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FRACTIONAL PROGRAMMING FOR STACKELBERG GAME UNDER TYPE-2 FUZZY ENVIRONMENT

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Abstract: This study intends to present a Stackelberg game model for design of the fractional type-2 fuzzy programming. In achieving this aspiration, we develop a probabilistic fuzzy multi-objective fractional linear programming where the entire parameters are of type-2 fuzzy numbers apart from the right-hand side of the constraints are follow Weibull distribution. In the projected approach, the membership function allied with each objective function is generated by using the first-order Taylor series approximation and converted into a single objective function by assuming the weights of the objective functions are equal. Type conversion is made in two ways by existing methods, and using stochastic programming, the probabilistic constraints are transformed into a deterministic form. The accessible model incorporates the non-linear programming viewpoint of the decision-maker and is solved with the help of intuitionistic fuzzy programming (IFS). A comparison study on the optimum results by genetic algorithm (GA) and particle swarm optimization (PSO) with the LINGO 15.0 iterative scheme is offered to resolve the created bi-level programming problem (BLPP) in the course of the Stackelberg game. To make obvious the feasibility of the projected representation and solution methodology, realistic data are measured and results are presented through several discussions.

Keywords: Bi-level fractional programming, Taylor series, GA and PSO, type-2 fuzzy number, stackelberg game, intuitionistic fuzzy programming.

MSC: 91A10.

1. INTRODUCTION

Bi-level programming problem (BLPP) has a hierarchical association flanked by upper and lower levels. It is urbanized for decentralized scheduling systems in which the upper level is designated as the leader and the lower level belongs to the objective of the follower. Most of the previous studies [1], [2], [3], [4] functional to build the paramount assessment with the leader and follower hierarchical dealings in an association. In a hierarchical association, the Stackelberg game played a significant issue. H. von Stackelberg [5] first originated it, as an economist to explain a non-cooperative assessment problem. In this game, the leader has the potential to employ his/her assessment of the followers. Roy [6] solved a Stackelberg game using a fuzzy programming technique and the solution has been compared with the Kuhn-Tucker transformation procedure. In recent times, Roy and Maiti [17] presented an optimal solution for the Stackelberg game involving a stochastic programming approach.

Zadeh [8] first initiated the type-2 fuzzy set (T2FS) in addition to the type-1 fuzzy set (T1FS). The membership rank of a T2FS is a fuzzy number followed by the interval $[0, 1]$, but the membership rank of a T1FS is a real number in $[0, 1]$. In concrete applications, suitable to requiring of a key in order, clamor in data etc., a foremost problem happens to establish the precise membership ranks and consequently, to prepare, the problem in requisites of TIFS. At the same time as T2FS has prepared owing to fuzziness in the membership function. Mizumoto and Tanaka [9] designed consistent operations on T2FS. Afterward, a group of hypothetical research works completed on the belongings of T2FS [10], [12], [13] and its numerous applications [14], [15], [16], [17], [18], [19] have been developed.

Due to the large measurable intricacy of a T2FS [20], the T2FS is considered an interval T2FS. In T2FS defuzzification procedure, are categories of two steps such as type lessening and defuzzification proper. In the type-lessening process, a T2FS is condensed into a type-reduced set (TRS). After that, TRS be able to effortlessly resolve by several recognized defuzzify methods, say the centroid method. Kernik and Mendal [21] measured a centroid-type lessening procedure to convert interval T2FS towards the region of T1FS. Many researchers contemplated exclusively on interval secondary membership functions [22], [23], [24] for which an increasing number of implementations are being urbanized [25], [26], [27]. Karmakar et al. [28] solve a type-2 fuzzy matrix games and applied to biogas-plant implementation problem. Dong and Wan [29] developed a matrix game with type-2 interval-valued intuitionistic fuzzy payoffs and discussed some Hamacher aggregation operators for these fuzzy payoffs. Furthermore, Seikh and Dutta [30] designed matrix games with picture fuzzy payoffs and solve by using a non-linear mathematical approach and applied to cyberterrorism attack.

Application Type-2 fuzzy environments are precious as it allow for additional refined modeling of uncertainty by integrating uncertainty not only in the membership function but also in the membership grades themselves. Other generalizations of fuzzy sets may not receive this level of complexity, creating Type-2 fuzzy environments mainly useful for confident applications demanding higher reliability in uncertainty modeling. In uncertainty supposition, randomness and fuzziness are the two courses of uncertainty. Based on these courses, fuzzy programming has been developed by the parameters as fuzzy sets while stochastic programming has been considered by the parameters as random variables. Due to versatility, Weibull distribution is extensively used in reliability and life data investigation.

A proportion of two non-linear programming functions is to be maximized or minimized. In former implementations, the objective function entangles exceeding one proportion of functions. Proportion improvement problems are usually called fractional programming (FP). In literature [31], [32], the multi-objective linear fractional programming problem (MOLFP) is measured. Yano and Sakawa [33] designed a fuzzy approach for solving MOLFP. Pal *et al.* [34] studied a goal programming procedure for a fuzzy multi-objective linear fractional programming problem (FMOLFP). However, many procedures are accessible for solving FMOLFP in the literature [35]. An outline of the research contributions by numerous authors in this province is provided in Table 1.

In what follows, the most notable contributions of this study are outlined:

- By Kahraman et al. [36] approach, type-2 fuzzy number is transformed to crisp value.
- Right-hand side parameters of the constraints are taken as Weibull distribution.
- Based on the Taylor series approximation, fractional multi-objective functions are converted to a single objective.
- IFS is applied established on membership function as exponential and non-membership function as parabolic.
- A computational experiment among LINGO, GA and PSO is presented.

In this study, we consider a bi-level linear fractional programming (BLFP) for the Stackelberg game with T2FVs. Kahraman et al. [36] defuzzification approach is applied for T2FVs. In BLFP, each objective function is associated with the membership function in each level and converted into a fractional membership function. Using the first-order Taylor series approximation, multiple fractional membership functions are reduced into a single objective function corresponding to each level

Table 1: A briefly reviewed literature on the proposed study.

References	Nature of problem	Environments	Additional function	No. of Objectives
Anandalingam and Apprey [37]	-	fuzzy	multi-level	single
Barkat et al. [22]	-	T2FS	interval	single
Shimizu and Aiyushi [38]	non-cooperative	fuzzy	bi-level	two
Kumbasar [28]	-	T2FS	fractional	single
Maiti and Roy [2]	cooperative	fuzzy	multi-choice	two
Maiti and Roy [3]	non-cooperative	IFS	ranking	two
Ren and Wang [39]	-	fuzzy random	bi-level	two
Roy and Maiti [17]	non-cooperative	T2FS	bi-level	two
Roy and Maiti [40]	non-cooperative	T2FS	bi-level	two
Sakawa and Katagiri [41]	-	fuzzy	fuzzy random	single
Wang et al. [26]	non-cooperative	T2FS	bi-level	two
Wang and Chen [27]	-	T2FS	interval	single
Youness et al. [42]	-	fuzzy	fractional	single
Our proposed model	non-cooperative	T2FS	fractional	multi

by considering an equal weight. Right-hand side parameters of the constraints are obtained into crisp numbers by applying Weibull distribution. Subsequently, using stochastic programming, the probabilistic constraints are transformed into deterministic form and the corresponding crisp problem becomes a non-linear problem and is solved by LINGO 15.0 iterative scheme, GA and PSO respectively, and then compared the results.

The contribution of this study is to develop the Stackelberg game under a type-2 fuzzy and probabilistic environment [43], [44]. The proposed work incorporated Kahraman *et al.* [36] defuzzification process for T2FVs, first-order Taylor series approximation for a linear form of membership functions, stochastic programming for probabilistic constraints, and intuitionistic fuzzy programming for designing a single objective problem. Some enviable properties and special cases of these T2FVs are also investigated to reduce the crisp form of T2FVs. In addition, we define three types of membership functions simple linear, triangular, and trapezoidal with their meticulous cases. At long last, the application of the proposed model is studied in multi-objective Stackelberg game problems and developed a qualified study with these membership functions by using LINGO 15.0 iterative scheme, GA and PSO respectively.

The rest of this paper is organized as follows: Section 2 introduces some basic knowledge and concepts of type-1 and type-2 fuzzy sets. In Section 3, the formulation of BLFP for the Stackelberg game is presented. The solution procedure is discussed in Section 4, and the effectiveness of the intuitionistic fuzzy programming is illustrated by a numerical example in Section 5. Implications and insights are displayed in Section 6. The conclusion, limitations, and subsequent direction are described in Section 7.

2. PRELIMINARIES

Here, we recall some basic knowledge of type-1 and type-2 fuzzy sets, which will be required for our subsequent developments.

2.1. T1FS

Definition 1. *[11] A T1FS, denoted as $\tilde{A}_1$, is defined on X, the universal set. It is constituted by $\tilde{A}_1 = (x, \mu_{\tilde{A}_1}(x)) : x \in X$ and $\mu_{\tilde{A}_1} : X \to [0,1]$ is the membership function. The membership function $\mu_{\tilde{A}_1}(x)$ assigns a value of 0 if $x \notin \tilde{A}_1$ and a value of 1 if $x \in \tilde{A}_1$.*

2.2. T2FS

Definition 2. *[12] A T2FS $\tilde{\tilde{D}}$ is described as a set of pairs $((x,r), \mu_{\tilde{\tilde{D}}}(x,r))$ rewarding the conditions: $\forall x$ in the domain X, the universal set and all r in the subset M_x of the interval $[0,1]$; $\mu_{\tilde{\tilde{D}}}(x,r)$, type-2 membership function satisfies $0 \leq \mu_{\tilde{D}}(x,r) \leq 1$. Here, M_x acts for the primary membership function of x, and $\mu_{\tilde{\tilde{D}}}(x)$ acts for the secondary membership function respectively. It ensures that $\forall r$ in M_x corresponds to the point x, the primary membership grades. The province* *of* $\mu_{\tilde{\tilde{D}}}(x)$ *is denoted by* X. *Alternatively,* $\tilde{\tilde{D}}$ *can be expressed as the integral of* $\mu_{\tilde{\tilde{D}}}(x)$ *over* x: $\tilde{\tilde{D}} = \int_{x\in X} \mu_{\tilde{\tilde{D}}}(x)/x = \int_{x\in X} \left[\int_{r\in M_x} \mu_{\tilde{\tilde{D}}}(x,r)/r \right]/x$.

Remark 3. *If the value* $\mu_{\tilde{\tilde{D}}}(x,r)$ *is equal to 1,* $\forall x, r$, *at that time* tilde$\tilde{D}$ *is referred to as an interval type-2 fuzzy set (IT2FS).*

Definition 4. *The uncertainty associated with the primary membership of an IT2FS is represented by the footprint of uncertainty (FOU), which is a bounded region. The FOU is formed by combining all the primary memberships, denoted as* M_x. *The upper and lower membership functions are represented by* $\overline{\mu}_{\tilde{\tilde{D}}}(x)$ *and* $\underline{\mu}_{\tilde{\tilde{D}}}(x)$, *respectively. Thus, the primary memberships* M_x *can be expressed as the interval* $[\underline{\mu}_{\tilde{\tilde{D}}}(x), \overline{\mu}_{\tilde{\tilde{D}}}(x)]$.

Example 5. *Consider* $\tilde{\tilde{D}}$ *be a type-2 fuzzy variable (T2FV) in which* $X = \{6, 8, 9\}$ *and* M_x, *the primary membership functions of* X *are given by* $M_6 = \{0.3, 0.5, 0.7\}$, $M_8 = \{0.4, 0.8, 0.9\}$ *and* $M_9 = \{0.1, 0.7, 0.8\}$ *separately. At that moment,* $\tilde{\mu}_{\tilde{\tilde{D}}}(6)$, *the secondary membership function of 6 and interpreted by* $\tilde{\mu}_{\tilde{\tilde{D}}}(6) = (0.4/0.3) + (0.7/0.5) + (1/0.7) \sim \begin{pmatrix} 0.3 & 0.5 & 0.7 \\ 0.4 & 0.7 & 1 \end{pmatrix}$
Here, $\tilde{\mu}_{\tilde{\tilde{D}}}(6, 0.3) = 0.4$ *characterizes that point 6, secondary membership grade can have the point 0.4, primary membership grade. Thus,* $\tilde{\tilde{D}}$ *incorporates point 6 with the membership:*

$$\begin{pmatrix} 0.3 & 0.5 & 0.7 \\ 0.4 & 0.7 & 1 \end{pmatrix}$$

, *which represents a RFV. Similarly,* $\tilde{\mu}_{\tilde{D}}(8) = (1/0.4) + (0.4/0.8) + (0.5/0.9) \sim \begin{pmatrix} 0.4 & 0.8 & 0.9 \\ 1 & 0.4 & 0.5 \end{pmatrix}$

$$\tilde{\mu}_{\tilde{D}}(9) = (0.4/0.1) + (1/0.7) + (0.6/0.8) \sim \begin{pmatrix} 0.1 & 0.7 & 0.8 \\ 0.4 & 1 & 0.6 \end{pmatrix}$$

Consequently, $\tilde{\tilde{D}}$ *can be written as* $\tilde{\tilde{D}} = \begin{cases} 6, & \textit{together with membership } \tilde{\mu}_{\tilde{\tilde{D}}}(6,r), \\ 8, & \textit{together with membership } \tilde{\mu}_{\tilde{\tilde{D}}}(8,r), \\ 9, & \textit{together with membership} \tilde{\mu}_{\tilde{\tilde{D}}}(9,r). \end{cases}$

Example 6. *Let us consider* $\tilde{\tilde{\xi}}$ *be a T2FV, defined as:*

$$\tilde{\tilde{\xi}} = \begin{cases} 5, & \textit{together with possibility } (0.2, 0.5, 0.6, 0.8), \\ 6, & \textit{together with possibility } (0.1, 0.3, 0.4, 0.7), \\ 7, & \textit{together with possibility } (0.3, 0.7, 0.8, 0.9). \end{cases}$$

As $\mu_{\tilde{\tilde{\xi}}}(5,s)$ *is a trapezoidal RFV, we have* $\mu_{\tilde{\tilde{\xi}}}(5,s) = \begin{cases} 0, & \text{if } s < 0.2, \\ \frac{s-0.2}{0.3}, & \text{if } 0.2 \le s \le 0.5, \\ 1, & \text{if } 0.5 \le s \le 0.6, \\ \frac{0.8-s}{0.2}, & \text{if } 0.6 \le s \le 0.8, \\ 0, & \text{if } s > 0.8 \end{cases}$

$\mu_{\tilde{\tilde{\xi}}}(6,s) = \begin{cases} 0, & \text{if } s < 0.1, \\ \frac{s-0.1}{0.2}, & \text{if } 0.1 \le s \le 0.3, \\ 1, & \text{if } 0.3 \le s \le 0.4, \\ \frac{0.7-s}{0.3}, & \text{if } 0.4 \le s \le 0.7, \\ 0, & \text{if } s > 0.7 \end{cases}$ *and* $\mu_{\tilde{\tilde{\xi}}}(7,s) = \begin{cases} 0, & \text{if } s < 0.3, \\ \frac{s-0.3}{0.4}, & \text{if } 0.3 \le s \le 0.7, \\ 1, & \text{if } 0.7 \le s \le 0.8, \\ \frac{0.9-s}{0.1}, & \text{if } 0.8 \le s \le 0.9, \\ 0, & \text{if } s > 0.9 \end{cases}$

The graphical representation of $\tilde{\tilde{\xi}}$ is depicted in Figure 1.

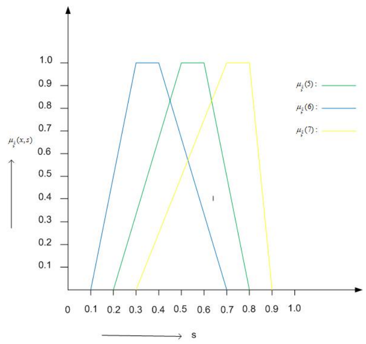

Figure 1: Type-2 fuzzy variable $\tilde{\tilde{\xi}}$

Definition 7. *[12] A trapezoidal interval type-2 fuzzy variable (TrIT2FV)* $\tilde{\tilde{A}}$ *can be written as* $\tilde{\tilde{A}} = \big((a_1^u, a_2^u, a_3^u, a_4^u; H_1(A^u), H_2(A^u)), (a_1^l, a_2^l, a_3^l, a_4^l; H_1(A^l), H_2(A^l))\big)$, *where* a_1^u, a_2^u, a_3^u *and* a_4^u *are real numbers associated with the upper membership function taking the membership values* $0,\ H_1(A^u),\ H_2(A^u)$ *and* 0 *respectively, whereas* $a_1^l, a_2^l, a_3^l, a_4^l$ *are allied with the inferior membership function captivating the membership values* $0,\ H_1(A^l), H_2(A^l)$ *and* 0 *respectively.*

So the FOU of $\tilde{\tilde{A}}$ is characterized in Figure 2.

