Yugoslav Journal of Operations Research
35 (2025), Number 4, 775–795
DOI: https://doi.org/10.2298/YJOR240315055K

Research Article

ENTROPY-BASED ANALYSIS USING LINEAR DIOPHANTINE MULTI FUZZY SOFT SETS: A DEA APPROACH FOR IMPROVED DECISION SYSTEMS

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Received: March 2024 / Accepted: October 2024

Abstract: This article unveils an innovative approach to improving the entropy measure analysis of Decision Making Units(DMUs) in the context of linear Diophantine multi-fuzzy soft sets. Though multi-fuzzy soft sets combine multi-dimensional values and parameters to create a hybrid model with considerable versatility, linear diophantine fuzzy sets, a noteworthy extension of conventional fuzzy sets, are also utilized to ease prior constraints. Entropy is a fundamental concept in fuzzy set theory and a useful tool for quantifying the level of fuzziness seen in fuzzy sets. We employ entropy measurements to quantify the weights of input and output components in data envelopment analysis,

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a non-parametric method frequently used in multi-criteria decision-making. The novelty of this study is integrating the weight determination in Data Envelopment Analysis (DEA) by introducing novel entropy measures with linear Diophantine multi-fuzzy soft sets. The significance of DEA is found in its strong analytical capabilities, which facilitate improved decision-making, boost operational effectiveness, and encourage ongoing development in a variety of industries. To illustrate the significance of our suggested approach, we offer a numerical example of building energy efficiency using a DEA model. This work contributes to fuzzy set theory and DEA techniques, offering a helpful tool for evaluating and enhancing complex decision systems.

Keywords: Linear diophantine multi-fuzzy soft set, entropy measure, DEA, energy efficiency.

MSC: 06D72,03E72, 08A72, 15B15.

1. INTRODUCTION

Data envelopment analysis (DEA) is an empirical approach that ranks, classifies, and benchmarks a group of homogeneous decision-making units (DMUs) by the desired inputs and outputs. It is based on mathematical programming and multi-criteria decision-making (MCDM). The Charnes Cooper and Rhodes (CCR) DEA model, which establishes the DMUs' performance efficiencies, was proposed by Charnes et al. [1] in 1978. The following are the main benefits of the DEA approach: it can use numerous outputs and inputs concurrently, without requiring knowledge of the production function or its constraints [2]. It can also use different inputs and outputs with different measurement scales, compare inefficient DMUs with reference sets directly, rank decision-making units, and create objectives for inefficient DMUs [3]. The following succinctly describes the primary drawbacks of the DEA approach: Due to its high computational value, it is challenging to solve complex tasks and measures relative efficiency rather than absolute efficiency [4, 5, 6]. It has several variances in results as a result of measurement error and the outcomes of performance evaluations may change as a result of changes in the kind and quantity of inputs and outputs. The ratio of a DMU's performance efficiency to the highest performance efficiency is known as its relative performance efficiency. A DMU's relative performance efficiency falls between 0 and 1. Studies on crisp DEA have been conducted in several fields [7, 8].

Real-world applications might not always have access to the clear input and output data needed for conventional DEA. The significance of DEA is found in its strong analytical capabilities, which facilitate improved decision-making, boost operational effectiveness, and encourage ongoing development in a variety of industries. However, inputs and outputs are frequently imprecise in real-world problems. The input/output data imprecision might be displayed as fuzzy numbers or ordinal relations. The concept of fuzziness has been established in DEA to deal with fuzzy data by Sengupta [9]. An important method for managing fluctuations and uncertainties in real-world problems is fuzzy set theory [10]. Several studies have examined fuzzy DEA (FDEA) in various fields [11, 12, 13, 14, 15]. According to fuzzy set theory, the rejection value is equal to one less than the acceptance value when the sum of an element's degree of non-membership (rejection) and degree of membership (acceptance) is equal to one [16]. However, the total of an element's acceptance and rejection values can be less than one in real-world situations. As a result, there is still some hesitation. Without a doubt, intuitionistic fuzzy set (IFS) theory is more appropriate than fuzzy set theory. An extension of fuzzy sets, intuitionistic fuzzy sets (IFS) [17] have been proven to be very helpful in handling vagueness. When evaluating an element, IFS takes into account its levels of acceptance and rejection, ensuring that the total of these values is less than or equal to 1.

Additionally, language data can be utilized directly in the DEA models because of fuzzy set theory. Fuzzy linear programming models are the shape that fuzzy DEA models take. This paper's primary goal is to investigate how to rank DMUs using fuzzy entropy and the fuzzy CCR model to find common sets of weights. The fuzzy set's degree of fuzziness is described by entropy. Numerous academics have examined it from various angles. Puri et al. [18] apply the DEA methodology to the banking industry, where two inputs labor and operating expenses have intuitionistic fuzzy essences at the branch level and are represented as TIFNs. The fuzzy CCR model and the fuzzy entropy of the DMUs were used by Tavakkoli-Moghaddam [19] to calculate the efficiency scores of DMUs. In an IF context, A. Arya et al. [20] provide a method to determine IF input-output targets that help convert inefficient DMUs into efficient DMUs. Some recent studies on DEA with fuzzy integration are discussed in the Table 1.

