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ANALYSIS OF A DEMAND-DRIVEN PRODUCTION INVENTORY MODEL IN A TRAPEZOIDAL NEUTROSOPHIC NUMBER-RULED DECISION ENVIRONMENT

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Abstract: An Economic Production Quantity (EPQ) model is built with fundamental intuitions that time influences demand pattern and demand can control production in manufacturing process. Also, existence of deterioration of produced items during

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warehousing is considered. Preservation technology is installed for lowering the deterioration rate as much as possible. The manufacturing-warehousing process includes ambiguities in several pockets of decision-making phenomena. Trapezoidal neutrosophic number describes the imprecise environment for decision phenomena in this paper. Numerical results reveal that the cost reduction goal is impacted negatively for large size of production potential and for much reliance of the production process on demand pattern. On contrary, incorporations of preservation measure and neutrosophic decision phenomena favor the cost minimization objective.

Keywords: Production inventory, EPQ model, deterioration, differential equation, neutrosophic derivative, preservation technology, optimization.

MSC: 90B05.

1. INTRODUCTION

1.1. Background of research

Inventory management is nothing but strategy for maintaining buffer stocks aiming at the uninterrupted supply subject to consumption of items by customers. Therefore, inventory management becomes an integral component of decision process both for retail or manufacturing bodies. Among several distinguished approaches in inventory management policies, lot size based mathematical modelling are celebrated and significant for its simplicity and ability to address the economic transaction from supplier to consumers through intermediate stock holders. Economic order and production quantity models are two primary modelling approaches connected to the retail and manufacturing oriented study, respectively.

1.2. Motivation

In this paper, we focused on a manufacturing and supply based inventory phenomena for which demand and production came as two impacting parameters on the managerial decision. Time a very crucial independent variable impacting on the demand pattern. It is to be noted that the demand grows as time forwards in a newly established manufacturing hub. In this regard, the demand rate can be treated as function of time in an economic production model. The production rate is controlled in a manufacturing phenomenon after complete or partial prediction on demand. Thus, production rate should be demand driven. These two are the fundamental issues regarding formulation of the EPQ model in this paper. The produced items may loss its utility due to deterioration. Therefore, we also addressed the constant rate of deterioration during complete decision cycle. The primary model in this paper was built in a deterministic phenomenon. However, a real-world business phenomenon includes lots of vagueness and imperfection with decision making. During demand prediction, production optimization and smart warehousing, the whole process goes through an uncertain environment. There are numerous approaches to deal uncertainty with mathematical tools. Probabilistic uncertainty is connected with the randomness of occurrence of the events. Interval number represents the vulnerability of imprecise data between bounds. Fuzzy sets and numbers are associated with the uncertainty of a given data in terms of degree of belongingness. The degree of both belongingness and non-belongingness are addressed by intuitionistic fuzzy set. In this regard, Neutrosophic sets and numbers provide the best structured mathematical tools regarding uncertainty having degree of acceptance, hesitance and rejection simultaneously. Therefore, we discussed the proposed EPQ model in Trapezoidal neutrosophic numbers ruled uncertain phenomena.

1.3. Novelties

The current article contributes to some novel perspectives on theoretical advancement. They are:

- In this paper, we constructed a demand driven production function. Also, the demand is taken as a function of time. Thus, both the demand and production are time controlled. The proposed model is novel in construction in the production rate related point of view.
- To tackle the uncertainty, we used neutrosophic environments and neutrosophic valued calculus for the proposed model. The neutrosophic valued differential equation approach is very rare in the existing literature.
- We considered the Trapezoidal neutrosophic numbers for uncertain parameters controlling the demand, production and deterioration rates. The incorporation of Trapezoidal neutrosophic numbers and de-neutrosophication formula for ultimate outcomes of the numerical encounters are also afresh advancements in this paper.

1.4. Structure of the paper

The remaining text in this paper is structured in pockets as follows: Section 2 summarizes the literature survey and gaps in the existing literature. Section 3 provides the mathematical preliminaries, which help the reader understand the mathematical foundation of this paper. Section 4 describes the notations and symbols in the paper and their meanings. Also, the same section discusses the assumptions for the mathematical formulation of the proposed model. Section 5 details the proposed model in a crisp environment. Subsequently, Section 6 reconsiders the proposed model under neutrosophic uncertainty. The crispification of the neutrosophic model is described in Section 7. Section 8 is about the numerical results of the proposed model in different environments and approaches. Section 9 lists significant findings and managerial interpretations. The concluding remarks on the investigation and findings of the paper are given in Section 10.

2. LITERATURE REVIEW

The literature review is made on keywords like recent literature on “production inventory model”, “recent advancement under different types of uncertainty”, “Inventory models under different types of uncertainty” and “Inventory model under neutrosophic uncertainty”. The subsections related to each of the identified keywords are then followed by additional subsections that address the research gaps, the underlying motivations for the study, and the specific contributions made by the paper.

2.1. Production inventory model

A manufacturing inventory model that incorporates projected backorders was proposed by Cárdenas-Barrón [1]. This method generates item of imperfect quality, all of which are reworked throughout the same production cycle. A production inventory system was developed by Taleizadeh et al. [2] that considers shortages, repair failures, random defective goods, and the existence of a single machine, all of which result in limited production capacity. An EPQ model that takes partial backlog shortages and imperfect manufacturing batches into account has been presented by Cunha et al. [3]. They show that it is better to sell defective things as soon as possible because doing so lowers holding expenses, which in turn lowers overall costs. Jawla and Singh [4] designed a multi-item production planning system considering the effects of preservation technology investment. Namakshenas and Mazdeh [5] introduced a production inventory planning problem for radiopharmaceuticals that are sensitive to time. The proposed framework aims to optimize both production processes and inventory control, ensuring that the radiopharmaceuticals, which have a limited shelf life, are efficiently managed to meet the demands of the imaging center. An EPQ model taking sustainability into account and three distinct shortage scenarios—partial backordered shortage, lost sale, and full backorder shortage—was presented by Taleizadeh et al. [6]. They discovered that the most accurate and practical model is a partially backlogged shortage case. Taleizadeh et al. [7] presented a remanufacturing based production model featuring price-dependent demand with price, producing quantity, and back-ordered quantity as decision variables. In their study, Farahbakhsh and Kheirkhah et al. [8] introduced a complex multi-period inventory routing problem, which involves optimizing both inventory management and transportation logistics over multiple time periods. To tackle this challenging problem, they developed and applied an innovative genetic algorithm. Nobil et al. [9] conducted a detailed investigation into an EPQ model, focusing on a production system that includes defective items with strict inspection policy. Their study specifically addressed a scenario in which remanufacturing processes are applied to the defective products, and the system also experiences shortages. An imperfect production-based production inventory problem is developed by Rahaman et al. [10] with promotional frequency, the environmental friendliness of the product, and its selling price dependent demand. Haque et al. [11] proposed a sustainable production planning model with remanufacturing rates dependent on demand, demonstrating that manufacturers may opt for remanufacturing over traditional manufacturing due to its lower cost and environmental benefits. Recently, various studies [12-14] have enhanced the understanding of inventory planning problems. This research focuses on analyzing a deteriorating EPQ model, where manufacturing rate is influenced by demand pattern, and demand is influenced by time with preservation technology investment to mitigate deterioration.

2.2. Recent advancement under different types of uncertainty

In many domains, including decision-making and mathematical modelling, uncertainty is essential because it captures the natural variety and unpredictability of real-world systems. By recognising and controlling uncertainty, one can solve the problems of accurate measurements or insufficient knowledge and produce more reliable forecasts and solutions. In the existing literature, a variety of tools and frameworks such as fuzzy [15], type-2 fuzzy [16], interval valued [17], interval valued fuzzy [18], intuitionistic fuzzy [19], neutrosophic [20] etc. have been developed to manage and address uncertainty in complex systems. Regarding Hukuhara and generalised Hukuhara differences of fuzzy numbers, Singh et al. [21] addressed the existence and uniqueness criteria for solving the systems of linear equations under fuzzy governed uncertainty. Alamin et al. [22] presented a strategy for solving first order linear fuzzy difference equation using fuzzy geometric approach. Biswas et al. [23] introduced the highway restaurant site selection problem and addressed it through two fuzzy Multi-Criteria Decision Making (MCDM) methodologies: the Analytic Hierarchy Process (AHP) and the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), utilising trapezoidal fuzzy numbers as instruments to manage uncertainty. Using the pentagonal fuzzy decision-making trial and evaluation laboratory (DEMATEL) Methodology, Gazi et al. [24] suggested a method for determining the most significant criterion in women's empowerment for sports. In an interval-valued intuitionistic fuzzy (IVIF) environment, Imran et al. [25] suggested a hybrid structure of the Aczel-Alsina (AA) and Bonferroni mean (BM) operators for robot selection. Ghrissi et al. [26] conducted an in-depth study focusing on the existence, uniqueness, and different forms of stability, particularly Ulam-Hyers stability, for the solution of a nonlinear fuzzy fractional differential equation (FFDE). This equation was defined using the Caputo generalized Hukuhara differentiability, which played a key role in their analysis. Later, Mukherjee [27] carried out a numerical investigation of a fuzzy fractional differential equation (FFDE) by employing the Homotopy Analysis Transform Method (HATM) with the Caputo fractional derivative. Zubair [28] introduced a single-valued neutrosophic uncertain linguistic set and applied this concept on Multiple Attribute Decision-Making (MADM) problem. To address and solve multi-objective probabilistic linear programming problems Edalatpanah et al. [29] employed a neutrosophic fuzzy goal programming approach. Naseem et al. [30] presented Aczel-Alsina aggregation operators which are based on complex single-valued neutrosophic set. Edalatpanah and Smarandache [31] performed an input-oriented data envelopment analysis using simplified neutrosophic set. Bodur et al. [32] analysed ten questions measuring distributive justice by using neutrosophic Likert scales. Rasinojehdehi and Valami [33] conducted an evaluation of airline performance using the Slack-Based Measure (SBM) framework. This framework is designed to assess the efficiency of interconnected decision-making units within a networked structure.

2.3. Inventory models under different types of uncertainty

Park [34] was the first to integrate fuzzy ideas into an Economic Order Quantity (EOQ) model following the introduction of fuzzy set theory in the literature. After that several researchers used fuzzy set theoretic concept in inventory planning problem such as Bag et al. [35], De and Sana [36], and Mahata et al. [37]. Jain er al. [38] studied an inventory transportation problem using trapezoidal fuzzy number. Barman et al. [39] analysed a production inventory planning system considering preservation technology investment in fuzzy environment. In a type-2 fuzzy uncertain environment, where the demand pattern is reliant on inventory level and selling price and the production rate is based on the demand rate, Debnath et al. [40] studied a sustainable EPQ model. Garai et al. [41] applied intuitionistic fuzzy numbers as a mathematical approach to quantify the uncertainty in an inventory system where the demand for stock varies based on the level of inventory. Momena et al. [42] explored an EOQ model where the demand for a product depends on its price, and the holding cost varies over time under all unit price discount policies in a densely fuzzy environment. Karmakar et al. [43] have created a sustainable production and rework model in a dense-lock fuzzy environment. Maiti [44] investigated an EPQ model with imperfect production and production rates dependent on demand in an uncertain environment. The study used the price of the produced item as a fuzzy cloud number and applied Particle Swarm Optimization (PSO) to solve the problem. Rahaman et al. [45] designed a production inventory model of deteriorating items in lock fuzzy environment. To control the deterioration, they used preservation technology and found that the lock fuzzy strategy is an intelligent methodin comparison to the conventional crisp and fuzzy approaches. Das [46] developed deteriorating multi objective inventory system with demand dependent setup cost, deterioration cost and production cost in fuzzy environment. An EPQ model that was worsening in the cloud fuzzy phenomenon was examined by Barman et al. [47] with a time-dependent demand and a partially backlogged shortage. Rahaman et al. [48] explored a production inventory model that focuses on items that deteriorate over time. In this model, the production rate is influenced by the current level of stock, while the demand rate depends on both the unit selling price and the available stock. The study employs the generalized Hukuhara derivative approach to analyze and solve the model, providing insights into how these factors interact within the inventory system. A production inventory model with green-level-dependent demand was examined by Manna et al. [49], taking into account carbon emissions during production in an unpredictable setting. Maity et al. [50] presented a comprehensive study on a green lot-sizing model that incorporates various factors such as green level, selling price, and stock-dependent demand. Their research considers the impact of carbon emissions and investments in emission reduction technologies, all within a complex framework characterized by a pentagonal intuitionistic dense fuzzy uncertain environment. Rahaman et al. [51] analyzed an EOQ model where the demand depends on both price and stock levels, set within a type-2 interval uncertainty framework.

2.4. Inventory model under neutrosophic uncertainty

The emergence of neutrosophic philosophy has sparked a great deal of interest and excitement within the global research community. As a result, researchers around the world are increasingly drawn to studying and incorporating neutrosophic ideas into their work. Neutrosophic numbers have been classified into several categories, such as triangular [52,53], trapezoidal [54], pentagonal [55,56] etc. Edalatpanah [57] introduced the notion of neutrosophic structured element in the neutrosophic philosophy. The notion of the neutrosophic derivative was first proposed by Smarandache [58] as an extension of the fuzzy derivative inside the neutrosophic domain. A neutrosophic differential equation is discussed through the parametric representation of the neutrosophic number by Sumathi and Priya [59] and Sumathi and Sweety [60]. Moi et al. [61] proposed the concept of a novel type of neutrosophic derivative, which they referred to as a generalized neutrosophic derivative. A number of researchers have applied neutrosophic logic within the field of inventory planning problem. Mullai and Surya [62, 63] utilized triangular neutrosophic numbers in their lot-sizing model. Mondal et al. [64] investigated a lot-sizing model for deteriorating periodic goods, incorporating partial backlogging and time-influenced demand into their analysis. Momena et al. [65] established existence and uniqueness condition for solving NDE used this to discuss a lot-sizing model in neutrosophic uncertain environment considering stock level, price, and warranty time dependent demand. Haque et al. [66] proposed neutrosophic Laplace transform method and applied this method to discuss neutrosophic inventory problem with price and deterioration dependent demand in neutrosophic arena. In this article, the Neutrosophic Differential Equation (NDE) method is employed to address a production quantity model where the demand varies over time, and the production rate is influenced by the level of demand. Preservation technology is also used to protect item from deterioration. Furthermore, parameters related to demand and the deterioration rates are considered in the context of neutrosophic numbers, which represent uncertainty and vagueness in the data. The model incorporates these neutrosophic elements to handle the complexities of time-dependent demand and its effect on production processes.

2.5. Research gaps and contributions

The comparison between the proposed work and the published work is displayed in Table 1 below:

Table 1: Comparing the contributions made by various authors that are relevant to this article

Author(s)	Types of the model	Produc-tion rate	Demand rate	Deter iorati on	Preserva-tion technology	Type of uncertainty
Haque et al. [11]	EPQ	CN	PD, GLD			Crisp
Jawla & Singh [4]	EPQ	CN	PD	√	√	Crisp
Barman et al. [39]	EPQ	DD	PD, GLD	√	√	Fuzzy
Rahaman et al. [48]	EPQ	SD	PD, SD	√		Fuzzy
Debnath et al. [40]	EPQ	DD	PD, SD			Type-2 fuzzy
Rahaman et al. [51]	EOQ	-	PD, SD	√		Type -2 interval
Rahaman et al. [45]	EPQ	SD	PD, SD	√	√	Lock fuzzy
Maiti [44]	EPQ	DD	CN			Cloudy fuzzy
Barman et al. [47]	EPQ	CN	TD	√		Cloudy fuzzy
Mondal et al. [63]	EOQ	-	TD	√		Neutrosophic
Momena et al. [64]	EOQ	-	PD, SD, WTD			Neutrosophic
Haque et al. [62]	EOQ	-	PD, DTD	√		Neutrosophic
This article	EPQ	DD	TD	√	√	Neutrosophic

CN: Constant; PD: Price Dependent; GLD: Green Level Dependent; DD: Demand Dependent; SD: Stock Dependent; TD: Time Dependent: WTD: Warranty time dependent; DTD: Deterioration dependent

From the detailed research survey on the above-mentioned keywords, we have found the following research gaps which are targeted to be overcome in the present article:

- Demand with time dependency was discussed in many existing models. However, the demand-dependent production rate is rarely considered in the existing literature. In this paper, the production rate is hypothesized to be demand driven.
- Numerous mathematical models with uncertain scenario used fuzzy or neutrosophic ruled uncertain data in the existing literature. However, the uncertain rule arithmetic and calculus were neglected by those models where the crisp data were after de-imprecision of the uncertain data in crisp ruled dynamical systems. Only a few literatures were evident with the application uncertain differential equation on lot size model. In this scenario, the neutrosophic differential equation approach in used in this uncertain model.
- The neutrosophic environments ensure the most structured mathematical tool dealing with uncertainty having sense of acceptance, hesitant and rejection. In the existing literature, we find only few papers with neutrosophic ruled uncertainty. However, in those model, triangular neutrosophic numbers were taken. In this chapter, Trapezoidal neutrosophic number ruled environment is considered.