2.3. Arithmetic operations on T2FS

Let $\tilde{\tilde{A}} = (A^u, A^l) = \big((a_{11}^u, a_{12}^u, a_{13}^u, a_{14}^u; H_1(A^u), H_2(A^u)), (a_{11}^l, a_{12}^l, a_{13}^l, a_{14}^l; H_1(A^l)$ $, H_2(A^l))\big)$ and $\tilde{\tilde{B}} = (B^u, B^l) = \big((b_{11}^u, b_{12}^u, b_{13}^u, b_{14}^u; H_1(B^u), H_2(B^u)), (b_{11}^l, b_{12}^l, b_{13}^l,$ $b_{14}^l; H_1(B^l),$

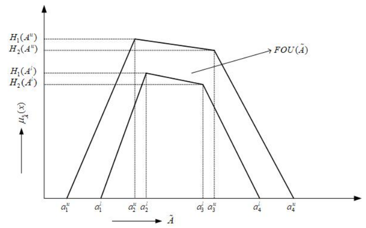

Figure 2: FOU of $\tilde{\tilde{A}}$

$H_2(B^l))\big)$ be two TrIT2FVs. Then the arithmetic operations are as follows:
Addition: The addition of TrIT2FVs is given by

$$\begin{aligned}\tilde{\tilde{A}}+\tilde{\tilde{B}} &= (A^u,A^l)+(B^u,B^l)\\ &= \big((a^u_{11}+b^u_{11},a^u_{12}+b^u_{12},a^u_{13}+b^u_{13},a^u_{14}+b^u_{14};H_1(A^u)+H_2(B^u)\\ &\quad -H_1(A^u).H_2(B^u)),(a^l_{11}+b^l_{11},a^l_{12}+b^l_{12},a^l_{13}+b^l_{13},a^l_{14}+b^l_{14};\\ &\quad H_1(A^l)+H_2(B^l)-H_1(A^l).H_2(B^l))\big).\end{aligned}$$

Subtraction: The subtraction of TrIT2FVs is given by

$$\begin{aligned}\tilde{\tilde{A}}-\tilde{\tilde{B}} &= (A^u,A^l)-(B^u,B^l)\\ &= \big((a^u_{11}-b^l_{11},a^u_{12}-b^l_{12},a^u_{13}-b^l_{13},a^u_{14}-b^l_{14};H_1(A^u)+H_2(B^l)-\end{aligned}$$

$$H_1(A^u).H_2(B^l)),(a^l_{11}-b^u_{11},a^l_{12}-b^u_{12},a^l_{13}-b^u_{13},a^l_{14}-b^u_{14};\\ H_1(A^l)+H_2(B^u)-H_1(A^l).H_2(B^u))\big).$$

Multiplication by a scalar quantity: The multiplication of $\tilde{\tilde{A}}$ by k (>0), is given by

$$\begin{aligned}k\tilde{\tilde{A}} &= k(A^u,A^l)\\ &= \big((ka^u_{11},ka^u_{12},ka^u_{13},ka^u_{14};1-(1-H_1(A^u))^k,1-(1-H_2(B^u))^k),\\ &\quad (kb^u_{11},kb^u_{12},kb^u_{13},kb^u_{14};1-(1-H_1(A^l))^k,1-(1-H_2(B^l))^k))\big).\end{aligned}$$

3. MATHEMATICAL MODEL

In this study, we consider the following bi-level fractional programming with probabilistic constraints in the Stackelberg game involving type-2 fuzzy numbers.
Model 1

$$\text{maximize} \qquad Z_{1rj}(\mathbf{x}) = \frac{\sum_{j=1}^{n}\left(\tilde{\tilde{a}}_{1rj}x_j+\tilde{\tilde{b}}_{1rj}\right)}{\sum_{j=1}^{n}\left(\tilde{\tilde{c}}_{1rj}x_j+\tilde{\tilde{d}}_{1rj}\right)},\ r=1,2,\ldots,l$$

$$\text{maximize} \quad Z_{2rj}(\mathbf{x}) = \frac{\sum_{j=1}^{n} (\tilde{\tilde{a}}_{2rj}xj + \tilde{\tilde{b}}_{2rj})}{\sum_{j=1}^{n} (\tilde{\tilde{c}}_{2rj}x_j + \tilde{\tilde{d}}_{2rj})}, \ r = 1, 2, \ldots, l$$

$$\text{subject to} \quad P_r\left(\sum_{j=1}^{n} \tilde{\tilde{A}}_{ij}x_j \leq q_i\right) \geq 1 - \gamma_i, \ i = 1, 2, \ldots, m$$

$$x_j \geq 0, \ \forall \ j.$$

where $0 < \gamma_i < 1, \ \forall \ i$ are the specified probabilities. The coefficients $\tilde{\tilde{a}}_{grj}, \ \tilde{\tilde{b}}_{grj}, \ \tilde{\tilde{c}}_{grj}, \ \tilde{\tilde{d}}_{grj}$ and $\tilde{\tilde{A}}_{ij}, \ \forall \ i, \ j, \ r$ are all type-2 fuzzy numbers, and only $q_i, \ \forall \ i$ is a random variable follows Weibull distribution. Also, Z_{1rj} and Z_{2rj} are the $(rj)^{th}$ objective functions corresponding to the upper-level and lower-level DMs respectively.

3.1. Defuzzification of T2FV

In this section, we present a defuzzification procedure for T2FVs. It consists of two ways: type conversion process and any existing defuzzification method for type-1 fuzzy set. According to Kahraman *et al.* [36] defuzzification approach, the defuzzified value of $\tilde{\tilde{A}}$ is given by

$$A' = \frac{1}{2}\Bigg\{\frac{1}{4}\Big((a_4^u - a_1^u) + \big(H_2(A^u) \times a_2^u - a_1^u\big) + \big(H_1(A^u) \times a_3^u - a_1^u\big)\Big) + a_1^u + \frac{1}{4}\Big((a_4^l - a_1^l) + \big(H_2(A^l) \times a_2^l - a_1^l\big) + \big(H_1(A^l) \times a_3^l - a_1^l\big)\Big) + a_1^l\Bigg\} \quad (1)$$

Using the equation (1), Model 1 can be written as follows :

Model 2

$$\text{maximize} \quad Z_{1rj}(\mathbf{x}) = \frac{\sum_{j=1}^{n} (a'_{1rj}x_j + b'_{1rj})}{\sum_{j=1}^{n} (c'_{1rj}x_j + d'_{1rj})}$$

$$\text{maximize} \quad Z_{2rj}(\mathbf{x}) = \frac{\sum_{j=1}^{n} (a'_{2rj}x_j + b'_{2rj})}{\sum_{j=1}^{n} (c'_{2rj}x_j + d'_{2rj})}$$

$$\text{subject to} \quad P_r\left(\sum_{j=1}^{n} A'_{ij}x_j \leq q_i\right) \geq 1 - \gamma_i, \ i = 1, 2, \ldots, m \quad (2)$$

$$x_j \geq 0, \ \forall \ j; \ r = 1, 2, \ldots l.$$

3.2. Crisp form of probabilistic constraint

Here, we consider the random variable q_i, $\forall\ i$ which follows Weibull distribution. Then the probability density function (pdf) of q_i is given by

$$f(q_i) = \begin{cases} \left(\frac{\alpha_i}{\beta_i}\right)\left(\frac{q_i}{\beta_i}\right)e^{-\left(\frac{q_i}{\beta_i}\right)^{\alpha_i}}, & \text{if } q_i > 0,\ \alpha_i > 0,\ \beta_i > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

with mean= $\beta_i\Gamma\left(1+\frac{1}{\alpha_i}\right)$, $\alpha_i > 0,\ \beta_i > 0$ and variance= $\beta_i^2\Gamma\left(1+\frac{2}{\alpha_i}\right) - \left[\beta_i\Gamma\left(1+\frac{1}{\alpha_i}\right)\right]^2$, $\alpha_i > 0,\ \beta_i > 0$.

Now the probabilistic constraint can be written with the help of equation (3) as follows:

$$\int_{u_i}^{\infty} f(q_i)dq_i \geq 1 - \gamma_i, \tag{4}$$

where $u_i = \sum\limits_{j=1}^{n} A'_{ij}x_j$ and $u_i \geq 0$.

Using equation (3), it can be further written the equation (4) as

$$\int_{u_i}^{\infty} \left(\frac{\alpha_i}{\beta_i}\right)\left(\frac{q_i}{\beta_i}\right)e^{-\left(\frac{q_i}{\beta_i}\right)^{\alpha_i}} dq_i \geq 1 - \gamma_i. \tag{5}$$

On integration, we obtain from the equation (5)

$$e^{-\left(\frac{u_i}{\beta_i}\right)^{\alpha_i}} \geq 1 - \gamma_i$$

$$\text{i.e.,} \quad u_i \leq \frac{\beta_i}{\Big(ln\big(1-\gamma_i\big)\Big)^{\frac{1}{\alpha_i}}} \tag{6}$$

$$\text{i.e.,} \quad \sum_{j=1}^{n} A'_{ij}x_j \leq \frac{\beta_i}{\Big(ln\big(1-\gamma_i\big)\Big)^{\frac{1}{\alpha_i}}} \tag{7}$$

Then Model 2 can be written as the following model as:

Model 3

$$\text{maximize} \qquad Z_{1rj}(\mathbf{x}) = \frac{\sum\limits_{j=1}^{n}\left(a'_{1rj}x_j + b'_{1rj}\right)}{\sum\limits_{j=1}^{n}\left(c'_{1rj}x_j + d'_{1rj}\right)}$$

$$\text{maximize} \qquad Z_{2rj}(\mathbf{x}) = \frac{\sum\limits_{j=1}^{n}\left(a'_{2rj}x_j + b'_{2rj}\right)}{\sum\limits_{j=1}^{n}\left(c'_{2rj}x_j + d'_{2rj}\right)}$$

$$\text{subject to} \quad \sum_{j=1}^{n} A'_{ij}x_j \leq \frac{\beta_i}{\Big(ln\big(1-\gamma_i\big)\Big)^{\frac{1}{\alpha_i}}}$$

$$x_j \geq 0,\ \forall\ j,\ \alpha_i > 0,\ \beta_i > 0 \text{ and } 0 < \gamma_i < 1,\ \forall\ i,\ r.$$

3.3. Bi-level fractional programming in Stackelberg game

In this section, we introduce fuzzy goals to each of the objective functions for both levels respectively. Then, Model 3 can be written as:

Model 4

$$\begin{aligned}
\text{maximize} \quad & Z_{1rj}(\mathbf{x}) \tilde{\gtrsim} f_{rj},\ r = 1, 2, \ldots, r_o \\
\text{or,} \quad & Z_{1rj}(\mathbf{x}) \tilde{\lesssim} f_{rj},\ r = r_{o+1}, r_{o+2}, \ldots, l \\
\text{maximize} \quad & Z_{2rj}(\mathbf{x}) \tilde{\gtrsim} h_{rj},\ r = 1, 2, \ldots, r_t \\
\text{or,} \quad & Z_{2rj}(\mathbf{x}) \tilde{\lesssim} h_{rj},\ r = r_{t+1}, r_{t+2}, \ldots, l \\
\text{subject to} \quad & \sum_{j=1}^{n} A'_{ij}x_j \leq \frac{\beta_i}{\Big(ln\big(1-\gamma_i\big)\Big)^{\frac{1}{\alpha_i}}} \\
& x_j \geq 0,\ \forall\ j,\ \alpha_i > 0,\ \beta_i > 0 \text{ and } 0 < \gamma_i < 1,\ \forall\ i.
\end{aligned}$$

where f_{rj} and h_{rj} are the aspiration levels of the $(rj)^{th}$ objective function corresponding to both level DMs respectively. Here the symbols $\tilde{\lesssim}$ and $\tilde{\gtrsim}$ represent "essentially less than" and "essentially more than" fuzziness of the aspiration levels. Now the membership functions of both level objective functions for each goal can be written as follows:

For $Z_{1rj} \tilde{\lesssim} f_{rj}$,

$$\mu_{1rj}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{1rj}(\mathbf{x}) \leq f_{rj}, \\ \frac{\overline{s}_{rj} - Z_{1rj}(\mathbf{x})}{\overline{s}_{rj} - f_{rj}}, & \text{if } f_{rj} \leq Z_{1rj}(\mathbf{x}) \leq \overline{s}_{rj}, \\ 0, & \text{if } Z_{1rj}(\mathbf{x}) \geq \overline{s}_{rj}. \end{cases}$$

For $Z_{1rj} \tilde{\gtrsim} f_{rj}$,

$$\mu_{1rj}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{1rj}(\mathbf{x}) \geq f_{rj}, \\ \frac{Z_{1rj}(\mathbf{x}) - \underline{s}_{rj}}{f_{rj} - \underline{s}_{rj}}, & \text{if } \underline{s}_{rj} \leq Z_{1rj}(\mathbf{x}) \leq f_{rj}, \\ 0, & \text{if } Z_{1rj}(\mathbf{x}) \leq \underline{s}_{rj}. \end{cases}$$

where $\overline{s}_{rj}$ and $\underline{s}_{rj}$ are the upper and lower tolerance limits for $(rj)^{th}$ fuzzy goal of the upper-level DM.

For $Z_{2rj} \tilde{\lesssim} h_{rj}$,

$$\mu_{2rj}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{2rj}(\mathbf{x}) \leq h_{rj}, \\ \frac{\overline{v}_{rj} - Z_{2rj}(\mathbf{x})}{\overline{v}_{rj} - h_{rj}}, & \text{if } h_{rj} \leq Z_{2rj}(\mathbf{x}) \leq \overline{v}_{rj}, \\ 0, & \text{if } Z_{2rj}(\mathbf{x}) \geq \overline{v}_{rj}. \end{cases}$$

For $Z_{2rj}\tilde{\gtrsim}h_{rj}$,

$$\mu_{2rj}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{2rj}(\mathbf{x}) \geq h_{rj}, \\ \frac{Z_{2rj}(\mathbf{x})-\underline{v}_{rj}}{h_{rj}-\underline{v}_{rj}}, & \text{if } \underline{v}_{rj} \leq Z_{2rj}(\mathbf{x}) \leq h_{rj}, \\ 0, & \text{if } Z_{2rj}(\mathbf{x}) \leq \underline{v}_{rj}. \end{cases}$$

where $\overline{v}_{rj}$ and $\underline{v}_{rj}$ are the upper and lower tolerance limits for $(rj)^{th}$ fuzzy goal of the lower-level DM.

