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Research Article

OPTIMAL CONTROL, CONSTRUCTION, AND ANALYSIS OF λ-BERNSTEIN BÉZIER SURFACES FOR QUASI-HARMONIC FUNCTIONAL

Mariyam Ehsan BUTTAR
Department of Mathematics, University of the Punjab, Lahore, Pakistan
mariyambuttar786@gmail.com, ORCID: 0009-0008-5395-5886

Daud AHMAD*
Department of Mathematics, University of the Punjab, Lahore, Pakistan
daud.math@pu.edu.pk, ORCID: 0000-0003-0966-8237

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Abstract: In this article, we investigate a novel construction scheme for λ-Bernstein Bézier surfaces and illustrate it with biquadratic and bicubic cases to demonstrate their geometric characteristics and their applications. The primary objective is to investigate how the shape parameter λ improves control over surface smoothness and facilitates an optimal solution in surface design. We examine the geometric properties of these surfaces, including mean and Gaussian curvature, shape operator coefficients, and Gauss-Weingarten coefficients. We also analyze the extremal conditions for λ-Bernstein Bézier surfaces derived from the vanishing condition for the gradient of the quasi-harmonic functional. Integral formulations based on Bernstein polynomials enable precise computation of the vanishing gradient condition, allowing us to determine the constraints on interior control points in terms of known boundary control points. Graphical illustrations validate the approach by providing a better understanding of the geometric properties of these surfaces, including improved surface smoothness and design flexibility. They effectively showcase the behavior of λ-Bernstein polynomials and their corresponding surfaces. Computational results demonstrate the effectiveness of this method for applications in computer graphics, computational geometry, computer science, and engineering, offer-

*Corresponding author

ing a robust framework for analyzing and generating optimal surfaces, contributing to advancements across various scientific disciplines.

Keywords: Optimization, minimal surfaces, mean curvature, variational minimization, computer graphics, computational geometry, engineering, λ-Bernstein Bézier surface.

MSC: 49Q05, 53A10, 58E12, 68U05.

1. INTRODUCTION

Minimal surfaces [1, 2], originating from Lagrange's work in the 18th century, have profoundly influenced diverse scientific fields. Characterized by their minimal area property [3], they find their applications in various fields [4, 5, 6], including approximation theory, computer graphics, engineering and numerical analysis. Minimal surfaces lie at the intersection of differential geometry [7, 8], complex analysis, and topology, and have spurred the development of new areas such as geometric measure theory and conformal geometry. These fascinating objects have had a significant impact on various fields, including robotics [9, 10, 11] and physics. Ai et al. [12] investigated the time-like minimal surface equation in Minkowski space, addressing conjectures on low regularity solutions for nonlinear wave equations. The study of the minimal surfaces has introduced new mathematical tools for better understanding of the surfaces with fixed or free boundary curves. These surfaces can be obtained as the variational improvement to the surfaces by utilizing variational techniques [13, 14, 15, 16, 17, 18, 19] for quasi-minimal and quasi-harmonic surfaces, obtained as the extremal of some suitable energy functional. These surfaces are used to construct and model surfaces in computer-aided geometric design (CAGD) [20, 21, 22], which can be used for various representations such as the movement of objects, the behavior of fluids and solids, and the interaction of light with surfaces.