3. MATHEMATICAL PRELIMINARIES

Definition 1. *[54] A neutrosophic set is represented by the ordered triplet* $\left(\zeta, T(\zeta), I(\zeta), F(\zeta)\right)$, *where* ζ *is an element in the universal set* X, $T(\zeta)$, $I(\zeta)$ *and* $F(\zeta)$ *respectively signify the degrees of truthiness, indeterminacy, and falsity of* ζ *in* X. *Each of* $T(\zeta)$, $I(\zeta)$ *and* $F(\zeta)$ *lies in the range [0, 1], fulfilling the condition*

$$0 \leq T(\zeta) + I(\zeta) + F(\zeta) \leq 3.$$

Remark 1. There are different types of neutrosophic number in the literature to deal with uncertainty involved in real life problems. In this study, various parameters related to the real-life inventory problem are considered as a trapezoidal neutrosophic number. Next, trapezoidal neutrosophic number are defined.

Definition 2. *[54] A Single-Valued Trapezoidal Neutrosophic (SVTpN) number, denoted as* $\tilde{G}_{TpN} = \langle (i,k,l,n); \mu_{\tilde{G}}, \nu_{\tilde{G}}, \kappa_{\tilde{G}} \rangle$ *is a specific type of neutrosophic set on the real numbers* $\mathbb{R}$. *It is characterized by its truth, indeterminacy, and falsity membership functions, which are defined as follows:*

$$T_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{(\zeta - i)\mu_{\tilde{G}}}{k-i} & when\ i \leq \zeta < k \\ \mu_{\tilde{G}} & when\ k \leq \zeta \leq l \\ \frac{(n-\zeta)\mu_{\tilde{G}}}{n-l} & when\ l < \zeta \leq n \\ 0 & otherwise \end{cases}$$

$$I_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{k-\zeta+\nu_{\tilde{G}}(\zeta-i)}{k-i} & when\ i \leq \zeta < k \\ \nu_{\tilde{G}} & when\ k \leq \zeta \leq l \\ \frac{\zeta-l+\nu_{\tilde{G}}(n-\zeta)}{n-l} & when\ l < \zeta \leq n \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{k-\zeta+\kappa_{\tilde{G}}(\zeta-i)}{k-i} & when\ i \leq \zeta < k \\ \kappa_{\tilde{G}} & when\ k \leq \zeta \leq l \\ \frac{\zeta-l+\kappa_{\tilde{G}}(n-\zeta)}{n-l} & when\ l < \zeta \leq n \\ 1 & otherwise \end{cases}$$

with $0 \leq T_{\tilde{G}_{TpN}}(\zeta) + I_{\tilde{G}_{TpN}}(\zeta) + F_{\tilde{G}_{TpN}}(\zeta) \leq 3$.

Definition 3. *[54] A Single-Valued Trapezoidal Neutrosophic (SVTpN) number of Type 1, denoted as* $\tilde{G}_{TpN} = \left(i_1, i_2, i_3, i_4; k_1, k_2, k_3, k_4; l_1, l_2, l_3, l_4\right)$ *is a specific type of neutrosophic set on the real numbers* $\mathbb{R}$. *It is characterized by its truth, indeterminacy, and falsity membership functions, which are defined as follows:*

$$T_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{\zeta - i_1}{i_2 - i_1} & when\ i_1 \leq \zeta < i_2 \\ 1 & when\ i_2 \leq \zeta \leq i_3 \\ \frac{i_4 - \zeta}{i_4 - i_3} & when\ i_3 < \zeta \leq i_4 \\ 0 & otherwise \end{cases}$$

$$I_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{k_2-\zeta}{k_2-k_1} & when\ k_1 \le \zeta < k_2 \\ 0 & when\ \ k_2 \le \zeta \le k_3 \\ \frac{\zeta-k_3}{k_4-k_3} & when\ k_3 < \zeta \le k_4 \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{G}_{TpN}}(\zeta) = \begin{cases} \frac{l_2-\zeta}{l_2-l_1} & when\ l_1 \le \zeta < l_2 \\ 0 & when\ \ l_2 \le \zeta \le l_3 \\ \frac{\zeta-l_3}{l_4-l_3} & when\ l_3 < \zeta \le l_4 \\ 1 & otherwise \end{cases}$$

with $0 \le T_{\tilde{G}_{TpN}}(\zeta) + I_{\tilde{G}_{TpN}}(\zeta) + F_{\tilde{G}_{TpN}}(\zeta) \le 3$.

Remark 2. Suppose, $i_1 = k_1 = l_1 = i$; $i_2 = k_2 = l_2 = k$; $i_3 = k_3 = l_3 = l$, $i_4 = k_4 = l_4 = n$ and taking the extreme value of the membership grades for truthiness, indeterminacy, and falseness as the generalized values $\mu_{\tilde{G}}$, $\nu_{\tilde{G}}$ and $\kappa_{\tilde{G}}$ instead of using 1. Then the definition 3.3 is obtained from definition 3.4.

Definition 4. *[54] The (α,β,γ)-cut of a neutrosophic set $\tilde{G}_{TpN} = \left(\zeta, T_{\tilde{G}_{TpN}}(\zeta),\ I_{\tilde{G}_{TpN}}(\zeta), F_{\tilde{G}_{TpN}}(\zeta)\right)$ over X is indicated by $\left[\tilde{G}_{TpN}\right]_{(\alpha,\beta,\gamma)}$ and is defined by $\left[\tilde{G}_{TpN}\right]_{(\alpha,\beta,\gamma)} = \left\{\langle \zeta, T_{\tilde{G}_{TpN}}(\zeta), I_{\tilde{G}_{TpN}}(\zeta), F_{\tilde{G}_{TpN}}(\zeta)\rangle : T_{\tilde{G}_{TpN}}(\zeta) \ge \alpha,\ I_{\tilde{G}_{TpN}}(\zeta) \le \beta, F_{\tilde{G}_{TpN}}(\zeta) \le \gamma\right\}$. It is also known as a parametric representation or parametric form of the neutrosophic set.*

Remark 3. The parametric representation of an SVTpN number $\tilde{G}_{TpN} = \left(i_1, i_2, i_3, i_4; k_1, k_2, k_3, k_4; l_1, l_2, l_3, l_4\right)$ includes six components. These six components are written as $\langle [G_1(\alpha), G_2(\alpha)], [G_1'(\beta), G_2'(\beta)], [G_1''(\gamma), G_2''(\gamma)]\rangle$, where $G_1(\alpha) = i_1 + \alpha(i_2 - i_1)$, $G_2(\alpha) = i_4 - \alpha(i_4 - i_3)$, $G_1'(\beta) = k_2 - \beta(k_2 - k_1)$, $G_2'(\beta) = k_3 + \beta(k_4 - k_3)$, $G_1''(\gamma) = l_2 - \gamma(l_2 - l_1)$ and $G_2''(\gamma) = l_3 + \gamma(l_4 - l_3)$.

De-neutrosophication is a key method used to interpret the results derived from neutrosophic systems and to make meaningful comparisons between different outcomes expressed in neutrosophic numbers. This technique emphasizes the need to translate neutrosophic numbers into specific, definitive values. By assigning a clear and precise value to these numbers, de-neutrosophication helps to simplify and make sense of the neutrosophic outcomes, ensuring that the results can be effectively analyzed and compared in a meaningful way.

Definition 5. *[66] Suppose $\widetilde{\mathcal{N}}$ be a neutrosophic number whose parametric form is $\langle [\mathcal{N}_1(\alpha), \mathcal{N}_2(\alpha)], [\mathcal{N}_1'(\beta), \mathcal{N}_2'(\beta)], [\mathcal{N}_1''(\gamma), \mathcal{N}_2''(\gamma)]\rangle$. Then, the de-neutrosophication value of $\widetilde{\mathcal{N}}$ is represented by $D\left(\widetilde{\mathcal{N}}\right)$, and it is defined as*

$$De\left(\widetilde{\mathcal{N}}\right) = \frac{\alpha\mathcal{N}_1(\alpha) + (1-\alpha)\mathcal{N}_2(\alpha) + \beta\mathcal{N}_1'(\beta) + (1-\beta)\mathcal{N}_2'(\beta) + \gamma\mathcal{N}_1''(\gamma) + (1-\gamma)\mathcal{N}_2''(\gamma)}{3}$$

where $0 \le \alpha, \beta, \gamma \le 1$.

Definition 6. *[61] Let $\tilde{g}: I \to \mathcal{N}$ be a neutrosophic-valued function given in the parametric representation by $\tilde{g}(t) = \langle [g_1(t;\alpha), g_2(t;\alpha)],\ [g_1'(t;\beta), g_2'(t;\beta)], [g_1''(t;\gamma), g_2''(t;\gamma)]\rangle$,*

$\forall t \in I$. *The generalized neutrosophic derivative of* $\tilde{g}(t)$ *at* $t = c \in I$ *is written as* $\tilde{\dot{g}}(c) = \langle \dot{g}_T(c), \dot{g}_I(c), \dot{g}_F(c) \rangle$ *in which* $\dot{g}_T(c)$, $\dot{g}_I(c)$ *and* $\dot{g}_F(c)$ *are defined as the following*

1. $\dot{g}_T(c) = [min\{\dot{g}_1(c;\alpha), \dot{g}_2(c;\alpha)\}, max\{\dot{g}_1(c;\alpha), \dot{g}_2(c;\alpha)\}]$
2. $\dot{g}_I(c) = [min\{\dot{g}_1'(c;\beta), \dot{g}_2'(c;\beta)\}, max\{\dot{g}_1'(c;\beta), \dot{g}_2'(c;\beta)\}]$
3. $\dot{g}_F(c) = [min\{\dot{g}_1''(c;\gamma), \dot{g}_2''(c;\gamma)\}, max\{\dot{g}_1''(c;\gamma), \dot{g}_2''(c;\gamma)\}]$

provided $\dot{g}_1(c;\alpha)$, $\dot{g}_2(c;\alpha)$, $\dot{g}_1'(c;\beta)$, $\dot{g}_2'(c;\beta)$, $\dot{g}_1''(c;\gamma)$ *and* $h_2'(t_0;\theta)$ *are all exists.* $\tilde{\dot{g}}(c)$ *is said to be a type-1 derivative if the parametric representation of* $\tilde{\dot{g}}(c)$ *is given by.*

$$\tilde{\dot{g}}(c) = \langle [\dot{g}_1(c;\alpha), \dot{g}_2(c;\alpha)], [\dot{g}_1'(c;\beta), \dot{g}_2'(c;\beta)], [\dot{g}_1''(c;\gamma), \dot{g}_2''(c;\gamma)] \rangle$$

and type-2 derivative if the parametric representation of $\tilde{\dot{g}}(c)$ *is given by*

$$\tilde{\dot{g}}(c) = \langle [\dot{g}_2(c;\alpha), \dot{g}_1(c;\alpha)], [\dot{g}_2'(c;\beta), \dot{g}_1'(c;\beta)], [\dot{g}_2''(c;\gamma), \dot{g}_1''(c;\gamma)] \rangle.$$

4. NOTATIONS AND ASSUMPTIONS OF THE PROPOSED MODEL

In order to explain the proposed model, the following symbols and assumptions are utilized throughout the explanation. These elements are essential for understanding the structure and application of the model.

4.1. Notations

Table 2 represents the notations and their explanation which are used in the proposed model.

Table 2: Notations and their explanation

Notations	Explanation	Unit
r	The component of the demand pattern that remains unchanged over time	Constant
s	The multiplier that represents the influence of time on the demand pattern	Constant
c_0	The portion of the production rate that remains unchanged or consistent throughout the process.	Constant
c_1	The coefficient that represents the pattern of demand in the production rate, which quantifies how the production rate is influenced by fluctuations in demand	Constant
τ	Rate of deterioration	Constant
η	Rate of preservation	Constant
h_c	The per unit cost associated with storing or keeping an item per unit time	\$/Unit time
p_r	Per unit cost of preservation per unit time	\$/Unit time
p_c	Production cost per unit item	\$/Unit item
S_c	Ordering cost	\$/Cycle
t_1	Length of the production phase (decision variable)	Unit time
T	Length of the inventory cycle (decision variable)	Unit time
Π	Total average cost (objective function)	\$

4.2. Assumptions

The proposed model is built upon the following assumptions:

a) The rate at which the produced item is demanded is influenced by the passage of time. In other words, as time progresses, the demand for the item does not remain constant but instead increases steadily. This increase follows a linear pattern, meaning that the rate of demand grows proportionally with time. Thus, as each unit of time passes, the demand rate continues to rise in a consistent and predictable manner. Consequently, the demand pattern is taken as $D(t) = r + ts$, where r and s are positive constants.

b) The production rate of the item is directly influenced by the rate of demand. This means that the amount of the item being produced is adjusted in response to how much demand there is at any given time. When the demand rate increases, the production rate is adjusted upward to meet this higher demand, ensuring that supply aligns with market needs. Conversely, if the demand rate decreases, the production rate may also be reduced accordingly. Essentially, the production rate is not fixed but varies in response to changes in the demand rate, ensuring that the supply of the item matches the current level of demand. So, the production rate is taken as $K = c_0 + c_1 D(t) = c_0 + c_1 r + c_1 st$, where c_0 and c_1 are positive constants.

c) The items in stock have deterioration at a constant rate τ $(0 \leq \tau \leq 1)$ throughout the whole lot-sizing cycle.

d) To protect the item from deterioration, producer imposes a technology which is popularly known as preservation technology. The rate of preservation is taken as η.

e) There is no lead time involved, meaning the time delay between ordering, and receiving the item is nonexistent.

f) The replenishment of items can occur instantly or at an unlimited rate, allowing for immediate restocking. However, the size of each replenishment batch or lot remains limited and cannot be infinite, indicating that items are added in specific quantities rather than endlessly.

g) Throughout the entire lot-sizing cycle, shortages are not factored in, implying that at no point is there a lack of available items.

h) The planning or operational period is limited to a specific duration. This time frame is not infinite; it has a clearly defined start and end, within which all activities, such as production and replenishment, are planned and executed.

5. FORMULATION OF THE PROPOSED CRISP EPQ MODEL

The production process begins at time $t = 0$ with production rates that depend on the demand. During the production phase, the stock level decreases due to the time-dependent demand pattern, $D(t)$. Additionally, the stock level is reduced by a constant deterioration rate τ. However, this deterioration rate is mitigated by investing in preservation technology at a rate η, which effectively reduces the overall deterioration rate to $\tau - \eta$. The production phase concludes at time t_1, resulting in the maximum stock level Q_{max}. In this phase, the governing differential equation is

$$\frac{dI(t)}{dt} = \{c_0 + c_1 r + c_1 st\} - \{r + st\} - (\tau - \eta)I(t) \tag{1}$$

with $I(0) = 0, I(t_1) = Q_{max}$.

The non-production phase begins at time $t = t_1$. During this phase, no production activities take place. The stock level continues to decrease due to the time-dependent demand $D(t)$ and the net deterioration rate of $\tau - \eta$, which accounts for both natural decay and the effect of preservation technology. The entire lot-sizing cycle concludes at time $t = T$. The differential equation that governs this phase is as follows:

$$\frac{dJ(t)}{dt} = -\{r + st\} - (\tau - \eta)J(t) \tag{2}$$

with $J(T) = 0$.