3.4. Taylor series approximation for membership linearization

The linearization process of a fractional bi-level multi-objective programming in the Stackelberg game is to be summarized through the following steps:

Step 1: We find x^*_{1rj} and x^*_{2rj} from $(rj)^{th}$ membership function associated with the objective functions Z_{1rj} and Z_{2rj} respectively, where $r = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, n$.
Step 2: Now apply the 1^{st} order Taylor series approximation, the membership functions corresponding to both level objective functions are transformed as follows:

$$\begin{aligned} \mu_{1rj}(\mathbf{x}) \quad &\cong \widehat{\mu_{1rj}}(\mathbf{x}) \\ &= \sum_{j=1}^{n} \mu_{1rj}(x^*_{1rj}) + \left(\big(x_1 - x^*_{1r1}\big) \frac{\partial \mu_{1r1}(x^*_{1r1})}{\partial x_{1r1}} + \ldots + \big(x_n - x^*_{1rn}\big) \right. \\ &\left. \frac{\partial \mu_{1rn}(x^*_{1rn})}{\partial x_{1rn}} \right) \\ &= \sum_{k=1}^{n} \left(\mu_{1rk}(x^*_{1rk}) + \big(x_k - x^*_{1rk}\big) \frac{\partial \mu_{1rk}(x^*_{1rk})}{\partial x_{1rk}} \right). \end{aligned} \tag{8}$$

$$\begin{aligned} \mu_{2rj}(\mathbf{x}) \quad &\cong \widehat{\mu_{2rj}}(\mathbf{x}) \\ &= \mu_{2rj}(x^*_{2rj}) + \left(\big(x_1 - x^*_{2r1}\big) \frac{\partial \mu_{2r1}(x^*_{2r1})}{\partial x_{2r1}} + \ldots + \big(x_n - x^*_{2rn}\big) \right. \\ &\left. \quad \frac{\partial \mu_{2rn}(x^*_{2rn})}{\partial x_{2rn}} \right) \\ &= \sum_{k=1}^{n} \left(\mu_{2rk}(x^*_{2rk}) + \big(x_k - x^*_{2rk}\big) \frac{\partial \mu_{2rk}(x^*_{2rk})}{\partial x_{2rk}} \right). \end{aligned} \tag{9}$$

Step 3: Next, add these membership values with equal weights and transform them into a single objective function corresponding to each level i.e.,

$$Z_1(\mathbf{x}) \quad = \sum_{j=1}^{n} \widehat{\mu_{1rj}}(\mathbf{x}),\ r = 1, 2, \ldots, l \tag{10}$$

$$Z_2(\mathbf{x}) \quad = \sum_{j=1}^{n} \widehat{\mu_{2rj}}(\mathbf{x}),\ r = 1, 2, \ldots, l \tag{11}$$

Using the equations (10) and (11), Model 4 can be written as follows:
Model 5

$$\begin{aligned} &\text{maximize} && Z_1(\mathbf{x}) \\ &\text{maximize} && Z_2(\mathbf{x}) \\ &\text{subject to} && \sum_{j=1}^{n} A'_{ij} x_j \leq \frac{\beta_i}{\Big(ln\big(1-\gamma_i\big)\Big)^{\frac{1}{\alpha_i}}} \\ & && x_j \geq 0,\ \forall\ j,\ \alpha_i > 0,\ \beta_i > 0 \text{ and } 0 < \gamma_i < 1,\ \forall\ i. \end{aligned}$$

4. SOLUTION METHODOLOGY

In this section, we develop an algorithm deliberated to accomplish a satisfactory solution for fractional bi-level programming in Stackelberg games under an intuitionistic fuzzy environment. The algorithm outlines the steps involved in intuitionistic fuzzy programming as follows:

4.1. Intuitionistic fuzzy programming

In the Stackelberg game, the upper-level DM will first select x_1 variable, after that the lower-level DM will specify x_2 variable with the complete familiarity of the upper-level DM. Keeping this tip of observation, we build up the subsequent algorithm for attaining a satisfactory way out of a BLFP for the Stackelberg game. The IFS is fundamentally grounded on membership and non-membership functions respectively. Here, we consider the membership function as exponential and the non-membership function as parabolic in nature. These are interpreted as follows:

$$\mu_{Z_g} = \begin{cases} 1, & \text{if } Z_g(x) \leq Z_g^0, \\ \frac{e^{-t\left(\frac{Z_g(\mathbf{x}) - Z_g^0}{Z_g^1 - Z_g^0}\right)} - e^{-t}}{1 - e^{-t}}, & \text{if } Z_g^0 \leq Z_g(x) \leq Z_g^1, \\ 0, & \text{if } Z_g(x) \geq Z_g^1. \end{cases} \tag{12}$$

$$\omega_{Z_g} = \begin{cases} 0, & \text{if } Z_g(x) \leq Z_g^0, \\ \left(\frac{Z_g(\mathbf{x}) - Z_g^0}{Z_g^1 - Z_g^0}\right)^2, & \text{if } Z_g^0 \leq Z_g(x) \leq Z_g^1, \\ 1, & \text{if } Z_g(x) \geq Z_g^1. \end{cases} \tag{13}$$

where $g = 1, 2$ and Z_g^0, Z_g^1 are the lower bound (worst solution) and upper bound (best solution) of the objective function $Z_g(x)$. The algorithm sketches out the steps involved in IFS as follows:
Step 1: Independently resolve both level objectives, taking the crisp correspondence constraints of Model 5 and calculating both bounds i.e. lower and upper of the objective functions respectively.
Step 2: With the help of the equations (12) and (13), computing the membership and non-membership functions for both level objective functions and also the upper-level decision variable x_1.
Step 3: According to Zimmerman [11], Model 5 is converted to the following crisp model as:
Model 6

$$\begin{aligned}
\text{maximize} \quad & (\zeta - \xi) \\
\text{subject to} \quad & \frac{e^{-t\left(\frac{Z_g'(\mathbf{x}) - Z_g'^0}{Z_g'^1 - Z_g'^0}\right)} - e^{-t}}{1 - e^{-t}} \geq \zeta, \ \frac{e^{-t\left(\frac{x_1 - x_1^u + d}{d}\right)} - e^{-t}}{1 - e^{-t}} \geq \zeta, \\
& \left(\frac{Z_g'(x) - Z_g'^0}{Z_g'^1 - Z_g'^0}\right)^2 \leq \xi, \left(\frac{x_1 - x_1^u + d}{d}\right)^2 \leq \xi, A'x \leq \frac{\beta_i}{\left(ln(1-\gamma_i)\right)^{\frac{1}{\alpha_i}}}, \\
& x \geq 0, \ \zeta \geq \xi, \ \zeta + \xi \leq 1, \ 0 < \gamma_i < 1, \ \alpha_i > 0, \ \beta_i > 0, \ g = 1, 2.
\end{aligned}$$

Step 4: Model 6 can be resolved by LINGO, GA and PSO, and subsequent to that optimal solutions ζ^* and ξ^* are obtained corresponding to ζ and ξ respectively.
Step 5: If the upper-level DM is convinced with the explanation recognized in Step 4, subsequently go to Step 6, or else update the membership and non-membership functions accordingly and next go to Step 2.
Step 6: Stop.

4.2. GA

GA ([45], [46]) is a biological evolution procedure that can solve both constrained and unconstrained optimization problems. It produces various solutions in one run, so it can be easily applied to a large number of data in dissimilar areas. In GA, a population is a set of probable explanations of a problem and a component of the population is called a genotype. Reproduction, mutation and crossover are the three main operators in GA.
Parameters: It depends on unlike parameters like the population size (PSize), maximum number of generations (MAXGen), probability of crossover (PCross) and probability of mutation (PMut). In this study, consider PSize= 500, PCross= 0.5, MAXGen= 500 and PMut =0.5.
Reproduction: It is a significant step in GA that regulates even if the exacting string will contribute to the reproduction procedure or not. In the population initialization process, settle on the limits of all types of dependent and independent variables in that order. The renowned selection procedures are rank, tournament,

roulette wheel, stochastic universal sampling and Boltzmann. Roulette wheel selection procedures depict all the achievable strings onto a wheel with a segment of the wheel owed to them conforming to their strength value. It is then rotated arbitrarily to choose definite solutions which will contribute to the arrangement of the subsequent generation.

Crossover: Crossover is the process of generating offspring by combining the genetic information of two or more parents. Subsequent to the selection development, the population is upgraded with improved individuals. The recognized crossover operators are partially matched, precedence preserving crossover, single-point, two-point, k-point, uniform, order, shuffle, cycle and abridged surrogate. An arbitrary crossover point is particular in a single-point crossover. Two parents' genetic information away from that point will be exchanged with each other.

Mutation: It is a genetic operator that is utilized to sustain genetic diversity from one production of a population to the next generation. The recognized mutation operators are simple inversion, displacement and mix-up mutation. The banishment mutation operator changes a sub-string of a specified creature solution contained by itself. The place is arbitrarily preferred from the specified sub-string for dislocation such that the consequent solution is valid as well as an arbitrary banishment mutation. Exchange mutation and insertion mutation are the two variants of banishment mutation. In insertion mutation and exchange mutation operators, a part of an individual solution is either inserted in another location or exchanged with another part, correspondingly.

Evaluation: In chromosome generations, the loop is terminated when assured conditions are met. Subsequent to that, the preferred chromosome is rebounded while the most excellent solution is established. The general terminating circumstances are:

- Reached the permanent number of generations: This termination process ends the progression when the user-defined highest numbers of progressions have been run. So this process is constantly active.
- Fitness Threshold: This process ends the progression when the most excellent fitness in the present population inclines a lesser amount of the user-defined strength threshold and the intention is to locate to diminish the fitness.
- Evolution Time: This process ends the progression when the onward progression time excels the user-defined maximum progression time.

In our study, for reaching the subsequently enhanced chromosomes, consider the roulette wheel selection method, uniform mutation, and arithmetic crossover. The above steps have been characterized with the assistance of a flowchart in Figure 3.

4.3. PSO

PSO [47] is a metaheuristic global optimization method that has been gained from the information exchange (behavior) of the birds in a swarm. Due to its

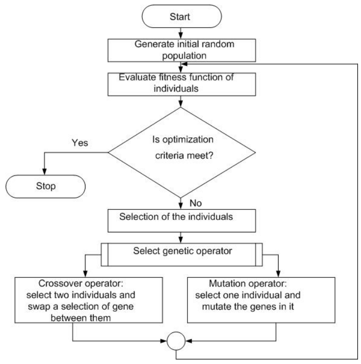

Figure 3: General structure of GA

cleanness of application in unsupervised, complex multidimensional problems that cannot be solved using conventional deterministic algorithms. In the PSO procedure, the population is said to be a swarm and the subsequent individuals are named particles. Each particle is a probable solution and is inclined by the experiences of its neighbors as well as itself in search space. It consists of three operators which are inertia weight, personal particle best (Pbest), and global particle best (Gbest). In the iteration process, the particle utilizes a memory in instruct to keep its finest position and take as a whole the best particle positions which are saved as the Pbest. The Pbest is assigned to region particles, whereas the overall finest particle position is restored as the Gbest, which is assigned to all particles in the swarm.Each particle is denoted as $X_i = (X_{i1}, X_{i2}, \ldots, X_{iq})$, where q is the q-dimensional space. The velocity of each particle is $V_i = (V_{i1}, V_{i2}, \ldots, V_{iq})$ and the velocity of the initial population is generated arbitrarily. The best local position of each particle is given by $P_i = (P_{i1}, P_{i2}, \ldots, P_{iq})$. Every particle fits the best local and global position respectively. On each iteration, the particle adjusts its Pbest and Gbest among particles in its neighbourhood as follows:

$$V_i^{(k+1)} = V_i^{(k)} + c_1 \times \mathrm{T}_{i1} \times \left(\mathrm{Pbest}^{(k)} - X_i^{(k)}\right) + c_2 \times \mathrm{T}_{i2} \times \left(\mathrm{Gbest}^{(k)} - X_i^{(k)}\right) X_i^{(k+1)} = X_i^{(k)} + V_i^{(k+1)}, \ i = 1, 2, \ldots, P.$$

where c_1 and c_2 are called cognitives and social parameters; T_{i1}, T_{i2} are random vectors $\in (0, 1)$.

5. NUMERICAL EXPERIMENT

In this section, we include a numerical example and real-life application to illustrate the efficiency of the proposed approach.

5.1. Example

In the experiment, we consider a probabilistic fuzzy multi-objective fractional linear bi-level programming in the Stackelberg game. The cost parameters of both level multi-objective functions and the constraint parameters are T2FNs. Also, the right-hand side of the constraints follows the Weibull distribution and all these data are tabulated.

5.1.1. Input data

Here all the relevant costs i.e., $\tilde{\tilde{a}}_{grj},\ \tilde{\tilde{b}}_{grj},\ \tilde{\tilde{c}}_{grj},\ \tilde{\tilde{d}}_{grj},\ \forall\ r,\ j;\ g = 1,2$ and the constraints parameters i.e., $\tilde{\tilde{A}}_{ij},\ \forall\ i,\ j$ which are constituted by trapezoidal T2FVs (TrT2FV) respectively, specified in Tables 2-6. Again, the shape and scale parameters of the Weibull distribution are given in Table 7.

Table 2: Input data for TrT2F $\tilde{\tilde{a}}_{grj}$

$\tilde{\tilde{a}}_{113} = ((41,43,50,54;0.8,0.7),(42,46,48,52;0.5,0.4))$
$\tilde{\tilde{a}}_{114} = ((43,45,52,56;1,0.9),(44,48,50,54;0.8,0.7))$
$\tilde{\tilde{a}}_{111} = ((37,39,46,50;0.8,0.7),(38,42,44,48;0.6,0.5))$
$\tilde{\tilde{a}}_{112} = ((39,41,48,52;0.9,0.6),(40,44,46,50;0.5,0.4))$
$\tilde{\tilde{a}}_{211} = ((40,42,49,53;1,0.9),(41,45,47,51;0.8,0.7))$
$\tilde{\tilde{a}}_{212} = ((48,50,57,61;1,0.8),(49,53,55,59;0.6,0.5))$
$\tilde{\tilde{a}}_{213} = ((56,58,65,69;1,0.8),(57,61,63,67;0.7,0.6))$
$\tilde{\tilde{a}}_{214} = ((64,66,73,77;0.9,0.7),(65,69,71,75;0.6,0.4))$
$\tilde{\tilde{a}}_{121} = ((84,86,93,97;1,0.8),(85,89,91,95;0.7,0.6))$
$\tilde{\tilde{a}}_{221} = ((83,85,92,96;0.8,0.7),(84,88,90,94;0.6,0.4))$
$\tilde{\tilde{a}}_{122} = ((109,111,118,122;0.8,0.6),(110,114,116,120;0.5,0.4))$
$\tilde{\tilde{a}}_{123} = ((134,136,143,147;0.7,0.5),(135,139,141,145;0.4,0.3))$
$\tilde{\tilde{a}}_{124} = ((159,161,168,172;1,0.7),(160,164,166,170;0.5,0.4))$
$\tilde{\tilde{a}}_{222} = ((118,120,127,131;0.7,0.6),(119,123,125,129;0.5,0.2))$
$\tilde{\tilde{a}}_{223} = ((153,155,162,166;0.7,0.6),(154,158,160,164;0.5,0.4))$
$\tilde{\tilde{a}}_{224} = ((188,190,197,201;1,0.7),(189,193,195,199;0.6,0.5))$

Table 3: Input data for TrT2F $\tilde{\tilde{b}}_{grj}$

$\tilde{\tilde{b}}_{11} = ((25,27,34,38;1.0,0.8),(26,30,32,36;0.7,0.6))$
$\tilde{\tilde{b}}_{21} = ((35,37,44,48;0.8,0.6),(36,40,42,46;0.5,0.4))$
$\tilde{\tilde{b}}_{12} = ((157,159,166,170;0.9,0.8),(158,162,164,168;0.6,0.5))$
$\tilde{\tilde{b}}_{22} = ((257,259,266,270;0.7,0.6),(258,262,264,268;0.5,0.4))$

Table 4: Input data for TrT2F $\tilde{\tilde{d}}_{grj}$

$\tilde{\tilde{d}}_{11} = ((15, 17, 24, 28; 0.9, 0.8), (16, 20, 22, 26; 0.7, 0.6))$
$\tilde{\tilde{d}}_{21} = ((18, 20, 27, 31; 0.8, 0.7), (19, 23, 25, 29; 0.6, 0.5))$
$\tilde{\tilde{d}}_{12} = ((107, 109, 116, 120; 0.8, 0.7), (108, 112, 114, 118; 0.6, 0.4))$
$\tilde{\tilde{d}}_{22} = ((167, 169, 176, 180; 0.9, 0.8), (168, 172, 174, 178; 0.7, 0.3))$

Table 5: Input data for TrT2F $\tilde{\tilde{d}}_{grj}$

$\tilde{\tilde{c}}_{111} = ((47, 49, 56, 60; 1.0, 0.8), (48, 52, 54, 58; 0.7, 0.6))$
$\tilde{\tilde{c}}_{112} = ((51, 53, 60, 64; 0.9, 0.8), (52, 56, 58, 62; 0.6, 0.5))$
$\tilde{\tilde{c}}_{113} = ((55, 57, 64, 68; 0.9, 0.7), (56, 60, 62, 66; 0.6, 0.5))$
$\tilde{\tilde{c}}_{114} = ((59, 61, 68, 72; 1.0, 0.7), (60, 64, 66, 70; 0.6, 0.4))$
$\tilde{\tilde{c}}_{211} = ((44, 46, 53, 57; 0.9, 0.6), (45, 49, 51, 55; 0.5, 0.4))$
$\tilde{\tilde{c}}_{212} = ((47, 49, 56, 60; 0.8, 0.7), (48, 52, 54, 58; 0.5, 0.4))$
$\tilde{\tilde{c}}_{213} = ((50, 52, 59, 63; 0.8, 0.6), (51, 55, 57, 61; 0.5, 0.3))$
$\tilde{\tilde{c}}_{214} = ((53, 55, 62, 66; 0.9, 0.6), (54, 58, 60, 64; 0.5, 0.3))$
$\tilde{\tilde{c}}_{121} = ((28, 30, 37, 41; 0.9, 0.8), (29, 33, 35, 39; 0.6, 0.5))$
$\tilde{\tilde{c}}_{122} = ((30, 32, 39, 43; 0.9, 0.7), (31, 35, 37, 41; 0.6, 0.5))$
$\tilde{\tilde{c}}_{123} = ((32, 34, 41, 45; 0.8, 0.6), (33, 37, 39, 43; 0.5, 0.4))$
$\tilde{\tilde{c}}_{124} = ((34, 36, 43, 47; 0.8, 0.6), (35, 39, 41, 45; 0.4, 0.3))$
$\tilde{\tilde{c}}_{221} = ((33, 35, 42, 46; 0.9, 0.8), (34, 38, 40, 44; 0.7, 0.6))$
$\tilde{\tilde{c}}_{222} = ((40, 42, 49, 53; 0.9, 0.7), (41, 45, 47, 51; 0.5, 0.3))$
$\tilde{\tilde{c}}_{223} = ((47, 49, 56, 60; 0.8, 0.6), (48, 52, 54, 58; 0.5, 0.3))$
$\tilde{\tilde{c}}_{224} = ((54, 56, 63, 67; 0.6, 0.5), (55, 59, 61, 65; 0.4, 0.2))$