Table 1: Literature Review on Fuzzy DEA approaches

Article	**MCDM Approach**	**Application**
A. Mahmoodirad et al. [21]	Intuitionistic Fuzzy DEA	Evaluating health center
M. A. Sahil et al. [22]	Intuitionistic Fuzzy DEA	Efficiency Analysis of Public sector bank
M. A. Pereira et al. [23]	Fuzzy DEA	Healthcare Access and Quality Index
M. A. Sahil et al. [24]	Sin Shaped Pythagorean Fuzzy DEA	Efficiency Analysis of Public sector bank
L.Hhang & L. Chen [25]	Fuzzy DEA	Efficiency Evaluation of DMUs
J. Zhu et al. [26]	Entropy based Cross efficiency DEA	Efficiency analysis of Enterprise integration
K. K. Raj et al. [27]	Fuzzy DEA	Cost efficiency Analysis of insurers
K.K.Mahanta & D.S. Sharanappa [28]	Spherical Fuzzy DEA	Efficiency Analysis
R.K. Ghalehno et al. [29]	Fuzzy DEA	Ranking of bank branch

1.1. Literature Review

The presence of various forms of uncertainties in the data, which might result from human error or ignorance, makes it challenging for people to select the best option in numerous circumstances. To evaluate these risks and evaluate the process, a wide range of theories are employed, including the fuzzy set theory and its extensions, such as the intuitionistic fuzzy set (IFS) [17], Pythagorean fuzzy set (PFS) [30], and q-rung orthopair fuzzy set (q-ROFS) [31]. Each of these theories states that an object's two membership degrees are used to evaluate it by professionals to ensure that its sum, square sum, and qth power of sum are all equal to one. The linear diophantine fuzzy set (LDFS) [32], which is the precursor of a novel and unique fuzzy concept, dispenses all these restrictions due to the existence of reference parameters. In light of the LDF environment, Jeevitha [33] promoted and applied the DEMATEL strategy in the context of climate change. The development of the linear diophantine multi-fuzzy aggregation operators and their use in digital transformation has been attributed to Jeevitha et al. [34]. To choose the right Agri-Drone, Vimala [35] built the intricate complex LDF soft set.

Aiyared Iampam [36] discussed LDFS for MCDM problems using multiple Einstein aggregation techniques. These operators are capable of deriving ranking information and identifying the optimal choice. Subsequently, Saya Ayub [37] linked algebraic aspects with LDF relations by using decision-making. Riaz [38] extended LDFS by introducing the idea of soft rough sets for application in material handling equipment. By presenting soft rough sets and their possible application in material handling equipment, Riaz[38] expanded the scope of the LDFS. In order to choose third-party logistic service providers, Riaz [39] developed aggregation operators (AOs) that used linear Diophantine fuzzy numbers (LDFNs) in priority order. Einstein's prioritized linear Diophantine fuzzy AOs with applications were proposed by Farid [40]. Riaz [41] recently created Frank AOs for linear Diophantine fuzzy numbers with interval values. Vimala et al. [42] not only explained the characteristics of LDFS and its uses, but they also created the MARCOS technique for LDFS. Jeevitha et al. [43] described the LDFS clustering approach using the LDF correlation coefficient. Petchimuthu [44] aimed to use its AOs to solve the supplier selection problem using IVLDF data. Jeevitha et al. [34] proposed the novel integration of LDF-CODAS and investigated the logistic provider selection.

Even though academics use the existing theories a lot, they have limits since they don't work well with parameterization tools, which keeps the decision-maker(s) from coming to the right conclusion. To avoid these problems, Molodtsov [45] developed the soft set (SS) theory, in which ratings are given based on specific variables. The ideas of fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS) were developed by Maji et al. [46, 47] by fusing this theory with the pre-existing FS and IFS theoretical methods. The hybrid model of the q-ROF multi-fuzzy soft set was derived by Mahalakshmi [48, 49]. Vimala [50, 51] developed and used the notion of soft sets to extend fuzzy sets. Jeevitha [52] implemented the multi-fuzzy soft set in LDFS and discussed its application in the tender selection process. Also, the LDMFS similarity measure was introduced and used to find the suitable alternative for the petrol [53]. Soft sets are extended into the field of hesitant fuzzy sets [54].

A crucial subject in fuzzy set theory is entropy[55]. The entropy of fuzzy sets serves to describe both their degree and fuzziness. Luca and Termini presented the axiom formulation of fuzzy set entropy [56], who also mentioned Shannon's probability entropy and interpreted it as an information-gathering metric. While Higashi and Klir [57] used the distance from a fuzzy set to its complement, Kaufmann [58] noted that the entropy of a fuzzy set can be obtained by measuring the distance between the fuzzy set and its nearest non-fuzzy set. Trillas and Riera proposed general expressions for this entropy [59], while Loo [60] proposed a definition of entropy that incorporates the definitions provided by Kaufmann [58] and Luca and Termini [56]. The well-recognized definitions of fuzzy similarity, distance, and entropy were provided by Liu [61]. Based on the axiom definitions of fuzzy entropy and distance measure, Fan and Ma [62] provided a general conclusion on fuzzy entropy caused by distance measure. Some writers have studied the entropy of interval-valued and intuitionistic fuzzy sets. By using the axiomatic definition of entropy of a fuzzy set presented by Luca and Termini [56] and Liu [61], Zhang et al. [63] suggested a novel definition of entropy for interval-valued fuzzy sets based on distance.