In optimization theory, one seeks optimal conditions [23] for a curve or a surface to represent extremals of a curve or surface within a domain D, by maximizing or minimizing the values of an objective function, a function to be optimized. These extreme values, known as critical points of a surface, can be derived through variational improvements in surfaces for the corresponding quasi-minimal surfaces [14, 16, 19, 18]. These objective functionals can represent various useful geometric properties of the objects under consideration. In differential geometry, this may represent the area functional, Dirichlet functional, or quasi-harmonic functional. In general relativity, for instance, extremals of the Einstein-Hilbert action under different conditions result in EFEs within modified gravity theories, reflecting the symmetries of spacetime [24]. These symmetries involve solving equations related to the Lie derivative of specific tensors in spherically symmetric space-times [25, 26, 27]. The solutions to these nonlinear partial differential equations have garnered attention across various fields, providing insights applicable beyond their original domain [28, 29, 30]. One significant focus in differential geometry is the construction and analysis of minimal surfaces, which are surfaces that minimize their area subject to specific constraints. Instead of minimizing the area functional to obtain these minimal surfaces, one can consider the Dirichlet functional or other energy functionals like the quasi-harmonic or bending energy functional with a given boundary of control points, in the context of Bézier surfaces [1, 2, 13, 15, 17]. These surfaces can be obtained as the extremals of various energy functionals [31, 32, 13, 33, 14, 15, 34, 16, 19, 17, 18, 35, 36, 37, 38, 39, 40, 35, 38].

Bézier surfaces model the surfaces depending on the basis functions and the control points, however, more control can be achieved by introducing a parameter(s) in its basis functions, usually referred to as the shape parameter(s). The Bézier surfaces can model the new shapes by generalizing the Bernstein polynomials [41]. The modifications in the Bernstein polynomials include shifted-knots Bernstein polynomials [42, 35], q-Bernstein polynomials [43, 44, 45, 46, 47, 48, 49, 46], their extension typically represented as (p, q)-Bernstein polynomials [50, 51, 47, 49] and other modifications like introducing a parameter(s) in the Bernstein functions. Cai et al. [52] studied a class of λ-Bernstein operators, establishing a Korovkin-type theorem and a Voronovskaja-type asymptotic formula that demonstrate improved convergence. Mursaleen et al. [53] investigated modified λ-Bernstein polynomials using shifted knots for Bézier basis functions, establishing Korovkin's theorem, a convergence theorem for Lipschitz functions, and an asymptotic Voronovskaja-type formula. Delgado [54] analyzed the geometric properties and algorithms of q-Bézier curves and surfaces. Zhou and Cai [55] investigated a class of bivariate λ-Bernstein operators, $B_m^{\lambda_1,\lambda_2}(f; x, y)$, on triangular domains, developing a Korovkin-type approximation theorem and a Voronovskaja-type asymptotic formula. Turhan et al. [56] investigated Kantorovich-Stancu type (α, λ, s)-Bernstein operators based on adapted Bézier bases, focusing on convergence properties with shape parameters $\lambda \in [-1, 1]$, $\alpha \in [0, 1]$, and a positive parameter s. Lin et al. [57] developed modified λ-Bernstein-Stancu operators with improved symmetrical properties and studied both a Korovkin-type approximation theorem and the Voronovskaja-type theorem. Yan and Liang [58] introduced basis functions to generate Bézier-like curves and surfaces that allow shape adjustments through a shape parameter λ without changing control points, retaining the classical Bézier properties when the shape parameter λ is set equal to zero. By introducing a parameter(s) into the basis functions one can find more control over the shape of these surfaces, as has been done by numerous mathematicians in the field. Hu et al. [59] introduce a shape-adjustable generalized Bézier curve with multiple shape parameters, demonstrating its applications in engineering surface modeling, including the construction of various complex surfaces and the derivation of necessary conditions for continuity.

Minimal surfaces and surfaces resulting from the extremals of the quasi-harmonic functional are interconnected. A minimal surface is defined by having zero mean curvature across all its parameterizations. Classical examples include the plane, catenoid, and helicoid, with other notable surfaces like the Enneper surface, Henneberg's surface, and Catalan's surface. A minimal surface is obtained by minimizing its area functional,

$$A(\boldsymbol{x}) = \int\int_D |\boldsymbol{x}_u(u,v) \times \boldsymbol{x}_v(u,v)| \, du \, dv, \tag{1}$$

