price (s_p^*).

Theorem 3. *For fix t_b, t_r,and s_p, the difference of total profit function with promotional efforts and without promotional efforts is strictly positive.*

$$[\pi(t_b,t_r,s_p)]_{\rho>1} - [\pi(t_b,t_r,s_p)]_{\rho=1} > 0 \tag{43}$$

Proof. From equation (16)

$$\begin{aligned}
&[\pi(t_b,t_r,s_p)]_{\rho>1} = s_p\rho\left\{\int_0^{t_r}(D(s_p,x)+\mu)dx + \int_0^{t_b}(D(s_p,x)+\mu)\beta(t_b-x)dx\right\}\\
&-\left(O_a + c_s\rho\left[\int_0^{t_b}\left\{\int_0^{t}(D(s_p,x)+\mu)\beta(t_b-x)dx\right\}dt\right]\right.\\
&+c_1\rho\int_0^{t_b}(D(s_p,x)+\mu)\,(1-\beta(t_b-x))\,dx\\
&+(c_h+c_d\theta)\,\rho\left[\int_0^{t_r}e^{g(x)}\left\{\int_t^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx\right\}dt\right]\\
&\left.+c_0\rho\left\{\int_0^{t_r}(D(s_p,x)+\mu)e^{g(x)}dx + \int_0^{t_b}\beta(t_b-x)(D(s_p,x)+\mu)dx\right\}\right)\\
&+K(\rho-1)^2\left[\int_0^{(t_b+t_r)}(D(s_p,t)+\mu)dt\right]^{\eta}
\end{aligned} \tag{44}$$

and For $\rho = 1$,

$$\begin{aligned}
[\pi(t_b,t_r,s_p)]_{\rho=1} = -O_a - (c_h + c_d\theta)\int_0^{t_r} e^{g(t)} \int_t^{t_r} e^{g(x)} \left(D(s_p,x)+\mu\right) dx\,dt \\
-c_s \int_0^{t_b} \left(\int_0^{t} \beta(t_b-x)\left(D(s_p,x)+\mu\right) dx\right) dt \\
-c_l \int_0^{t_b} \left(\beta(t_b-x)-1\right)\left(D(s_p,x)+\mu\right) dx \\
-c_0 \left(\int_0^{t_r} e^{g(x)}\left(D(s_p,x)+\mu\right) dx - \int_0^{t_b} \beta(t_b-x)\left(D(s_p,x)+\mu\right) dx\right) \\
+s_p \left(\int_0^{t_r} \left(D(s_p,x)+\mu\right) dx + \int_0^{t_b} \beta(t_b-x)\left(D(s_p,x)+\mu\right) dx\right.
\end{aligned} \tag{45}$$

$$\begin{aligned}
[\pi]_{\rho>1} - [\pi]_{\rho=1} = (\rho-1)\Bigg[& s_p \int_0^{t_r} \left(D(s_p,x)+\mu\right) dx \\
& + \int_0^{t_b} \left(D(s_p,x)+\mu\right)\beta(t_b-x)\,dx\Bigg] \\
& - c_s\rho \left[\int_0^{t_b} \left\{\int_0^{t} \left(D(s_p,x)+\mu\right)\beta(t_b-x)\,dx\right\} dt\right. \\
& - c_1 \int_0^{t_b} \left(D(s_p,x)+\mu\right)\left[1-\beta(t_b-x)\right] dx \\
& - (c_h+c_d\theta)\,\rho \left[\int_0^{t_r} e^{g(x)} \left\{\int_t^{t_r} \left(D(s_p,x)+\mu\right) e^{g(x)}\,dx\right\} dt\right. \\
& - c_0\rho \left\{\int_0^{t_r} \left(D(s_p,x)+\mu\right) e^{g(x)}\,dx\right. \\
& \left.\left. + \int_0^{t_b} \beta(t_b-x)\left(D(s_p,x)+\mu\right) dx\right\}\right] \\
& + K(\rho-1)^2 \left[\int_0^{(t_1+t_r)} \left(D(s_p,t)+\mu\right) dt\right]^{\eta}
\end{aligned}$$

$$[\pi]_{\rho>1} - [\pi]_{\rho=1} = [\pi]_{\rho=1} + O_a + K(\rho-1)^2 \left[\int_0^{(t_1+t_2)} (D(s_p,t)+\mu)dt\right]^{\eta} > 0$$

Since, $[\pi(t_b,t_r,s_p)]_{\rho=1} > 0$, $O_a > 0$, $K(\rho-1)^2 \left[\int_0^{(t_1+t_2)} (D(s_p,t)+\mu)dt\right]^{\eta} > 0$

Therefore, the profit funcion with promotoinal efforts is greater than the profit funcion without promotoinal efforts.