Some researchers have studied the entropy of interval-valued and intuitionistic fuzzy sets. An intuitionistic fuzzy set's degree of fuzziness can be determined thanks to the idea of entropy of intuitionistic fuzzy sets, which was established by Burillo and Bustince [64]. An entropy measure of a non-probabilistic form with a geometric interpretation of intuitionistic fuzzy sets was proposed by Szmidt and Kacprzyk [65]. Using the concept of interval-valued fuzzy sets, Zeng and Li [66] articulated the axioms of Szmidt and Kacprzyk [65] and explored the connection between the entropy and similarity measures. To define the fuzzy entropy of intuitionistic fuzzy sets, Hung and Yang [67] took advantage of the idea of probability. They also developed the definition of the axiom and its properties, as well as two families of entropy measures for intuitionistic fuzzy sets. A unified paradigm for interval-valued fuzzy set's entropy and cardinality was introduced by Vlachos et al. [68].

However, the entropy metrics for the soft sets were introduced in [69]. The proposed entropy measures for IFSSs were made by Jiang et al. [70]. The similarity and entropy measurements for FSS were defined by Liu et al. [71]. The generalized IFSS's entropy and distance measurements were defined by Selvachandran et al. [72]. To calculate the degree of fuzziness of the set, Athira et al. [73] introduce some new entropy measures for PFSS. A. AydoÄdu [74] examines the features of newly proposed information measures for linear Diophantine fuzzy sets.

Several fuzzy number extensions have reportedly been combined with the entropy approach under uncertainty, according to the literature. The focus of this work is on the linear Diophantine multi-fuzzy soft extension of the entropy approach, which has not yet been investigated in the literature. This work therefore establishes the foundation for the energy efficiency calculation in the DEA model using LDMFSS entropy measurements.

1.2. Key Insights and Essence of the research

1. This study presents an unconventional approach to enhance the analysis of entropy measures in the context of linear Diophantine multi-fuzzy soft sets (LDMFSS).
2. This study suggests two different entropy measures for LDMFSS that can be used to quickly figure out the weights of input and output components in DEA.
3. Using information from the UCI machine learning repository, a case study on building energy efficiency is used to illustrate the practical implications of the findings.
4. It demonstrates the robustness of the suggested entropy measure through a comparative analysis, proving its dependability and efficiency in intricate decision-making systems.
5. The Multi-fuzzy soft sets and linear Diophantine fuzzy sets are used to overcome previous limitations in fuzzy set theory and DEA approaches.

6. Our proposed theory improves the assessment and optimization of complex decision systems by offering a more reliable and flexible model for decision-making situations.
7. This theory supplies professionals and academics with an effective tool for a range of applications, advancing DEA methods and linear diophantine fuzzy set theory.

1.3. Research Gap

There is no existing research that integrates the extension of fuzzy sets with entropy measures within the DEA model framework, highlighting a significant gap in the literature. Additionally, despite the widespread use of fuzzy entropy in various algorithms, only a few studies have incorporated fuzzy entropy for the specific purpose of evaluating energy efficiency. This underscores a notable area of opportunity for further exploration. To address these gaps, we introduce novel entropy measures specifically designed for linear Diophantine multi-fuzzy soft sets (LDMFSS). Our research demonstrates how these new entropy measures can be effectively applied to enhance the DEA model, thereby providing a more robust and nuanced approach to decision-making and efficiency evaluation.

This article has been split into four components. The fundamental definitions required for our proposed theory are covered in the first section. Two new definitions of the entropy measure for LDMFSS and its attributes are presented in the second section. The final section addresses the numerical example and the new, innovative entropy implementation approach for the DEA model. The final section of the conclusion contains an overview of this research.

2. PRELIMINARIES

The symbols and their description are given in Table 2. Throughout the paper, these symbols are used as per the description.

Table 2: Nomenclature

Symbol	Description
Λ	Universal Set
$\eth$	A element in Universal Set
μ	Membership Score(MS)
ν	Non-Membership Score(NMS)
α, β	Reference Parameters respect to MS and NMS
Ξ, ϕ	Parameter Set
ε_i	Elements in Parameter Set
$LDMFS(\Lambda)$	The set of all LDMFS over Λ
$LDMFSS(\Lambda)$	The set of all LDMFSS over Λ

Definition 1. *[10] The $\mathbb{FS}$ $\mathfrak{F}$ on Λ is specified as*

$$\mathfrak{F} = \{(\eth, \mu(\eth)) : \eth \in \Lambda\}$$

where $\mu(\eth)$ characterize the MS of $\eth$ and μ is a mapping from Λ to [0,1].

Definition 2. *[32] The* $\mathbb{LDFS}$ $\mathfrak{L}$ *on* Λ *is specified as*

$$\mathfrak{L} = \{(\mathfrak{d}, \langle \mu_{\mathfrak{L}}(\mathfrak{d}), \nu_{\mathfrak{L}}(\mathfrak{d})\rangle, \langle \alpha_{\mathfrak{L}}(\mathfrak{d}), \beta_{\mathfrak{L}}(\mathfrak{d})\rangle,) : \forall \mathfrak{d} \in \Lambda\}$$

where $\mu_{\mathfrak{L}}(\mathfrak{d}), \nu_{\mathfrak{L}}(\mathfrak{d}), \alpha_{\mathfrak{L}}(\mathfrak{d}), \beta_{\mathfrak{L}}(\mathfrak{d})$ *characterized the MS, NMS and its respective reference parameters respectively. Additionally, it restricted to the conditions that* $0 \leq \mu_{\mathfrak{L}}(\mathfrak{d})\alpha_{\mathfrak{L}}(\mathfrak{d}) + \nu_{\mathfrak{L}}(\mathfrak{d})\beta_{\mathfrak{L}}(\mathfrak{d}) \leq 1$ *with* $0 \leq \alpha_{\mathfrak{L}}(\mathfrak{d}) + \beta_{\mathfrak{L}}(\mathfrak{d}) \leq 1$.