Table 6: Input data for TrT2F $\tilde{\tilde{A}}_{ij}$

$\tilde{\tilde{A}}_{11} = ((28, 33, 43, 58; 0.9, 0.8), (30, 34, 40, 57; 0.6, 0.5))$
$\tilde{\tilde{A}}_{12} = ((35, 40, 70, 90; 0.8, 0.6), (37, 45, 58, 80; 0.5, 0.4))$
$\tilde{\tilde{A}}_{13} = ((38, 44, 73, 96; 0.7, 0.6), (40, 48, 61, 83; 0.5, 0.4))$
$\tilde{\tilde{A}}_{14} = ((42, 44, 49, 56; 0.9, 0.8), (43, 45, 47, 51; 0.7, 0.3))$
$\tilde{\tilde{A}}_{21} = ((20, 27, 40, 59; 0.9, 0.7), (25, 30, 32, 54; 0.6, 0.2))$
$\tilde{\tilde{A}}_{22} = ((40, 55, 63, 67; 0.8, 0.4), (48, 61, 62, 64; 0.3, 0.2))$
$\tilde{\tilde{A}}_{23} = ((30, 38, 46, 70; 0.7, 0.4), (35, 40, 42, 67; 0.3, 0.2))$
$\tilde{\tilde{A}}_{24} = ((24, 41, 49, 53; 0.6, 0.5), (38, 43, 45, 52; 0.4, 0.2))$
$\tilde{\tilde{A}}_{31} = ((32, 44, 52, 56; 1.0, 0.9), (41, 46, 48, 55; 0.8, 0.7))$
$\tilde{\tilde{A}}_{32} = ((36, 47, 55, 59; 0.8, 0.5), (44, 49, 51, 58; 0.4, 0.3))$
$\tilde{\tilde{A}}_{33} = ((16, 24, 29, 31; 0.5, 0.4), (22, 25, 26, 30; 0.3, 0.1))$
$\tilde{\tilde{A}}_{34} = ((23, 27, 32, 54; 0.9, 0.8), (25, 28, 29, 44; 0.7, 0.6))$
$\tilde{\tilde{A}}_{41} = ((40, 50, 55, 59; 0.8, 0.7), (45, 51, 58, 67; 0.6, 0.5))$
$\tilde{\tilde{A}}_{42} = ((61, 90, 97, 99; 0.9, 0.6), (78, 95, 96, 98, 0.5, 0.4))$
$\tilde{\tilde{A}}_{43} = ((25, 38, 50, 97; 0.6, 0.5), (34, 40, 41, 90; 0.4, 0.1))$
$\tilde{\tilde{A}}_{44} = ((28, 67, 89, 99; 0.9, 0.8), (37, 78, 80, 97; 0.7, 0.6))$

Table 7: Input data for α_i, β_i and γ_i

$\alpha_1 = 0.167$	$\beta_1 = 0.25$	$\gamma_1 = 0.28$
$\alpha_2 = 0.25$	$\beta_2 = 0.23$	$\gamma_2 = 0.22$
$\alpha_3 = 0.5$	$\beta_3 = 0.17$	$\gamma_3 = 0.065$
$\alpha_4 = 0.25$	$\beta_4 = 0.14$	$\gamma_4 = 0.18$

5.1.2. Optimal result

Utilizing all Tables i.e., Tables 1 to 7, Model 3 can be rewritten as follows:

Model 7

$$\text{maximize} \quad Z_{111} = \frac{35.56x_1 + 36.18x_2 + 37.69x_3 + 45.39x_4 + 27.63}{47.15x_1 + 48.53x_2 + 51.21x_3 + 54.61x_4 + 18.45}$$

$$\text{maximize} \quad Z_{112} = \frac{79.99x_1 + 90.7x_2 + 103.4x_3 + 136.29x_4 + 138.63}{28.98x_1 + 30.28x_2 + 30.06x_3 + 30.64x_4 + 91.91}$$

$$\text{maximize} \quad Z_{211} = \frac{42.61x_1 + 46.69x_2 + 55.14x_3 + 57.89x_4 + 32.43}{40.18x_1 + 42.49x_2 + 43.55x_3 + 46.65x_4 + 19.89}$$

$$\text{maximize} \quad Z_{212} = \frac{72.41x_1 + 93.13x_2 + 123.33x_3 + 165.06x_4 + 203.93}{34.2x_1 + 36.94x_2 + 41.23x_3 + 42.88x_4 + 145}$$

$$\left.\begin{aligned} \text{subject to} \quad & 34.89x_1 + 46.13x_2 + 48.03x_3 + 39.71x_4 \leq 198.93, \\ & 29.76x_1 + 40.28x_2 + 33.75x_3 + 30.44x_4 \geq 65.60, \\ & 43.28x_1 + 37.45x_2 + 16.68x_3 + 29.19x_4 \geq 50.92, \\ & 41.29x_1 + 70.41x_2 + 39.43x_3 + 62.19x_4 \leq 109.61, \\ & x_j \geq 0, \ \forall\, j \end{aligned}\right\} \tag{14}$$

Now, we consider two objective functions of the upper-level DM being employed to be more than 1.0, 2.6 respectively and for the lower-level to be more than 1.4, 2.2 respectively. In other words, the fuzzy aspiration levels and tolerance limits of two objective goals corresponding to both level DMs are $(1.0, 2.6), (1.4, 2.2)$ and $(-0.9, 0.01), (-1.09, -9.8)$ respectively. We now examine the solutions based on three different membership functions and these are defined as follows.

5.1.3. Linear membership function

In this case, the membership functions of the goals corresponding to both levels are obtained and their graphical representations are in Figures 4 and 5 respectively.

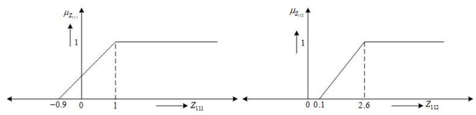

Figure 4: Upper-level membership functions defined as simple linear

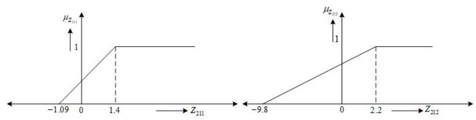

Figure 5: Lower-level membership functions defined as simple linear

$$\mu_{111}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{111}(\mathbf{x}) \geq 1, \\ R_1, & \text{if } -0.9 \leq Z_{111}(\mathbf{x}) \leq 1, \ \& \\ 0, & \text{if } Z_{111}(\mathbf{x}) \leq \ -0.9 \end{cases}$$

$$\mu_{112}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{112}(\mathbf{x}) \geq 2.6, \\ R_2, & \text{if } 0.1 \leq Z_{112}(\mathbf{x}) \leq 2.6, \\ 0, & \text{if } Z_{112}(\mathbf{x}) \leq \ 0.01 \end{cases} \tag{15}$$

$$\mu_{211}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{211}(\mathbf{x}) \geq 1.4, \\ R_3, & \text{if } -1.09 \leq Z_{211}(\mathbf{x}) \leq 1.4, \ \& \\ 0, & \text{if } Z_{211}(\mathbf{x}) \leq \ -1.09 \end{cases}$$

$$\mu_{212}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{212}(\mathbf{x}) \geq 2.2, \\ R_4, & \text{if } -9.8 \leq Z_{212}(\mathbf{x}) \leq 2.2, \\ 0, & \text{if } Z_{212}(\mathbf{x}) \leq \ -9.8 \end{cases} \tag{16}$$

where $R_1 = \frac{77.995x_1+79.86x_2+83.78x_3+94.54x_4+44.24}{89.58x_1+90.21x_2+97.3x_3+103.76x_4+35.06}$,

$$R_2 = \frac{79.7x_1 + 90.4x_2 + 103.01x_3 + 135.98x_4 + 137.71}{75.06x_1 + 78.42x_2 + 77.86x_3 + 79.36x_4 + 238.05},$$

$$R_3 = \frac{86.41x_1+93x_2+102.61x_3+108.74x_4+54.11}{100.05x_1+105.8x_2+108.44x_3+116.16x_4+49.53},$$

$$R_4 = \frac{407.57x_1 + 455.14x_2 + 527.38x_3 + 585.28x_4 + 1624.93}{410.4x_1 + 443.28x_2 + 494.76x_3 + 514.56x_4 + 1740}$$

In Model 4, each objective function is associated with the corresponding membership functions (15)-(16) respectively. Then by 1^{st} order Taylor series approximation as discussed in subsection 3.4, the membership functions given by the equations (15)-(16) are transformed as:

$$\widehat{\mu_{111}}(\mathbf{x}) = -0.0273x_1 - 0.0298x_2 - 0.0335x_3 - 0.0144x_4 + 0.99 \tag{17}$$
$$\widehat{\mu_{112}}(\mathbf{x}) = 0.0164x_1 + 0.0349x_2 + 0.068x_3 + 0.1464x_4 + 0.7397 \tag{18}$$
$$\widehat{\mu_{211}}(\mathbf{x}) = -0.036x_1 - 0.0324x_2 - 0.0074x_3 - 0.0121x_4 + 0.9878 \tag{19}$$
$$\widehat{\mu_{212}}(\mathbf{x}) = -0.0009x_1 + 0.0044x_2 + 0.0119x_3 + 0.0255x_4 + 0.95 \tag{20}$$

Now, both level objective functions are obtained by adding the pair of equations (17)-(20) respectively. Then the objective functions are obtained as follows:

$$Z_1(\mathbf{x}) = -0.0109x_1 + 0.0051x_2 + 0.0345x_3 + 0.132x_4 + 1.7298$$
$$Z_2(\mathbf{x}) = -0.0369x_1 - 0.028x_2 + 0.0045x_3 + 0.0134x_4 + 1.9461$$

Thus, Model 7 can be stated as below:
Model 8

$$\begin{aligned} \text{maximize} \quad & Z_1(\mathbf{x}) = -0.0109x_1 + 0.0051x_2 + 0.0345x_3 + 0.132x_4 + 1.7298 \\ \text{maximize} \quad & Z_2(\mathbf{x}) = -0.0369x_1 - 0.028x_2 + 0.0045x_3 + 0.0134x_4 + 1.9461 \\ \text{subject to} \quad & \text{the constraints (14).} \end{aligned}$$

Solve the Model 8 by LINGO software. We calculate Z_g^0 and Z_g^1 values for $g = 1, 2$. These values are $Z_1^0 = 1.7009$, $Z_1^1 = 1.9192$, $Z_2^0 = 1.8481$, $Z_2^1 = 1.9647$ at $x_1 = 0.0392$, $x_2 = 0.0$, $x_3 = 0.801$ and $x_4 = 1.2286$. So, we now rewrite Model 8 with the help of Model 6 as follows:

Model 9

$$\begin{aligned}
\text{maximize} \quad & (\zeta - \xi) \\
\text{subject to} \quad & \frac{e^{-t\left(\frac{Z_1(\mathbf{x})-1.7009}{1.9192-1.7009}\right)} - e^{-t}}{1-e^{-t}} \geq \zeta, \ \frac{e^{-t\left(\frac{Z_2(\mathbf{x})-1.8481}{1.9647-1.8481}\right)} - e^{-t}}{1-e^{-t}} \geq \zeta, \\
& \frac{e^{-t\left(\frac{x_1-0.0392+x_2+d}{d}\right)} - e^{-t}}{1-e^{-t}} \geq \zeta, \ \left(\frac{Z_1(\mathbf{x})-1.7009}{1.9192-1.7009}\right)^2 \leq \xi, \\
& \left(\frac{Z_2(\mathbf{x})-1.8481}{1.9647-1.8481}\right)^2 \leq \xi, \ \left(\frac{x_1-0.0392+x_2+d}{d}\right)^2 \leq \xi, \\
& \text{the constraints (14)} \\
& \zeta \geq \xi, \ \zeta + \xi \leq 1, \ 0 \leq \zeta, \xi \leq 1.
\end{aligned}$$

Solve the Model 9 by LINGO software, GA and PSO respectively and the consequences are listed in Table 8. Table 9 shows the corresponding parameter set for these two algorithms: GA and PSO.

Table 8: Optimum results for linear membership function

Result	Method		
	LINGO	GA	PSO
Z_1	1.8357	1.0182	1.7009
Z_2	1.9201	1.0223	1.8482
x_1	0.0000	0.5152	2.6529
x_2	0.0000	1.6054	0.0000
x_3	0.7642	0.5182	0.0000
x_4	1.3077	2.0600	0.0000
ζ	0.4813	0.8520	0.9985
ξ	0.2815	0.1431	0.0001

Table 9: Parameter set up

GA	PSO
Population: 100	Population: 100
Crossover rate: 0.8	V_{max} : 10
Mutation rate: 0.1	Inertial weight: 0.2-0.9
Generation: 500	Iteration: 100

5.1.4. Triangular membership function

In this case, the membership functions of the goals are obtained as follows, and their graphical representations are in Figures 6 and 7 respectively.

$$\mu_1(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{111}(\mathbf{x}) \geq 3, \\ R_5, & \text{if } 2 \leq Z_{111}(\mathbf{x}) \leq 3, \\ R_6, & \text{if } 0 \leq Z_{111}(\mathbf{x}) \leq 2, \\ 0, & \text{if } Z_{111}(\mathbf{x}) \leq \ 0 \end{cases} \ \& \ \mu_2(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{112}(\mathbf{x}) \geq 5, \\ R_7, & \text{if } 3 \leq Z_{112}(\mathbf{x}) \leq 5, \\ R_8, & \text{if } 1 \leq Z_{112}(\mathbf{x}) \leq 3, \\ 0, & \text{if } Z_{112}(\mathbf{x}) \leq \ 1 \end{cases}$$

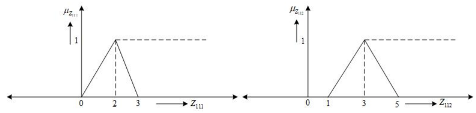

Figure 6: Upper-level membership functions defined as triangular

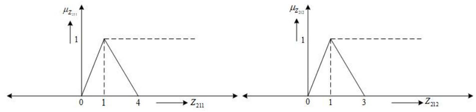

Figure 7: Lower-level membership functions defined as triangular

$$\mu_3(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{211}(\mathbf{x}) \geq 4, \\ R_9, & \text{if } 1 \leq Z_{211}(\mathbf{x}) \leq 4, \\ R_{10}, & \text{if } 0 \leq Z_{211}(\mathbf{x}) \leq 1, \\ 0, & \text{if } Z_{211}(\mathbf{x}) \leq \ 0 \end{cases} \ \& \ \mu_4(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{212}(\mathbf{x}) \geq 3, \\ R_{11}, & \text{if } 1 \leq Z_{212}(\mathbf{x}) \leq 3, \\ R_{12}, & \text{if } 0 \leq Z_{212}(\mathbf{x}) \leq 1, \\ 0, & \text{if } Z_{212}(\mathbf{x}) \leq \ 0 \end{cases}$$

$$\text{where } R_5 = \frac{105.89x_1+109.41x_2+115.94x_3+118.44x_4+27.72}{47.15x_1+48.53x_2+51.21x_3+54.61x_4+18.45},$$
$$R_6 = \frac{35.56x_1+36.18x_2+37.69x_3+45.39x_4+27.63}{24.3x_1+97.06x_2+102.42x_3+109.22x_4+36.9}$$
$$R_7 = \frac{64.91x_1+60.7x_2+46.9x_3+16.91x_4+320.92}{57.96x_1+60.56x_2+60.12x_3+61.28x_4+183.82},$$
$$R_8 = \frac{51.01x_1+60.42x_2+73.34x_3+75.01x_4+46.72}{57.96x_1+60.56x_2+60.12x_3+61.28x_4+367.64}$$
$$R_9 = \frac{118.11x_1+123.27x_2+119.06x_3+128.71x_4+47.13}{120.54x_1+127.47x_2+130.65x_3+139.95x_4+59.67},$$
$$R_{10} = \frac{42.61x_1+46.69x_2+55.14x_3+57.89x_4+32.43}{40.18x_1+42.49x_2+43.55x_3+46.65x_4+59.67}$$
$$R_{11} = \frac{30.19x_1+17.69x_2+0.36x_3-36.42x_4+231.07}{68.4x_1+73.88x_2+82.46x_3+85.76x_4+290},$$
$$R_{12} = \frac{72.41x_1+93.13x_2+123.33x_3+165.06x_4+203.93}{34.2_1+36.94x_2+41.23x_3+42.88x_4+145}$$

Proceeding the same way as discussed in subsection 3.4, we formulate Model 10 as follows:

Model 10

$$\begin{aligned} &\text{maximize} && Z_3(\mathbf{x}) = 0.1422x_1 + 0.0304x_2 + 0.0514x_3 + 0.0717x_4 + 0.7218 \\ &\text{maximize} && Z_4(\mathbf{x}) = -0.0268x_1 - 0.0673x_2 - 0.1388x_3 - 0.2252x_4 + 1.68 \\ &\text{subject to} && \text{the constraints (14).} \end{aligned}$$

Model 10 represents a linear programming problem and by LINGO software it has been solved. The solutions are achieved through the proposed intuitionistic fuzzy programming in subsection 4.1 by LINGO software, GA and PSO and the consequences are summarized in Table 10.