$$i.e. \quad [\pi(t_b,t_r,s_p)]_{\rho>1} > [\pi(t_b,t_r,s_p)]_{\rho=1} \tag{46}$$

Hence, the above result shows that, we can maximize our profit function by applying promotional efforts and Figure 4, represents the impact of with and without promotional efforts in optimum profit.. □

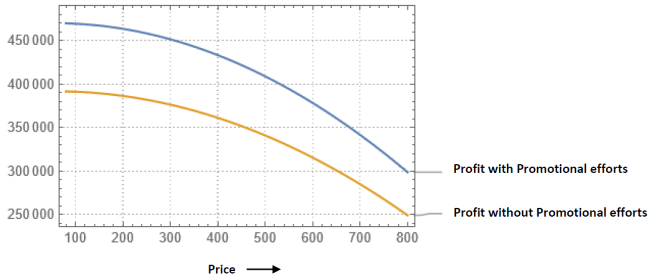

Figure 4: The graph shows the impact of with and without promotional efforts in optimum profit

Theorem 4. *If the selling price s_p is a function of t_b and t_r, then the function demonstrates concavity and reaches its global maximum at the point $\left(t_b^*, t_r^*\right)$.*

Proof. Refer by Lemma 1, Theroem 1, and Theroem 2. we have the optimal points, $\left(t_b^*, t_r^*\right)$, Now we have selling price function,
Assume, $s_p\left(t_b, t_r\right) = \psi_p\left(t_b, t_r\right)$

$$\psi_p\left(t_b, t_r\right) = \frac{1}{4b\rho\left(t_b\delta_b - 1\right)\left(t_b + t_r + t_b t_r \delta_b\right)}\left(t_b\delta_b - 1\right)\left[2a\left(t_b + t_r + t_b t_r \delta_b\right)\right.$$
$$-c\left(2t_b^2 + t_r^2 + t_b t_r^2 \delta_b\right) + 2\left(t_b + t_r + t_b t_r \delta_b\right)\mu\Big] - 2b\rho\left[c_l t_b^2 \delta_b\left(1 - t_b\delta\right)\right.$$
$$\left. -c_p\left(t_b - t_r - t_b^2\delta_b\right) + K\left(t_b + t_r\right)\left(\rho - 1\right)^2\right] \tag{47}$$

Taking the partial derivative of $\psi_p\left(t_b, t_r\right)$ with respect to t_b and t_r and we get,

$$\begin{aligned} \frac{\partial \psi_p\left(t_b, t_r\right)}{\partial t_b} = \quad & \frac{-1}{4b\rho\left(t_b\delta - 1\right)^2\left(t_b + t_r + t_b t_r \delta_b\right)^2}\left[c\left(t_b\delta - 1\right)^2\left(4t_b t_r - t_r^2\right.\right. \\ & \left.+2t_b^2\left(1 + t_r\delta\right)\right) \\ & -2b\left[C_p t_r\left(2 + t_b^2\delta^2 - 2t_b\delta_b\left(2 + t_r\delta_b\right)\right)\right. \\ & -t_b\delta_b\left[K\left(t_b + 2t_r\right)\left(1 + t_r\delta_b\right)\left(\rho - 1\right)^2\right. \\ & \left.\left.\left. -C_l\left(t_b\delta - 1\right)^2\left(t_b + 2t_r + t_b t_r \delta_b\right)\rho\right]\right]\right] < 0 \end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial \psi_p(t_b,t_r)}{\partial t_r} = {} & \frac{1}{4b\rho(t_b\delta_b-1)(t_b+t_r+t_bt_r\delta_b)^2}\left[c\rho\left(t_b^2\delta^2-1\right)\right. \\ & \times\left(2t_b^2-t_r^2-t_bt_r(2+t_r\delta_b)\right) \\ & +2bt_b\left[C_p\left(t_b^2\delta^2-2\right)\right. \\ & \left.\left.+t_b\delta_b\left[K(\rho-1)^2-C_l\rho\left(t_b^2\delta^2-1\right)\right]\right]\right]\end{aligned} \tag{49}$$

Again, taking partial derivative with respect to t_b and t_r, we get

$$\begin{aligned}\left[\frac{\partial^2\psi_p(t_b,t_r)}{\partial t_b^2}\right]_{(t_b^*,t_r^*)} = {} & \frac{-1}{2(t_b+t_r+t_bt_r\delta_b)^3}\left[\frac{ct_r^2(3+t_r\delta_b)}{b}\right. \\ & +\frac{2}{\rho(t_b\delta_b-1)^3}\left[C_pt_r\left(2+2t_r\delta_b+t_r^2\delta^2\right)\right. \\ & -6t_b\delta_b(1+t_r\delta)-t_b^3\delta^3(t_r\delta_b+1) \\ & \left.\left.+3t_b^2\delta^2\left(2+3t_r\delta_b+t_r^2\delta^2\right)\right]\right]<0\end{aligned} \tag{50}$$