Definition 3. *[34] Let K be the set of indices. The* $\mathbb{LDMFS}$ $\mathfrak{M}$ *on* Λ *is specified as*

$$\mathfrak{M} = \{(\mathfrak{d}, \langle \mu^j_{\mathfrak{M}}(\mathfrak{d}), \nu^j_{\mathfrak{M}}(\mathfrak{d})\rangle, \langle \alpha^j_{\mathfrak{M}}(\mathfrak{d}), \beta^j_{\mathfrak{M}}(\mathfrak{d})\rangle) : \forall \mathfrak{d} \in \Lambda\}$$

such that $0 \leq \mu^j_{\mathfrak{M}}(\mathfrak{d})\alpha^j_{\mathfrak{M}}(\mathfrak{d}) + \nu^j_{\mathfrak{M}}(\mathfrak{d})\beta^j_{\mathfrak{M}}(\mathfrak{d}) \leq 1$ *with* $0 \leq \alpha^j_{\mathfrak{M}}(\mathfrak{d}) + \beta^j_{\mathfrak{M}}(\mathfrak{d}) \leq 1$

Definition 4. *[14] The pair* $(\mathfrak{J}, \Xi)$ *is specified as*

$$(\mathfrak{J}, \Xi) = \{(\varepsilon_i, \mathfrak{J}(\varepsilon_i)) : \forall \varepsilon_i \in \Xi\}$$

where $\mathfrak{J}$ *is a mapping from* Ξ *to* $LDMFS(\Lambda)$ *and* $\mathfrak{J}(\varepsilon_i)$ *is a* $\mathbb{LDMFSS}$.

Example 5. *Let* $\{\mathfrak{d}_1, \mathfrak{d}_2\}$ *be two alternatives in* Λ *and* $\{\varepsilon_1, \varepsilon_2\} \in \Xi$. *Then the* $\mathbb{LDMFSS}(\Lambda)$ *is characterized as*

$$(\mathfrak{J}, \Xi) = \left\{ \begin{array}{l} \mathfrak{J}(\varepsilon_1) = \left\{ \begin{array}{l} \{(\mathfrak{d}_1, \langle(0.7, 0.8), (0.5, 0.6)\rangle, \langle(0.6, 0.9), (0.3, 0.1)\rangle) \\ (\mathfrak{d}_2, \langle(0.4, 0.7), (0.7, 0.6)\rangle, \langle(0,7, 0.8), (0.3, 0, 1)\rangle)\} \end{array} \right\} \\ \mathfrak{J}(\varepsilon_2) = \left\{ \begin{array}{l} \{(\mathfrak{d}_1, \langle(0.5, 0.4), (0.6, 0.7)\rangle, \langle(0.7, 0.6), (0.2, 0.3)\rangle) \\ (\mathfrak{d}_2, \langle(0.9, 0.7), (0.2, 0.5)\rangle, \langle(0.7, 0.6), (0.2, 0.3)\rangle) \end{array} \right\} \end{array} \right\}$$

3. ENTROPY MEASURES OF LDMFSS

Definition 6. *Let* $\mathfrak{E}$ *be a real valued function from* $LDMFSS(\Lambda)$ *to [0,1]. Then the function* $\mathfrak{E}$ *is stated as an entropy of* $\mathbb{LDMFSS}$, *if it satisfies the following axioms:*

- ✓ **(C1) Minimality:** $\mathfrak{E}(\mathfrak{J}, \Xi) = 0 \Leftrightarrow \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, *or* $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1, \forall \varepsilon_i \in \Xi$
- ✓ **(C2) Maximality:** $\mathfrak{E}(\mathfrak{J}, \Xi) = 1$, $\Leftrightarrow$ $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\forall \varepsilon_i \in \Xi$
- ✓ **(C3) Resolution:** $\mathfrak{E}(\mathfrak{J}, \Xi) = \mathfrak{E}((\mathfrak{J}, \Xi)^c)$
- ✓ **(C4) Symmetry:** $\mathfrak{E}(\mathfrak{J}, \Xi) \leq \mathfrak{E}(\mathfrak{J}, \Phi)$ *if* $\mathfrak{J}(\Xi) \leq \mathfrak{J}(\Phi)$ *for* $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \leq \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \leq \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$ *and if* $\mathfrak{J}(\Xi) \geq \mathfrak{J}(\Phi)$ *for* $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \geq \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \geq \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$

Theorem 7. *Let* $\Lambda = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, ..., \mathfrak{d}_p\}$ *be the universal set and* $\Xi = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_n\}$. *Let* $(\mathfrak{J}, \Xi) = \{\mathfrak{J}(\varepsilon_i) = \langle \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}), \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\rangle, \langle \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}), \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\rangle : \forall \varepsilon_i \in \Xi\}_{j \in K}$ *be the family of* $\mathbb{LDMFSS}(\Lambda)$. *Define an* $\mathfrak{E}_1(\mathfrak{J}, \Xi)$ *as follows:*

$$\mathfrak{E}_1(\mathfrak{J}, \Xi) = \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1 - |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1 + |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big] \tag{1}$$