Table 10: Optimum results for triangular membership function

Result	Method		
	LINGO 15.0	GA	PSO
Z_3	0.9397	0.9505	0.8503
Z_4	1.4571	1.4427	1.3277
x_1	1.0834	1.1163	0.0000
x_2	0.2183	0.4357	1.8160
x_3	0.0231	0.0000	0.0000
x_4	0.7814	0.7908	1.0217
ζ	0.4954	0.6992	0.7734
ξ	0.2533	0.2018	0.0572

5.1.5. Trapezoidal membership function

In this case, the membership functions (see Figures 8 and 9) of the goals depend on four parameters and are computed as follows:

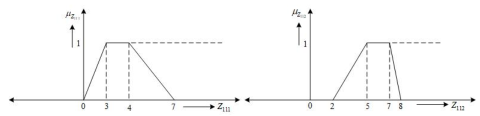

Figure 8: Upper-level membership functions defined as trapezoidal

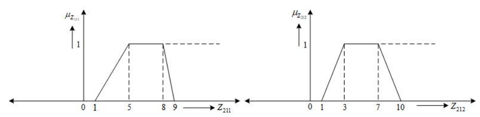

Figure 9: Lower-level membership functions defined as trapezoidal

$$\mu_5(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{111}(\mathbf{x}) \geq 7, \\ P_1, & \text{if } 4 \leq Z_{111}(\mathbf{x}) \leq 7, \\ 1 & \text{if } 3 \leq Z_{111}(\mathbf{x}) \leq 4, \\ P_2, & \text{if } 0 \leq Z_{111}(\mathbf{x}) \leq 3, \\ 0, & \text{if } Z_{111}(\mathbf{x}) \leq \ 0 \end{cases} \& \ \mu_6(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{112}(\mathbf{x}) \geq 8, \\ P_3, & \text{if } 7 \leq Z_{112}(\mathbf{x}) \leq 8, \\ 1 & \text{if } 5 \leq Z_{112}(\mathbf{x}) \leq 7, \\ P_4, & \text{if } 2 \leq Z_{112}(\mathbf{x}) \leq 5, \\ 0, & \text{if } Z_{112}(\mathbf{x}) \leq \ 2 \end{cases}$$

$$\mu_7(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{211}(\mathbf{x}) \geq 9, \\ P_5, & \text{if } 8 \leq Z_{211}(\mathbf{x}) \leq 9, \\ 1 & \text{if } 5 \leq Z_{211}(\mathbf{x}) \leq 8, \\ P_6, & \text{if } 1 \leq Z_{211}(\mathbf{x}) \leq 5, \\ 0, & \text{if } Z_{211}(\mathbf{x}) \leq \ 1 \end{cases} \& \ \mu_8(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{212}(\mathbf{x}) \geq 10, \\ P_7, & \text{if } 7 \leq Z_{212}(\mathbf{x}) \leq 10, \\ 1 & \text{if } 3 \leq Z_{212}(\mathbf{x}) \leq 7, \\ P_8, & \text{if } 1 \leq Z_{212}(\mathbf{x}) \leq 3, \\ 0, & \text{if } Z_{212}(\mathbf{x}) \leq \ 1 \end{cases}$$

where

$$
\begin{aligned}
P_1 &= \tfrac{294.49x_1+303.53x_2+320.78x_3+336.88x_4+101.52}{141.45x_1+145.6x_2+153.63x_3+163.83x_4+55.35},\\
P_2 &= \tfrac{35.56x_1+36.18x_2+37.69x_3+45.39x_4+27.63}{141.45x_1+145.6x_2+153.63x_3+163.83x_4+55.35},\\
P_3 &= \tfrac{151.85x_1+151.54x_2+137.08x_3+108.83x_4+596.65}{28.98x_1+30.28x_2+30.06x_3+30.64x_4+91.91},\\
P_4 &= \tfrac{22.03x_1+30.14x_2+43.28x_3+75.01x_4-45.19}{86.94x_1+90.84x_2+90.18x_3+91.92x_4+275.73},\\
P_5 &= \tfrac{319.01x_1+335.72x_2+336.81x_3+361.96x_4+146.58}{40.18x_1+42.49x_2+43.55x_3+46.65x_4+19.89},\\
P_6 &= \tfrac{2.43x_1+4.2x_2+11.59x_3+11.24x_4+12.54}{160.72x_1+169.96x_2+174.2x_3+186.6x_4+79.56},\\
P_7 &= \tfrac{269.59x_1+276.27x_2+288.97x_3+263.74x_4+1246.07}{102.6x_1+110.82x_2+123.69x_3+128.64x_4+435},\\
P_8 &= \tfrac{38.21x_1+56.19x_2+82.1x_3+122.18x_4+58.93}{68.4x_1+73.88x_2+82.46x_3+85.76x_4+290}.
\end{aligned}
$$

Proceeding the same way in subsection 3.4, we develop the following model as:
Model 11

$$
\begin{aligned}
\text{maximize} \quad & Z_5(\mathbf{x}) = -0.0031x_1 + 0.0113x_2 + 0.0375x_3 + 0.1173x_4 + 0.2917\\
\text{maximize} \quad & Z_6(\mathbf{x}) = -0.0278x_1 + 0.0062x_2 + 0.0665x_3 + 0.1457x_4 + 0.4428\\
\text{subject to} \quad & \text{the constraints (14).}
\end{aligned}
$$

Using the intuitionistic fuzzy programming described in subsection 4.1, Model 11 is solved by LINGO software, GA and PSO algorithms and the consequences are listed in Table 11.

Table 11: Optimum results for trapezoidal membership function

Result	Method		
	LINGO 15.0	GA	PSO
Z_5	0.3885	0.4575	0.3992
Z_6	0.6105	0.6514	0.4736
x_1	0.1763	0.0000	4.3711
x_2	0.0000	0.0000	0.0000
x_3	0.2.5952	0.1323	0.1000
x_4	0.0000	1.3713	1.0000
ζ	0.4527	0.6396	0.5632
ξ	0.3319	0.3358	0.2147

A comparison of the results for three membership functions are presented under three different algorithms i.e., LINGO software, GA and PSO respectively in Table 12.

Table 12: Comparison results under LINGO, GA and PSO

Membership function	Method	Z_{111}	Z_{112}	Z_{211}	Z_{212}
Simple linear	LINGO	0.8976	2.5549	1.3161	2.2101
	GA	0.8359	2.8178	1.2185	2.3963
	PSO	1.099	1.6089	1.1501	1.6800
Triangular	LINGO	0.8904	2.2903	1.2282	1.9349
	GA	0.8787	2.3207	1.2162	1.9550
	PSO	0.8604	2.4836	1.2187	2.1169
Trapezoidal	LINGO 15.0	0.8249	2.4057	1.3075	2.0802
	GA	0.9475	2.4597	1.3290	2.1342
	PSO	0.8169	2.5172	1.1442	2.0435

5.2. Real-life application

Sugar cane is an ancient crop of the Austronesian and Papuan people. It is the world's largest crop by production quantity, and India's sugar cane farmers 32.2 million tonnes had been produced by the end of 2017-18. Four types of basic products and by-products are obtained from sugar cane processing. These are listed below:

- Sucrose: Refined sugar, white sugar, raw sugar, jaggery etc., all come in this category. It depends on the process employed and purified of the final product. Basically, these are all sweetening agents used as food additives.
- Bagasse: Bagasse is a fibrous object that is leftovers after the juice is extracted from the sugar cane. It is utilized in the thermal power plant as fuel for the creation of power. Dry bagasse has a calorific value of around 4600 Kcal/kg.
- Molasses: Molasses are the ultimate viscous liquid that is leftover once the practically possible sugar has been extracted from the juice. It is a by-product of the sugar industry. It is fermented and distilled to produce ethanol or ENA. Ethanol is utilized in petrol blending which ENA is for making alcoholic beverages/cosmetics etc.
- Impurities/press mud: Apart from this impurities like wax, non-sugars, inorganic compounds etc. whatever was removed during the juice explaining process, is utilized as manufacture/manure of fertilizers.

Let us consider two sugar mill companies XYZ and PQR which are located in India under the India Sugar Mills Association. The companies produce the above-mentioned four products. Due to weather problems sugar cane cannot produce good quality products or by-products. Furthermore, it would have the same scenario for labour problems. Henceforth, the input data for PQR and XYZ companies are not crisp values. They are considered type-2 fuzzy variables and shown in Tables 13-16 respectively.

Table 13: Sucrose and Bagasse in PQR company

Capacity available	Demand per unit of product	
	Sucrose	Bagasse
Sugar cane (units of quantity)	((77, 90, 130, 140; 0.7, 0.6), (85, 95, 120, 135; 0.5, 0.4))	((35, 67, 89, 99; 1, 0.8), (44, 78, 80, 97; 0.7, 0.6))
Machines (hours)	((26, 37, 47, 56; 0.7, 0.4), (35, 40, 42, 55; 0.3, 0.2))	((20, 23, 29, 31; 0.7, 0.6), (22, 25, 26, 30; 0.5, 0.4))
Profit per unit	((742, 747, 760, 786; 1, 0.9), (746, 748, 750, 770; 0.8, 0.7))	((170, 290, 305, 314; 0.9, 0.8), (180, 295, 300, 310; 0.3, 0.1))
Owned capital	((234, 250, 262, 280; 0.8, 0.7), (245, 255, 260, 276; 0.4, 0.3))	((131, 147, 200, 287; 0.8, 0.6), (143, 150, 156, 268; 0.5, 0.3))
Inventory cost per unit	((69, 77, 97, 100; 0.9, 0.7), (75, 80, 90, 98; 0.5, 0.4))	((40, 45, 60, 96; 0.4, 0.3), (44, 47, 55, 87; 0.2, 0.1))

Table 14: Molasses and Impurities/press mud in PQR company

Capacity available	Demand per unit of product	
	Molasses	Impurities/press mud
Sugar cane (units of quantity)	((40, 44, 52, 80; 1, 0.9), (41, 46, 48, 76; 0.8, 0.7))	((24, 42, 49, 53; 0.6, 0.5), (38, 43, 45, 52; 0.4, 0.1))
Machines (hours)	(7, 9, 15, 28; 0.4, 0.3), (8, 10, 12, 25; 0.2, 0.1))	((1, 3, 6, 9; 0.4, 0.3), (2, 4, 5, 8; 0.2, 0.1))
Profit per unit	((120, 130, 144, 195; 0.5, 0.4), (125, 135, 140, 167; 0.3, 0.2))	((137, 146, 198, 282; 0.9, 0.7), (140, 150, 158, 267; 0.4, 0.2))
Owned capital	((100, 131, 140, 170; 0.5, 0.4), (128, 136, 138, 145; 0.3, 0.1))	((58, 70, 97, 100; 0.6, 0.5), (63, 80, 90, 98; 0.4, 0.2))
Inventory cost per unit	((25, 28, 45, 55; 0.5, 0.3), (27, 30, 35, 52; 0.2, 0.1))	((9, 15, 20, 42; 0.4, 0.3), (12, 16, 17, 40; 0.2, 0.1))

Table 15: Sucrose and Bagasse in XYZ company

Capacity available	Demand per unit of product	
	Sucrose	Bagasse
Sugar cane (units of quantity)	((45, 52, 60, 84; 1, 0.9), (50, 55, 57, 70; 0.8, 0.7))	((24, 55, 63, 62; 0.9, 0.8), (53, 58, 60, 44; 0.7, 0.6))
Machines (hours)	((21, 37, 47, 66; 0.5, 0.4), (35, 40, 42, 47; 0.3, 0.1))	((9, 13, 28, 47; 0.6, 0.4), (10, 14, 25, 34; 0.2, 0.1))
Profit per unit	((234, 250, 262, 280; 0.8, 0.7), (245, 255, 260, 276; 0.4, 0.3))	((123, 137, 146, 177; 0.8, 0.6), (130, 138, 142, 165; 0.5, 0.4))
Owned capital	((76, 79, 97, 110; 1, 0.9), (77, 80, 90, 98; 0.7, 0.6))	((41, 45, 60, 103; 0.5, 0.4), (44, 47, 55, 98; 0.3, 0.2))
Inventory cost per unit	((25, 41, 49, 53; 0.6, 0.5), (38, 43, 45, 52; 0.4, 0.1))	((19, 24, 29, 31; 0.6, 0.5), (22, 25, 26, 30; 0.3, 0.2))

Table 16: Molasses and Impurities/press mud in XYZ company

Capacity available	Demand per unit of product	
	Molasses	Impurities/press mud
Sugar cane (units of quantity)	$((32, 37, 47, 66; 0.6, 0.4),$ $(35, 40, 42, 47; 0.3, 0.1))$	$((20, 23, 29, 31; 0.7, 0.6),$ $(22, 25, 26, 30; 0.5, 0.4))$
Machines (hours)	$((9, 13, 28, 46; 0.4, 0.3),$ $(10, 14, 25, 34; 0.2, 0.1))$	$((7, 9, 15, 28; 0.4, 0.3),$ $(8, 10, 12, 25; 0.2, 0.1))$
Profit per unit	$((63, 70, 97, 110; 1, 0.8),$ $(65, 80, 90, 98; 0.7, 0.6))$	$((33, 40, 50, 60; 0.8, 0.6),$ $(36, 45, 48, 54; 0.4, 0.3))$
Owned capital	$((11, 25, 33, 45; 0.6, 0.5),$ $(20, 30, 31, 42; 0.2, 0.1))$	$((7, 25, 31, 33; 0.4, 0.3),$ $(20, 26, 29, 32; 0.2, 0.1))$
Inventory cost per unit	$((10, 18, 22, 24; 0.4, 0.3),$ $(17, 20, 21, 23; 0.2, 0.1))$	$((5, 8, 18, 20; 0.4, 0.3),$ $(6, 14, 16, 19; 0.2, 0.1))$

Again, the input parameters as sugar cane and machines corresponding to both companies are treated as random variables, and the data are supplied in Table 17.

Table 17: Input parameters for sugar cane and machines

Companies		Parameters		
		α	β	γ
PQR	Sugar cane	0.167	0.25	0.2455
	Machines	0.5	0.23	0.0372
XYZ	Sugar cane	0.167	0.618	003
	Machines	0.25	0.88	0.064

By assumption in both companies, the rising cost and the capital demands are needed which comparative to the individual performances and they have a fixed capital demand of $((1038, 1066, 1190, 1200; 1, 0.9), (1040, 1180, 1186, 1198; 0.8, 0.7))$ and $((745, 747, 760, 790;$
$1, 0.8), (746, 748, 750, 787; 0.3, 0.2))$ respectively. Also, they decide that the inventory cost is additionally to 20% of the whole production in order to promise safety level.

Let the quantities of the four products be $x_{11},\ x_{12},\ x_{13}$ and x_{14} respectively whereas $x_{15},\ x_{16},\ x_{17},\ x_{18}$ are the inventory quantities for them. Both companies have tried to maximize the profitability of the return on investment as well as the general marginal return on investment. Therefore, PQR company has the objective functions $Z_{311},\ Z_{312}$ which appeared in the first level whereas $Z_{411},\ Z_{412}$ are the objective functions of XYZ company in the second level. Now using Tables 12-16, a new model (Model 12) can be formulated with the help of Model 3 as follows:

Model 12

$$\text{maximize} \quad Z_{311} = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14}}{200.01x_{11} + 150.03x_{12} + 90.05x_{13} + 58.03x_{14} + 1050.03}$$

$$\text{maximize} \quad Z_{312} = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14}}{70.03x_{15} + 40.03x_{16} + 24.99x_{17} + 15.06x_{18}}$$

$$\text{maximize} \quad Z_{411} = \frac{200.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14}}{80.01x_{11} + 44.99x_{12} + 19.94x_{13} + 15.04x_{14} + 600.03}$$

$$\text{maximize} \quad Z_{412} = \frac{200.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14}}{30.03x_{15} + 18.03x_{16} + 11.8x_{17} + 8.03x_{18}}$$

$$\text{subject to} \quad \left.\begin{aligned} 0.2(x_{11} + x_{12} + x_{13} + x_{14}) \le (x_{15} + x_{16} + x_{17} + x_{18}), \\ 85x_{11} + 65.05x_{12} + 49.9x_{13} + 29.96x_{14} \le 500.29, \\ 30.04x_{11} + 20.01x_{12} + 10.01x_{13} + 3.09x_{14} \le 160.04, \\ 54.99x_{11} + 45.06x_{12} + 29.95x_{13} + 20.01x_{14} \le 300.16, \\ 27.99x_{11} + 16.05x_{12} + 15.06x_{13} + 10.01x_{14} \le 201.17, \\ x_{11} \ge x_{15},\ x_{12} \ge x_{16},\ x_{13} \ge x_{17},\ x_{14} \ge x_{18}, \\ x_{1j} \ge 0,\ \forall\, j \end{aligned}\right\} \quad (21)$$

Clearly, Model 12 is a bi-level fractional programming problem. We now study the optimal solutions based on three different membership functions and these are discussed as follows:

5.2.1. Linear membership function

In this case, the membership functions of the goals corresponding to both levels are obtained, and their graphical representations are depicted in Figures 10 and 11 respectively.