$$\begin{aligned}\left[\frac{\partial^2\psi_p(t_b,t_r)}{\partial t_r^2}\right]_{(t_b^*,t_r^*)} = {} & -\left[\frac{ct_b^2(\delta_b+1)}{\rho(1-\delta_bt_b)(\delta_bt_rt_b+t_b+t_r)^3}\right. \\ & \times\left[\delta_bt_b\left(\rho C_l\left(1-\delta_b^2t_b^2\right)\right.\right. \\ & \left.+k(\rho-1)^2\right) \\ & \left.+C_p\left(\delta_b^2t_b^2-2\right)\right] \\ & +\frac{ct_b^2(\delta_bt_b+1)}{2b(\delta_bt_b-1)} \\ & \times\left[2\delta_b^2t_b^2+\delta_bt_b-3\right] \\ & \left.\Big/ (\delta_bt_rt_b+t_b+t_r)^3\right]<0\end{aligned} \tag{51}$$

$$\begin{aligned}\left[\frac{\partial^2\psi_p(t_b,t_r)}{\partial t_bt_r}\right]_{(t_b^*,t_r^*)} = {} & \frac{-1}{2b\rho(\delta_bt_b-1)^2(\delta_bt_bt_r+t_b+t_r)^3} \\ & \times\left[b\left(C_p\left(\delta_b^2t_b^3(1-\delta_bt_b)-\delta_bt_b^2(\delta_bt_r+4)\right.\right.\right. \\ & \left.+2t_b(\delta_bt_r+1)+2t_r\right) \\ & -\delta_bt_b\left(\rho C_l(\delta_bt_b-1)^2\left(t_r\left(\delta_b^2t_b^2+3\delta_bt_b+2\right)+\delta_bt_b^2\right)\right. \\ & \left.\left.\left.-K(\rho-1)^2\left(t_r\left(\delta_b^2t_b^2-\delta_bt_b+2\right)+\delta_bt_b^2\right)\right)\right)\right] \\ & -c\rho t_b\left(t_r\left(\delta_b^2t_b^2+3\delta_bt_b+3\right)+\delta_bt_b^2\right) \\ & \times(\delta_bt_b-1)^2<0\end{aligned} \tag{52}$$

Next, we check the optimality of price function at point say, (t_b^*, t_r^*) by using Hessian matrix.

$$\begin{vmatrix} \frac{\partial^2 \psi_p(t_b^*, t_r^*)}{\partial t_b^2} & \frac{\partial^2 \psi_p(t_b^*, t_r^*)}{\partial t_r t_b} \\ \frac{\partial^2 \psi_p(t_b^*, t_r^*)}{\partial t_b t_r} & \frac{\partial^2 \psi_p(t_b^*, t_r^*)}{\partial t_r^2} \end{vmatrix} > 0$$

Since, $\left[\frac{\partial^2 \psi_p(t_b,t_r)}{\partial t_b t_r}\right]_{(t_b^*,t_r^*)} = \left[\frac{\partial^2 \psi_p(t_b,t_r)}{\partial t_r t_b}\right]_{(t_b^*,t_r^*)}$ and $\left[\frac{\partial^2 \psi_p(t_b,t_r)}{\partial t_b^2}\right]_{(t_b^*,t_r^*)} < 0$

Thus, the determinant of Hessian matrix at (t_b^*, t_r^*) is positive definite and the pair of (t_b^*, t_r^*) is global maximum for optimal selling price. □

Theorem 5. *The total profit function $\pi(s_p, t_b, t_r)$ is concave with respect to s_p, t_b, and t_r. Therefore, the global maximum of $\pi(s_p, t_b, t_r)$ is attained at the optimal point s_p^*, t_b^*, t_r^*.*

7. ALGORITHM FOR COMPUTATIONAL PROCEDURES

Step 1. Initialize: Begin the algorithm.
Step 2. Input Parameters: Enter all the assumed inventory parameters.
Step 3. Parameter Validation: Verify if the assumed parameters satisfy the necessary hypotheses. If they do, proceed to the next step; if not, return to Step 2 and adjust the parameters.
Step 4. Calculate Optimal Values: Determine the optimal values of t_b^* and t_r^* using Equations (21) to (27) .
Step 5. Check Theorem Compliance: If t_b^* and t_r^* satisfy the conditions outlined in Theorem 2, proceed to Step 6. Otherwise, return to Step 4 to recalculate.
Step 6. Optimize Selling Price: Calculate the optimal selling price s_p^* by substituting t_b^* and t_r^* into Equation (42).
Step 7. Maximize Profit: Substitute the optimal values of t_b^*, t_r^*, and s_p^* into Equation (17) to (19) to determine the maximum profit.
Step 8. Verify Optimality: Confirm the optimality of the solution using Theorem 4, ensuring that the maximum profit is achieved at the point (t_b^*, t_r^*, s_p^*).
Step 9. Complete: End the algorithm.