Proof. (C1) **Minimality:** $\mathfrak{E}_1(\mathfrak{J}, \Xi) = 0 \Leftrightarrow \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1,\ \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0,\ \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1,$ $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0, \forall \varepsilon_i \in \Xi$

If it is a crisp set, $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$

$\mathfrak{E}_1(\mathfrak{J}, \Xi)$ as follows:

$$\begin{aligned}\mathfrak{E}_1(\mathfrak{J}, \Xi) =& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1 - |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1 + |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big]\\ =& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1 - |1-0|}{1 + |1-0|}\Big]\\ =& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1-1}{1+1}\Big]\\ =& 0\end{aligned}$$

If $\mathfrak{E}_1(\mathfrak{J}, \Xi) = 0$

$$\frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1 - |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1 + |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big] = 0$$

$$\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1 - |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1 + |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big] = 0$$

$$\frac{1 - |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1 + |\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|} = 0$$

$$\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$$

Also by the definition of $\mathbb{LDFS}$ $0 \leq \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \leq 1$

$\Rightarrow \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$

The proof is similar for the other condition.

(C2) **Maximality:** Let $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$ and $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, then

$$\begin{aligned}
\mathfrak{E}_1(\mathfrak{J},\Xi) =& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1-|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1+|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big]\\
=& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1-|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1+|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big]\\
=& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}1\\
=& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{pk}(pk)\\
=& 1
\end{aligned}$$

(C3) **Resolution:**

$$\begin{aligned}
\mathfrak{E}_1((\mathfrak{J},\Xi)^c) =& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1-|\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1+|\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big]\\
=& \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{pk}\sum_{r=1}^{p}\sum_{j=1}^{k}\frac{1-|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}{1+|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|}\Big]\\
=& \mathfrak{E}(\mathfrak{J},\Xi)
\end{aligned}$$

(C4) **Symmetry:** If $\mathfrak{J}(\Xi) \le \mathfrak{J}(\Phi)$ for $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$
we get, $\mu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d}) \le \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \nu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$ and
$\alpha^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d}) \le \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le \beta^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$
If $\mathfrak{J}(\Xi) \ge \mathfrak{J}(\Phi)$ for $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$
we get, $\mu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d}) \ge \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \nu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$ and
$\alpha^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d}) \ge \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \ge \beta^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$
In both cases, we have $\mathfrak{E}_1(\mathfrak{J},\Xi) \le \mathfrak{E}_1(\mathfrak{J},\Phi)$ □

Theorem 8. *Let $\Lambda = \{\mathfrak{d}_1,\mathfrak{d}_2,\mathfrak{d}_3,...,\mathfrak{d}_p\}$ be the universal set and $\Xi = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_n\}$. Let $(\mathfrak{J},\Xi) = \{\mathfrak{J}(\varepsilon_i) = \langle\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}),\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\rangle,\langle\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}),\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\rangle : \forall\varepsilon_i \in \Xi\}_{j\in K}$ be the family of $\mathbb{LDMFSS}(\Lambda)$. Define an $\mathfrak{E}_2(\mathfrak{J},\Xi)$ as follows:*

$$\mathfrak{E}_2(\mathfrak{J},\Xi) = \frac{1}{n}\sum_{i=1}^{n}\Big[1-\frac{1}{2p}\sum_{r=1}^{p}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|+\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\Big] \quad (2)$$

Proof. (C1) **Minimality:** $\mathfrak{E}(\mathfrak{J},\Xi) = 0 \Leftrightarrow \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0, \forall\varepsilon_i \in \Xi$

If it is a crisp set, $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$
$\mathfrak{E}(\mathfrak{J},\Xi)$ as follows:

$$\begin{aligned}\mathfrak{E}_2(\mathfrak{J},\Xi) =& \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right]\\ =&\frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}[1+1]\right] = \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}(2p)\right]=0\end{aligned}$$

If $\mathfrak{E}_2(\mathfrak{J},\Xi) = 0$

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right] =& 0\\ \left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right] =& 0\\ \frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\} =& 1\\ \max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|+\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})| =& 2p\end{aligned}$$

Also we know that, $0 \le \alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) - \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) \le 1 \Rightarrow \mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 1$, $\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = 0$
(C2) **Maximality:** Let $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$ and $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d}) = \beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, then

$$\begin{aligned}\mathfrak{E}_2(\mathfrak{J},\Xi) =& \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right]\\ =& \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right]\\ =& \frac{1}{n}\sum_{i=1}^{n}(1)\\ =& 1\end{aligned}$$

(C3) **Resolution** :

$$\begin{aligned}
\mathfrak{E}_2((\mathfrak{J},\Xi)^c) =& \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})| \\ +\max_{j\in K}|\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right] \\
=& \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2p}\sum_{r=1}^{p}\left\{\begin{array}{c}\max_{j\in K}|\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})| \\ +\max_{j\in K}|\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})-\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})|\end{array}\right\}\right] \\
=& \mathfrak{E}(\mathfrak{J},\Xi)
\end{aligned}$$

(C4) **Symmetry:** If $\mathfrak{J}(\Xi)\leq\mathfrak{J}(\Phi)$ for $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$ we get,

$$\mu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})\leq\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\nu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$$

and

$$\alpha^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})\leq\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\leq\beta^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$$