Solving the equation (1) with using the initial conditions, the stock level in production phase is calculated as

$$I(t) = \frac{(\tau-\eta)\{c_0 - r(1-c_1)\} + s(1-c_1)}{(\tau-\eta)^2}\left\{1 - e^{-(\tau-\eta)t}\right\} - \frac{s(1-c_1)}{\tau-\eta}t \tag{3}$$

Solving the equation (2) with using the initial conditions, the stock level in non-production phase is calculated as

$$J(t) = \frac{s-(\tau-\eta)(r+st)}{(\tau-\eta)^2} - \frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^2}e^{(\tau-\eta)(T-t)} \tag{4}$$

The highest inventory level at the end of the production phase is attained as

$$Q_{max} = \frac{(\tau-\eta)\{c_0 - r(1-c_1)\} + s(1-c_1)}{(\tau-\eta)^2}\left\{1 - e^{-(\tau-\eta)t_1}\right\} - \frac{s(1-c_1)t_1}{\tau-\eta} \tag{5}$$

From equations (3) and (4), the following constraint is attained by applying the continuity condition of the inventory level functions in productive phase and non-productive phase as

$$\frac{s-(\tau-\eta)(r+st_1)}{(\tau-\eta)^2} - \frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^2}e^{(\tau-\eta)(T-t_1)} = \frac{(\tau-\eta)\{c_0 - r(1-c_1)\} + s(1-c_1)}{(\tau-\eta)^2}\left\{1 - e^{-(\tau-\eta)t_1}\right\} - \frac{s(1-c_1)t_1}{\tau-\eta} \tag{6}$$

The model incorporates several key costs that are important for its formulation. These costs are detailed and calculated in the following manner.

Holding Cost (HC): The total cost of holding is attained as

$$HC = h_c\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} J(t)dt\right]$$

$$HC = h_c\left[\frac{(\tau-\eta)\{c_0 - r(1-c_1)\} + s(1-c_1)}{(\tau-\eta)^3}\left\{e^{-(\tau-\eta)t_1} + (\tau-\eta)t_1 - 1\right\} - \frac{s(1-c_1)t_1^2}{2(\tau-\eta)} + \frac{s-(\tau-\eta)r}{(\tau-\eta)^2}(T - t_1) - \frac{s}{2(\tau-\eta)}(T^2 - t_1^2) + \frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^3}\left\{1 - e^{(\tau-\eta)(T-t_1)}\right\}\right]$$

Production Cost (PC): The total cost of production is attained by

$$PC = p_c\left[\int_0^{t_1}\{c_0 + c_1 r + c_1 st\}dt\right] = p_c\left[(c_0 + c_1 r)t_1 + \frac{c_1 s t_1^2}{2}\right]$$

Preservation Cost (PRC): Preservation cost is defined as the cost of preserving the produced items from natural deterioration. The total cost for preservation is obtained as

$$PRC = p_r\eta\left[\int_0^{t_1}\{I(t) - D(t)\}dt + \int_{t_1}^{T}\{J(t) - D(t)\}dt\right]$$

$$= p_r\eta\left[\frac{(\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)}{(\tau-\eta)^3}\left\{e^{-(\tau-\eta)t_1}+(\tau-\eta)t_1-1\right\}-\frac{s(1-c_1)t_1^2}{2(\tau-\eta)}+\frac{s-(\tau-\eta)r}{(\tau-\eta)^2}(T-t_1)-\frac{s}{2(\tau-\eta)}(T^2-t_1^2)+\frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^3}\left\{1-e^{(\tau-\eta)(T-t_1)}\right\}-rT-\frac{s}{2}T^2\right]$$

Setup Cost (SC): The setup cost, denoted as S_c, is assumed to remain constant and unchanged throughout the entire duration of the inventory cycle.

The total average cost (TAC) provides a comprehensive measure of the production system's cost efficiency throughout the entire cycle. TAC of the proposed production inventory system is calculated by averaging these costs over the duration of the cycle, which is obtained as

$$\Pi(t_1,T)=\frac{SC+PC+HC+PRC}{T}$$

$$\Pi(t_1,T)=\frac{1}{T}\left[S_c+p_c\left[(c_0+c_1r)t_1+\frac{c_1st_1^2}{2}\right]+(h_c+p_r\eta)\left\{\left[\frac{(\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)}{(\tau-\eta)^3}\left\{e^{-(\tau-\eta)t_1}+(\tau-\eta)t_1-1\right\}-\frac{s(1-c_1)t_1^2}{2(\tau-\eta)}+\frac{s-(\tau-\eta)r}{(\tau-\eta)^2}(T-t_1)-\frac{s}{2(\tau-\eta)}(T^2-t_1^2)+\frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^3}\left\{1-e^{(\tau-\eta)(T-t_1)}\right\}\right]\right\}-p_r\eta\left(rT+\frac{s}{2}T^2\right)\right]$$

Consequently, the cost-minimization problem can be expressed in the following ways:

$$\begin{cases} Min\quad \Pi(t_1,T)=\frac{SC+PC+HC+PRC}{T} \\ Subject\ to\quad Q_{max}=\frac{(\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)}{(\tau-\eta)^2}\left\{1-e^{-(\tau-\eta)t_1}\right\}-\frac{s(1-c_1)t_1}{\tau-\eta} \\ equation\ (6) \\ 0<t_1<T \end{cases} \tag{7}$$

The objective in this context is to conduct a thorough examination of the convexity properties of the average cost function, denoted as $\Pi(t_1,T)$, with respect to the decision variables t_1 and T. This involves analyzing how the function behaves and whether it maintains a convex shape when varying these specific variables, t_1 and T, which are critical to the decision-making process.

Theorem 1. *The average cost function $\Pi(t_1,T)$ is characterized as strictly pseudo-convex with respect to both t_1 and T hence $\Pi(t_1,T)$ exhibits the minimum value at some point, provided*

$$\frac{(h_c+p_r\eta)\left((\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)\right)}{\tau-\eta}e^{-(\tau-\eta)t_1}+p_cc_1s>\frac{(h_c+p_r\eta)s}{\tau-\eta}\left\{e^{(\tau-\eta)(T-t_1)}-c_1\right\} \tag{8}$$

Proof: To simplify our analysis, let's consider the average cost function $\Pi(t_1,T)$ as $\Pi(t_1,T)=\frac{\Pi_1(t_1,T)}{\Pi_2(t_1,T)}$, where

$$\Pi_1(t_1,T)=S_c+p_c\left[(c_0+c_1r)t_1+\frac{c_1st_1^2}{2}\right]+(h_c+p_r\eta)\left\{\left[\frac{(\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)}{(\tau-\eta)^3}\left\{e^{-(\tau-\eta)t_1}+(\tau-\eta)t_1-1\right\}-\frac{s(1-c_1)t_1^2}{2(\tau-\eta)}+\frac{s-(\tau-\eta)r}{(\tau-\eta)^2}(T-t_1)-\frac{s}{2(\tau-\eta)}(T^2-t_1^2)+\frac{s-(\tau-\eta)(r+sT)}{(\tau-\eta)^3}\left\{1-e^{(\tau-\eta)(T-t_1)}\right\}\right]\right\}-p_r\eta\left(rT+\frac{s}{2}T^2\right)$$

and $\Pi_2(t_1, T) = T$. Now, $\phi_1(t_1, T)$ can be written as

$$\phi_1(t_1, T) = S_c + \frac{A_1}{(\tau-\eta)^3}\left(e^{-(\tau-\eta)t_1} + (\tau-\eta)t_1 - 1\right) + B_1 t_1 + C_1 T + D_1 \frac{t_1^2}{2} - E_1 \frac{t_1^2}{2} - F_1 \frac{T^2}{2} + \frac{(h_c+p_r\eta)}{(\tau-\eta)^3}\{s - (\tau-\eta)(r+sT)\}\left\{1 - e^{(\tau-\eta)(T-t_1)}\right\}$$

where $A_1 = (h_c + p_r\eta)[(\tau-\eta)\{c_0 - r(1-c_1)\} + s(1-c_1)]$, $B_1 = p_c(c_0 + rc_1) - (h_c + p_r\eta)\frac{s-r(\tau-\eta)}{(\tau-\eta)^2}$, $C_1 = (h_c + p_r\eta)\frac{s-(\tau-\eta)r}{(\tau-\eta)^2} - rp_r\eta$, $D_1 = c_1 s p_c + (h_c + p_r\eta)\frac{s}{(\tau-\eta)}$, $E_1 = (h_c + p_r\eta)\frac{s(1-c_1)}{(\tau-\eta)}$ and $F_1 = (h_c + p_r\eta)\frac{s}{(\tau-\eta)} + sp_r\eta$.

To build the Hessian matrix for $\Pi_1(t_1, T)$, it is necessary to compute all the first-order and second-order partial derivatives of $\Pi_1(t_1, T)$ with respect to both t_1 and T. This includes evaluating the partial derivatives of the function in relation to each variable individually, as well as their mixed partial derivatives.

$$\frac{\partial \Pi_1(t_1,T)}{\partial t_1} = \frac{A_1}{(\tau-\eta)^2}\left(1 - e^{-(\tau-\eta)t_1}\right) + B_1 + D_1 t_1 - E_1 t_1 + \frac{(h_c+p_r\eta)}{(\tau-\eta)^2}\{s - (\tau-\eta)(r + sT)\}e^{(\tau-\eta)(T-t_1)}$$

$$\frac{\partial \Pi_1(t_1,T)}{\partial T} = C_1 - F_1 T - \frac{(h_c+p_r\eta)}{(\tau-\eta)^2}\left\{s - (\tau-\eta)(r+sT)e^{(\tau-\eta)(T-t_1)}\right\}$$

$$\frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1^2} = \frac{A_1}{\tau-\eta}e^{-(\tau-\eta)t_1} + D_1 - E_1 - \frac{(h_c+p_r\eta)}{\tau-\eta}\{s - (\tau-\eta)(r+sT)\}e^{(\tau-\eta)(T-t_1)}$$

$$\frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1 \partial T} = -(h_c + p_r\eta)(r + sT)e^{(\tau-\eta)(T-t_1)} = \frac{\partial^2 \Pi_1(t_1,T)}{\partial T \partial t_1}$$

$$\frac{\partial^2 \Pi_1(t_1,T)}{\partial T^2} = -F_1 + \frac{s(h_c+p_r\eta)}{\tau-\eta}e^{(\tau-\eta)(T-t_1)} + (h_c + p_r\eta)(r + sT)e^{(\tau-\eta)(T-t_1)}$$

The Hessian matrix associated with $\Pi_1(t_1, T)$ can therefore be represented in the following manner:

$$\begin{bmatrix} \frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1^2} & \frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1 \partial T} \\ \frac{\partial^2 \Pi_1(t_1,T)}{\partial T \partial t_1} & \frac{\partial^2 \Pi_1(t_1,T)}{\partial T^2} \end{bmatrix}$$

The first principal minor is defined as $|H_{11}| = \frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1^2} = \frac{A_1}{(\tau-\eta)^2}\left(1 - e^{-(\tau-\eta)t_1}\right) + B_1 + D_1 t_1 - E_1 t_1 + \frac{(h_c+p_r\eta)}{(\tau-\eta)^2}\{s - (\tau-\eta)(r+sT)\}e^{(\tau-\eta)(T-t_1)}$.

$$|H_{11}| = (h_c + p_r\eta)(r + sT)e^{(\tau-\eta)(T-t_1)} + \frac{A_1}{\tau-\eta}e^{-(\tau-\eta)t_1} + D_1 - E_1 - \frac{(h_c+p_r\eta)s}{\tau-\eta}e^{(\tau-\eta)(T-t_1)}$$

Clearly, $\tau - \eta > 0$ and $(h_c + p_r\eta) > 0$. As $0 < l < 1$, $s(1-l) > 0$. Once more, the constant component of the production rate exceeds the constant component of the market demand, therefore, $(c_0 - r) + c_1 r > 0$, and hence A_1, D_1 and E_1 are all positive. Hence $|H_{11}| > 0$ if

$$\frac{A_1}{\tau-\eta}e^{-(\tau-\eta)t_1} + D_1 > E_1 + \frac{(h_c+p_r\eta)s}{\tau-\eta}e^{(\tau-\eta)(T-t_1)}$$

The second principal minor is defined as

$|H_{22}| = \frac{\partial^2 \Pi_1(t_1,T)}{\partial t_1^2}\frac{\partial^2 \Pi_1(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 \Pi(t_1,T)}{\partial T \partial t_1}\right)^2 = \Big[(h_c + p_r\eta)(r + sT)e^{(\tau-\eta)(T-t_1)} + \frac{A_1}{\tau-\eta}e^{-(\tau-\eta)t_1} + D_1 - E_1 - \frac{(h_c+p_r\eta)s}{\tau-\eta}e^{(\tau-\eta)(T-t_1)}\Big]\Big[\frac{s(h_c+p_r\eta)}{\tau-\eta}e^{(\tau-\eta)(T-t_1)} - F_1\Big] + (h_c + p_r\eta)(r + sT)e^{(\tau-\eta)(T-t_1)}\Big[\frac{A_1}{\tau-\eta}e^{-(\tau-\eta)t_1} + D_1 - E_1 - \frac{(h_c+p_r\eta)s}{\tau-\eta}e^{(\tau-\eta)(T-t_1)}\Big]$

It is evident that the absolute value of $|H_{22}|$ is positive as long as condition (8) is maintained. As a result, the function $\Pi_1(t_1,T)$ is both positive definite and convex with respect to the variables t_1 and T. Additionally, $\Pi_1(t_1,T)$ is non-negative and it is differentiable with respect to both t_1 and T, allowing for smooth changes in the function. On the other hand, the function $\Pi_2(t_1,T)$ is positive and affine. As a result of these properties, the average cost function $\Pi(t_1,T)$ is strictly pseudo-convex with respect to t_1 and T, meaning it has a unique minimum value. ■

To identify the points that satisfy the conditions required to achieve the minimum value of the total cost function $\Pi(t_1,T)$, set the first-order partial derivatives of $\Pi(t_1,T)$ with respect to both t_1 and T equal to zero.

$\frac{\partial \Pi(t_1,T)}{\partial t_1} = \frac{1}{T}\Big[\frac{(h_c+p_r\eta)\left((\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)\right)}{(\tau-\eta)^2}\left(1 - e^{-(\tau-\eta)t_1}\right) + p_c(c_0 + rc_1) - (h_c + p_r\eta)\frac{s-(\tau-\eta)r}{(\tau-\eta)^2} + \left(p_c c_1 s + (h_c + p_r\eta)\frac{s}{(\tau-\eta)}\right)t_1 - (h_c + p_r\eta)\frac{s(1-c_1)}{\tau-\eta}t_1 + \frac{(h_c+p_r\eta)}{(\tau-\eta)^2}\{s - (\tau-\eta)(r + sT)\}e^{(\tau-\eta)(T-t_1)}\Big] = 0$

i.e.,

$$\frac{(h_c+p_r\eta)\left((\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)\right)}{(\tau-\eta)^2}\left(1 - e^{-(\tau-\eta)t_1}\right) + p_c(c_0 + rc_1) - (h_c + p_r\eta)\frac{s-(\tau-\eta)r}{(\tau-\eta)^2} + \left(p_c c_1 s + (h_c + p_r\eta)\frac{s}{(\tau-\eta)}\right)t_1 - (h_c + p_r\eta)\frac{s(1-c_1)}{\tau-\eta}t_1 + \frac{(h_c+p_r\eta)}{(\tau-\eta)^2}\{s - (\tau-\eta)(r + sT)\}e^{(\tau-\eta)(T-t_1)} = 0 \tag{9}$$

and $\frac{\partial \Pi(t_1,T)}{\partial T} = -\frac{\Pi_1(t_1,T)}{T^2} + \frac{1}{T}\frac{\partial \Pi_1(t_1,T)}{\partial T} = 0$

i.e.,

$$S_c + \frac{(h_c+p_r\eta)\left((\tau-\eta)\{c_0-r(1-c_1)\}+s(1-c_1)\right)}{(\tau-\eta)^3}\left(e^{-(\tau-\eta)t_1} + (\tau-\eta)t_1 - 1\right) + \left(p_c(c_0 + rc_1) - (h_c + p_r\eta)\frac{s-(\tau-\eta)r}{(\tau-\eta)^2}\right)t_1 + \left(p_c c_1 s + (h_c + p_r\eta)\frac{s}{\tau-\eta}\right)\frac{t_1^2}{2} - \frac{(h_c+p_r\eta)k_2}{\tau-\eta}\frac{t_1^2}{2} + \left((h_c + p_r\eta)\frac{s}{\tau-\eta} + sp_r\eta\right)\frac{T^2}{2} + \frac{(h_c+p_r\eta)}{(\tau-\eta)^3}\{s - (\tau-\eta)(r + sT)\}\left\{1 - e^{(\tau-\eta)(T-t_1)}\right\} + \frac{(h_c+p_r\eta)T}{(\tau-\eta)^2}\left\{s - (\tau-\eta)(r + sT)e^{(\tau-\eta)(T-t_1)}\right\} = 0. \tag{10}$$

6. PROPOSED EPQ MODEL IN A NEUTROSOPHIC ENVIRONMENT

In this section, we aim to reconstruct the production-supply inventory system within the context of the neutrosophic framework. To do this, we introduce three parameters, denoted as $\tilde{r}$, $\tilde{s}$ and $\tilde{\tau}$, which are treated as neutrosophic numbers. By incorporating these neutrosophic numbers, we can develop a neutrosophic version of the existing equations.