The process of the algorithm is also illustrate in Figure 5.

8. NUMERICAL EXPERIENCE

The objectives of the numerical applications are two folds:

- To derive the optimal solutions for the retailer's expected profit functions.
- To conduct a sensitivity analysis, demonstrating the impact of various parameters on the model's outcomes.

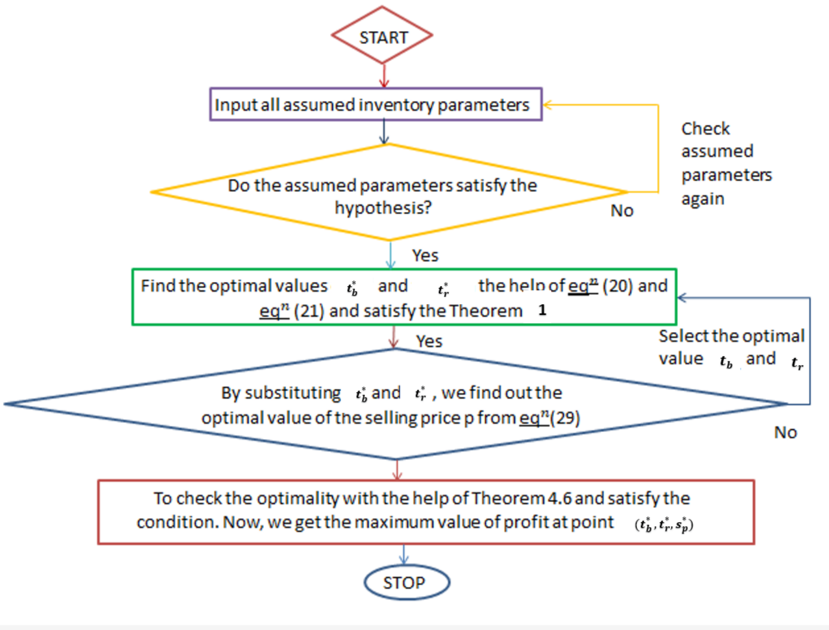

Figure 5: Flowchart showing the steps to optimize the total profit

8.1. Numerical Illustrations

This section presents numerical examples to validate the proposed inventory model. We illustrate the model with a specific example to substantiate the mathematical formulation. The majority of parameters are drawn from previous studies, including Soni & Chauhan [13] and Maihimi & Karimi [9], while others are tailored to suit the context of our problem.

Example 1. *The following parameter-values are considered:* $a_m = 500$ *units,* $b = 0.1$, $c = 0.15$, $c_p = \$200$ *per unit,* $c_d = \$0.1$ *per unit,* $c_h = \$0.1$ *per unit,* $c_b = \$150$ *per unit per year,* $c_l = \$50$ *per unit,* $\mu = 20$, $\rho = 2$, $\theta = 0.01\%$ *per unit,* $K = 1$, $\eta = 1$, $\delta_b = 0.2$, $\mu = 20$, $O_a = 1200$ *per unit*
The numerical values of t_b^*, t_r^*, s_p^* *are computed by using wolfram MATHEMATICA software. The optimum values are:* $t_b^* = 0.0410749$, $t_r^* = 0.0717368$, $s_p^* = \$354.078$, $\pi_{avg} = \$403527.00$

Example 2. *The following parameter-values are considered:* $a = 550$ *units,* $b = 4$, $c = 6.5$, $c_p = \$25$, $c_d = \$0.01$, $c_h = \$3$ *per unit,* $c_b = \$6$, $c_l = \$2$ *per unit,* $\mu = 20$, $\theta = 0.001\%$, $K = 1$, $\eta = 1$, $\delta_b = 0.001$, $\varepsilon = 20$, $O_a = 200$, *which shows the impact of with and without promotional efforts in proposed model.*

9. SENSITIVITY ANALYSIS

The sensitivity analysis examines how variations in the input parameters a_m, b, c, μ, and ρ affect the optimal decision variables and the total profit per unit time for Example 1 (see Tables 3 and 4). By varying one parameter at a time while holding the others constant, this analysis reveals how each parameter influences the outcomes. The results offer valuable insights and practical recommendations for managers seeking to optimize organizational profit. Notably, it is found that parameters such as holding costs and purchasing costs have a negligible impact on the decision variables, and as a result, these are not included in the tables.