If $\mathfrak{J}(\Xi)\geq\mathfrak{J}(\Phi)$ for $\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\nu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$, $\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})$ we get,

$$\mu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})\geq\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\mu^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\nu^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$$

and

$$\alpha^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})\geq\alpha^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\beta^j_{\mathfrak{J}(\varepsilon_i)}(\mathfrak{d})\geq\beta^j_{\mathfrak{J}(\lambda_i)}(\mathfrak{d})$$

In both cases, we have $\mathfrak{E}_2(\mathfrak{J},\Xi)\leq\mathfrak{E}_2(\mathfrak{J},\Phi)$ $\square$

Example 9. *Let*

$$(\mathfrak{J},\Xi)=\left\{\begin{array}{l}\mathfrak{J}(\varepsilon_1)=\left\{\begin{array}{l}\{(\mathfrak{d}_1,\langle(0.7,0.8),(0.5,0.6)\rangle,\langle(0.6,0.9),(0.3,0.1)\rangle) \\ (\mathfrak{d}_2,\langle(0.4,0.7),(0.7,0.6)\rangle,\langle(0,7,0.8),(0.3,0,1)\rangle)\}\end{array}\right\} \\ \mathfrak{J}(\varepsilon_2)=\left\{\begin{array}{l}\{(\mathfrak{d}_1,\langle(0.5,0.4),(0.6,0.7)\rangle,\langle(0.7,0.6),(0.2,0.3)\rangle) \\ (\mathfrak{d}_2,\langle(0.9,0.7),(0.2,0.5)\rangle,\langle(0.7,0.6),(0.2,0.3)\rangle)\end{array}\right\}\end{array}\right\}$$

Then

$$\begin{aligned}
\mathfrak{E}_1(\mathfrak{J},\Xi) =& \frac{1}{2}\sum_{1=1}^{2}\left[\frac{1}{2.2}\sum_{r=1}^{2}\left\{\begin{array}{c}\frac{1-|(0.7)(0.6)-(0.5)(0.3)}{1+|(0.7)(0.6)-(0.5)(0.3)}+\\ \frac{1-|(0.8)(0.9)-(0.6)(0.1)}{1+|(0.8)(0.9)-(0.6)(0.1)}\end{array}\right\}\right]\\
=& \frac{1}{2}\sum_{1=1}^{2}\left[\frac{1}{2.2}\left\{\begin{array}{c}\frac{1-|0.42-0.15|}{1+|0.42-0.15|}+\frac{1-|0.72-0.06|}{1+|0.72-0.06|}+\\ \frac{1-|0.28-0.21|}{1+|0.28-0.21|}+\frac{1-|0.42-0.08|}{1+|0.42-0.08|}\end{array}\right\}\right]\\
=& \frac{1}{2}\sum_{1=1}^{2}\left[\frac{1}{2.2}[\frac{0.73}{1.27}+\frac{0.14}{1.86}+\frac{0.93}{1.07}+\frac{0.64}{1.36}]\right]\\
=& \frac{1}{2}\sum_{1=1}^{2}[\frac{1}{4}(1.99)]\\
=& \frac{1}{2}[\frac{1}{4}(1.99)+\frac{1}{4}(3.025)]\\
=& \frac{1}{2}(0.4975+0.75625)\\
=& 0.626875
\end{aligned}$$

Also,

$$\mathfrak{E}_2(\mathfrak{J},\Xi) = \frac{1}{2}[1-\frac{2}{4}]+[1-\frac{2}{4}] = \frac{1}{2}[0.5+0.5] = 0.5.$$

4. EFFICIENCY EVALUATION WITH ENTROPY-WEIGHTED LDMFSS IN DEA

4.1. LDMFS Entropy implementation strategy for DEA model

Consider m DMUs ($\mathfrak{D}_1,\mathfrak{D}_2,...,\mathfrak{D}_m$) and p inputs and q outputs for each DMUs. Let $\mathfrak{V} = \{\mathfrak{v}_1,\mathfrak{v}_2,...,\mathfrak{v}_n\}$ be the parameter set.

Step 1: Construct the decision matrix for each input and output such that the rows represent DMUs and the columns represent parameters.

Step 2: Calculate the entropy ε_j for each decision matrix.

Step 3: Calculate the diversity index $\mathfrak{d}_j$ for each alternative, where

$$\mathfrak{d}_j = 1-\varepsilon_j \tag{3}$$

Step 4: Determine the weight of each LDMFSS using a specific formula.

$$\mathfrak{w}_j = \frac{\mathfrak{d}_j}{\sum\mathfrak{d}_j} \tag{4}$$

Step 5: Compute the energy efficiency(ζ_i) of each DMU by utilizing the obtained weights.

$$\zeta_i = \frac{\sum_{r=1}^{n}\sum_{j=p+1}^{q} \mathfrak{w}_j y_{rj}}{\sum_{r=1}^{n}\sum_{j=1}^{p} \mathfrak{w}_j y_{rj}} \tag{5}$$

The pictorial representation of the workflow is demonstrated in Figure 1.