(1) and (2). The resulting neutrosophic counterparts of equation (1) for the production phase ($0 \le t \le t_1$) is presented as follows:

$$\begin{cases} \frac{d\tilde{I}(t)}{dt} = c_0 - (1-c_1)\tilde{r} - (1-c_1)\tilde{s}t - (\tilde{\tau}-\eta)\tilde{I}(t) \\ \qquad\qquad with\ \tilde{I}(t) = 0, \tilde{I}(t_1) = \tilde{Q} \end{cases} \tag{11}$$

The resulting neutrosophic counterparts of equation (2) for the non-production phase ($t_1 \le t \le T$) is presented as follows:

$$\begin{cases} \frac{d\tilde{J}(t)}{dt} = -\{\tilde{r} + \tilde{s}t\} - (\tilde{\tau}-\eta)\tilde{J}(t), \\ \qquad\qquad with\ \tilde{J}(T) = 0 \end{cases} \tag{12}$$

The notion of generalized neutrosophic differentiability is now utilized to provide an explanation for the neutrosophic differential equations, specifically equations (11) and (12). To proceed, let's consider that the (α,β,γ)-cut of the neutrosophic-valued function $\tilde{I}(t)$ is $[\tilde{I}(t)]_{(\alpha,\beta,\gamma)} = \langle [I_1(t;\alpha), I_2(t;\alpha)], [I_1'(t;\beta), I_2'(t;\beta)], [I_1''(t;\gamma), I_2''(t;\gamma)]\rangle$. In addition, consider the parametric representations, specifically the (α,β,γ)-cut, of the neutrosophic numbers denoted as $\tilde{a}$, $\tilde{b}$ and $\tilde{\tau}$ as

$[\tilde{r}]_{(\alpha,\beta,\gamma)} = \langle [r_1(\alpha), r_2(\alpha)],\ [r_1'(\beta), r_2'(\beta)],\ [r_1''(\gamma), r_2''(\gamma)]\rangle$,

$[\tilde{s}]_{(\alpha,\beta,\gamma)} = \langle [s_1(\alpha), s_2(\alpha)], [s_1'(\beta), s_2'(\beta)], [s_1''(\gamma), s_2''(\gamma)]\rangle$ and

$[\tilde{\tau}]_{(\alpha,\beta,\gamma)} = \langle [\tau_1(\alpha), \tau_2(\alpha)],\ [\tau_1'(\beta), \tau_2'(\beta)], [\tau_1''(\gamma), \tau_2''(\gamma)]\rangle$.

The analysis considers two distinct cases based on the two types of generalized neutrosophic derivatives that are applicable to the neutrosophic valued function denoted as $\tilde{I}(t)$. Each case is examined in relation to one of the specific forms of these generalized derivatives, allowing for a comprehensive exploration of the function.

Case 1: When $\tilde{I}(t)$ is type-1 neutrosophic differentiable

In this case, the equation (11), which characterizes the productive phase occurring within the time interval from $0 \le t \le t_1$, is subsequently transformed into its corresponding parametric form as follows:

$$\begin{gathered}\left\langle\left[\dot{I}_1(t;\alpha), \dot{I}_2(t;\alpha)\right],\left[\dot{I}_1'(t;\beta), \dot{I}_2'(t;\beta)\right]\left[\dot{I}_1''(t;\gamma), \dot{I}_2''(t;\gamma)\right]\right\rangle = c_0 - (1-c_1)\langle [r_1(\alpha), r_2(\alpha)], \\ [r_1'(\beta), r_2'(\beta)], [r_1''(\gamma), r_2''(\gamma)]\rangle - (1-c_1)\langle [s_1(\alpha), s_2(\alpha)],[s_1'(\beta), s_2'(\beta)], [s_1''(\gamma), s_2''(\gamma)]\rangle t \\ -\{\langle [\tau_1(\alpha), \tau_2(\alpha)], [\tau_1'(\beta), \tau_2'(\beta)], [\tau_1''(\gamma), \tau_2''(\gamma)]\rangle - \eta\}\langle [I_1(t;\alpha), I_2(t;\alpha)],[I_1'(t;\beta), \\ I_2'(t;\beta)], [I_1''(t;\gamma), I_2''(t;\gamma)]\rangle\end{gathered}$$

The expression presented above can be expanded into a system of crisp differential equations, structured as follows:

$$\dot{I}_1(t;\alpha) = c_0 - (1-c_1)r_2(\alpha) - (1-c_1)s_2(\alpha)t - \{\tau_2(\alpha) - \eta\}I_2(t;\alpha)$$

$$\dot{I}_2(t;\alpha) = c_0 - (1-c_1)r_1(\alpha) - (1-c_1)s_1(\alpha)t - \{\tau_1(\alpha) - \eta\}I_1(t;\alpha)$$

$$\dot{I}_1'(t;\beta) = c_0 - (1-c_1)r_2'(\beta) - (1-c_1)s_2'(\beta)t - \{\tau_2'(\beta) - \eta\}I_2'(t;\beta)$$

$$\dot{I}_2'(t;\beta) = c_0 - (1-c_1)r_1'(\beta) - (1-c_1)s_1'(\beta)t - \{\tau_1'(\beta) - \eta\}I_1'(t;\beta)$$

$$\dot{I}_1''(t;\gamma) = c_0 - (1-c_1)r_2''(\gamma) - (1-c_1)s_2''(\gamma)t - \{\tau_2''(\gamma) - \eta\}I_2''(t;\gamma)$$

$$\dot{I}_2''(t;\gamma) = c_0 - (1-c_1)r_1''(\gamma) - (1-c_1)s_1''(\gamma)t - \{\tau_1''(\gamma) - \eta\}I_1''(t;\gamma)$$

with the initial condition $I_1(0;\alpha) = I_2(0;\alpha) = I_1'(0;\beta) = I_2'(0;\beta) = I_1''(0;\gamma) = I_2''(0;\gamma) = 0$. Solving the above equations by using the initial conditions we get

$$I_1(t;\alpha) = \lambda_1 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \lambda_2 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{1}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}[(\tau_2(\alpha) - \eta)\{c_0 - (1-c_1)r_1(\alpha) - (1-c_1)s_1(\alpha)t\} + (1-c_1)s_2(\alpha)],$$

$$I_2(t;\alpha) = -\sqrt{\frac{\tau_1(\alpha)-\eta}{\tau_2(\alpha)-\eta}}\lambda_1 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \sqrt{\frac{\tau_1(\alpha)-\eta}{\tau_2(\alpha)-\eta}}\lambda_2 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{1}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}[(\tau_1(\alpha) - \eta)\{c_0 - (1-c_1)r_2(\alpha) - (1-c_1)s_2(\alpha)t\} + (1-c_1)s_1(\alpha)],$$

$$I_1'(t;\beta) = \lambda_3 e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \lambda_4 e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \frac{1}{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}[(\tau_2'(\beta) - \eta)\{c_0 - (1-c_1)r_1'(\beta) - (1-c_1)s_1'(\beta)t\} + (1-c_1)s_2'(\beta)]$$

$$I_2'(t;\beta) = -\sqrt{\frac{\tau_1'(\beta)-\eta}{\tau_2'(\beta)-\eta}}\lambda_3 e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \sqrt{\frac{\tau_1'(\beta)-\eta}{\tau_2'(\beta)-\eta}}\lambda_4 e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \frac{1}{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}[(\tau_1'(\beta) - \eta)\{c_0 - (1-c_1)r_2'(\beta) - (1-c_1)s_2'(\beta)t\} + (1-c_1)s_1'(\beta)]$$

$$I_1''(t;\gamma) = \lambda_5 e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \lambda_6 e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \frac{1}{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}[(\tau_2''(\gamma) - \eta)\{c_0 - (1-c_1)r_1''(\gamma) - (1-c_1)s_1''(\gamma)t\} + (1-c_1)s_2''(\gamma)]$$

$$I_2''(t;\gamma) = -\sqrt{\frac{\tau_1''(\gamma)-\eta}{\tau_2''(\gamma)-\eta}}\lambda_5 e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \sqrt{\frac{\tau_1''(\gamma)-\eta}{\tau_2''(\gamma)-\eta}}\lambda_6 e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \frac{1}{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}[(\tau_1''(\gamma) - \eta)\{c_0 - (1-c_1)r_2''(\gamma) - (1-c_1)s_2''(\gamma)t\} + (1-c_1)s_1''(\gamma)]$$

where

$$\lambda_1 = -\frac{1}{2\sqrt{\tau_1(\alpha)-\eta}(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\left[\sqrt{\tau_1(\alpha)-\eta}\big((\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\} + (1-c_1)s_2(\alpha)\big) - \sqrt{\tau_2(\alpha)-\eta}\big((\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\} + (1-c_1)s_1(\alpha)\big)\right]$$

$$\lambda_2 = -\frac{1}{2\sqrt{\tau_1(\alpha)-\eta}(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\left[\sqrt{\tau_1(\alpha)-\eta}\big((\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\} + (1-c_1)s_2(\alpha)\big) + \sqrt{\tau_2(\alpha)-\eta}\big((\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\} + (1-c_1)s_1(\alpha)\big)\right]$$

$$\lambda_3 = -\frac{1}{2\sqrt{\tau_1'(\beta)-\eta}(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\left[\sqrt{\tau_1'(\beta)-\eta}\big((\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\} + (1-c_1)s_2'(\beta)\big) - \sqrt{\tau_2'(\beta)-\eta}\big((\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\} + (1-c_1)s_1'(\beta)\big)\right]$$

$$\lambda_4 = -\frac{1}{2\sqrt{\tau_1'(\beta)-\eta}(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\left[\sqrt{\tau_1'(\beta)-\eta}\big((\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\} + (1-c_1)s_2'(\beta)\big) + \sqrt{\tau_2'(\beta)-\eta}\big((\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\} + (1-c_1)s_1'(\beta)\big)\right]$$

$$\lambda_5 = -\frac{1}{2\sqrt{\tau_1''(\gamma)-\eta}(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\left[\sqrt{\tau_1''(\gamma)-\eta}\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\} + (1-c_1)s_2''(\gamma)\big) - \sqrt{\tau_2''(\gamma)-\eta}\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\} + (1-c_1)s_1''(\gamma)\big)\right]$$

$$\lambda_6 = \frac{1}{2\sqrt{\tau_1''(\gamma)-\eta}(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\left[\sqrt{\tau_1''(\gamma)-\eta}\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_2''(\gamma)\big)+\sqrt{\tau_2''(\gamma)-\eta}\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_1''(\gamma)\big)\right]$$

Again, the equation (12), which characterizes the non-productive phase occurring within the time interval $t_1 \le t \le T$, is subsequently transformed into its corresponding parametric form as follows:

$$\langle[\dot{J}_1(t;\alpha),\dot{J}_2(t;\alpha)],[\dot{J}_1'(t;\beta),\dot{J}_2'(t;\beta)][\dot{J}_1''(t;\gamma),\dot{J}_2''(t;\gamma)]\rangle = -\langle[r_1(\alpha),r_2(\alpha)],[r_1'(\beta),r_2'(\beta)],[r_1''(\gamma),r_2''(\gamma)]\rangle - \langle[s_1(\alpha),s_2(\alpha)],[s_1'(\beta),s_2'(\beta)],[s_1''(\gamma),s_2''(\gamma)]\rangle t - \{[\tau_1(\alpha),\tau_2(\alpha)],[\tau_1'(\beta),\tau_2'(\beta)],[\tau_1''(\gamma),\tau_2''(\gamma)]\rangle - \eta\}\langle[J_1(t;\alpha),J_2(t;\alpha)],[J_1'(t;\beta),J_2'(t;\beta)],[J_1''(t;\gamma),J_2''(t;\gamma)]\rangle$$

The expression presented above can be expanded into a system of crisp differential equations, structured as follows:

$$\dot{J}_1(t;\alpha) = -r_2(\alpha) - s_2(\alpha)t - \{\tau_2(\alpha)-\eta\}J_2(t;\alpha)$$

$$\dot{J}_2(t;\alpha) = -r_1(\alpha) - s_1(\alpha)t - \{\tau_1(\alpha)-\eta\}J_1(t;\alpha)$$

$$\dot{J}_1'(t;\beta) = -r_2'(\beta) - s_2'(\beta)t - \{\tau_2'(\beta)-\eta\}J_2'(t;\beta)$$

$$\dot{J}_2'(t;\beta) = -r_1'(\beta) - s_1'(\beta)t - \{\tau_1'(\beta)-\eta\}J_1'(t;\beta)$$

$$\dot{J}_1''(t;\gamma) = -r_2''(\gamma) - s_2''(\gamma)t - \{\tau_2''(\gamma)-\tau\}J_2''(t;\gamma)$$

$$\dot{J}_2''(t;\gamma) = -r_1''(\gamma) - s_1''(\gamma)t - \{\tau_1''(\gamma)-\tau\}J_1''(t;\gamma)$$

with $J_1(T;\alpha) = J_2(T;\alpha) = J_1'(T;\beta) = J_2'(T;\beta) = J_1''(T;\gamma) = J_2''(T;\gamma) = 0$. Solving the above equations by using the initial conditions we get

$$J_1(t;\alpha) = \lambda_7 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \lambda_8 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}$$

$$J_2(t;\alpha) = -\sqrt{\frac{\tau_1(\alpha)-\eta}{\tau_2(\alpha)-\eta}}\lambda_7 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \sqrt{\frac{\tau_1(\alpha)-\eta}{\tau_2(\alpha)-\eta}}\lambda_8 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_2(\alpha)t+r_2(\alpha))}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}$$

$$J_1'(t;\beta) = \lambda_9 e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \lambda_{10} e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_1'(\beta)t+r_1'(\beta))}{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}$$

$$J_2'(t;\beta) = -\sqrt{\frac{\tau_1'(\beta)-\eta}{\tau_2'(\beta)-\eta}}\lambda_9 e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \sqrt{\frac{\tau_1'(\beta)-\eta}{\tau_2'(\beta)-\eta}}\lambda_{10} e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t} + \frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_2'(\beta)t+r_2'(\beta))}{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}$$

$$J_1''(t;\gamma) = \lambda_{11} e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \lambda_{12} e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)(s_1''(\gamma)t+r_1''(\gamma))}{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}$$

$$J_2''(t;\gamma) = -\sqrt{\frac{\tau_1''(\gamma)-\eta}{\tau_2''(\gamma)-\eta}}\lambda_{11} e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \sqrt{\frac{\tau_1''(\gamma)-\eta}{\tau_2''(\gamma)-\eta}}\lambda_{12} e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t} + \frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_2''(\gamma)t+r_2''(\gamma))}{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}$$

where

$\lambda_7 = -\frac{e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}}{2\sqrt{\tau_1(\alpha)-\eta}(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\left[\sqrt{\tau_1(\alpha)-\eta}\{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))\}-\right.$
$\left.\sqrt{\tau_2(\alpha)-\eta}\{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_2(\alpha)T+r_2(\alpha))\}\right]$

$\lambda_8 = -\frac{e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}}{2\sqrt{\tau_1(\alpha)-\eta}(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\left[\sqrt{\tau_1(\alpha)-\eta}\{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))\}+\right.$
$\left.\sqrt{\tau_2(\alpha)-\eta}\{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_2(\alpha)T+r_2(\alpha))\}\right]$

$\lambda_9 = -\frac{e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}}{2\sqrt{\tau_1'(\beta)-\eta}(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\left[\sqrt{\tau_1'(\beta)-\eta}\{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_1'(\beta)T+r_1'(\beta))\}-\right.$
$\left.\sqrt{\tau_2'(\beta)-\eta}\{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_2'(\beta)T+r_2'(\beta))\}\right]$

$\lambda_{10} = -\frac{e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}}{2\sqrt{\tau_1'(\beta)-\eta}(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\left[\sqrt{\tau_1'(\beta)-\eta}\{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_1'(\beta)T+\right.$
$\left.r_1'(\beta))\}+\sqrt{\tau_2'(\beta)-\eta}\{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_2'(\beta)T+r_2'(\beta))\}\right]$

$\lambda_{11} = -\frac{e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}}{2\sqrt{\tau_1''(\gamma)-\eta}(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\left[\sqrt{\tau_1''(\gamma)-\eta}\{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)(s_1''(\gamma)T+\right.$
$\left.r_1''(\gamma))\}-\sqrt{\tau_2''(\gamma)-\eta}\{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_2''(\gamma)T+r_2''(\gamma))\}\right]$