Table 3: Sensitivity analysis for different demand parameters

Parameters	values	Shortage (t_b^*) Period (in years)	Optimal Inventory (t_r^*) Period (in years)	Price (s_p^*) (\$)	Profit (π_{avg}^*) (\$)
a	480	0.0410749	0.0717368	354.08	403527.00
	500	0.0389683	0.0670058	363.71	455621.00
	520	0.0370215	0.0627289	373.35	512534.00
	540	0.0352186	0.0588499	383.02	574549.00
	560	0.0334378	0.0554821	392.84	641951.00
b	0.14	0.0534788	0.1175230	262.12	188567.00
	0.13	0.0505226	0.1047260	279.52	224138.00
	0.12	0.0474752	0.0928916	300.03	269126.00
	0.11	0.0443287	0.0819238	324.48	327106.00
	0.10	0.0410749	0.0717368	354.08	403527.00
c	01	0.0410105	0.0718472	353.99	403573.00
	02	0.0409349	0.0719775	353.88	403626.00
	03	0.0408597	0.0721081	353.78	403677.00
	04	0.0407159	0.0723603	353.75	403727.00
	05	0.0407102	0.0723703	353.57	403772.00

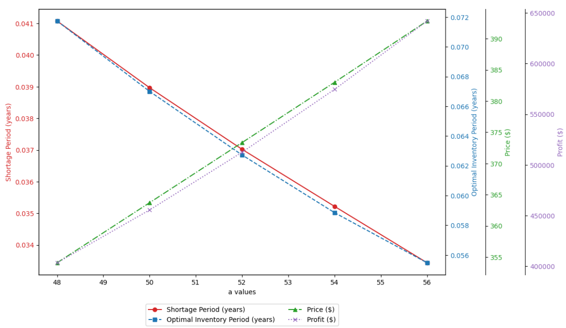

Figure 6: Impact of Changing Parameter a_m

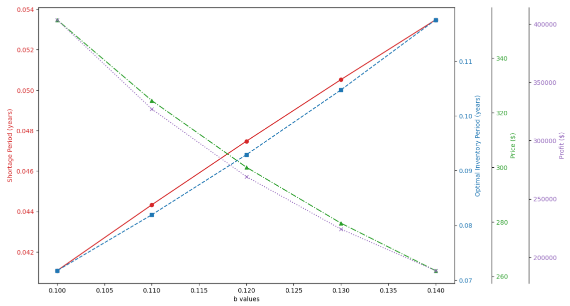

Figure 7: Impact of Changing Parameter b

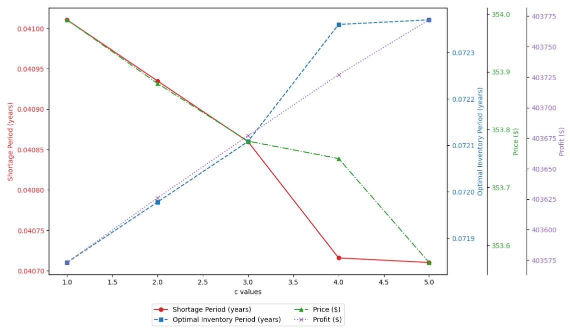

Figure 8: Impact of Changing Parameter c

Table 4: Sensitivity analysis of optimal results through a probability distribution

Random variable(ε)	Shortage (t_b^*) Period (in years)	Optimal Inventory (t_r^*) Period (in years)	Price(s_p^*) (\$)	Profit (π_{avg}^*) (\$)
$\varepsilon \sim N(2,1)$	0.0664	0.1940	206.16	093574.30
$\varepsilon \sim N(6,1)$	0.0586	0.1704	226.16	120412.00
$\varepsilon \sim N(9,1)$	0.0539	0.1561	241.16	142800.00
$\varepsilon \sim U(12,4)$	0.0518	0.1237	246.16	186417.00
$\varepsilon \sim exp(12)$	0.0489	0.1123	256.16	217220.00

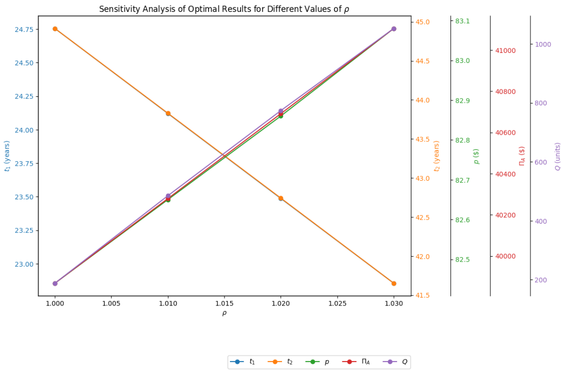

Figure 9: Sensitivity Analysis of Optimal Results for Different Values of ρ

Table 5: Sensitivity analysis of the optimal result for different values of ρ

ρ	t_b (in years)	Δt_b (in years)	t_r (in years)	Δt_r (in years)	s_p (\$)	Δs_p (\$)	π_{avg} (\$)	$\Delta\pi_{avg}$ (\$)	q $(units)$	Δq $(units)$
1.00	24.755	-	44.914	-	82.44	-	39867.30	-	188.22	-
1.01	24.121	0.6346	43.829	1.0845	82.65	0.207	40278.60	0411.30	484.91	296.68
1.02	23.487	1.2682	42.742	2.1716	82.86	0.420	40691.20	0823.90	773.60	585.38
1.03	22.854	1.9016	41.652	3.2623	83.08	0.638	41105.20	1237.90	1052.19	863.96