where $D \subset \mathbb{R}^2$ represents a domain over which the surface $\boldsymbol{x}(u,v)$ is defined as a mapping, with the boundary condition $\boldsymbol{x}(\partial D) = \Gamma$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Here, $\boldsymbol{x}_u(u,v)$ and $\boldsymbol{x}_v(u,v)$ denote the partial derivatives of $\boldsymbol{x}(u,v)$ with respect to u and v, respectively. The integrand of the area functional (1) involves a square root, making it solvable in only some straightforward cases. To address this, the area functional can be replaced by some other suitable energy functional, such as the Dirichlet functional or the quasi-harmonic functional. The minimal surfaces and the harmonic surfaces are interconnected. In this article, we utilize the quasi-harmonic functional to find the extremal constraints for λ-Bézier surfaces, following the construction scheme outlined by Yan and Liang [58]. These surfaces have been further explored for their geometric characteristics, particularly for the biquadratic and bicubic cases.

This paper presents the following key contributions:

- We investigate a construction scheme for λ-Bernstein Bézier surfaces, illustrated by the biquadratic and bicubic cases.
- We provide graphical illustrations to explore the geometric characteristics of these surfaces, including mean and Gaussian curvature, shape operator coefficients, and Gauss-Weingarten coefficients.
- We present integral formulations based on Bernstein polynomials, enabling the computation of interior control points from known boundary points.
- We investigate λ-Bernstein Bézier surfaces as extremals of the quasi-harmonic functional, offering enhanced control over surface smoothness through the shape parameter λ.
- We validate our approach through computational results that demonstrate its potential applications in computer graphics, computational geometry, and engineering.

In this paper, we have presented the construction scheme for λ-Bernstein Bézier surfaces and the corresponding quasi-harmonic surfaces that depend on λ-Bernstein polynomials. We have provided graphical illustrations of these polynomials and the corresponding surfaces, along with the geometric characteristics of these surfaces, including mean and Gaussian curvature, shape operator coefficients, and Gauss-Weingarten coefficients. To visually explore the geometric aspects of these surfaces, we have incorporated examples of both biquadratic and bicubic cases, each characterized by the shape parameter λ. Furthermore, we present the constraint integrals in terms of Bernstein polynomials, which can be solved for the known control points and Bernstein polynomials. This allows us to obtain the vanishing gradient condition for the quasi-harmonic functional, thereby solving for the unknown interior control points and obtaining the surface as the extremal of the quasi-harmonic functional.

The rest of the article is organized as follows. In section 2, we discuss λ-Bernstein Bézier surfaces, their derivatives, and integrals, which are useful for the subsequent sections. In section 3, we present the geometric characteristics of λ-Bernstein Bézier surfaces, specifically for biquadratic and bicubic λ-Bernstein Bézier surfaces with graphical representation of geometric quantities. In section 4, we present a result for the λ-Bernstein Bézier surfaces as extremals of the quasi-harmonic functional. Finally, in Section 5, the concluding remarks and future prospects of the study are presented.

2. PRELIMINARIES

In this section, we introduce λ-Bernstein polynomials. These polynomials are shown graphically for various degrees and they serve as the basis for λ-Bernstein Bézier curves and surfaces. We also present the construction scheme for λ-Bernstein Bézier surfaces (biquadratic and bicubic). Partial derivatives of these polynomials and surfaces are provided, as they will be used in the forthcoming section when computing the gradient of the quasi-harmonic functional. These curves and surfaces play an important role in constructing the corresponding Bézier curves and surfaces, which are significant in geometric modeling and computer graphics. The λ-Bernstein polynomials and surfaces (Yan and Liang [58]) are the λ-extension of Bernstein polynomials and the corresponding λ-Bernstein polynomial-based Bézier surfaces with shape-adjustable features. These basis functions of any order m are derived from a set of second-order initial basis functions by utilizing a recursive technique for $m > 2$, using λ as a shape parameter. They are Bernstein-like functions. For the same set of control points, the shape parameter λ can control the shape of such curves and surfaces. The λ-Bernstein basis functions of order $m > 2$, with u as an independent parameter, can be expressed as,