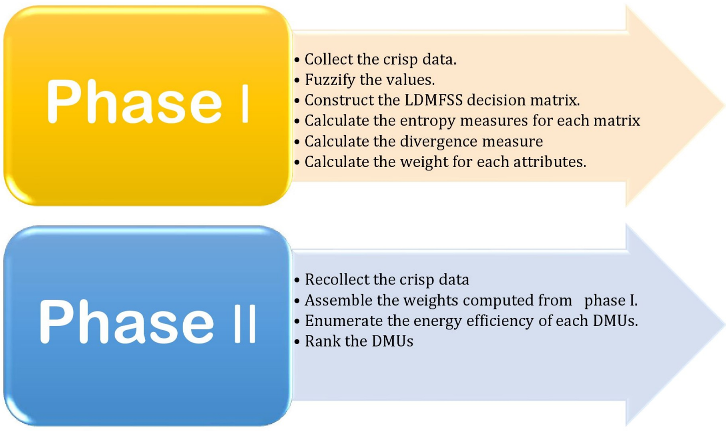

Figure 1: Diagrammatic representation of algorithmic workflow

4.2. Case Study

We gather the data for the case study from the UCI Machine Learning Repository [75]. Twelve buildings underwent an energy analysis with six inputs ($\mathfrak{X}_i$(i = 1,2,3,4,5,6)) and two outputs($\mathfrak{Y}_i$(i=1,2)). The heating load and cooling load are the outputs, and the six inputs are relative compactness, surface area, wall area, roof area, overall height, and glazing area. We build sixteen distinct DMUs based on the different distributions of glazing areas.

4.2.1. Description of Parameters

- ✓ The term "**Relative Compactness**" describes how compact the shape of the building is. In comparison to a less compact building (such as a long, thin building), a more compact building (such as a cube) has less surface area relative to its volume. This is an important measure because a compact design can limit heat gain or loss by minimizing the surface area exposed to the outside environment.
- ✓ **Surface Area**, expressed in square meters, is the building's entire external surface area. It consists of the foundation, the roof, and every wall. The area available for heat exchange with the environment increases with surface area, which affects both heating and cooling loads.
- ✓ **Wall Area** is the total area of the building's external walls. It is important for thermal performance since, depending on the insulation, materials, and solar exposure, walls can either gain or lose heat.

✓ **Roof Area** is the overall area of the building's roof. Because of its exposure to sunlight, rain, and wind, the roof plays a crucial role in the building's thermal envelope, influencing both the building's heating and cooling loads.

✓ The building's **overall height** is measured from the ground to the peak of the roof. Because of things like air stratification and wind exposure, taller buildings could have different heating and cooling dynamics.

✓ The term "**orientation**" describes the direction that a building faces, such as North, South, East, or West. This influences the building's exposure to sunlight at different times of the day and year, which in turn impacts the amount of heating and cooling that is needed.

✓ The **distribution of the glazing area** among the building's various orientations is referred to as glazing area distribution. For instance, adding extra glazing to the southern (northern hemisphere) side of a building can boost solar gain in the winter but also raise the cooling load in the summer.

✓ The amount of energy required during cold weather to keep a comfortable temperature inside is known as the "**heating load**." Usually, it is expressed in kilowatts (kW). The building's overall thermal envelope, window performance, and insulation all have an impact on this load.

✓ The amount of energy needed to maintain a pleasant indoor temperature in hot weather is known as the **cooling load**, and it is expressed in kilowatts (kW). The building's exposure to sunshine, internal heat gains, ventilation, and cooling system efficiency are some of the factors that affect this load.

4.2.2. Decision-Making process

Owing to the substantial volume of data, all gathered and calculated data are included in the supplemental files. The collected crisp data is fuzzified for phase 1, and the crisp data is used exactly as it is for phase 2.

Phase I: Computing the weight of inputs and outputs

1. For each set of inputs and outputs, the LDMFSS decision matrix is constructed (Given in the supplementary file). As a parameter set, the twelve distinct structures are considered, and the multi-dimensional value is classified according to various orientations.
2. We estimate and display the entropy for every LDMFSS decision matrix in Table 3.
3. The weights of each input and output, as well as the divergence measure, are calculated and presented in Table 3.

Phase II: Generating Energy Efficiency

1. We compute each DMU's energy efficiency using the weights specified in phase I (Refer 4).
2. We rank the DMUs according to their energy efficiency as follows.

Table 3: Weight of each input and output

Alternatives	Entropy measure	Divergence measure	Weight
$\mathfrak{X}_1$	0.465722684	0.534277316	0.113789795
$\mathfrak{X}_2$	0.447277108	0.552722892	0.117718313
$\mathfrak{X}_3$	0.573687975	0.426312025	0.090795466
$\mathfrak{X}_4$	0.260313689	0.739686311	0.157537576
$\mathfrak{X}_5$	0.207547417	0.792452583	0.168775679
$\mathfrak{X}_6$	0.474654848	0.525345152	0.111887432
$\mathfrak{Y}_1$	0.431822003	0.568177997	0.121009924
$\mathfrak{Y}_2$	0.443673453	0.556326547	0.118485816

Table 4: Energy Efficiency of DMUs

DMUs	Energy Efficiency	Rank	DMUs	Energy Efficiency	Rank
$\mathfrak{D}_1$	0.029708579	16	$\mathfrak{D}_9$	0.041284824	10
$\mathfrak{D}_2$	0.037895249	14	$\mathfrak{D}_{10}$	0.041732179	8
$\mathfrak{D}_3$	0.037811269	15	$\mathfrak{D}_{11}$	0.04129712	9
$\mathfrak{D}_4$	0.037830987	13	$\mathfrak{D}_{12}$	0.046318645	1
$\mathfrak{D}_5$	0.037867782	12	$\mathfrak{D}_{13}$	0.046174687	2
$\mathfrak{D}_6$	0.037961643	11	$\mathfrak{D}_{14}$	0.045135838	4
$\mathfrak{D}_7$	0.042303342	6	$\mathfrak{D}_{15}$	0.04612022	3
$\mathfrak{D}_8$	0.041813536	7	$\mathfrak{D}_{16}$	0.045135766	5

Result and Discussion:

As can be seen from the observations, the D12 has the maximum energy efficiency. The results are represented diagrammatically in the Figure 2.