$\lambda_{12} = -\frac{e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}}{2\sqrt{\tau_1''(\gamma)-\eta}(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\left[\sqrt{\tau_1''(\gamma)-\eta}\{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)\big(s_1''(\gamma)T+\right.$
$\left.r_1''(\gamma)\big)\}+\sqrt{\tau_2''(\gamma)-\eta}\{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)\big(s_2''(\gamma)T+r_2''(\gamma)\big)\}\right]$

Some relevant costs:
Therefore, the holding cost,
$\widetilde{HC} = \langle[HC_1(\alpha), HC_2(\alpha)], [HC_1'(\beta), HC_2'(\beta)], [HC_1''(\gamma), HC_2''(\gamma)]\rangle$ given by

$$HC_1(\alpha) = h_c\left[\int_0^{t_1} I_1(t;\alpha)\,dt + \int_{t_1}^{T} J_1(t;\alpha)\,dt\right] = h_c\left[\int_0^{t_1}\left\{\lambda_1 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \lambda_2 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{(\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)-(1-c_1)s_1(\alpha)t\}+(1-c_1)s_2(\alpha)}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\}dt + \int_{t_1}^{T}\left\{\lambda_7 e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \lambda_8 e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t} + \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\}dt\right] =$$
$$h_c\left[\left\{\frac{\lambda_1}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-1\right) - \frac{\lambda_2}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-1\right) + \frac{2\big((\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_2(\alpha)\big)t_1-(\tau_2(\alpha)-\eta)(1-c_1)s_1(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\} + \left\{\frac{\lambda_7}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right) + \frac{\lambda_8}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}-e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right) - \frac{(\tau_2(\alpha)-\eta)s_1(\alpha)(T^2-t_1^2)+2\big((\tau_2(\alpha)-\eta)r_1(\alpha)-s_2(\alpha)\big)(T-t_1)}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\}\right]$$

Similarly,

$$HC_2(\alpha) =$$
$$h_c \left[\left\{\frac{\lambda_1}{\tau_2(\alpha)-\eta}\left(1-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)+\frac{\lambda_2}{\tau_2(\alpha)-\eta}\left(1-e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)+\right.\right.$$
$$\left.\frac{2\left((\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\}+(1-c_1)s_1(\alpha)\right)t_1-(\tau_1(\alpha)-\eta)(1-c_1)s_2(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\}+$$
$$\left\{\frac{\lambda_7}{\tau_2(\alpha)-\eta}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}\right)+\frac{\lambda_8}{\tau_2(\alpha)-\eta}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-\right.\right.$$
$$\left.\left.\left.e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}\right)-\frac{(\tau_1(\alpha)-\eta)s_2(\alpha)\left(T^2-t_1^2\right)+2\left((\tau_1(\alpha)-\eta)r_2(\alpha)-s_1(\alpha)\right)(T-t_1)}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\right\}\right]$$

$$HC_1'(\beta) =$$
$$h_c \left[\left\{\frac{\lambda_3}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-1\right)-\right.\right.$$
$$\frac{\lambda_4}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-1\right)+$$
$$\left.\frac{2\left((\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\}+(1-c_1)s_2'(\beta)\right)t_1-(\tau_2'(\beta)-\eta)(1-c_1)s_1'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\right\}+$$
$$\left\{\frac{\lambda_9}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+\right.$$
$$\frac{\lambda_{10}}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)-$$
$$\left.\left.\frac{(\tau_2'(\beta)-\eta)s_1'(\beta)\left(T^2-t_1^2\right)+2\left((\tau_2'(\beta)-\eta)r_1'(\beta)-s_2'(\beta)\right)(T-t_1)}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\right\}\right]$$

$$HC_2'(\beta) =$$
$$h_c \left[\left\{\frac{\lambda_3}{\tau_2'(\beta)-\eta}\left(1-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+\frac{\lambda_4}{\tau_2'(\beta)-\eta}\left(1-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+\right.\right.$$
$$\left.\frac{2\left((\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\}+(1-c_1)s_1'(\beta)\right)t_1-(\tau_1'(\beta)-\eta)(1-c_1)s_2'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\right\}+$$
$$\left\{\frac{\lambda_9}{\tau_2'(\beta)-\eta}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}\right)+\right.$$
$$\frac{\lambda_{10}}{\tau_2'(\beta)-\eta}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}\right)-$$
$$\left.\left.\frac{(\tau_1'(\beta)-\eta)s_2'(\beta)\left(T^2-t_1^2\right)+2\left((\tau_1'(\beta)-\eta)r_2'(\beta)-s_1'(\beta)\right)(T-t_1)}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\right\}\right]$$

$$HC_1'(\gamma) =$$
$$h_c \left[\left\{\frac{\lambda_5}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-1\right)-\right.\right.$$
$$\frac{\lambda_6}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t}-1\right)+$$

$$\left.\frac{2\Big(\big((\tau_2''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_2''(\gamma)\big)t_1-(\tau_2''(\gamma)-\eta)(1-c_1)s_1''(\gamma)t_1^2\Big)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\right\} +$$
$$\left\{\frac{\lambda_{11}}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+\right.$$
$$\frac{\lambda_{12}}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)-$$
$$\left.\left.\frac{(\tau_2''(\gamma)-\eta)s_1''(\gamma)(T^2-t_1^2)+2\big((\tau_2''(\gamma)-\eta)r_1''(\gamma)-s_2''(\gamma)\big)(T-t_1)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\right\}\right]$$

$$HC_2''(\gamma)=$$
$$h_c\left[\left\{\frac{\lambda_5}{\tau_2''(\gamma)-\eta}\left(1-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+\frac{\lambda_6}{\tau_2''(\gamma)-\eta}\left(1-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+\right.\right.$$
$$\left.\frac{2\Big(\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_1''(\gamma)\big)t_1-(\tau_1''(\gamma)-\eta)(1-c_1)s_2''(\gamma)t_1^2\Big)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\right\} +$$
$$\left\{\frac{\lambda_{11}}{\tau_2''(\gamma)-\eta}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}\right)+\right.$$
$$\frac{\lambda_{12}}{\tau_2''(\gamma)-\eta}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}\right)-$$
$$\left.\left.\frac{(\tau_1''(\gamma)-\eta)s_2''(\gamma)(T^2-t_1^2)+2\big((\tau_1''(\gamma)-\eta)r_2''(\gamma)-s_1''(\gamma)\big)(T-t_1)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\right\}\right]$$

Therefore, the production cost $\widetilde{PC}=\langle[PC_1(\alpha),PC_2(\alpha)],\ [PC_1'(\beta),PC_2'(\beta)],\ [PC_1''(\gamma),PC_2''(\gamma)]\rangle$ during the entire circle is given by

$$PC_1(\alpha)=p_c\left[\int_0^{t_1}\{(c_0+c_1r_1(\alpha))+c_1s_1(\alpha)t\}dt\right]=p_c\left[\big(c_0+c_1r_1(\alpha)\big)t_1+c_1s_1(\alpha)\frac{t_1^2}{2}\right],$$
$$PC_2(\alpha)=p_c\left[\int_0^{t_1}\{\big(c_0+la_2(\alpha)\big)+lb_2(\alpha)t\}dt\right]=p_c\left[\big(c_0+c_1r_2(\alpha)\big)t_1+c_1s_2(\alpha)\frac{t_1^2}{2}\right],$$
$$PC_1'(\beta)=p_c\left[\big(c_0+c_1r_1'(\beta)\big)t_1+c_1s_1'(\beta)\frac{t_1^2}{2}\right],$$
$$PC_2'(\beta)=p_c\left[\big(c_0+c_1r_2'(\beta)\big)t_1+c_1s_2'(\beta)\frac{t_1^2}{2}\right],$$
$$PC_1''(\gamma)=p_c\left[\big(c_0+c_1r_1''(\gamma)\big)t_1+c_1s_1''(\gamma)\frac{t_1^2}{2}\right],$$
$$PC_2''(\gamma)=p_c\left[\big(c_0+c_1r_2''(\gamma)\big)t_1+c_1s_2''(\gamma)\frac{t_1^2}{2}\right]$$

Total cost for preservation is $\widetilde{PRC}=\langle[PRC_1(\alpha),PRC_2(\alpha)],\ [PRC_1'(\beta),PRC_2'(\beta)],\ [PRC_1''(\gamma),PRC_2''(\gamma)]\rangle$ obtained as

$$PRC_1(\alpha)=p_r\eta\left[\int_0^{t_1}\{I_1(t;\alpha)-\widetilde{D}(t)\}dt+\int_{t_1}^{T}\{J_1(t;\alpha)-\widetilde{D}(t)\}dt\right]=$$
$$p_r\eta\left[\int_0^{t_1}\left\{\lambda_1e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t}+\lambda_2e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t}+\right.\right.$$
$$\left.\frac{(\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)-(1-c_1)s_1(\alpha)t\}+(1-c_1)s_2(\alpha)}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}-a_2(\alpha)-b_2(\alpha)t\right\}dt+$$
$$\int_{t_1}^{T}\left\{\lambda_7e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t}+\lambda_8e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t}+\frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}-\right.$$
$$\left.\left.a_2(\alpha)-b_2(\alpha)t\right\}dt\right]=p_r\eta\left[\left\{\frac{\lambda_1}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-1\right)-\right.\right.$$

$$\frac{\lambda_2}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-1\right)+ \frac{2\big((\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_2(\alpha)\big)t_1-(\tau_2(\alpha)-\eta)(1-c_1)s_1(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\Bigg\}+ \Bigg\{\frac{\lambda_7}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)+ \frac{\lambda_8}{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}-e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)- \frac{(\tau_2(\alpha)-\eta)s_1(\alpha)(T^2-t_1^2)+2\big((\tau_2(\alpha)-\eta)r_1(\alpha)-s_2(\alpha)\big)(T-t_1)}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\Bigg\}-r_2(\alpha)T-\frac{s_2(\alpha)}{2}T^2\Bigg]$$

Similarly,

$$PRC_2(\alpha)= p_r\eta\Bigg[\Bigg\{\frac{\lambda_1}{\tau_2(\alpha)-\eta}\left(1-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)+\frac{\lambda_2}{\tau_2(\alpha)-\eta}\left(1-e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}\right)+ \frac{2\big((\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\}+(1-c_1)s_1(\alpha)\big)t_1-(\tau_1(\alpha)-\eta)(1-c_1)s_2(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\Bigg\}+ \Bigg\{\frac{\lambda_7}{\tau_2(\alpha)-\eta}\left(e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}-e^{\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}\right)+\frac{\lambda_8}{\tau_2(\alpha)-\eta}\left(e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}t_1}- e^{-\sqrt{(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}T}\right)-\frac{(\tau_1(\alpha)-\eta)s_2(\alpha)(T^2-t_1^2)+2\big((\tau_1(\alpha)-\eta)r_2(\alpha)-s_1(\alpha)\big)(T-t_1)}{2(\tau_1(\alpha)-\eta)(\tau_2(\alpha)-\eta)}\Bigg\}-r_1(\alpha)T- \frac{s_1(\alpha)}{2}T^2\Bigg]$$

$$PRC_1'(\beta)= p_r\eta\Bigg[\Bigg\{\frac{\lambda_3}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-1\right)- \frac{\lambda_4}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-1\right)+ \frac{2\big((\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\}+(1-c_1)s_2'(\beta)\big)t_1-(\tau_2'(\beta)-\eta)(1-c_1)s_1'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\Bigg\}+ \Bigg\{\frac{\lambda_9}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+ \frac{\lambda_{10}}{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)- \frac{(\tau_2'(\beta)-\eta)s_1'(\beta)(T^2-t_1^2)+2\big((\tau_2'(\beta)-\eta)r_1'(\beta)-s_2'(\beta)\big)(T-t_1)}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\Bigg\}-r_2'(\beta)T-\frac{s_2'(\beta)}{2}T^2\Bigg]$$

$$PRC_2'(\beta)= p_r\eta\Bigg[\Bigg\{\frac{\lambda_3}{\tau_2'(\beta)-\eta}\left(1-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+\frac{\lambda_4}{\tau_2'(\beta)-\eta}\left(1-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}\right)+ \frac{2\big((\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\}+(1-c_1)s_1'(\beta)\big)t_1-(\tau_1'(\beta)-\eta)(1-c_1)s_2'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\Bigg\}+ \Bigg\{\frac{\lambda_9}{\tau_2'(\beta)-\eta}\left(e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-e^{\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}\right)+$$

$$\frac{\lambda_{10}}{\tau_2'(\beta)-\eta}\left(e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}t_1}-e^{-\sqrt{(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}T}\right)-$$

$$\frac{(\tau_1'(\beta)-\eta)s_2'(\beta)\,(T^2-t_1^2)+2\big((\tau_1'(\beta)-\eta)r_2'(\beta)-s_1'(\beta)\,\big)(T-t_1)}{2(\tau_1'(\beta)-\eta)(\tau_2'(\beta)-\eta)}\Bigg\}-r_1'(\beta)T-\frac{s_1'(\beta)}{2}T^2\Bigg]$$

$$PRC_1''(\gamma)=$$

$$p_r\eta\Bigg[\Bigg\{\frac{\lambda_5}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-1\right)-$$

$$\frac{\lambda_6}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t}-1\right)+$$

$$\frac{2\big((\tau_2''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_2''(\gamma)\big)t_1-(\tau_2''(\gamma)-\eta)(1-c_1)s_1''(\gamma)t_1^2}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\Bigg\}+$$

$$\Bigg\{\frac{\lambda_{11}}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+$$

$$\frac{\lambda_{12}}{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)-$$

$$\frac{(\tau_2''(\gamma)-\eta)s_1''(\gamma)(T^2-t_1^2)+2\big((\tau_2''(\gamma)-\eta)r_1''(\gamma)-s_2''(\gamma)\big)(T-t_1)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\Bigg\}-r_2''(\gamma)T-\frac{s_2''(\gamma)}{2}T^2\Bigg]$$

$$PRC_2''(\gamma)=$$

$$p_r\eta\Bigg[\Bigg\{\frac{\lambda_5}{\tau_2''(\gamma)-\eta}\left(1-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+\frac{\lambda_6}{\tau_2''(\gamma)-\eta}\left(1-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}\right)+$$

$$\frac{2\big((\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_1''(\gamma)\big)t_1-(\tau_1''(\gamma)-\eta)(1-c_1)s_2''(\gamma)t_1^2}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\Bigg\}+$$

$$\Bigg\{\frac{\lambda_{11}}{\tau_2''(\gamma)-\eta}\left(e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-e^{\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}\right)+$$

$$\frac{\lambda_{12}}{\tau_2''(\gamma)-\eta}\left(e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}t_1}-e^{-\sqrt{(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}T}\right)-$$

$$\frac{(\tau_1''(\gamma)-\eta)s_2''(\gamma)(T^2-t_1^2)+2\big((\tau_1''(\gamma)-\eta)r_2''(\gamma)-s_1''(\gamma)\big)(T-t_1)}{2(\tau_1''(\gamma)-\eta)(\tau_2''(\gamma)-\eta)}\Bigg\}-r_1''(\gamma)T-\frac{s_1''(\gamma)}{2}T^2\Bigg]$$

Thus, the parametric form of the inventory system's overall average cost over the whole inventory cycle may be found as

$$[\widetilde{\Pi}]_{(\alpha,\beta,\gamma)}=\langle[\Pi_1(\alpha),\Pi_2(\alpha)],\ [\Pi_1'(\beta),\Pi_2'(\beta)],\ [\Pi_1''(\gamma),\Pi_2''(\gamma)]\rangle,$$

where $\Pi_1(\alpha)=\frac{S_c+PC_1(\alpha)+HC_1(\alpha)+PRC_1(\alpha)}{T}$, $\Pi_2(\alpha)=\frac{S_c+PC_2(\alpha)+HC_2(\alpha)+PRC_2(\alpha)}{T}$, $\Pi_1'(\beta)=\frac{S_c+PC_1'(\beta)+HC_1'(\beta)+PRC_1'(\beta)}{T}$, $\Pi_2'(\beta)=\frac{S_c+PC_2'(\beta)+HC_2'(\beta)+PRC_2'(\beta)}{T}$, $\Pi_1''(\gamma)=\frac{S_c+PC_1''(\gamma)+HC_1''(\gamma)+PRC_1''(\gamma)}{T}$ and $\Pi_2''(\gamma)=\frac{S_c+PC_2''(\gamma)+HC_2''(\gamma)+PRC_2''(\gamma)}{T}$.