$$b_i^{m,\lambda}(u) = B_i^m(u)\Lambda_i^m(u), \tag{2}$$

where

$$\Lambda_i^m(u) = [1 + (\frac{3C_{m-2}^{i-1} + C_{m-1}^i - C_m^i}{C_m^i})\lambda - \frac{2C_{m-1}^i}{C_m^i}\lambda u + \lambda u^2], \tag{3}$$

with $B_i^m(u)$ as the classical Bernstein polynomials

$$B_i^m(u) = C_m^i u^i (1-u)^{m-i}, \tag{4}$$

$\lambda \in [-1, 1]$, $u \in [0, 1]$, $i = 0, 1, 2, 3, ..., m$, $m \geq 2$ and C_m^i, the binomial coefficients defined as $C_i^m = \frac{m!}{i!(m-i)!}$. The λ-Bernstein polynomials [58], $b_{i,n}^{\lambda}(u)$, exhibit properties analogous to classical Bernstein polynomials $B_i^n(u)$. These polynomials are linearly independent and can be expressed as a linear combination of classical Bernstein basis functions $B_i^n(u)$ of degree n and $n + 2$,

$$\begin{aligned} b_i^{n,\lambda}(u) =& (1-\lambda)B_i^n(u) + \lambda(\frac{3C_{n-2}^{i-1} + C_{n-1}^i}{C_{n+2}^i}B_i^{n+2}(u) + \frac{6C_{n-2}^{i-1}}{C_{n+2}^{i+1}} \times \\ & B_{i+1}^{n+2}(u) + \frac{3C_{n-2}^{i-1} - C_{n-1}^i + C_n^i}{C_{n+2}^{i+2}}B_{i+2}^{n+2}(u)). \end{aligned} \tag{5}$$

For $\lambda \in [-1,1]$, $b_i^{n,\lambda}(u) \geq 0$ $(i = 0,1,\ldots,n;\ n \geq 2)$, and $\sum_{k=0}^{n} b_k^{n,\lambda}(u) = 1$. These polynomials satisfy symmetry properties: $b_k^{n,\lambda}(1-u) = b_{n-k}^{n,\lambda}(u)$ for $k = 0,1,\ldots,n$ and $u \in [0,1]$. At the end points, for $k = 0,1,\ldots,n$, for $n \geq 2$, $b_k^{n,\lambda}(0) = 1$ if $k = 0$, and $b_k^{n,\lambda}(0) = 0$ if $k \neq 0$. Similarly, $b_k^{n,\lambda}(1) = 1$ if $k = n$, and $b_k^{n,\lambda}(1) = 0$ if $k \neq n$, whereas the their derivatives at the end points are: $b_k'^{n,\lambda}(0)$ is equal to $-n+2\lambda$ if $k = 0$, $n+2\lambda$ if $k = 1$, and 0 otherwise, (for $k = 0,1,\ldots,n$ and $n > 2$). Similarly, $b_k'^{n,\lambda}(1)$ is equal to $-n+2\lambda$ if $k = n-1$, $n+2\lambda$ if $k = n$, and 0 otherwise. Note that the product of two Bernstein polynomials of degrees m and n depending on the same variable u can be written as,

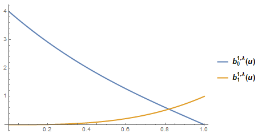
$$B_i^n(u)B_j^m(u) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j}^{n+m}(u). \tag{6}$$

Similarly the product of two λ-Bernstein polynomials of the degrees m and n depending on the same variable u can be expressed as,

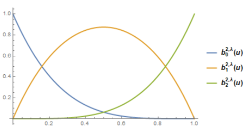
$$b_i^{n,\lambda}(u)b_j^{m,\lambda}(u) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j}^{n+m}(u)\Lambda_i^n(u)\Lambda_j^m(u). \tag{7}$$