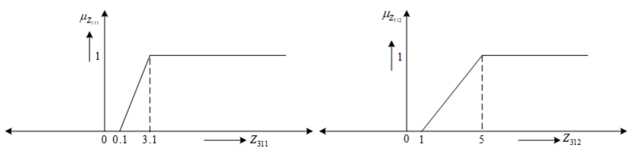

Figure 10: Upper-level membership functions defined as simple linear

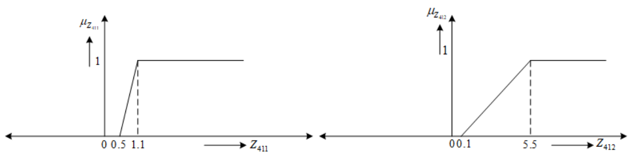

Figure 11: Lower-level membership functions defined as simple linear

$$\mu_9(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{311}(\mathbf{x}) \geq 3.1, \\ Q_1, & \text{if } 0.1 \leq Z_{311}(\mathbf{x}) \leq 3.1, \\ 0, & \text{if } Z_{311}(\mathbf{x}) \leq \ 0.1 \end{cases} \ \& \ \mu_{10}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{312}(\mathbf{x}) \geq 5, \\ Q_2, & \text{if } 1 \leq Z_{312}(\mathbf{x}) \leq 5, \\ 0, & \text{if } Z_{312}(\mathbf{x}) \leq \ 1 \end{cases} \tag{22}$$

$$\mu_{11}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{411}(\mathbf{x}) \geq 1.1, \\ Q_3, & \text{if } 0.5 \leq Z_{411}(\mathbf{x}) \leq 1.1, \\ 0, & \text{if } Z_{411}(\mathbf{x}) \leq \ 1.1 \end{cases} \ \& \ \mu_{12}(\mathbf{x}) = \begin{cases} 1, & \text{if } Z_{412}(\mathbf{x}) \geq 5.5, \\ Q_4, & \text{if } 0.1 \leq Z_{412}(\mathbf{x}) \leq 5.5, \\ 0, & \text{if } Z_{412}(\mathbf{x}) \leq \ 0.1 \end{cases} \tag{23}$$

$$\text{where } Q_1 = \frac{679.989x_{11} + 184.997x_{12} + 90.995x_{13} + 144.147x_{14} - 105.29}{600.03x_{11} + 450.09x_{12} + 270.15x_{13} + 174.09x_{14} + 3150.09},$$

$$Q_2 = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 150x_{14} - 70.03x_{15} - 40.03x_{16} - 25x_{17} - 15.06x_{18}}{280.12x_{15} + 160.12x_{16} + 99.96x_{17} + 60.24x_{18}},$$

$$Q_3 = \ = \frac{160.005x_{11} + 92.535x_{12} + 65.03x_{13} + 27.44x_{14} - 300.015}{48.006x_{11} + 26.994x_{12} + 11.964x_{13} + 9.024x_{14} + 360.018} \text{ and}$$

$$Q_4 = \frac{200.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14} - 3x_{15} - 1.8x_{16} - 1.18x_{17} - 0.803x_{18}}{162.162x_{15} + 97.36x_{16} + 63.72x_{17} + 43.362x_{18}}.$$

Using equations (22)-(23) and 1^{st} order Taylor series approximation as discussed in subsection 3.4 (For comprehensive calculation, we refer to Appendix A), Model 5 can be written as Model 13 which is shown below:

Model 13

$$\begin{aligned} \text{maximize} \quad & Z_7(\mathbf{x}) = 17.3079x_{11} + 4.9196x_{12} + 2.4556x_{13} + 3.6959x_{14} - \\ & 338.112x_{15} - 193.269x_{16} - 120.6543x_{17} - 72.7112x_{18} + 48.9819 \\ \text{maximize} \quad & Z_8(\mathbf{x}) = 4.729x_{11} + 2.7215x_{12} + 1.7904x_{13} + 0.8248x_{14} - 70.871x_{15} \\ & -42.5511x_{16} - 27.8482x_{17} - 18.9509x_{18} + 19.1253 \\ \text{subject to} \quad & \text{the constraints (21).} \end{aligned}$$

Solve the Model 13 by LINGO software. We derive the values $Z_7^0 = 49.2524$, $Z_7^1 = 14.536$, $Z_8^0 = 20.1133$, $Z_8^1 = 5.942$ at $x_{11} = 5.0033$, $x_{12} = 0.0$, $x_{13} = 0.0$, $x_{14} = 1.2508$, $x_{15} = 0.0$, $x_{16} = 0.0$, $x_{17} = 0.0$ and $x_{18} = 1.2508$. With the help of the above results, Model 9 is solved by using LINGO software, GA and PSO respectively and the optimal consequences are shown in Table 18.

Table 18: Optimum results for linear membership function

Result	Method		
	LINGO 15.0	GA	PSO
Z_7	4.3807	4.6713	4.4674
Z_8	7.1491	7.1906	7.1425
x_{11}	1.1918	1.2018	1.1576
x_{12}	0.0000	0.0000	0.0000
x_{13}	0.0000	0.0000	0.0000
x_{14}	4.4160	4.6788	4.5729
x_{15}	0.0000	0.0000	0.0000
x_{16}	0.0000	0.0000	0.0000
x_{17}	0.0000	0.0000	0.0000
x_{18}	1.1216	1.1333	1.1202
ζ	0.9144	0.9725	0.9875
ξ	0.0856	0.0258	0.0003

5.2.2. Triangular membership function

In this case, the membership functions of the goals corresponding to both levels are obtained and their graphical representations are shown in Figures 12 and 13 respectively.

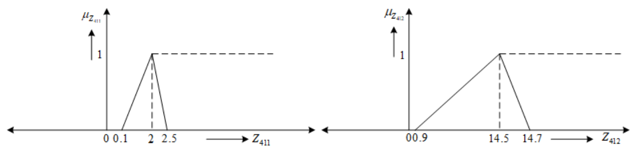

Figure 12: Upper-level membership functions defined as triangular

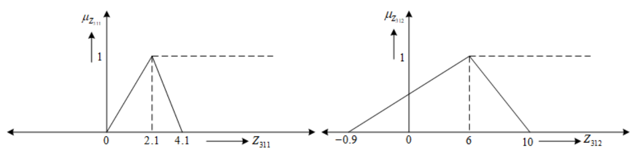

Figure 13: Lower-level membership functions defined as triangular

$$\mu_{13} = \begin{cases} 0, & \text{if } Z_{311}(\mathbf{x}) \geq 4.1, \\ S_1, & \text{if } 2.1 \leq Z_{311}(\mathbf{x}) \leq 4.1, \\ S_2, & \text{if } 0 \leq Z_{311}(\mathbf{x}) \leq 2.1, \\ 0, & \text{if } Z_{311}(\mathbf{x}) \leq \ 0 \end{cases} \text{ and } \mu_{14} = \begin{cases} 0, & \text{if } Z_{312}(\mathbf{x}) \geq 10, \\ S_3, & \text{if } 6 \leq Z_{312}(\mathbf{x}) \leq 10, \\ S_4, & \text{if } -0.9 \leq Z_{312}(\mathbf{x}) \leq 6, \\ 0, & \text{if } Z_{312}(\mathbf{x}) \leq \ -0.9 \end{cases} \tag{24}$$

$$\mu_{15} = \begin{cases} 0, & \text{if } Z_{411}(\mathbf{x}) \geq 2.5, \\ S_5, & \text{if } 2 \leq Z_{411}(\mathbf{x}) \leq 2.5, \\ S_6, & \text{if } 0.1 \leq Z_{411}(\mathbf{x}) \leq 2, \\ 0, & \text{if } Z_{411}(\mathbf{x}) \leq \ 0.1 \end{cases} \text{and } \mu_{16} = \begin{cases} 0, & \text{if } Z_{412}(\mathbf{x}) \geq 14.7, \\ S_7, & \text{if } 14.5 \leq Z_{412}(\mathbf{x}) \leq 14.7, \\ S_8, & \text{if } 0.9 \leq Z_{412}(\mathbf{x}) \leq 14.5, \\ 0, & \text{if } Z_{412}(\mathbf{x}) \leq \ 0.9 \end{cases} \tag{25}$$

$$\begin{aligned}
\text{where } S_1 &= -\frac{120.051x_{11} + 415.123x_{12} + 269.205x_{13} + 87.973x_{14} + 4305.12}{400.02x_{11} + 300.06x_{12} + 180.1x_{13} + 116.06x_{14} + 2100.06}, \\
S_2 &= \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14}}{420.021x_{11} + 315.063x_{12} + 189.105x_{13} + 121.863x_{14} + 2205.063}, \\
S_3 &= \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 149.94x_{17} + 90.36x_{18}}{280.12x_{15} + 160.12x_{16} + 99.96x_{17} + 60.24x_{18}}, \\
S_4 &= \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 63.027x_{15} + 13.554x_{18}}{483.207x_{15} + 276.207x_{16} + 172.431x_{17} + 103.914x_{18}},
\end{aligned}$$

$$\begin{aligned}
S_5 &= \frac{39.99x_{11} + 25.05x_{12} + 35.12x_{13} + 4.88x_{14} - 1200.06}{40.005x_{11} + 22.495x_{12} + 9.97x_{13} + 7.52x_{14} + 300.015}, \\
S_6 &= \frac{192.009x_{11} + 110.531x_{12} + 73.006x_{13} + 33.456x_{14} - 60.003}{152.019x_{11} + 85.481x_{12} + 37.886x_{13} + 28.576x_{14} + 1140.057}, \\
S_7 &= \frac{200.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14} + 171.1x_{17} + 116.435x_{18}}{6.006x_{15} + 3.606x_{16} + 2.36x_{17} + 1.606x_{18}}, \\
S_8 &= \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 16.227x_{16} + 10.62x_{17} + 7.227x_{18}}{462.462x_{15} + 277.662x_{16} + 181.72x_{17} + 123.662x_{18}}.
\end{aligned}$$

Proceeding the same way as in subsection 5.1.4 (For comprehensive calculation, we refer to Appendix B), we formulate Model 14 as follows:

Model 14

$$\begin{aligned}
\text{maximize} \quad & Z_9(\mathbf{x}) = 9.8372x_{11} + 2.7741x_{12} + 1.3811x_{13} + 2.1013x_{14} - 191.264x_{15} - \\
& \qquad 109.329x_{16} - 68.2521x_{17} - 41.1315x_{18} + 28.9451 \\
\text{maximize} \quad & Z_{10}(\mathbf{x}) = 6.0245x_{11} + 1.7392x_{12} + 0.88x_{13} + 1.2876x_{14} - 94.0039x_{15} - \\
& \qquad 56.4399x_{16} - 36.9379x_{17} - 25.1366x_{18} + 24.0883 \\
\text{subject to} \quad & \text{the constraints (21).}
\end{aligned}$$

Using the proposed intuitionistic fuzzy programming which is discussed in subsection 4.1, the solutions are obtained by LINGO software, GA and PSO and the consequences are summarized in Table 19.

Table 19: Optimum results for triangular membership function

Result	Method		
	LINGO	GA	PSO
Z_9	2.4701	2.5459	1.6472
Z_{10}	7.9374	7.9845	7.4352
x_{11}	1.0218	1.0209	1.0013
x_{12}	0.0000	0.0000	0.0000
x_{13}	0.0000	0.0000	0.0000
x_{14}	4.5912	4.7901	4.8652
x_{15}	0.0000	0.0000	0.0000
x_{16}	0.0000	0.0000	0.0000
x_{17}	0.0000	0.0000	0.0000
x_{18}	1.1226	1.1307	1.1517
ζ	0.7209	0.9254	0.8579
ξ	0.0796	0.0212	0.0586

5.2.3. Trapezoidal membership function

In this case, the membership functions (see Figures 14 and 15) of the goals depend on four parameters and are computed as follows:

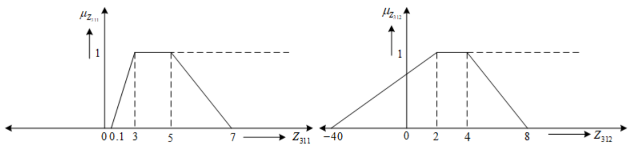

Figure 14: Upper-level membership functions defined as trapezoidal

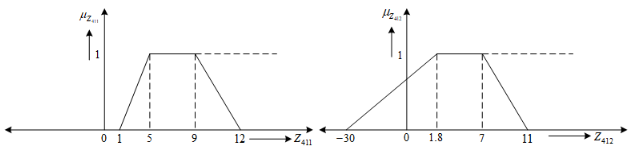

Figure 15: Lower-level membership functions defined as trapezoidal

$$\mu_{17}(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{311}(\mathbf{x}) \geq 7, \\ L_1, & \text{if } 5 \leq Z_{311}(\mathbf{x}) \leq 7, \\ 1 & \text{if } 3 \leq Z_{311}(\mathbf{x}) \leq 5, \\ L_2, & \text{if } 0.1 \leq Z_{311}(\mathbf{x}) \leq 3, \\ 0, & \text{if } Z_{311}(\mathbf{x}) \leq \ 0.1 \end{cases} \text{ and } \mu_{18}(\mathbf{x}) = \begin{cases} 0, & \text{if } Z_{312}(\mathbf{x}) \geq 8, \\ L_3, & \text{if } 4 \leq Z_{312}(\mathbf{x}) \leq 8, \\ 1 & \text{if } 2 \leq Z_{312}(\mathbf{x}) \leq 4, \\ L_4, & \text{if } -40 \leq Z_{312}(\mathbf{x}) \leq 2, \\ 0, & \text{if } Z_{312}(\mathbf{x}) \leq \ -40 \end{cases} \tag{26}$$

$$\mu_{19} = \begin{cases} 0, & \text{if } Z_{411}(\mathbf{x}) \geq 12, \\ L_5, & \text{if } 9 \leq Z_{411}(\mathbf{x}) \leq 12, \\ 1 & \text{if } 5 \leq Z_{411}(\mathbf{x}) \leq 9, \\ L_6, & \text{if } 1 \leq Z_{411}(\mathbf{x}) \leq 5, \\ 0, & \text{if } Z_{411}(\mathbf{x}) \leq \ 1 \end{cases} \text{ and } \mu_{20} = \begin{cases} 0, & \text{if } Z_{412}(\mathbf{x}) \geq 11, \\ L_7, & \text{if } 7 \leq Z_{412}(\mathbf{x}) \leq 11, \\ 1 & \text{if } 1.8 \leq Z_{412}(\mathbf{x}) \leq 7, \\ L_8, & \text{if } -30 \leq Z_{412}(\mathbf{x}) \leq 1.8, \\ 0, & \text{if } Z_{412}(\mathbf{x}) \leq \ -30 \end{cases} \tag{27}$$

where
$$L_1 = -\frac{300.06x_{11} + 550.15x_{12} + 350.25x_{13} + 140.2x_{14} + 5250.15}{400.02x_{11} + 300.06x_{12} + 180.1x_{13} + 116.06x_{14} + 2100.06},$$
$$L_2 = \frac{679.989x_{11} + 184.997x_{12} + 90.995x_{13} + 144.147x_{14} - 105.003}{580.029x_{11} + 435.087x_{12} + 261.145x_{13} + 168.287x_{14} + 3045.087},$$
$$L_3 = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 99.96x_{17} + 60.24x_{18}}{280.12x_{15} + 160.12x_{16} + 99.96x_{17} + 60.24x_{18}},$$
$$L_4 = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 999.6x_{17} + 602.4x_{18}}{2941.26x_{15} + 1681.26x_{16} + 1049.58x_{17} + 632.52x_{18}},$$
$$L_5 = -\frac{520.08x_{11} + 289.88x_{12} + 104.46x_{13} + 100.4x_{14} + 5400.27}{240.03x_{11} + 134.97x_{12} + 59.82x_{13} + 45.12x_{14} + 1800.09},$$
$$L_6 = \frac{184.008x_{11} + 106.032x_{12} + 71.012x_{13} + 31.95x_{14} - 120.006}{144.018x_{11} + 80.982x_{12} + 35.892x_{13} + 27.072x_{14} + 1080.054},$$

$$L_7 = \frac{200.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14} + 435.435x_{15} + 116.435x_{18}}{6.006x_{15} + 3.606x_{16} + 2.36x_{17} + 1.606x_{18}},$$
$$L_8 = \frac{201.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14} + 900.9x_{15} + 240.9x_{18}}{954.954x_{15} + 573.354x_{16} + 375.24x_{17} + 255.354x_{18}}.$$

Proceeding the same way which is described in subsection 5.1.5 (For comprehensive calculation, we refer to Appendix C), we formulate the following model as:

Model 15

$$\begin{aligned} \text{maximize} \quad & Z_{11}(\mathbf{x}) = 0.9408x_{11} + 0.2423x_{12} + 0.1169x_{13} + 0.1972x_{14} - \\ & \qquad 17.3386x_{15} - 9.911x_{16} - 6.1872x_{17} - 3.7287x_{18} + 5.8909 \\ \text{maximize} \quad & Z_{12}(\mathbf{x}) = 0.8341x_{11} + 0.4802x_{12} + 0.3185x_{13} + 0.1451x_{14} - \\ & \qquad 12.0347x_{15} - 7.2257x_{16} - 4.7289x_{17} - 3.2181x_{18} + 4.3423 \\ \text{subject to} \quad & \text{the constraints (21).} \end{aligned}$$

Using the intuitionistic fuzzy programming stated in subsection 4.1, Model 15 is solved by LINGO software, GA and PSO, and the consequences are listed in Table 20.