Hence, from a mathematical standpoint, the problem of minimizing the inventory system, when considering the case of type-1 neutrosophic differentiability of the inventory level function, can be formulated and expressed as follows:

$$\begin{cases} Min\ \Pi_1(\alpha) \\ Min\ \Pi_2(\alpha) \\ Min\ \Pi_1'(\beta) \\ Min\ \Pi_2'(\beta) \\ Min\ \Pi_1''(\gamma) \\ Min\ \Pi_2''(\gamma) \\ Subject\ to\ T > t_1 > 0 \\ 0 \le \alpha, \beta, \gamma \le 1\ with\ \alpha + \beta + \gamma \le 3 \end{cases} \tag{13}$$

De-neutrosophication: The de-neutrosophied single objective minimization problem in the of type-1 neutrosophic differentiability is given as

$$\begin{cases} Min\ De(\widetilde{\Pi}) \\ Subject\ to\ T > t_1 > 0 \\ 0 \le \alpha, \beta, \gamma \le 1\ with\ \alpha + \beta + \gamma \le 3 \end{cases} \tag{14}$$

where $De(\widetilde{\Pi}) = \frac{\alpha\Pi_1(\alpha)+(1-\alpha)\Pi_2(\alpha)+\beta\Pi_1'(\beta)+(1-\beta)\Pi_2'(\beta)+\gamma\Pi_1''(\gamma)+(1-\gamma)\Pi_2''(\gamma)}{3}$.

Case 2: When $\tilde{q}(t)$ is type-2 neutrosophic differentiable

In this case, the equation (11), which characterizes the productive phase occurring within the time interval from $0 \le t \le t_1$, is subsequently transformed into its corresponding parametric form as follows:

$$\langle[\dot{I}_2(t;\alpha), \dot{I}_1(t;\alpha)],[\dot{I}_2'(t;\beta), \dot{I}_1'(t;\beta)][\dot{I}_2''(t;\gamma), \dot{I}_1''(t;\gamma)]\rangle = c_0 - (1-c_1)\langle[r_1(\alpha), r_2(\alpha)], [r_1'(\beta), r_2'(\beta)], [r_1''(\gamma), r_2''(\gamma)]\rangle - (1-c_1)\langle[s_1(\alpha), s_2(\alpha)],[s_1'(\beta), s_2'(\beta)], [s_1''(\gamma), s_2''(\gamma)]\rangle t - \{\langle[\tau_1(\alpha), \tau_2(\alpha)], [\tau_1'(\beta), \tau_2'(\beta)], [\tau_1''(\gamma), \tau_2''(\gamma)]\rangle - \eta\}\langle[I_1(t;\alpha), I_2(t;\alpha)],[I_1'(t;\beta), I_2'(t;\beta)], [I_1''(t;\gamma), I_2''(t;\gamma)]\rangle$$

The expression presented above can be expanded into a system of crisp differential equations, structured as follows:

$$\dot{I}_2(t;\alpha) = c_0 - (1-c_1)r_2(\alpha) - (1-c_1)s_2(\alpha)t - \{\tau_2(\alpha) - \eta\}I_2(t;\alpha)$$

$$\dot{I}_1(t;\alpha) = c_0 - (1-c_1)r_1(\alpha) - (1-c_1)s_1(\alpha)t - \{\tau_1(\alpha) - \eta\}I_1(t;\alpha)$$

$$\dot{I}_2'(t;\beta) = c_0 - (1-c_1)r_2'(\beta) - (1-c_1)s_2'(\beta)t - \{\tau_2'(\beta) - \eta\}I_2'(t;\beta)$$

$$\dot{I}_1'(t;\beta) = c_0 - (1-c_1)r_1'(\beta) - (1-c_1)s_1'(\beta)t - \{\tau_1'(\beta) - \eta\}I_1'(t;\beta)$$

$$\dot{I}_2''(t;\gamma) = c_0 - (1-c_1)r_2''(\gamma) - (1-c_1)s_2''(\gamma)t - \{\tau_2''(\gamma) - \eta\}I_2''(t;\gamma)$$

$$\dot{I}_1''(t;\gamma) = c_0 - (1-c_1)r_1''(\gamma) - (1-c_1)s_1''(\gamma)t - \{\tau_1''(\gamma) - \eta\}I_1''(t;\gamma)$$

with the initial condition $I_1(0;\alpha) = I_2(0;\alpha) = I_1'(0;\beta) = I_2'(0;\beta) = I_1''(0;\gamma) = I_2''(0;\gamma) = 0$. Solving the above equations by using the initial conditions we get

$$I_1(t;\alpha) = \frac{(\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_1(\alpha)}{(\tau_1(\alpha)-\eta)^2}\left(1 - e^{-(\tau_1(\alpha)-\eta)t}\right) - \frac{(1-c_1)s_1(\alpha)}{\tau_1(\alpha)-\eta}t$$

$$I_2(t;\alpha) = \frac{(\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\}+(1-c_1)s_2(\alpha)}{(\tau_2(\alpha)-\eta)^2}\left(1 - e^{-(\tau_2(\alpha)-\eta)t}\right) - \frac{(1-c_1)s_2(\alpha)}{\tau_2(\alpha)-\eta}t$$

$$I_1'(t;\beta) = \frac{(\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\}+(1-c_1)s_1'(\beta)}{(\tau_1'(\beta)-\eta)^2}\left(1 - e^{-(\tau_1'(\beta)-\eta)t}\right) - \frac{(1-c_1)s_1'(\beta)}{\tau_1'(\beta)-\eta}t$$

$$I_2'(t;\beta) = \frac{(\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\}+(1-c_1)s_2'(\beta)}{(\tau_2'(\beta)-\eta)^2}\left(1-e^{-(\tau_2'(\beta)-\eta)t}\right) - \frac{(1-c_1)s_2'(\beta)}{\tau_2'(\beta)-\eta}t$$

$$I_1''(t;\gamma) = \frac{(\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_1''(\gamma)}{(\tau_1''(\gamma)-\eta)^2}\left(1-e^{-(\tau_1''(\gamma)-\eta)t}\right) - \frac{(1-c_1)s_1''(\gamma)}{\tau_1''(\gamma)-\eta}t$$

$$I_2''(t;\gamma) = \frac{(\tau_2''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_2''(\gamma)}{(\tau_2''(\gamma)-\eta)^2}\left(1-e^{-(\tau_2''(\gamma)-\eta)t}\right) - \frac{(1-c_1)s_1''(\gamma)}{\tau_2''(\gamma)-\eta}t$$

Again, the equation (12), which characterizes the non-productive phase occurring within the time interval $t_1 \leq t \leq T$, is subsequently transformed into its corresponding parametric form as follows:

$$\langle[\dot{J}_2(t;\alpha), \dot{J}_1(t;\alpha)],[\dot{J}_2'(t;\beta), \dot{J}_1'(t;\beta)][\dot{J}_2''(t;\gamma), \dot{J}_1''(t;\gamma)]\rangle = -\langle[r_1(\alpha), r_2(\alpha)], [r_1'(\beta), r_2'(\beta)], [r_1''(\gamma), r_2''(\gamma)]\rangle - \langle[s_1(\alpha), s_2(\alpha)], [s_1'(\beta), s_2'(\beta)], [s_1''(\gamma), s_2''(\gamma)]\rangle t - \{[\tau_1(\alpha), \tau_2(\alpha)], [\tau_1'(\beta), \tau_2'(\beta)], [\tau_1''(\gamma), \tau_2''(\gamma)]\rangle - \eta\}\langle[J_1(t;\alpha), J_2(t;\alpha)], [J_1'(t;\beta), J_2'(t;\beta)], [J_1''(t;\gamma), J_2''(t;\gamma)]\rangle$$

The expression presented above can be expanded into a system of crisp differential equations, structured as follows:

$$J_2(t;\alpha) = -r_2(\alpha) - s_2(\alpha)t - \{\tau_2(\alpha)-\eta\}\, J_2(t;\alpha)$$

$$J_1(t;\alpha) = -r_1(\alpha) - s_1(\alpha)t - \{\tau_1(\alpha)-\eta\}\, J_1(t;\alpha)$$

$$J_2'(t;\beta) = -r_2'(\beta) - s_2'(\beta)t - \{\tau_2'(\beta)-\eta\}J_2'(t;\beta)$$

$$J_1'(t;\beta) = -r_1'(\beta) - s_1'(\beta)t - \{\tau_1'(\beta)-\eta\}J_1'(t;\beta)$$

$$J_2''(t;\gamma) = -r_2''(\gamma) - s_2''(\gamma)t - \{\tau_2''(\gamma)-\tau\}J_2''(t;\gamma)$$

$$J_1''(t;\gamma) = -r_1''(\gamma) - s_1''(\gamma)t - \{\tau_1''(\gamma)-\tau\}J_1''(t;\gamma)$$

with $J_1(T;\alpha) = J_2(T;\alpha) = J_1'(T;\beta) = J_2'(T;\beta) = J_1''(T;\gamma) = J_2''(T;\gamma) = 0$. Solving the above equations by using the initial conditions we get

$$J_1(t;\alpha) = \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2} - \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2}e^{(\tau_1(\alpha)-\eta)(T-t)}$$

$$J_2(t;\alpha) = \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_2(\alpha)t+r_2(\alpha))}{(\tau_2(\alpha)-\eta)^2} - \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_2(\alpha)T+r_2(\alpha))}{(\tau_2(\alpha)-\eta)^2}e^{(\tau_2(\alpha)-\eta)(T-t)}$$

$$J_1'(t;\beta) = \frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_1'(\beta)t+r_1'(\beta))}{(\tau_1'(\beta)-\eta)^2} - \frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_1'(\beta)T+r_1'(\beta))}{(\tau_1'(\beta)-\eta)^2}e^{(\tau_1'(\beta)-\eta)(T-t)}$$

$$J_2'(t;\beta) = \frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_2'(\beta)t+r_2'(\beta))}{(\tau_2'(\beta)-\eta)^2} - \frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_2'(\beta)T+r_2'(\beta))}{(\tau_2'(\beta)-\eta)^2}e^{(\tau_2'(\beta)-\eta)(T-t)}$$

$$J_1''(t;\gamma) = \frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_1''(\gamma)t+r_1''(\gamma))}{(\tau_1''(\gamma)-\eta)^2} - \frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_1''(\gamma)T+r_1''(\gamma))}{(\tau_1''(\gamma)-\eta)^2}e^{(\tau_1''(\gamma)-\eta)(T-t)}$$

$$J_2''(t;\gamma) = \frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)\left(s_2''(\gamma)t+r_2''(\gamma)\right)}{(\tau_2''(\gamma)-\eta)^2} - \frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)\left(s_2''(\gamma)T+r_2''(\gamma)\right)}{(\tau_2''(\gamma)-\eta)^2}e^{(\tau_2''(\gamma)-\eta)(T-t)}$$

The model incorporates a variety of relevant costs, which are outlined and defined in the following sections. These costs play a crucial role in the overall structure and function of the proposed model.

The holding cost $\widetilde{HC} = \langle[HC_1(\alpha), HC_2(\alpha)], [HC_1'(\beta), HC_2'(\beta)], [HC_1''(\gamma), HC_2''(\gamma)]\rangle$ is derived as

$$HC_1(\alpha) = h_c\left[\int_0^{t_1} I_1(t;\alpha)\,dt + \int_{t_1}^{T} J_1(t;\alpha)\,dt\right]$$

$$= h_c\left[\int_0^{t_1}\left\{\frac{(\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_1(\alpha)}{(\tau_1(\alpha)-\eta)^2}\left(1-e^{-(\tau_1(\alpha)-\eta)t}\right)-\frac{(1-c_1)s_1(\alpha)}{\tau_1(\alpha)-\eta}t\right\}dt + \int_{t_1}^{T}\left\{\frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2}-\frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2}e^{(\tau_1(\alpha)-\eta)(T-t)}\right\}dt\right]$$

$$= h_c\left[\frac{(\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_1(\alpha)}{(\tau_1(\alpha)-\eta)^3}\left(e^{-(\tau_1(\alpha)-\eta)t_1}+(\tau_1(\alpha)-\eta)t_1-1\right)-\frac{(1-c_1)s_1(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)}+\frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)r_1(\alpha)}{(\tau_1(\alpha)-\eta)^2}(T-t_1)-\frac{s_1(\alpha)}{2(\tau_1(\alpha)-\eta)}(T^2-t_1^2)+\frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^3}\left(1-e^{(\tau_1(\alpha)-\eta)(T-t_1)}\right)\right]$$

Similarly,

$$HC_2(\alpha) = h_c\left[\frac{(\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\}+(1-c_1)s_2(\alpha)}{(\tau_2(\alpha)-\eta)^3}\left(e^{-(\tau_2(\alpha)-\eta)t_1}+(\tau_2(\alpha)-\eta)t_1-1\right)-\frac{(1-c_1)s_2(\alpha)t_1^2}{2(\tau_2(\alpha)-\eta)}+\frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)r_2(\alpha)}{(\tau_2(\alpha)-\eta)^2}(T-t_1)-\frac{s_2(\alpha)}{2(\tau_2(\alpha)-\eta)}(T^2-t_1^2)+\frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_2(\alpha)T+r_2(\alpha))}{(\tau_2(\alpha)-\eta)^3}\left\{1-e^{(\tau_2(\alpha)-\eta)(T-t_1)}\right\}\right]$$

$$HC_1'(\beta) = h_c\left[\frac{(\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\}+(1-c_1)s_1'(\beta)}{(\tau_1'(\beta)-\eta)^3}\left(e^{-(\tau_1'(\beta)-\eta)t_1}+(\tau_1'(\beta)-\eta)t_1-1\right)-\frac{(1-c_1)s_1'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)}+\frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)r_1'(\beta)}{(\tau_1'(\beta)-\eta)^2}(T-t_1)-\frac{s_1'(\beta)}{2(\tau_1'(\beta)-\eta)}(T^2-t_1^2)+\frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_1'(\beta)T+r_1'(\beta))}{(\tau_1'(\beta)-\eta)^3}\left\{1-e^{(\tau_1'(\beta)-\eta)(T-t_1)}\right\}\right]$$

$$HC_2'(\beta) = h_c\left[\frac{(\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\}+(1-c_1)s_2'(\beta)}{(\tau_2'(\beta)-\eta)^3}\left(e^{-(\tau_2'(\beta)-\eta)t_1}+(\tau_2'(\beta)-\eta)t_1-1\right)-\frac{(1-c_1)s_2'(\beta)t_1^2}{2(\tau_2'(\beta)-\eta)}+\frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)r_2'(\beta)}{(\tau_2'(\beta)-\eta)^2}(T-t_1)-\frac{s_2'(\beta)}{2(\tau_2'(\beta)-\eta)}(T^2-t_1^2)+\frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_2'(\beta)T+r_2'(\beta))}{(\tau_2'(\beta)-\eta)^3}\left\{1-e^{(\tau_2'(\beta)-\eta)(T-t_1)}\right\}\right]$$

$$HC_1''(\gamma) = h_c\left[\frac{(\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_1''(\gamma)}{(\tau_1''(\gamma)-\eta)^3}\left(e^{-(\tau_1''(\gamma)-\eta)t_1}+(\tau_1''(\gamma)-\eta)t_1-1\right)-\frac{(1-c_1)s_1''(\gamma)t_1^2}{2(\tau_1''(\gamma)-\eta)}+\frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)r_1''(\gamma)}{(\tau_1''(\gamma)-\eta)^2}(T-t_1)-\frac{s_1''(\gamma)}{2(\tau_1''(\gamma)-\eta)}(T^2-t_1^2)+\frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_1''(\gamma)T+r_1''(\gamma))}{(\tau_1''(\gamma)-\eta)^3}\left\{1-e^{(\tau_1''(\gamma)-\eta)(T-t_1)}\right\}\right]$$

$$HC_2''(\gamma) = h_c\left[\frac{(\tau_2''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_2''(\gamma)}{(\tau_2''(\gamma)-\eta)^3}\left(e^{-(\tau_2''(\gamma)-\eta)t_1}+(\tau_2''(\gamma)-\eta)t_1+1\right)-\frac{(1-c_1)s_2''(\gamma)t_1^2}{2(\tau_2''(\gamma)-\eta)}+\frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)r_2''(\gamma)}{(\tau_2''(\gamma)-\eta)^2}(T-t_1)-\frac{s_2''(\gamma)}{2(\tau_2''(\gamma)-\eta)}(T^2-t_1^2)+\frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)(s_2''(\gamma)T+r_2''(\gamma))}{(\tau_2''(\gamma)-\eta)^3}\left\{1-e^{(\tau_2''(\gamma)-\eta)(T-t_1)}\right\}\right]$$