The Bernstein-like functions $b_i^{n,\lambda}(u)$, as defined in eq. (5), for degrees $m = 1,2,3,4,5,6$, are shown in Fig. 1. In particular, FIG.1(a) illustrates the λ-Bernstein polynomials of degree $m = 1$, obtained from eq. (5), namely $b_0^{1,\lambda}(u)$ and $b_1^{1,\lambda}(u)$ defined by,

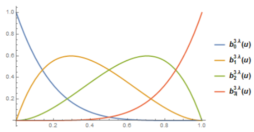
$$b_0^{1,\lambda}(u) = (1-u)(1+3\lambda-2u\lambda+u^2\lambda), b_1^{1,\lambda}(u) = u(1-\lambda+u^2\lambda). \tag{8}$$



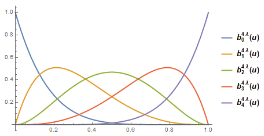
(a) λ-Bernstein basis function of degree $n = 1$



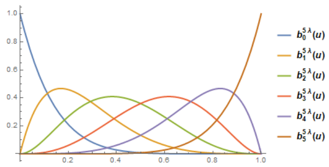
(b) λ-Bernstein basis function of degree $n = 2$



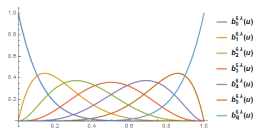
(c) λ-Bernstein basis function of degree $n = 3$



(d) λ-Bernstein basis function of degree $n = 4$



(e) λ-Bernstein basis function of degree $n = 5$



(f) λ-Bernstein basis function of degree $n = 6$

Figure 1: λ-Bernstein basis function of different degrees $n = 1,2,3,4,5,6$

Figure 1(b) represents the λ-Bernstein polynomials of degree $m = 2$ defined by eq. (5). These polynomials include $b_0^{2,\lambda}(u)$, $b_1^{2,\lambda}(u)$, and $b_2^{2,\lambda}(u)$, where,

$$
\begin{aligned}
b_0^{2,\lambda}(u) &= (1-u)^2(1-2u\lambda+u^2\lambda),\\
b_1^{2,\lambda}(u) &= 2(1-u)u(1+\lambda-u\lambda+u^2\lambda),\\
b_2^{2,\lambda}(u) &= u^2(1-\lambda+u^2\lambda).
\end{aligned}
\tag{9}
$$

Figure 1(c) represents the respective λ-Bernstein polynomials $b_0^{3,\lambda}(u)$, $b_1^{3,\lambda}(u)$, $b_2^{3,\lambda}(u)$, and $b_3^{3,\lambda}(u)$, of degree $m=3$ can be obtained from eq. (5), given by,

$$
\begin{aligned}
b_0^{3,\lambda}(u) &= (1-u)^3\left(1-2u\lambda+u^2\lambda\right),\\
b_1^{3,\lambda}(u) &= 3(1-u)^2u\left(1+\frac{2\lambda}{3}-\frac{4u\lambda}{3}+u^2\lambda\right),\\
b_2^{3,\lambda}(u) &= 3(1-u)u^2\left(1+\frac{\lambda}{3}-\frac{2u\lambda}{3}+u^2\lambda\right),\\
b_3^{3,\lambda}(u) &= u^3\left(1-\lambda+u^2\lambda\right).
\end{aligned}
\tag{10}
$$

The λ-Bernstein polynomials $b_0^{4,\lambda}(u)$, $b_1^{4,\lambda}(u)$, $b_2^{4,\lambda}(u)$, $b_3^{4,\lambda}(u)$, and $b_4^{4,\lambda}(u)$, of degree $m=4$, (obtained from the eq. (5)), are displayed in Figure 1(d). These polynomials are given by,

$$
\begin{aligned}
b_0^{4,\lambda}(u) &= (1-u)^4\left(1-2u\lambda+u^2\lambda\right),\\
b_1^{4,\lambda}(u) &= 4(1-u)^3u\left(1+\frac{\lambda}{2}-\frac{3u\lambda}{2}+u^2\lambda\right),\\
b_2^{4,\lambda}(u) &= 6(1-u)^2u^2\left(1+\frac{\lambda}{2}-u\lambda+u^2\lambda\right),\\
b_3^{4,\lambda}(u) &= 4(1-u)u^3\left(1-\frac{u\lambda}{2}+u^2\lambda\right),\\
b_4^{4,\lambda}(u) &= u^4\left(1-\lambda+u^2\lambda\right).
\end{aligned}
\tag{11}
$$