9.1. Sensitivity analysis of the three components of demand

In Table 3 and Figure 5, as parameter a_m increases, the selling price rises sharply. In this situation, the proposed model suggests higher selling prices, which assist management in decision-making. Furthermore, a higher selling price indicates the retailer's interest in the product. Thus, the profit increases accordingly.

- The impact of varying parameter a_m demonstrates that inventory depletes more rapidly and the shortage duration shortens. Consequently, the model encourages retailers to increase replenishment sizes and place orders earlier due to the continually decreasing inventory holding period. This adaptability highlights the model's robustness in responding to changes in demand dynamics.
- As the price dependency component of the demand function b decreases leads to a simultaneous increase in both the optimal selling price and total profit. Meanwhile, the shortage and replenishment periods decrease, as illustrated in Table 3 and Figure 6. This indicates that as demand grows, the model empowers retailers to adjust selling prices to sustain desired profit levels. Retailers should strategically place orders earlier to mitigate backlogging costs and lost sales, which otherwise diminish the profit. Furthermore, the model provides the flexibility to either boost replenishment sizes or order sooner, as both the shortage period t_r and replenishment period t_b are consistently reduced.

- An increase in the retarded growth rate c leads to a slight reduction in the optimal shortage period and selling price. Simultaneously, the optimal inventory period and total profit see a rise, as depicted in Table 3 and Figure 7. This trend suggests that as the time component of demand grows, overall profit improves even in a declining market. The model advocates for lowering product prices to attract more customers and stimulate higher demand, thereby enhancing overall profitability. Also illustrate the analysis graphically in Figures 7, 8, and 9.

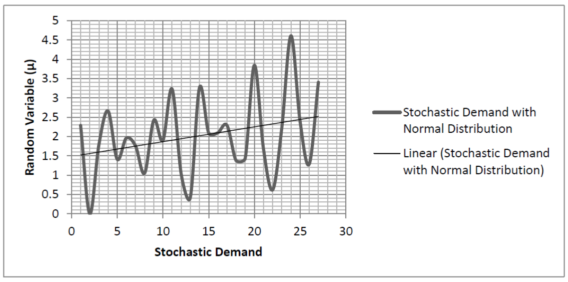

Figure 10: Random behaviour of demand with Normal Distribution *N(2,1)*

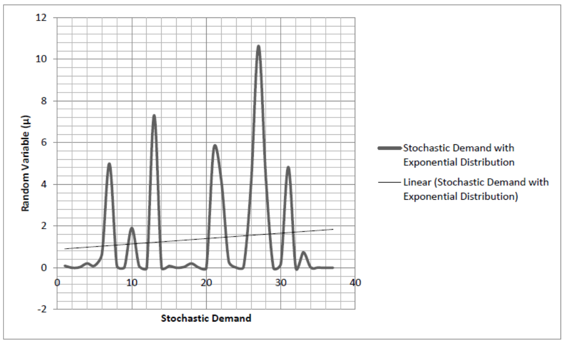

Figure 11: Random behaviour of demand with Exponential Distribution *exp(12)*

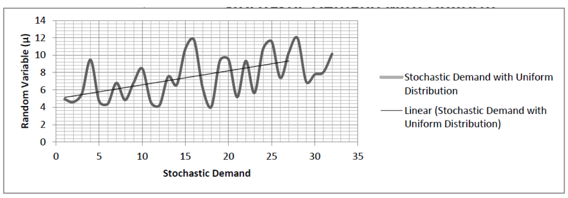

Figure 12: Random behaviour of demand with Uniform Distribution *U(12,4)*

9.2. Sensitivity Analysis of the Randomness in the Bivariate Demand Function

We utilized the same input data from Example 1 to explore how variations in the random variable $E(\varepsilon) = \mu$ impact the optimal solution. Table 4 summarizes the optimal solutions for different values of this random variable. The results underscore the significant influence that the distribution function of ε has on the model variables, highlighting the critical need for precise estimation of this distribution function. By accurately determining the distribution function for ε, retailers can refine their estimates of expected sales, pricing, profit, replenishment schedules, and ordering policies. Our sensitivity analysis demonstrates that as the value of the random variable ε increases, both the optimal selling price (s_p^*) and the total profit $\pi(t_b^*, t_r^*, s_p^*)$ rise, while the optimal shortage period (t_b^*) and replenishment period (t_r^*) decrease. These trends are illustrated for various normal, uniform, and exponential distribution functions of ε in Table 4. As the mean μ increases, so does the demand, which leads to higher ordering quantities and mitigates the impact of a declining market. Consequently, all types of distributions show increased demand and improved total profit, which aligns with rational expectations. Figures 10, 11, and 12 visually represent the random behavior of the demand function across these distributions.