Total production cost $\widetilde{PC} = \langle [PC_1(\alpha), PC_2(\alpha)],\ [PC_1'(\beta), PC_2'(\beta)],\ [PC_1''(\gamma), PC_2''(\gamma)]\rangle$ during the entire circle is obtained as

$$PC_1(\alpha) = p_c\left[\int_0^{t_1}\{(c_0+c_1r_1(\alpha))+c_1s_1(\alpha)t\}dt\right] = p_c\left[\left(c_0+c_1r_1(\alpha)\right)t_1+c_1s_1(\alpha)\frac{t_1^2}{2}\right],$$

$PC_2(\alpha) = p_c\left[\int_0^{t_1}\{(c_0 + la_2(\alpha)) + lb_2(\alpha)t\}dt\right] = p_c\left[(c_0 + c_1r_2(\alpha))t_1 + c_1s_2(\alpha)\frac{t_1^2}{2}\right],$

$PC_1'(\beta) = p_c\left[(c_0 + c_1r_1'(\beta))t_1 + c_1s_1'(\beta)\frac{t_1^2}{2}\right],$

$PC_2'(\beta) = p_c\left[(c_0 + c_1r_2'(\beta))t_1 + c_1s_2'(\beta)\frac{t_1^2}{2}\right],$

$PC_1''(\gamma) = p_c\left[(c_0 + c_1r_1''(\gamma))t_1 + c_1s_1''(\gamma)\frac{t_1^2}{2}\right],$

$PC_2''(\gamma) = p_c\left[(c_0 + c_1r_2''(\gamma))t_1 + c_1s_2''(\gamma)\frac{t_1^2}{2}\right]$

Total cost for preservation is $\widetilde{PRC} = \langle[PRC_1(\alpha), PRC_2(\alpha)],\ [PRC_1'(\beta), PRC_2'(\beta)],\ [PRC_1''(\gamma), PRC_2''(\gamma)]\rangle$ obtained as

$$PRC_1(\alpha) = p_r\eta\left[\int_0^{t_1}\{I_1(t;\alpha) - \widetilde{D}(t)\}dt + \int_{t_1}^{T}\{J_1(t;\alpha) - \widetilde{D}(t)\}dt\right]$$

$$= p_r\eta\left[\int_0^{t_1}\left\{\frac{(\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_1(\alpha)}{(\tau_1(\alpha)-\eta)^2}\left(1 - e^{-(\tau_1(\alpha)-\eta)t}\right) - \frac{(1-c_1)s_1(\alpha)}{\tau_1(\alpha)-\eta}t - r_2(\alpha) - s_2(\alpha)t\right\}dt + \int_{t_1}^{T}\left\{\frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)t+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2} - \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^2}e^{(\tau_1(\alpha)-\eta)(T-t)} - r_2(\alpha) - s_2(\alpha)t\right\}dt\right]$$

$$= p_r\eta\left[\frac{(\tau_1(\alpha)-\eta)\{c_0-(1-c_1)r_1(\alpha)\}+(1-c_1)s_1(\alpha)}{(\tau_1(\alpha)-\eta)^3}\left(e^{-(\tau_1(\alpha)-\eta)t_1} + (\tau_1(\alpha) - \eta)t_1 - 1\right) - \frac{(1-c_1)s_1(\alpha)t_1^2}{2(\tau_1(\alpha)-\eta)} + \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)r_1(\alpha)}{(\tau_1(\alpha)-\eta)^2}(T - t_1) - \frac{s_1(\alpha)}{2(\tau_1(\alpha)-\eta)}(T^2 - t_1^2) + \frac{s_1(\alpha)-(\tau_1(\alpha)-\eta)(s_1(\alpha)T+r_1(\alpha))}{(\tau_1(\alpha)-\eta)^3}\left(1 - e^{(\tau_1(\alpha)-\eta)(T-t_1)}\right) - r_2(\alpha)T - \frac{s_2(\alpha)}{2}T^2\right]$$

Similarly,

$$PRC_2(\alpha) = p_r\eta\left[\frac{(\tau_2(\alpha)-\eta)\{c_0-(1-c_1)r_2(\alpha)\}+(1-c_1)s_2(\alpha)}{(\tau_2(\alpha)-\eta)^3}\left(e^{-(\tau_2(\alpha)-\eta)t_1} + (\tau_2(\alpha) - \eta)t_1 - 1\right) - \frac{(1-c_1)s_2(\alpha)t_1^2}{2(\tau_2(\alpha)-\eta)} + \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)r_2(\alpha)}{(\tau_2(\alpha)-\eta)^2}(T - t_1) - \frac{s_2(\alpha)}{2(\tau_2(\alpha)-\eta)}(T^2 - t_1^2) + \frac{s_2(\alpha)-(\tau_2(\alpha)-\eta)(s_2(\alpha)T+r_2(\alpha))}{(\tau_2(\alpha)-\eta)^3}\left\{1 - e^{(\tau_2(\alpha)-\eta)(T-t_1)}\right\} - r_1(\alpha)T - \frac{s_1(\alpha)}{2}T^2\right]$$

$$PRC_1'(\beta) = p_r\eta\left[\frac{(\tau_1'(\beta)-\eta)\{c_0-(1-c_1)r_1'(\beta)\}+(1-c_1)s_1'(\beta)}{(\tau_1'(\beta)-\eta)^3}\left(e^{-(\tau_1'(\beta)-\eta)t_1} + (\tau_1'(\beta) - \eta)t_1 - 1\right) - \frac{(1-c_1)s_1'(\beta)t_1^2}{2(\tau_1'(\beta)-\eta)} + \frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)r_1'(\beta)}{(\tau_1'(\beta)-\eta)^2}(T - t_1) - \frac{s_1'(\beta)}{2(\tau_1'(\beta)-\eta)}(T^2 - t_1^2) + \frac{s_1'(\beta)-(\tau_1'(\beta)-\eta)(s_1'(\beta)T+r_1'(\beta))}{(\tau_1'(\beta)-\eta)^3}\left\{1 - e^{(\tau_1'(\beta)-\eta)(T-t_1)}\right\} - r_2'(\beta)T - \frac{s_2'(\beta)}{2}T^2\right]$$

$$PRC_2'(\beta) = p_r\eta\left[\frac{(\tau_2'(\beta)-\eta)\{c_0-(1-c_1)r_2'(\beta)\}+(1-c_1)s_2'(\beta)}{(\tau_2'(\beta)-\eta)^3}\left(e^{-(\tau_2'(\beta)-\eta)t_1} + (\tau_2'(\beta) - \eta)t_1 - 1\right) - \frac{(1-c_1)s_2'(\beta)t_1^2}{2(\tau_2'(\beta)-\eta)} + \frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)r_2'(\beta)}{(\tau_2'(\beta)-\eta)^2}(T - t_1) - \frac{s_2'(\beta)}{2(\tau_2'(\beta)-\eta)}(T^2 - t_1^2) + \frac{s_2'(\beta)-(\tau_2'(\beta)-\eta)(s_2'(\beta)T+r_2'(\beta))}{(\tau_2'(\beta)-\eta)^3}\left\{1 - e^{(\tau_2'(\beta)-\eta)(T-t_1)}\right\} - r_1'(\beta)T - \frac{s_1'(\beta)}{2}T^2\right]$$

$$PRC_1''(\gamma) = p_r\eta\left[\frac{(\tau_1''(\gamma)-\eta)\{c_0-(1-c_1)r_1''(\gamma)\}+(1-c_1)s_1''(\gamma)}{(\tau_1''(\gamma)-\eta)^3}\left(e^{-(\tau_1''(\gamma)-\eta)t_1}+(\tau_1''(\gamma)-\eta)t_1-1\right)-\frac{(1-c_1)s_1''(\gamma)t_1^2}{2(\tau_1''(\gamma)-\eta)}+\frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)r_1''(\gamma)}{(\tau_1''(\gamma)-\eta)^2}(T-t_1)-\frac{s_1''(\gamma)}{2(\tau_1''(\gamma)-\eta)}(T^2-t_1^2)+\frac{s_1''(\gamma)-(\tau_1''(\gamma)-\eta)(s_1''(\gamma)T+r_1''(\gamma))}{(\tau_1''(\gamma)-\eta)^3}\left\{1-e^{(\tau_1''(\gamma)-\eta)(T-t_1)}\right\}-r_2''(\gamma)T-\frac{s_2''(\gamma)}{2}T^2\right]$$

$$PRC_2''(\gamma) = p_r\eta\left[\frac{(\tau_2''(\gamma)-\eta)\{c_0-(1-c_1)r_2''(\gamma)\}+(1-c_1)s_2''(\gamma)}{(\tau_2''(\gamma)-\eta)^3}\left(e^{-(\tau_2''(\gamma)-\eta)t_1}+(\tau_2''(\gamma)-\eta)t_1+1\right)-\frac{(1-c_1)s_2''(\gamma)t_1^2}{2(\tau_2''(\gamma)-\eta)}+\frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)r_2''(\gamma)}{(\tau_2''(\gamma)-\eta)^2}(T-t_1)-\frac{s_2''(\gamma)}{2(\tau_2''(\gamma)-\eta)}(T^2-t_1^2)+\frac{s_2''(\gamma)-(\tau_2''(\gamma)-\eta)(s_2''(\gamma)T+r_2''(\gamma))}{(\tau_2''(\gamma)-\eta)^3}\left\{1-e^{(\tau_2''(\gamma)-\eta)(T-t_1)}\right\}-r_1''(\gamma)T-\frac{s_1''(\gamma)}{2}T^2\right]$$

Thus, the parametric form of the inventory system's overall average cost over the whole inventory cycle may be found as

$$[\widetilde{\Pi}]_{(\alpha,\beta,\gamma)} = \langle[\Pi_1(\alpha),\Pi_2(\alpha)],\ [\Pi_1'(\beta),\Pi_2'(\beta)],\ [\Pi_1''(\gamma),\Pi_2''(\gamma)]\rangle,$$

where $\Pi_1(\alpha)=\frac{S_c+PC_1(\alpha)+HC_1(\alpha)+PRC_1(\alpha)}{T}$, $\Pi_2(\alpha)=\frac{S_c+PC_2(\alpha)+HC_2(\alpha)+PRC_2(\alpha)}{T}$, $\Pi_1'(\beta)=\frac{S_c+PC_1'(\beta)+HC_1'(\beta)+PRC_1'(\beta)}{T}$, $\Pi_2'(\beta)=\frac{S_c+PC_2'(\beta)+HC_2'(\beta)+PRC_2'(\beta)}{T}$, $\Pi_1''(\gamma)=\frac{S_c+PC_1''(\gamma)+HC_1''(\gamma)+PRC_1''(\gamma)}{T}$ and $\Pi_2''(\gamma)=\frac{S_c+PC_2''(\gamma)+HC_2''(\gamma)+PRC_2''(\gamma)}{T}$.

Hence, from a mathematical standpoint, the problem of minimizing the inventory system, when considering the case of type-2 neutrosophic differentiability of the inventory level function, can be formulated and expressed as follows:

$$\begin{cases} Min\ \Pi_1(\alpha) \\ Min\ \Pi_2(\alpha) \\ Min\ \Pi_1'(\beta) \\ Min\ \Pi_2'(\beta) \\ Min\ \Pi_1''(\gamma) \\ Min\ \Pi_2''(\gamma) \\ Subject\ to\ T > t_1 > 0 \\ 0 \le \alpha,\beta,\gamma \le 1\ with\ \alpha+\beta+\gamma \le 3 \end{cases} \tag{15}$$

De-neutrosophication: The de-neutrosophied single objective minimization problem in the of tape-2 neutrosophic differentiability is given as

$$\begin{cases} Min\ De(\widetilde{\Pi}) \\ Subject\ to\ T > t_1 > 0 \\ 0 \le \alpha,\beta,\gamma \le 1\ with\ \alpha+\beta+\gamma \le 3 \end{cases} \tag{16}$$

where $De(\widetilde{\Pi})=\frac{\alpha\Pi_1(\alpha)+(1-\alpha)\Pi_2(\alpha)+\beta\Pi_1'(\beta)+(1-\beta)\Pi_2'(\beta)+\gamma\Pi_1''(\gamma)+(1-\gamma)\Pi_2''(\gamma)}{3}$.

7. NUMERICAL SIMULATION

7.1. Algorithm of the Numerical Solution

Step 1: Enter the respective value of the crisp parameters c_0, c_1, r, s, τ, η, p_c, h_c, p_r and S_c.

Step 2: Address the minimization problem described in equation (7) and note the optimum value of the decision variables and objective function.

Step 3: Consider the model in the neutrosophic environment by taking the parameters r, s and τ as the trapezoidal neutrosophic number. Go to step 4 and step 6.

Step 4: Determine the de-neutrosophication value of r, s and τ using area removal method.

Step 5: Solve the minimization problem (7) using the de-neutrosophication values of r, s and τ and other crisp parameter (Old method).

Step 6: Consider two types of generalised neutrosophic derivability of the neutrosophic valued function and solve the minimisation problem (14) and (16), which are referred to as Case 1 and Case 2, respectively.

Step 7: Determine the optimal average cost by comparing the outcomes of Case 1, Case 2, and the old method with the crisp method.

Step 9: End

The algorithm described above can be represented visually through a flowchart, which is illustrated in Figure 1. This flowchart provides a clear and systematic representation of the steps involved in the algorithm, making it easier to understand the overall process.

7.2. Numerical results and graphical representation

In this part of the paper, we analyzed the numerical outcomes for three cases concerned to the proposed model. Model 1 represents the crisp valued that is deterministic decision environment. Model 2 explains the numerical outcomes of the proposed model with Trapezoidal neutrosophic ruled imprecision under neutrosophic differentiability of second type. Also, we considered another imprecise model with traditional consideration of de-neutrosophication approach. For numerical simulation, the following inputs are considered:

(1) For the crisp model, we take $c_0 = 120$, $c_1 = 0.3$, $r = 70$, $s = 0.3$, $\tau = 0.4$, $\eta = 0.2$, $p_c = 2$, $h_c = 2$, $p_r = 1$, $S_c = 800$.

(2) For the neutrosophic model, the neutrosophic number $\tilde{a}$, $\tilde{b}$ and $\tilde{\tau}$ are taken as a single-valued trapezoidal neutrosophic number of Type 1 as
$\tilde{r} = (60,66,74,80; 64,68,72,76; 68,72,76,80)$,
$\tilde{s} = (0.22,0.27,0.33,0.38; 0.27,0.30,0.32.0.35; 0.28,0.30,0.35,0.37)$ and
$\tilde{\tau} = (0.32,0.36,0.44,0.48; 0.35,0.38,0.45,0.48; 0.38,0.41,0.44, 0.47)$
and the value of other parameters is taken as the same as in the crisp model.

The optimum value of the average cost (TAC^*) and the decision variables, namely, the total time cycle (T) and production time (t_1) is represented by Table 2. A graphical counterpart of the obtained results is also displayed through the bar diagram given in Figure 2.