The corresponding λ-Bernstein polynomials of degree $m=5$ are $b_0^{5,\lambda}(u)$, $b_1^{5,\lambda}(u)$, $b_2^{5,\lambda}(u)$, $b_3^{5,\lambda}(u)$, $b_4^{5,\lambda}(u)$, and $b_5^{5,\lambda}(u)$ obtained from eq. (5) are shown in Figure 1(e) and they are as follows,

$$
\begin{aligned}
b_0^{5,\lambda}(u) &= (1-u)^5(1-2u\lambda+u^2\lambda),\\
b_1^{5,\lambda}(u) &= 5(1-u)^4u\left(1+\frac{2\lambda}{5}-\frac{8u\lambda}{5}+u^2\lambda\right),\\
b_2^{5,\lambda}(u) &= 10(1-u)^3u^2\left(1+\frac{\lambda}{2}-\frac{6u\lambda}{5}+u^2\lambda\right),\\
b_3^{5,\lambda}(u) &= 10(1-u)^2u^3\left(1+\frac{3\lambda}{10}-\frac{4u\lambda}{5}+u^2\lambda\right),\\
b_4^{5,\lambda}(u) &= 5(1-u)u^4\left(1-\frac{\lambda}{5}-\frac{2u\lambda}{5}+u^2\lambda\right),\\
b_5^{5,\lambda}(u) &= u^5(1-\lambda+u^2\lambda).
\end{aligned}
\tag{12}
$$

The λ-Bernstein polynomials of degree $m = 6$ (obtained from eq. (5)) are shown in Figure 1(f), denoted by $b_0^{6,\lambda}(u)$, $b_1^{6,\lambda}(u)$, $b_2^{6,\lambda}(u)$, $b_3^{6,\lambda}(u)$, $b_4^{6,\lambda}(u)$, $b_5^{6,\lambda}(u)$, and $b_6^{6,\lambda}(u)$, where,

$$\begin{aligned}
b_0^{6,\lambda}(u) &= (1-u)^6\left(1-2u\lambda+u^2\lambda\right),\\
b_1^{6,\lambda}(u) &= 6(1-u)^5u\left(1+\frac{\lambda}{3}-\frac{5u\lambda}{3}+u^2\lambda\right),\\
b_2^{6,\lambda}(u) &= 15(1-u)^4u^2\left(1+\frac{7\lambda}{15}-\frac{4u\lambda}{3}+u^2\lambda\right),\\
b_3^{6,\lambda}(u) &= 20(1-u)^3u^3\left(1+\frac{2\lambda}{5}-u\lambda+u^2\lambda\right),\\
b_4^{6,\lambda}(u) &= 15(1-u)^2u^4\left(1+\frac{2\lambda}{15}-\frac{2u\lambda}{3}+u^2\lambda\right),\\
b_5^{6,\lambda}(u) &= 6(1-u)u^5\left(1-\frac{\lambda}{3}-\frac{u\lambda}{3}+u^2\lambda\right),\\
b_6^{6,\lambda}(u) &= u^6\left(1-\lambda+u^2\lambda\right).
\end{aligned} \tag{13}$$