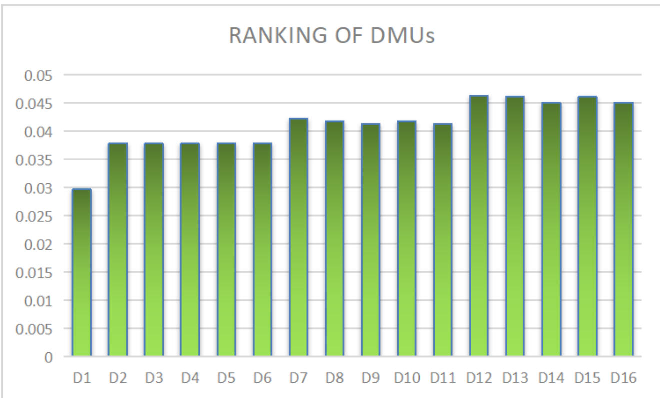

Figure 2: Ranking of DMUs

4.3. Comparative Analysis

Based on these studies and the comparative analysis, we conclude the present research that the results derived using the suggested technique are compatible with the existing research, making it conservative. The main advantage of the proposed technique over the current methods for decision-making, however, is the fact that it has a lot more data to interpret the ambiguities in the data. Since it conveys object information more accurately and objectively under LDMFSS, it is a useful tool for dealing with ambiguous and imprecise information throughout the decision-making process. Compared to the existing methods, the LDMFSS implementation in the DEA methodology with the designated entropy measures has the advantage of not relying just on unfavorable aspects when making decisions.

Table 5: Comparative Analysis

Method	Ranking
Fuzzy entropy	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
Fuzzy soft entropy	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
IF entropy	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
IF soft entropy[71]	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
PF entropy	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
PF soft Entropy[73]	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$
LDF entropy[74]	$\mathfrak{D}_{12} < \mathfrak{D}_{13} < \mathfrak{D}_{15} < \mathfrak{D}_{14} < \mathfrak{D}_{16} < \mathfrak{D}_7 < \mathfrak{D}_8 < \mathfrak{D}_{10} < \mathfrak{D}_{11} < \mathfrak{D}_9 < \mathfrak{D}_6 < \mathfrak{D}_5 < \mathfrak{D}_4 < \mathfrak{D}_2 < \mathfrak{D}_3 < \mathfrak{D}_1$

5. LIMITATION OF PROPOSED METHOD

Although our suggested approach removes some of the earlier restrictions, there are still some limits, which are listed below.

1. Our suggested theory cannot handle the case where the sum of reference parameter values is more than 1. Example: $0.7+0.5 \geq 1$. Thus, the theory breaks down when the total of the two reference parameters is more than 1.
2. In the event of a lot of data, the suggested model's computation may take a while.

6. CONCLUSION

This study introduces a novel approach to enhance entropy measure analysis within the framework of LDMFSS. Our proposed algorithm contributes to both fuzzy set theory and DEA techniques. By leveraging the versatility of multi-fuzzy soft sets and the flexibility offered by linear Diophantine fuzzy sets, our proposed methodology overcomes prior constraints, offering a more robust and adaptable model for decision-making contexts. In this study, we proposed two different entropy measures for LDMFSS.

This enables us to effectively quantify the weights of input and output components in DEA. The importance of DEA stems from its strong analytical powers, which facilitate improved decision-making, boost operational effectiveness, and encourage ongoing development in a number of industries. Through a case study focusing on building energy efficiency, we demonstrate the practical significance of our approach, highlighting its applicability in real-world scenarios. The data for the case study are utilized from the UCI machine learning repository. Further, we conduct a comparative analysis that showcases the robustness of our proposed entropy measure. This validation underscores the reliability and effectiveness of our methodology in addressing complex decision systems.

Our future research endeavors will extend this work to further explore the context of LDMFSS relations, providing additional insights and refinements. Moreover, we aim to broaden the applicability of our proposed model by exploring its integration with other decision-making methodologies such as TOPSIS, VIKOR, and AHP, as well as investigating its potential applications in information aggregation, correlation analysis, and distance and similarity measures.

Thus, this research provides a valuable contribution to the LDFS theory and DEA techniques, offering researchers and practitioners a powerful tool for evaluating and enhancing complex decision systems across various domains.

Funding: The article has been written with the joint financial support of RUSA-Phase 2.0 grant sanctioned vide letter No.F 24-51/2014-U, Policy (TN Multi-Gen), Dept. of Edn. Govt. of India, Dt. 09.10.2018, DST-PURSE 2nd Phase programme vide letter No. SR/PURSE Phase 2/38 (G) Dt. 21.02.2017 and DST (FIST - level I) 657876570 vide letter No.SR/FIST/MS-I/2018/17 Dt. 20.12.2018.

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