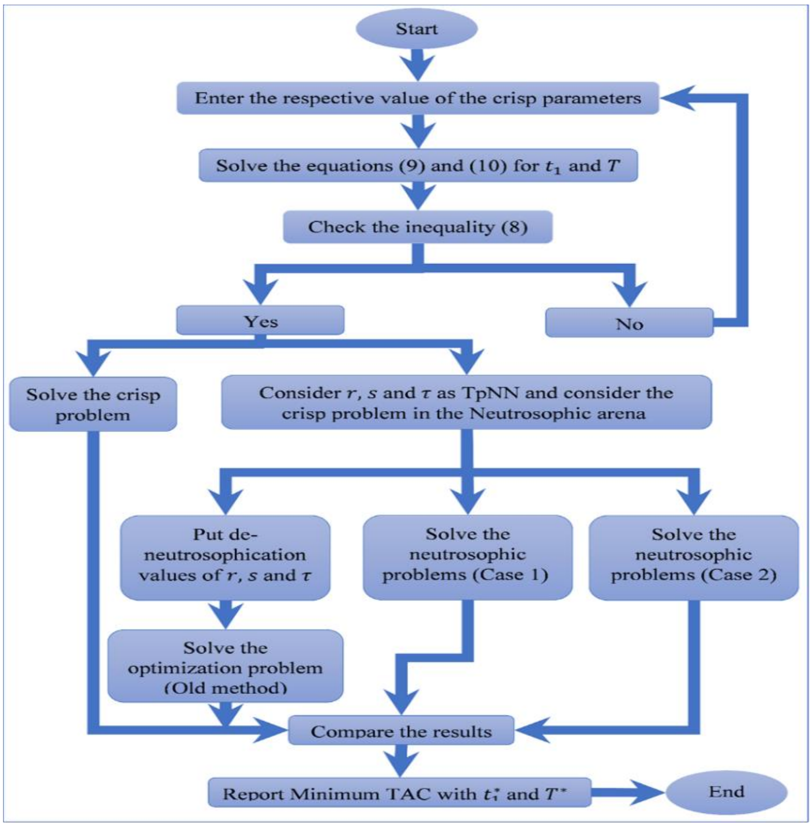

Figure 1: Flowchart of solution algorithm for numerical exploration of the proposed model

Table 2: Optimum results for three different methods for solving the proposed EPQ model in crisp and neutrosophic arena

Model	$t_1{}^*$	T^*	TAC^*
Crisp Model	1.535865	3.968183	476.41
Neutrosophic Model (Case 2)	1.359449	3.946628	463.14
Neutrosophic Model (Old Method)	1.604673	3.981577	479.79

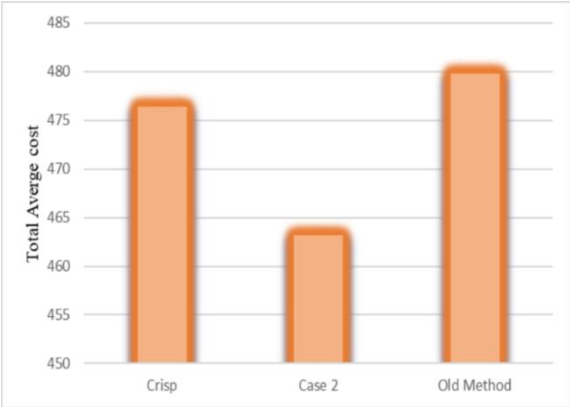

Figure 2: Total average costs in crisp method, neutrosophic analytic approach and neutrosophic old method

From Table 2 and Figure 2, it is perceived that the cost minimization objective is better fulfilled while considering the neutrosophic phenomena with neutrosophic calculus-oriented discussion and proposed de-neutrosophication technique. The optimum values of the objective function of cost reduction \$463.14 corresponds to the optimal values of two decision variables, the production cycle size of 1.359449 months and decision cycle size 3.946628 months. The old method with the removal of the area de-neutrosophication technique before going for the crisp calculus-oriented approach seems to give the most likely outcome of the crisp model. In old method of de-neutrosophication technique, the optimal result for cost minimization objective is obtained as \$479.79, for the optimal production cycle 1.604673 months and optimal decision cycle 3.981577. On the other hand, the crisp model corresponds the numerical outcomes with lowest average cost of \$ 476.41 at production cycle length 1.535865 months and decision cycle length 3.968183 months. Therefore, the proposed approach to dealing with the dynamic of the inventory is established through numerical outcomes.

Figure 3 shows the graph of the TAC concerning the lot-sizing cycle length. The initial trend of the graph of the TAC is decreasing against the total time cycle, and reaching the lowest value of \$ 476.41 at time 3.968183 months, the graph again increases.

Figure 4 shows the three-dimensional inter-dependence among the average cost, total time cycle, and production cycle. The locally convex nature of the graph around (1.535865, 3.968183, 476.41) is spotted clearly in the figure.

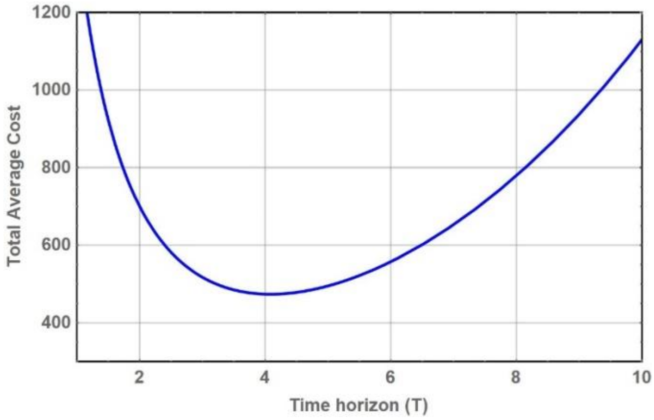

Figure 3: Graphical representation of the crisp cost function with respect to the inventory time horizon T

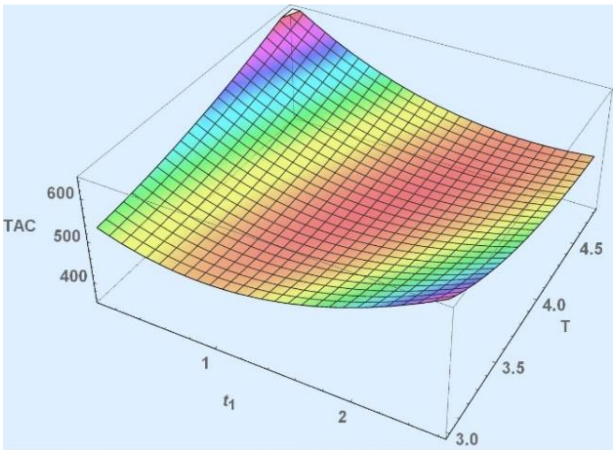

Figure 4: Three-dimensional interdependence of the cost function, productive phase time t_1, and time horizon

7.3. Sensitivity Analysis

The effect of the neutrosophic environment on the proposed economic production quantity model and the optimal results are depicted in the above discussion. In this section, a sensitivity analysis is performed for both cases on the crisp parameter by changing a parameter on a range of -20% to +20% while other parameters kept their original values. The sensitivity of the optimal results against the crisp parameters is given in Table 3. A graphical counterpart of the tabular display is presented in Figures 5, 6 and 7.

Table 3: Sensitivity of the optimum results concerning the crisp input

Crisp parameters	Change in %	New Value	t_1^*	T^*	TAC*
$c_0 = 120$	+30	156	0.6711159	3.404597	492.22
	+20	144	0.8483154	3.545230	484.90
	+10	132	1.070627	3.719955	475.43
	-10	108	1.769044	4.320030	447.06
	-20	96	2.472972	5.071292	426.46
	-30	84	3.531889	6.258762	400.32
$\eta = 0.2$	+30	0.26	1.247909	3.937183	455.26
	+20	0.24	1.284284	3.937972	458.05
	+10	0.22	1.321397	3.941101	460.67
	-10	0.18	1.398630	3.954611	465.47
	-20	0.16	1.439128	3.965119	467.69
	-30	0.14	1.481145	3.978237	469.79
$c_1 = 0.3$	+30	0.39	1.203856	3.827319	469.67
	+20	0.36	1.253011	3.864940	467.59
	+10	0.33	1.304797	3.904640	465.42
	-10	0.27	1.417236	3.991139	460.74
	-20	0.24	1.478458	4.038439	458.23
	-30	0.21	1.543455	4.088831	455.60
$p_c = 2$	+30	2.6	0.9135915	3.627145	488.28
	+20	2.4	1.063842	3.739431	480.77
	+10	2.2	1.212397	3.845744	472.37
	-10	1.8	1.505093	4.042459	453.09
	-20	1.6	1.649345	4.133463	442.28
	-30	1.4	1.792152	4.219739	430.73
$h_c = 2$	+30	2.6	1.374870	3.877278	510.52
	+20	2.4	1.365106	3.868859	494.79
	+10	2.2	1.355599	3.860616	479.06
	-10	1.8	1.364030	4.086572	446.37
	-20	1.6	1.362137	4.238661	428.55
	-30	1.4	1.351407	4.400231	409.55
$p_r = 1$	+30	1.3	1.356218	3.908763	463.38
	+20	1.2	1.357341	3.921273	463.31
	+10	1.1	1.358419	3.933894	463.22
	-10	0.9	1.360429	3.959476	463.04
	-20	0.8	1.361357	3.972439	462.93
	-30	0.7	1.362229	3.985520	462.82
$S_0 = 800$	+30	1040	1.803593	4.630437	519.12
	+20	960	1.659824	4.412041	501.43
	+10	880	1.512060	4.184643	482.81
	-10	720	1.202304	3.721255	442.22
	-20	640	1.027513	3.547593	420.22
	-30	560	0.8168831	3.316784	396.94

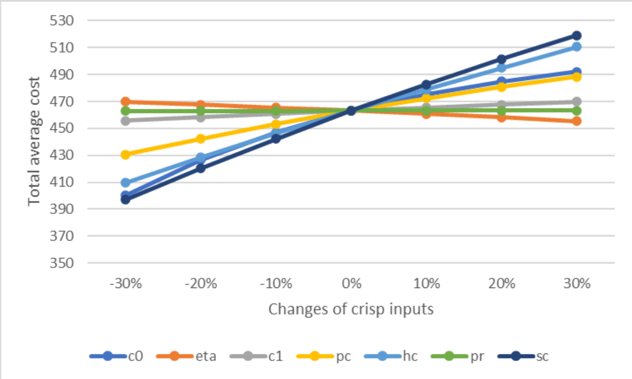

Figure 5: Sensitivity of crisp inputs on cost function

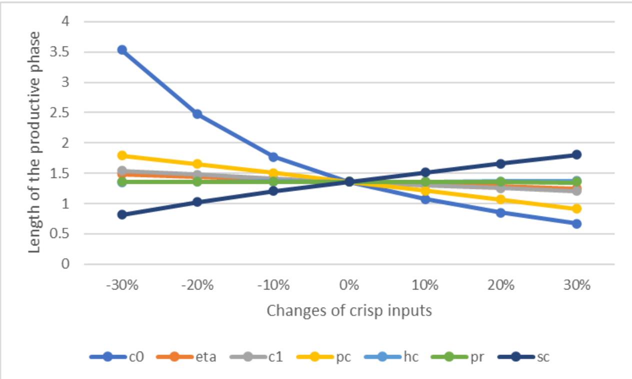

Figure 6: Sensitivity of crisp on the length of the productive phase

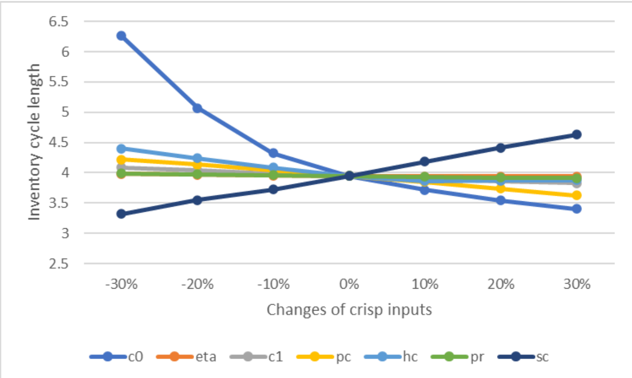

Figure 7: Sensitivity of crisp inputs on the length of the inventory time horizon

From Table 3 and Figures 5, 6 and 7, the following points can be summarized:

- In the mathematical formulation of the proposed model, c_0 is representing the primary part of the production part. It can be regarded as demand independent production potential of the manufacturing system. In Table 3, we analyzed the stability of the optimal results with respect to the variance of the production potential on a range of -20% to +20%. From the top of Table 3, it is perceived that lower value of c_0 corresponds the lower values of the average cost. This can be interpreted as the boosts in the production potential includes additional costs, which results in the increment of the average cost. Impacts of the variance of the production potential on the production and decision cycles are just opposite to that of average cost.

- Another significantly impacting parameter is c_1 representing the dependency of the production rate on the rate of consumption. In the hypothesis of model formulation, the linear dependence of between the production rate and demand was considered. In that relation, c_1 is the variational constant. In Table 3, we analyzed the stability of the optimal results with respect to the variance of c_1 on a range of -20% to +20%. From numerical results, it is obtained that cost minimization objective can be refined by making the variational constant c_1 lowered. This parameter also shows opposite impacts on the production and decision cycle to that of the average cost.
- Deterioration is natural occurrence related to the stocking of items. So, the deterioration of the produced items was considered in the hypothesis of the model formulation. Preservation technology is also considered for diminishing the deterioration of the items in warehouse. The deterioration was uncertain, because it is natural occurrence. The preservation technology is a part of the managerial policy. So, we considered the rate of preservation η to be deterministic. In Table 3, we analyzed the stability of the optimal results with respect to the variance of preservation measure c_1 on a range of -20% to +20%. The incorporation of the preservation measure must add some additional cost. However, it preserves the utility of the produced items. Therefore, the overall impact is noted that the cost can be minimized by integrating preservation technology in managerial policies.
- The sensitivity of optimal solution against the other cost controlling parameters shows the expected out comes concerned to the proposed model.

8. MAJOR RESEARCH FINDINGS AND MANAGERIAL INTUITIONS

The overall research findings related to the proposed model and its association with Trapezoidal neutrosophic decision environment can be summarized as follows:

(i) In Table 3, it is noted that the average cost increases with the production potential. The average cost can be minimized with lowering the dependency of the production rate on the consumption pattern. The incorporation preservation major favour the cost minimization goal.

(ii) From Table 2, it is noted that incorporation of Trapezoidal neutrosophic number ruled uncertain environment seems to be beneficial over the deterministic environment for achieving the cost minimization goal.

(iii) Neutrosophic valued calculus may describe the EPQ model in a more comprehensive approach compared to the approach of utilization of de-neutrosophication technique for deterministic model. The numerical results also perceive advancement of the neutrosophic differential equation approach.

The results and perceptions carry the following managerial intuitions:

(i) Numerical outcomes include one of the significant issues is that average cost increases with the production potential. This observation can be decoded that more production efficiency results additional cost. Since, the model was based on the cost minimization goal only, we did not address whether this phenomena favour profit maximization goal or not. The relationship between average cost minimization and average profit maximization is not straightforward in this context. It may happen that revised and incremented production potential add handsome revenue so the average profit maximization goal would be achieved suppressing its negative impacts on cost minimization gaol. In paper, we only talked about the negative influence of the production potential in the context of cost minimization. The managerial decision should be taken on the overall profit maximization objective.

(ii) The influence of the consumption rate and enthusiasms towards produced items on the production rate was incorporated in the model formulation. Numerical result suggested the minimized cost can be achieved when the dependence of the production on demand be belittled. This observation can be interpreted the managerial policy should ensure the advancement demand prediction so that production process can be free from the volatility about demand pattern.

(iii) Preservation technology minimizes the natural decay or deterioration of the produced items in inventory. Additional costs are required for installing and maintaining the preservation environment for the product in warehouse. It is noticed in the numerical outcomes that incorporation of preservation measures favours the cost minimization objective suppressing the additional cost required for preservation maintenances. The result signifies that preservation measure has a robustly impacting role to diminish deterioration during warehousing.

(iv) Real-world economic transactions and communications cannot be free from impreciseness. In conventional mathematical models, the impreciseness is overlooked and a deterministic counterpart of the actual phenomena is discussed. However, this paper shows that incorporation of neutrosophic logic-based vagueness and analysis of the proposed model using neutrosophic calculus favour the cost minimization goal.

9. CONCLUSIONS

In this paper, a demand driven production inventory model has been formulated and analyzed in neutrosophic environment. In the proposed EPQ model, demand has been taken time-influenced and production rate is demand-controlled. Deterioration of the produced items during storing has been considered and preservation measure has been taken to minimize the menace due to deterioration. Ambiguities regarding demand prediction, production efficiency optimization and deterioration has been tackled using the mathematical frames of Trapezoidal neutrosophic numbers and associated uncertain differential equations. Key findings in this paper can be decoded as follows:

- The higher production potential results negatively the cost minimization objective.
- More reliance of the production on demand rate also causes additional costs.
- Preservation technology favours cost reduction goal.
- Imprecise environment given by Trapezoidal neutrosophic number, neutrosophic differential equation and de-neutrosophication approach produces superior results compared to deterministic and traditional approaches of imprecise environment.

The concluding remarks regarding future scopes of research in this direction is listed as follows:

- The neutrosophic logic has been used in numerous decision-making problems of diverging contexts in contemporary time. Lot sizing approach is a celebrated managerial decision phenomenon in Operation Research which has not been addressed much with the philosophy of neutrosophy. There are open scopes for conducting research works on impacting EOQ and EPQ models, supply chain network with alignment of mentioned notion.
- Neutrosophic valued calculus, integral equation and differential equation have not been advanced yet. However, the study of dynamical systems in neutrosophic environment necessitates the introduction of the mentioned theories in deep rooted manner. So, analytical advancements of the imprecise calculus and fractional calculus with neutrosophy may be matter of enthusiasms for researchers in upcoming days.

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