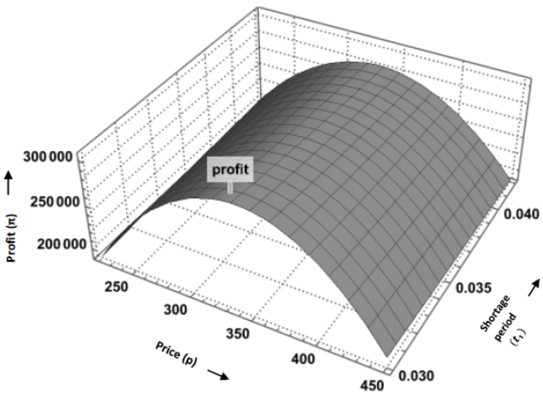

Figure 13: Concavity of the Profit function with respect to s_p and t_b

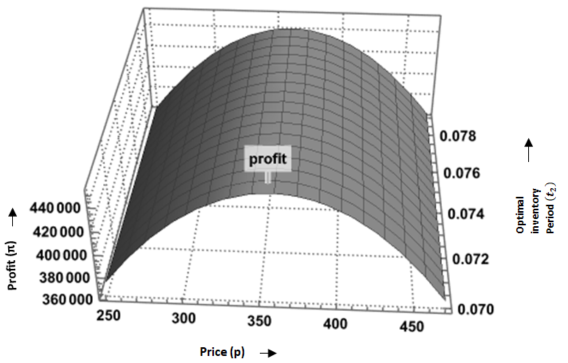

Figure 14: Concavity of the Profit function with respect to s_p and t_r

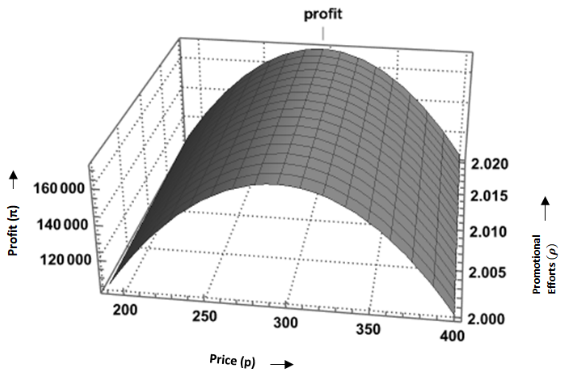

Figure 15: Concavity of the Profit function with respect to s_p and ρ

9.3. Sensitivity analysis of the impact of with and without promotional efforts

Table 4 presents the outcomes of the proposed model. In the absence of promotional efforts ($\rho = 1$), the expected total profit per unit time is $\pi_{avg}(t_b^*, t_r^*, s_p^*) = 39,867.30$, with an optimal order quantity of $q^* = 188.225$ units per cycle. When promotional efforts are introduced ($\rho = 1.01, 1.02, 1.03$), we observe a reduction in both t_b to 22.8543 and t_r to 41.6519, while the selling price rises to 83.09. Consequently, the optimal profit increases to 41,105.20, and the quantity ordered per cycle grows to 1,052.19 units. The decrease in both the shortage period and on-hand inventory period encourages earlier replenishment, and the increase in demand boosts overall profit. These findings highlight the positive impact of promotional efforts on business performance. They suggest that when using promotions to enhance market demand, inventory managers should incorporate these demand effects into their strategies to maximize profitability. Figures 13, 14, and 15 illustrate the optimal profit function across various decision variable combinations: with t_r fixed and t_b and s_p varied, t_b fixed with t_r and s_p varied, and variations in s_p and ρ. These visualizations confirm the robustness of the proposed model and its effectiveness in addressing uncertainties in the global market.

9.4. Managerial Efficiency of the Proposed Study and Strategic Management Suggestions

The proposed model, along with its numerical illustrations and sensitivity analyses, offers valuable managerial insights that can influence market strategies in the context of declining demand:

a) The analysis of the model reveals that when demand experiences a rapid initial surge, retailers have a strategic opportunity to increase product prices. This approach helps manage inventory more effectively while maximizing profits. Additionally, this surge in demand shortens both the shortage and replenishment periods, implying that retailers must place larger orders than usual to avoid lost sales and the costs associated with back-ordering.