It is well know that the integral of Bernstein polynomial function $B_j^n(u)$ is given by

$$\int_0^1 B_j^n(u)du = \frac{1}{1+n}, \tag{14}$$

whereas in this case, the integral of λ-Bernstein polynomial functions is as follows

$$\begin{aligned}
\int_0^1 b_j^{n,\lambda}(u)du &= \int_0^1 B_j^n(u)[1+(\frac{3C_{n-2}^{j-1}+C_{n-1}^j-C_n^j}{C_n^j})\lambda-\frac{2C_{n-1}^j}{C_n^j}\lambda u+\lambda u^2]du\\
&= \frac{1}{n+1}-2\frac{\left(6j^2-6jn+n^2-n\right)}{(n-1)n(n+1)(n+3)}\lambda.
\end{aligned} \tag{15}$$

It is to be noted that the integral of the Bernstein polynomial function $B_j^n(u)$ is independent of the index j and depends only on the degree n of the polynomial. However, the integral of λ-Bernstein polynomial functions depends on the index j, degree n of the polynomial, and the shape parameter λ. For instance, for the indices $j = 0, 1, 2, 3$, the integrals of the λ-Bernstein polynomials $b_0^{n,\lambda}(u)$, $b_1^{n,\lambda}(u)$, $b_2^{n,\lambda}(u)$ and $b_3^{n,\lambda}(u)$ are given by $(n^2+4n+3)^{-1}(n-2\lambda+3)$, $(n^3+4n^2+3n)^{-1}(12\lambda+n^2-2\lambda n+3n)$, $(n^4+3n^3-n^2-3n)^{-1}(-48\lambda+n^3-2\lambda n^2+2n^2+26\lambda n-3n)$ and $(n^4+3n^3-n^2-3n)^{-1}(-108\lambda+n^3-2\lambda n^2+2n^2+38\lambda n-3n)$, respectively. The λ-Bernstein Bézier curve of degree n, parameterized by $\lambda \in [-1, 1]$, is defined as

$$\mathbf{x}(u) = \sum_{i=0}^{m} b_i^{m,\lambda}(u)\,P_i, \tag{16}$$

where $b_i^{m,\lambda}(u)$ are the λ-Bernstein basis functions as defined in the eq. (2). Here, $u \in [0,1]$ and $i = 0,1,2,...,m$. The control points are denoted by P_i. The λ-Bernstein Bézier surface is a surface based on λ-Bernstein polynomials of degree m and n (as defined in eq. (2)) and it is the generalization of λ-Bernstein Bézier curves defined by the eq. (16). Let $b_i^{m,\lambda}(u)$ and $b_j^{n,\lambda}(v)$ be the m and n degree λ-Bernstein basis functions, respectively, then we can define the λ-Bernstein Bézier surface of degree $m \times n$ as follows

$$\mathbf{x}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_i^{m,\lambda}(u)\, b_j^{n,\lambda}(v) P_{ij}, \tag{17}$$

over the rectangular domain with the control points P_{ij}, $(0 \le i \le m, 0 \le j \le n)$ and $(u,v) \in [0,1] \times [0,1]$ such that $m,n \ge 2$. Utilizing eq. (2), for the λ-Bernstein basis functions, $b_i^{m,\lambda}(u)$ and $b_j^{n,\lambda}(v)$, of degree m and n, the λ-Bernstein Bézier surface $\mathbf{x}(u,v) = \sum_{i,j=0}^{m,n} B_i^m(u)\Lambda_i^m(u)\, B_j^n(v)\Lambda_j^n(v)\, P_{ij}$ can alternatively be written as in the following form,