Table 20: Optimum results for trapezoidal membership function

Result	Method		
	LINGO 15.0	GA	PSO
Z_{11}	1.3480	1.4202	1.3793
Z_{12}	0.2769	0.3760	0.3435
x_{11}	0.0000	0.0000	0.0000
x_{12}	0.0000	0.0000	0.0000
x_{13}	0.0000	0.0000	0.0000
x_{14}	5.7610	4.2920	4.1830
x_{15}	0.0000	0.0000	0.0000
x_{16}	0.0000	0.0000	0.0000
x_{17}	0.0000	0.0000	0.0000
x_{18}	1.5230	1.4260	1.4312
ζ	0.7209	0.9678	0.9257
ξ	0.0796	0.0213	0.0037

A comparison of the results for three membership functions is presented under LINGO software, GA and PSO respectively in Table 21.

Table 21: Comparison results under LINGO, GA and PSO

Membership function	Method	Z_{311}	Z_{312}	Z_{411}	Z_{412}
Simple linear	LINGO 15.0	0.9688	88.5916	0.5156	43.6082
	GA	0.9878	90.3960	0.5270	44.3874
	PSO	0.9671	88.6779	0.5140	43.5121
Triangular	LINGO 15.0	0.9230	83.0280	0.4860	40.4770
	GA	0.9352	84.1477	0.4931	40.9330
	PSO	0.9333	82.4716	0.4916	40.0466
Trapezoidal	LINGO 15.0	0.6240	37.6634	0.2933	16.4685
	GA	0.4954	29.9683	0.2258	13.1038
	PSO	0.4852	29.1011	0.2206	12.7246

5.3. Results implication and discussion

This section shows the discussion related to the optimal consequences for the crisp form of the proposed model procured by the various methodologies and algorithms (PSO and GA). Here, we present the symposium on the experimentation consequences performed on the proposed model under this study. In this study, we solve a BLFP with probabilistic constraints in the Stackelberg game under intuitionistic fuzzy programming. From Table 12, it is observed that the optimal objective function values can be procured using LINGO software, GA and PSO. In the case of GA, we have the best solutions as the maximum objective function values for both levels DMs $Z_{111} = 0.8359,\ Z_{112} = 2.8178$ and $Z_{211} = 1.2185,\ Z_{212} = 2.3963$ respectively (for simple linear membership). In the case of PSO, the results are obtained as $Z_{111} = 1.099,\ Z_{112} = 1.6086,\ Z_{211} = 1.1501$ and $Z_{212} = 1.68$ (for simple linear membership) which are less than the result of GA. Similarly, the best results are achieved by using GA only for considering the other two membership functions. Again, from Table 22, it is concluded that the maximum profitability of the return on investment and the marginal return on investment corresponding to PQR and XYZ companies are obtained for consideration of the simple linear membership function using GA. Also, Model 6 is solved using intuitionistic fuzzy programming discussed in subsection 4.1 corresponding to numerical example and real-life application with $t = 0.01$ and $d = 0.5$.

In the manner of Kuo and Huang [1], the relative error rate (RE) is given by RE=$\frac{|F_L^* - F_A^*|}{F_L^*} \times 100$, where F_A^* indicate the optimal solutions in accordance with GA/PSO whereas F_L^* signify the optimal solutions on the basis of LINGO software. In addition, the standard deviation (SD) is given by SD=$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(F_{A_i}^* - F_{A_f}^*\right)^2}$, where $i = 1, 2, \ldots, N$ and $F_{A_i}^*$ represents the optimal solution for GA/PSO at i^{th} run and $F_{A_f}^*$ indicates the average of N optimal solutions for GA/PSO. To study the order of convergence, we have established the RE and SD for each membership function subsequent to LINGO software, GA and PSO respectively. The consequences are exposed in Tables 22-24 and compare the results based on both levels of DMs respectively. Also, the results are displayed in Figures 16-19.

Table 22: Convergence results under LINGO

Membership function		LINGO			
		Z_{311}	Z_{312}	Z_{411}	Z_{412}
Linear	optimal cost	0.9688	88.5916	0.5156	43.6082
	RE	0.0206%	0.0005%	0.0776%	0.0005%
	SD	0.00	0.00	0.00	0.00
Triangular	optimal cost	0.9230	83.0280	0.4860	40.4770
	RE	0.0542%	0.0013%	0.1235%	0.0007%
	SD	0.00	0.00	0.00	0.00
Trapezoidal	optimal cost	0.6240	37.6634	0.2933	16.4685
	RE	0.0801%	0.0013%	0.1023%	0.0018%
	SD	0.00	0.00	0.00	0.00

Table 23: Convergence results under GA

Membership function		GA			
		Z_{311}	Z_{312}	Z_{411}	Z_{412}
Linear	optimal cost	0.9690	88.5920	0.5160	43.6084
	RE	0.00	0.00	0.00	0.00
	SD	0.00003	0.00002	0.0001	0.0002
Triangular	optimal cost	0.9235	83.0291	0.4866	40.4773
	RE	0.00	0.00	0.00	0.00
	SD	0.00004	0.00014	0.0002	0.0001
Trapezoidal	optimal cost	0.6245	37.6639	0.2936	16.4688
	RE	0.00	0.00	0.00	0.00
	SD	0.00004	0.00002	0.0001	0.0002

Table 24: Convergence results under PSO

Membership function		PSO			
		Z_{311}	Z_{312}	Z_{411}	Z_{412}
Linear	optimal cost	0.9689	88.5918	0.5154	43.6083
	RE	0.0103%	0.0002%	0.1164%	0.0002%
	SD	0.00004	0.00002	0.0001	0.0002
Triangular	optimal cost	0.9232	83.0278	0.4861	40.4772
	RE	0.0325%	0.0016%	0.1029%	0.0002%
	SD	0.00038	0.0018	0.0001	0.0001
Trapezoidal	optimal cost	0.6242	37.6635	0.2935	16.4686
	RE	0.0481%	0.0011%	0.0341%	0.0012%
	SD	0.00039	0.00017	0.0002	0.0016

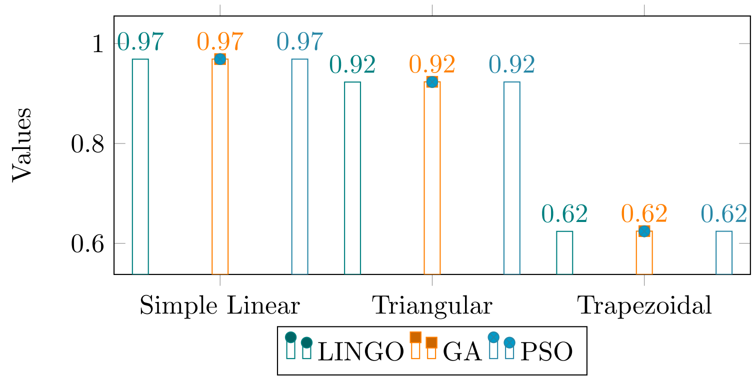

Figure 16: Comparison of results in Z_{311} under different membership function

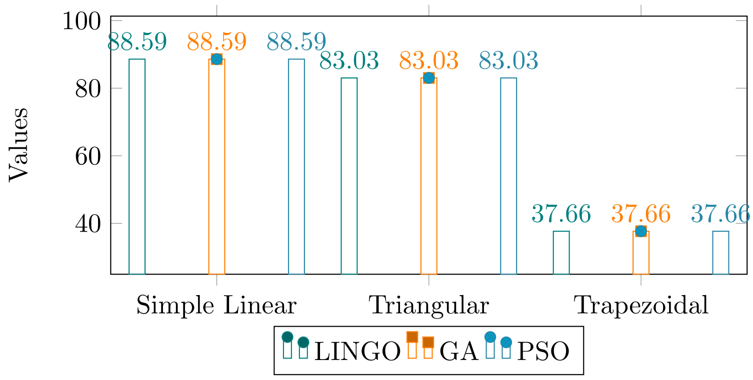

Figure 17: Comparison of results in Z_{312} under different membership function

It is noticed from Tables 22-24 that the GA contributes a superior solution in contrast to LINGO software /PSO in each case and the SD in LINGO software is

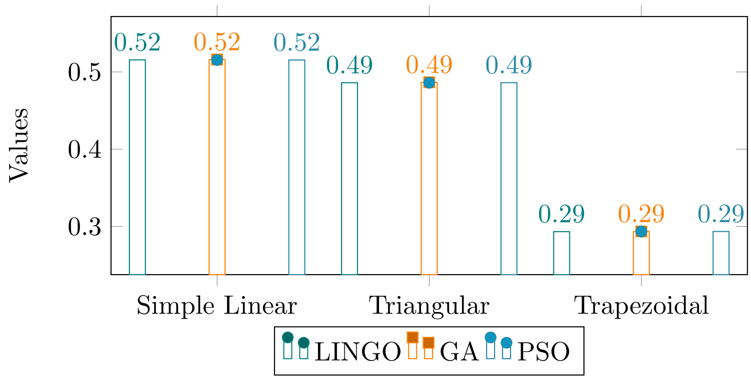

Figure 18: Comparison of results in Z_{411} under different membership function

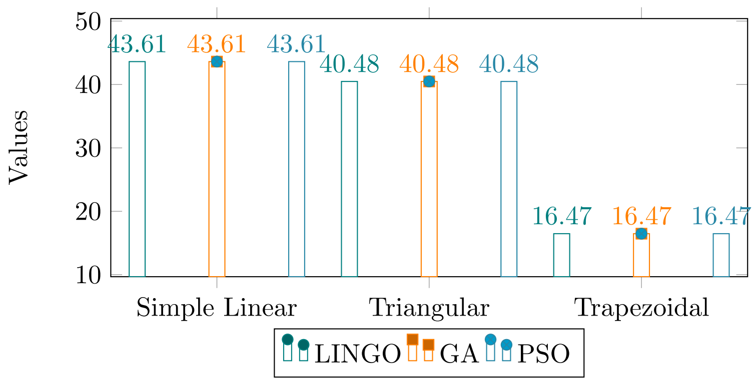

Figure 19: Comparison of results in Z_{412} under different membership function

zero which concludes to an established solution than GA/PSO.

6. IMPLICATIONS AND INSIGHTS

In this Section, we have presented some practical and managerial implications. These implications are discussed below:

6.1. Practical implications

The following practical implications are derived from this research:

- The proposed model incorporates the Stackelberg game and Weibull distribution invention in improving decision-making models in consequences with uncertainty.
- It allows for additional realistic demonstration of imprecise information, important to improve considered planning and compromise outcomes, principally in fields like economics, supply chain management, or multi-agent systems.
- The combination of the Stackelberg game, probabilistic fractional multi-objective programming, and type-2 fuzzy numbers provides a refined understanding of leadership dynamics, enabling supplementary vigorous strategies in complex, uncertain environments.

6.2. Managerial implications

The subsequent managerial implications are sketched from the proposed study as:

- To improve a Stackelberg game under a probabilistic fuzzy multi-objective fractional linear programming, the introduction of type-2 fuzzy numbers and Weibull distribution in the proposed model has different facilities from numerous sides.
- The proposed model will also be advantageous to demonstrate the uncertain parameters in problematical circumstances. Since uncertainty is predictable, only lone-type uncertainty cannot picture the real-life parameters appropriately in a substitute situation. Therefore the encountered varied uncertainty will help to figure out the parameters with the help of acceptance degrees as well as ancient data by interrelating among different situations.
- The results of the model can advantage the managers. The results that are achieved from three different methods can oblige the managers to use the additional suitable solution approach. For example, the proposed model with t=0.01 and d=0.5 can be beneficial to the managers to provide the maximum profitability of the return on investment and the marginal return on investment. Moreover, RE and SD under different membership functions will help the managers set the model parameters and satisfactory levels.

7. CONCLUSION, LIMITATION AND SUBSEQUENT DIRECTION

In this study, we have addressed a BLFP where the cost parameters and the parameters of the constraint except the right-hand side are type-2 fuzzy numbers whereas the right-hand side of the constraints follows the Weibull distribution. The most noteworthy findings and deeds of the study have been momentarily stated as follows:

- A BLFP in Stackelberg game framework, incorporating T2FVs has been formulated.
- The 1^{st} order Taylor series approximation has been utilized to solve in the Stackelberg game and considering the equal weights of the objective functions.
- Probabilistic constraints can be followed by Weibull distribution to convert into a deterministic form using stochastic programming.
- An analysis of the objective functions by LINGO 15.0 iterative scheme, GA, and PSO have been exhibited, utilizing real-life data.

This study has been structured for the original to establish the vital approach for investment and marginal return in Stackelberg game problems. The outcomes of this study have substantial consequences for employing Trapezoidal type-2 fuzzy variables at constraint parameters except the right-hand side parameters of the constraints are taken as Weibull distribution as well as both level fractional objective functions in order to not only meet general marginal return on investment but also maximize profitability of the return on investment.

In addition to all the advantages managed about by the proposed model, this study has a little of limitations that can be skilled in further studies. The greatest significant item on the list is assembling dependable, realistic, and precise data. Although peer-reviewed journals are the main data source of this study, the uncertainty that comes from the human aspect and software handling cannot be overlooked. Therefore, essential data of the model have to be estimated in the course of statistical or neutrosophic fuzzy logic-based approach in the future. However, the proposed model applies to different types of BLFP including production planning under Pythagorean fuzzy TOPSIS environment [48], and interval type-2 fuzzy sets [49] in a Stackelberg game context in further study.

As a search of this study, the model involves an extensive acceptance of functional limitations like rising costs and capital demands. The investment and marginal return problem in the Stackelberg game based on bi-level programming under the Trapezoidal type-2 fuzzy variable would be an inspiring enhancement to this study.

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Appendix A: A comprehensive calculation to show $Z_7(\mathbf{x})$ and $Z_8(\mathbf{x})$ (for Linear membership function) are given underneath:

Solve the equations (22) and (23) with the constraints (21) by using LINGO 15.0 iterative scheme, we have $\mu_9(\mathbf{x}) = 0.5551$, at $x_{11} = 5.2760$, $x_{14} = 0.5014$, $x_{15} = 0.9048$, $x_{18} = 0.2507$; $\mu_{10}(\mathbf{x}) = 48.7193$, at $x_{11} = 2.6941$, $x_{14} = 0.6735$, $x_{18} = 0.6735$; $\mu_{11}(\mathbf{x}) = 0.9126$, at $x_{11} = 5.1213$, $x_{13} = 0.6190$, $x_{15} = 5.1213, x_{17} = 0.6191$, and $\mu_{12}(\mathbf{x}) = 19.2380$, at $x_{11} = 4.0643$, $x_{14} = 1.0161$, $x_{18} = 1.0161$.