b) From a strategic management perspective, as the price sensitivity component of demand *b* decreases, both the optimal price and profit rise significantly. Retailers and manufacturers understand that pricing is a key lever in controlling customer demand. Therefore, as market demand increases, the model suggests raising prices accordingly. It also advises retailers to place orders earlier than usual and minimize shortages to reduce unnecessary costs, thereby optimizing profit margins.

c) As the time-sensitive parameter *c* increases, annual demand gradually decelerates, indicating that demand becomes more time-sensitive. In this scenario, the model recommends reducing product prices to stimulate demand. Sensitivity analysis of parameter *c* reveals an inverse relationship between price and profit, suggesting that even in a declining market, retailers can boost profits by adjusting prices strategically. This underscores the robustness of the proposed model.

d) The model also addresses stochastic demand by considering a non-negative continuous random variable in the additive case. By applying various distributions, we assessed the sensitivity of random variables. The analysis consistently shows that demand increases as the expected value of the random variables rises, suggesting that random factors can drive demand growth. The model advises retailers and manufacturers to adjust prices upward to maintain inventory levels and maximize profits while increasing order quantities to meet the heightened demand. This demonstrates the model's realistic and sophisticated approach to tackling the complexities of a declining market.

e) Furthermore, the impact of promotional efforts is highlighted through sensitivity analysis. The findings reveal that promotional activities significantly influence the profit function. We analytically demonstrate that the expected total profit with promotional efforts is consistently higher than without them. This suggests that promotional strategies can effectively boost market demand, and inventory managers should account for these endogenous demand effects when developing strategies to achieve optimal profit.

10. DISCUSSION OF CONCLUDING REMARKS AND MANAGERIAL SUGGESTIONS FOR MARKET PLAYERS

The proposed model presents an optimal pricing strategy for deteriorating items by integrating promotional efforts with price and time-dependent stochastic demand in a declining market. Through rigorous theoretical analysis, we established the existence and uniqueness of the optimal solution, ensuring that the model is both effective and reliable. Sensitivity analysis further validates the model's robustness by examining how decision variables influence the desired profit under various conditions. For inventory management, this model provides actionable insights, demonstrating that managers can achieve optimal profit by adopting the proposed policies and carefully considering key factors. The model is especially relevant for products that deteriorate over time, such as fruits, vegetables, medicine, grains, electronics, volatile liquids, and gas cylinders. It approaches the problem from a retailer's perspective within the framework of Economic Order Quantity (EOQ), ensuring consistency with established theories and practices. The model's validity and stability are demonstrated through two numerical examples, which underscore its significance for industries handling perishable goods. The results highlight the model's robustness, making it particularly attractive for businesses looking to optimize their systems in light of product deterioration.

From a managerial perspective, this model provides valuable insights for industries seeking to achieve financial gains at optimal levels. The sensitivity analysis offers different scenarios and solutions, providing an alternative approach to traditional inventory management methods and opening the door to broader applications. The key contribution of this work lies in its practical relevance for retail businesses dealing with perishable items, electronic components, fashionable clothing, domestic goods, and similar products. This study is the first to address time-dependent demand and time-varying holding costs in conjunction with shortages for non-instantaneously deteriorating items under a finite replenishment rate. In conclusion, the proposed model is highly applicable to industries where time-dependent demand rates and variable holding costs play a crucial role, offering a practical solution for optimizing inventory management.

10.1. Future Recommendations for the Extension of the Proposed Study

There are several promising avenues for extending our proposed model. One potential direction is to incorporate a stochastic deterioration rate, building on the variable deterioration rate already considered. Additionally, the time-dependent demand function could be replaced with a probabilistic demand function to better capture real-world uncertainties. For more comprehensive modeling, the framework could be expanded to include factors such as trade credits, warehouse management, quantity discounts, stochastic inflation, deteriorating costs, vdeterioration rates, and permissible delays in payments. Another intriguing area for future research is the impact of selling defective items at a reduced price on overall demand. Shifting the focus from single-item scenarios to multiple-item systems under stochastic demand constraints would further enhance the model's applicability. For a more realistic approach, extending the model to encompass elements like warehouse management, quality discounts, deteriorating costs, and time-dependent deterioration rates in a declining market would be valuable.

Investigating the model under time-, price-, and stock-dependent stochastic demand scenarios poses a challenge in a declining market, but it is a worthwhile pursuit. Exploring the model within a fuzzy environment could also provide deeper insights and more flexible decision-making tools. Additionally, considering variable lead times—potentially controlled through additional investment—along with variable holding and purchasing costs, opens up new possibilities for refinement. Extending the model to tackle the complexities of time-, price-, and stock-dependent stochastic demand scenarios in a declining market, though demanding, would be a significant contribution. Moreover, expanding the model to operate within a fuzzy environment offers an exciting research opportunity for further exploration.

Availability of data and material: For numerical validation of different cases are obtained by optimizing respective functions through MATHEMATICA and Python software based on classical optimization method.

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