Now, from the equation (22)

$$\mu_9 = \frac{679.989x_{11} + 184.997x_{12} + 90.995x_{13} + 144.147x_{14} - 105.29}{600.03x_{11} + 450.09x_{12} + 270.15x_{13} + 174.09x_{14} + 3150.09},$$

$$\mu_{10} = \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 150x_{14} - 70.03x_{15} - 40.03x_{16}}{280.12x_{15} + 160.12x_{16} + 99.96x_{17} + 60.24x_{18}}$$

Therefore,

$$\begin{aligned}\widehat{\mu_9}(\mathbf{x}) =& \quad 0.5551 + \Bigg(\big(x_{11} - 5.276\big)\frac{\partial\mu_9(x^*)}{\partial x_{11}} + x_{12}\frac{\partial\mu_9(x^*)}{\partial x_{12}} + x_{13}\frac{\partial\mu_9(x^*)}{\partial x_{13}} + \\ & \big(x_{14} - 0.5014\big)\frac{\partial\mu_9(x^*)}{\partial x_{14}}\Bigg) \\ =& \quad 0.5551 + (x_{11} - 5.276) \times 0.0547 - x_{12} \times 0.01 - x_{13} \times 0.0092 + \\ & (x_{14} - 0.5014) \times 0.0075 \\ =& \quad 0.0547x_{11} - 0.01x_{12} - 0.0092x_{13} + 0.0075x_{14} + 0.2627\end{aligned}$$

and

$$\begin{aligned}\widehat{\mu_{10}}(\mathbf{x}) =& \quad 48.7193 + \Bigg(\big(x_{11} - 2.6941\big)\frac{\partial\mu_{10}(x^*)}{\partial x_{11}} + x_{12}\frac{\partial\mu_{10}(x^*)}{\partial x_{12}} + \\ & x_{13}\frac{\partial\mu_{10}(x^*)}{\partial x_{13}} + \big(x_{14} - 0.6735\big)\frac{\partial\mu_{10}(x^*)}{\partial x_{14}} + x_{15}\frac{\partial\mu_{10}(x^*)}{\partial x_{15}} + \\ & x_{16}\frac{\partial\mu_{10}(x^*)}{\partial x_{16}} + x_{17}\frac{\partial\mu_{10}(x^*)}{\partial x_{17}} + \big(x_{18} - 0.6735\big)\frac{\partial\mu_{10}(x^*)}{\partial x_{18}}\Bigg) \\ =& \quad 48.7193 + (x_{11} - 2.6941) \times 17.2532 + x_{12} \times 4.9296 + x_{13} \times 2.4648 + \\ & (x_{14} - 0.6735) \times 3.6959 - x_{15} \times 333.112 - x_{16} \times 193.269 - x_{17} \times \\ & 120.6543 - (x_{18} - 0.6735) \times 72.7112 \\ =& \quad 17.2532x_{11} + 4.9296x_{12} + 2.4648x_{13} + 3.6959x_{14} - 338.112x_{15} - \\ & 193.269x_{16} - 120.6543x_{17} - 72.7112x_{18} + 48.7192\end{aligned}$$

Then

$$\begin{aligned}Z_7(\mathbf{x}) =& \quad \widehat{\mu_9}(\mathbf{x}) + \widehat{\mu_{10}}(\mathbf{x}) \\ =& \quad 17.3079x_{11} + 4.9196x_{12} + 2.4556x_{13} + 3.6959x_{14} - 338.112x_{15} - \\ & 193.269x_{16} - 120.6543x_{17} - 72.7112x_{18} + 48.7192\end{aligned}$$

Again, from the equation (23)

$$\begin{aligned}\mu_{11} =& \quad \frac{160.005x_{11} + 92.535x_{12} + 65.03x_{13} + 27.44x_{14} - 300.015}{48.006x_{11} + 26.994x_{12} + 11.964x_{13} + 9.024x_{14} + 360.018}, \\ \mu_{12} =& \quad \frac{200.01x_{11} + 115.1x_{12} + 75x_{13} + 34.96x_{14} - 1.2x_{17} - 0.8x_{18}}{162.162x_{15} + 43.362x_{18}}\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\mu_{11}}(\mathbf{x}) = {} & 0.9126 + \Bigg(\big(x_{11} - 5.1213\big)\frac{\partial \mu_{11}(x^*)}{\partial x_{11}} + x_{12}\frac{\partial \mu_{11}(x^*)}{\partial x_{12}} + \\ & \big(x_{13} - 0.6190\big)\frac{\partial \mu_{11}(x^*)}{\partial x_{13}} + x_{14}\frac{\partial \mu_{11}(x^*)}{\partial x_{14}} \Bigg) \\ = {} & 0.9126 + (x_{11} - 5.1213) \times 0.1895 - x_{12} \times 0.1107 + (x_{13} - \\ & 0.6190)x_{13} \times 0.0882 + x_{14} \times 0.0313 \\ = {} & 0.1895x_{11} + 0.1107x_{12} + 0.0882x_{13} + 0.0313x_{14} - 0.1125\end{aligned}$$

and

$$\begin{aligned}\widehat{\mu_{12}}(\mathbf{x}) = {} & 19.2380 + \Bigg(\big(x_{11} - 4.0643\big)\frac{\partial \mu_{12}(x^*)}{\partial x_{11}} + x_{12}\frac{\partial \mu_{12}(x^*)}{\partial x_{12}} + \\ & x_{13}\frac{\partial \mu_{12}(x^*)}{\partial x_{13}} + \big(x_{14} - 1.0161\big)\frac{\partial \mu_{12}(x^*)}{\partial x_{14}} + x_{15}\frac{\partial \mu_{12}(x^*)}{\partial x_{15}} + \\ & x_{16}\frac{\partial \mu_{12}(x^*)}{\partial x_{16}} + x_{17}\frac{\partial \mu_{12}(x^*)}{\partial x_{17}} + \big(x_{18} - 1.0161\big)\frac{\partial \mu_{12}(x^*)}{\partial x_{18}} \Bigg) \\ = {} & 19.238 + (x_{11} - 4.0643) \times 4.5395 + x_{12} \times 2.6108 + x_{13} \times 1.7022 \\ & +(x_{14} - 1.0161) \times 0.7935 - x_{15} \times 70.8713 - x_{16} \times 42.5511 - x_{17} \\ & \times 27.8482 - (x_{18} - 1.0161) \times 18.9509 \\ = {} & 4.5395x_{11} + 2.6108x_{12} + 1.7022x_{13} + 0.7935x_{14} - 70.8713x_{15} - \\ & 42.5511x_{16} - 27.8482x_{17} - 18.9509x_{18} + 19.2378\end{aligned}$$

Then

$$\begin{aligned}Z_8(\mathbf{x}) = {} & \widehat{\mu_{11}}(\mathbf{x}) + \widehat{\mu_{12}}(\mathbf{x}) \\ = {} & 4.7290x_{11} + 2.7215x_{12} + 1.7904x_{13} + 0.8248x_{14} - 70.8713x_{15} - \\ & 42.5511x_{16} - 27.8482x_{17} - 18.9509x_{18} + 19.1253\end{aligned}$$

Appendix B: A comprehensive calculation to show $Z_9(\mathbf{x})$ and $Z_{10}(\mathbf{x})$ (for Triangular membership function) are given underneath:
Solve the equations (24) and (25) with the constraints (21) by using LINGO 15.0 iterative scheme, we have $\mu_{13}(\mathbf{x}) = 0.8407,$ at $x_{11} = 5.2760,\ x_{14} = 0.5014,\ x_{15} = 0.9048,\ x_{18} = 0.2507;$ $\mu_{14}(\mathbf{x}) = 28.5180,$ at $x_{11} = 2.7609,\ x_{14} = 0.6902,\ x_{18} = 0.6902;$ $\mu_{15}(\mathbf{x}) = 0.4987,$ at $x_{11} = 5.1213,\ x_{13} = 0.6191,\ x_{15} = 0.5290, x_{17} = 0.6191,$ and $\mu_{16}(\mathbf{x}) = 23.9131,$ at $x_{11} = 3.7960,\ x_{14} = 0.9490,\ x_{18} = 0.9490.$
Now, from the equation (24)

$$\begin{aligned}\mu_{13} = {} & \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14}}{420.021x_{11} + 315.063x_{12} + 189.105x_{13} + 121.863x_{14} + 2205.063}, \\ \mu_{14} = {} & \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 63.027x_{15} + 13.554x_{18}}{483.207x_{15} + 276.207x_{16} + 172.431x_{17} + 103.914x_{18}}\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\mu_{13}}(\mathbf{x}) = & \quad 0.8407 + \Bigg(\big(x_{11} - 5.2760\big)\frac{\partial \mu_{13}(x^*)}{\partial x_{11}} + x_{12}\frac{\partial \mu_{13}(x^*)}{\partial x_{12}} + x_{13}\frac{\partial \mu_{13}(x^*)}{\partial x_{13}} + \\ & \big(x_{14} - 0.5014\big)\frac{\partial \mu_{13}(x^*)}{\partial x_{14}}\Bigg) \\ = & \quad 0.8407 + (x_{11} - 5.2760) \times 0.0774 - x_{12} \times 0.0145 - x_{13} \times 0.0132 + \\ & (x_{14} - 0.5014) \times 0.0106 \\ = & \quad 0.0774x_{11} - 0.0145x_{12} - 0.0132x_{13} + 0.0106x_{14} + 0.4270\end{aligned}$$

and

$$\begin{aligned}\widehat{\mu_{14}}(\mathbf{x}) = & \quad 28.5180 + (x_{11} - 2.7609) \times 9.7598 + x_{12} \times 2.7886 + x_{13} \times 1.3943 + \\ & (x_{14} - 0.6902) \times 2.0907 - x_{15} \times 191.2643 - x_{16} \times 109.3290 - x_{17} \times \\ & 68.2521 + (x_{18} - 0.6902) \times (-41.1315) \\ = & \quad 9.7598x_{11} + 2.7886x_{12} + 1.3943x_{13} + 2.090x_{14} - 191.2643x_{15} - \\ & 109.3290x_{16} - 68.2521x_{17} - 41.1315x_{18} + 28.5181\end{aligned}$$

Then

$$\begin{aligned}Z_9(\mathbf{x}) = & \quad \widehat{\mu_{13}}(\mathbf{x}) + \widehat{\mu_{14}}(\mathbf{x}) \\ = & \quad 9.8372x_{11} + 2.7741x_{12} + 1.3811x_{13} + 201013x_{14} - 191.2643x_{15} \\ & -109.3290x_{16} - 68.2521x_{17} - 41.1315x_{18} + 28.9451\end{aligned}$$

Again, from the equation (25)

$$\begin{aligned}\mu_{15} = & \quad \frac{192.009x_{11} + 110.531x_{12} + 73.006x_{13} + 33.456x_{14} - 60.003}{152.019x_{11} + 85.481x_{12} + 37.886x_{13} + 28.576x_{14} + 1140.057}, \\ \mu_{16} = & \quad \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 16.227x_{16} + 7.27x_{18}}{462.462x_{15} + 277.662x_{16} + 181.72x_{17} + 123.662x_{18}}\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\mu_{15}}(\mathbf{x}) = & \quad 0.4987 + (x_{11} - 5.1213) \times 0.0598 + x_{12} \times 0.0350 + (x_{13} - 0.6191)x_{13} \\ & \times 0.0279 + x_{14} \times 0.0099 \\ = & \quad 0.0598x_{11} + 0.0350x_{12} + 0.0279x_{13} + 0.0099x_{14} + 0.1751\end{aligned}$$

and

$$\begin{aligned}\widehat{\mu_{16}}(\mathbf{x}) = & \quad 23.9131 + (x_{11} - 3.7960) \times 5.9647 + x_{12} \times 1.7042 + x_{13} \times 0.8521 \\ & +(x_{14} - 0.9490) \times 1.2777 - x_{15} \times 94.0039 - x_{16} \times 56.4399 - x_{17} \times \\ & 36.9379 + (x_{18} - 0.9490) \times (-25.1366) \\ = & \quad 5.9647x_{11} + 1.7042x_{12} + 0.8521x_{13} + 1.2777x_{14} - 94.0039x_{15} - \\ & 56.4399x_{16} - 36.9379x_{17} - 25.1366x_{18} + 23.9132\end{aligned}$$

Then

$$\begin{aligned} Z_{10}(\mathbf{x}) = \quad & \widehat{\mu_{15}}(\mathbf{x}) + \widehat{\mu_{16}}(\mathbf{x}) \\ = \quad & 6.0245x_{11} + 1.7392x_{12} + 0.88x_{13} + 1.2876x_{14} - 94.0039x_{15} - \\ & 56.4399x_{16} - 36.9379x_{17} - 25.1366x_{18} + 24.0883 \end{aligned}$$

Appendix C: A comprehensive calculation to show $Z_{11}(\mathbf{x})$ and $Z_{12}(\mathbf{x})$ (for Trapezoida membership function) are given underneath:
Solve the equations (26) and (27) with the constraints (21) by using LINGO 15.0 iterative scheme, we have $\mu_{17}(\mathbf{x}) = 0.5743,$ at $x_{11} = 5.2760,\ x_{14} = 0.5014,\ x_{15} = 0.9048,\ x_{18} = 0.2507;$ $\mu_{18}(\mathbf{x}) = 5.6161,$ at $x_{11} = 5.0033,\ x_{14} = 1.2508,\ x_{18} = 1.2508;$ $\mu_{19}(\mathbf{x}) = 0.4709,$ at $x_{11} = 5.1213,\ x_{13} = 0.6191,\ x_{15} = 0.5290, x_{17} = 0.6191,$ and $\mu_{20}(\mathbf{x}) = 4.2134,$ at $x_{11} = 4.0643,\ x_{14} = 1.0161,\ x_{18} = 1.0161.$
Now, from the equation (26)

$$\begin{aligned} \mu_{17} = \quad & \frac{679.989x_{11} + 184.997x_{12} + 90.995x_{13} + 144.147x_{14} - 105.003}{580.029x_{11} + 435.087x_{12} + 261.145x_{13} + 168.287x_{14} + 3045.087}, \\ \mu_{18} = \quad & \frac{699.99x_{11} + 200x_{12} + 100x_{13} + 149.95x_{14} + 999.6x_{17} + 602.4x_{18}}{2941.26x_{15} + 1681.26x_{16} + 1049.58x_{17} + 632.52x_{18}} \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\mu_{17}}(\mathbf{x}) = \quad & 0.5743 + (x_{11} - 5.2760) \times 0.0560 - x_{12} \times 0.0105 - x_{13} \times 0.0095 + \\ & (x_{14} - 0.5014) \times 0.0077 \\ = \quad & 0.0560x_{11} - 0.0105x_{12} - 0.0095x_{13} + 0.0077x_{14} + 0.2749 \end{aligned}$$

and

$$\begin{aligned} \widehat{\mu_{18}}(\mathbf{x}) = \quad & 5.6161 + (x_{11} - 5.0033) \times 0.8848 + x_{12} \times 0.2528 + x_{13} \times 0.1264 + \\ & (x_{14} - 1.2508) \times 0.1895 - x_{15} \times 17.3386 - x_{16} \times 9.9110 - x_{17} \times \\ & 6.1872 + (x_{18} - 1.2508) \times (-3.7287) \\ = \quad & 0.8848x_{11} + 0.2528x_{12} + 0.1264x_{13} + 0.1895x_{14} - 17.3386x_{15} - \\ & 9.9110x_{16} - 6.1872x_{17} - 3.7287x_{18} + 5.6160 \end{aligned}$$

Then

$$\begin{aligned} Z_{11}(\mathbf{x}) = \quad & \widehat{\mu_{17}}(\mathbf{x}) + \widehat{\mu_{18}}(\mathbf{x}) \\ = \quad & 0.9408x_{11} + 0.2423x_{12} + 0.1169x_{13} + 0.1972x_{14} - 17.3386x_{15} - \\ & 9.9110x_{16} - 6.1872x_{17} - 3.7287x_{18} + 5.8909 \end{aligned}$$

Again, from the equation (27)

$$\begin{aligned} \mu_{19} = \quad & \frac{184.008x_{11} + 106.032x_{12} + 71.012x_{13} + 31.95x_{14} - 120.006}{144.018x_{11} + 80.982x_{12} + 35.892x_{13} + 27.072x_{14} + 1080.054}, \\ \mu_{20} = \quad & \frac{201.01x_{11} + 115.03x_{12} + 75x_{13} + 34.96x_{14} + 900.9x_{15} + 240.9x_{18}}{954.954x_{15} + 573.354x_{16} + 375.24x_{17} + 255.354x_{18}} \end{aligned}$$

Therefore,

$$
\begin{aligned}
\widehat{\mu_{19}}(\mathbf{x}) = \;& 0.4709 + (x_{11} - 5.1213) \times 0.0632 + x_{12} \times 0.0369 + (x_{13} - 0.6191)x_{13} \\
& \times 0.0294 + x_{14} \times 0.0104 \\
= \;& 0.0632x_{11} + 0.0369x_{12} + 0.02940x_{13} + 0.0104x_{14} + 0.1290
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mu_{20}}(\mathbf{x}) = \;& 4.2134 + (x_{11} - 4.0643) \times 0.7709 + x_{12} \times 0.4433 + x_{13} \times 0.2891 + \\
& (x_{14} - 1.0161) \times 0.1347 - x_{15} \times 12.0347 - x_{16} \times 7.2257 - x_{17} \times \\
& 4.7289 + (x_{18} - 1.0161) \times (-3.2181) \\
= \;& 0.7709x_{11} + 0.4433x_{12} + 0.2891x_{13} + 0.1347x_{14} - 12.0347x_{15} - \\
& 7.2257x_{16} - 4.7289x_{17} - 3.2181x_{18} + 4.2133
\end{aligned}
$$

Then

$$
\begin{aligned}
Z_{12}(\mathbf{x}) = \;& \widehat{\mu_{19}}(\mathbf{x}) + \widehat{\mu_{20}}(\mathbf{x}) \\
= \;& 0.8341x_{11} + 0.4802x_{12} + 0.3185x_{13} + 0.1451x_{14} - 12.0347x_{15} - \\
& 7.2257x_{16} - 4.7289x_{17} - 3.2181x_{18} + 4.3423
\end{aligned}
$$