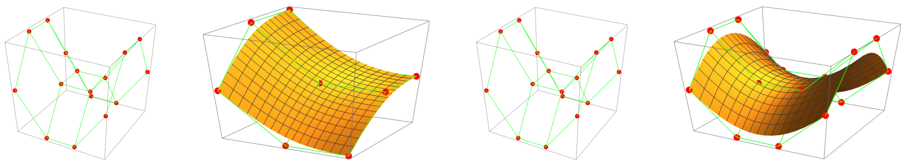
$$\begin{aligned}\mathbf{x}(u,v) = \sum_{i,j=0}^{m,n} B_i^m(u)B_j^n(v)\big[1 + (\frac{3C_{m-2}^{i-1} + C_{m-1}^{i} - C_m^i}{C_m^i})\lambda - \frac{2C_{m-1}^i}{C_m^i}\lambda u + \lambda u^2\big] \times \\ \big[1 + (\frac{3C_{n-2}^{j-1} + C_{n-1}^{j} - C_n^j}{C_n^j})\lambda - \frac{2C_{n-1}^j}{C_n^j}\lambda v + \lambda v^2\big] P_{ij}.\end{aligned} \tag{18}$$

The corresponding curves and surfaces of λ-Bernstein basis functions share many features with the Bézier curves and surfaces depending on the classical Bernstein basis functions, respectively. These basis functions degenerate to standard Bernstein basis functions when the shape parameter $\lambda = 0$. With the ability to be easily modified by adjusting the shape parameter with the same control points, these curves and surfaces offer significant improvement in the shape. By adjusting the shape parameter, one can move these curves and surfaces closer (or further away) from the control polygon. Consider the control points $P_{jk} = \left(j, k, (-1)^{j+k}\right)$. The biquadratic λ-Bernstein Bézier surface can be represented in parametric form using the surface parameters u and v, along with the shape parameter λ, as follows,

$$\begin{aligned}\mathbf{x}(u,v) = &\big(2u(\lambda + 2\lambda u^2 - 3\lambda u + 1), 2v(\lambda + 2\lambda v^2 - 3\lambda v + 1), \\ &(4\lambda u^4 - 8\lambda u^3 + 8\lambda u^2 + 4u^2 - 4\lambda u - 4u + 1)\times \\ &(4\lambda v^4 - 8\lambda v^3 + 8\lambda v^2 + 4v^2 - 4\lambda v - 4v + 1)\big),\end{aligned} \tag{19}$$

and it can be adjusted by varying the shape parameter $\lambda \in [-1,1]$. The Figure 2 represent the biquadratic and bicubic cases of λ-Bernstein Bézier surfaces along with the control points for given shape parameter λ.

In the following sections, we intend to investigate geometric constructs such as biquadratic and bicubic λ-Bernstein Bézier surfaces, examining their properties. We also explore the vanishing gradient of the quasi-harmonic functional for the λ-Bernstein Bézier surfaces $\mathbf{x}(u,v)$ as its extremum in another section. To achieve



(a) (b) (c) (d)

Figure 2: Illustration of biquadratic and bicubic λ-Bernstein Bezier surfaces with control points

this, we find the derivatives of the λ-Bernstein basis functions $b_j^{n,\lambda}(u)$ and those of the λ-Bernstein Bézier surfaces $\mathbf{x}(u,v)$ with respect to the surface parameters. To find the derivative of the λ-Bernstein basis function given in eq. (2), we denote $(b_i^{m,\lambda}(u))_u$ as its first derivative with respect to u, $(b_j^{n,\lambda}(v))_v$ as its first derivative with respect to v, $(b_i^{m,\lambda}(u))_{uu}$ as its second-order derivative with respect to u and $(b_i^{m,\lambda}(u))_{vv}$ as its second-order derivative with respect to v. It turns out that

$$(b_i^{m,\lambda}(u))_u = m(B_{i-1}^{m-1}(u) - B_i^{m-1}(u))\Lambda_i^m(u) + 2\lambda B_i^m(u)\big(u - \frac{C_{m-1}^i}{C_m^i}\big), \quad (20)$$

$$(b_j^{n,\lambda}(v))_v = n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))\Lambda_j^n(v) + 2\lambda B_j^n(v)\big(v - \frac{C_{n-1}^j}{C_n^j}\big), \quad (21)$$

$$\begin{aligned}(b_i^{m,\lambda}(u))_{uu} =& 4\lambda m(B_{i-1}^{m-1}(u) - B_i^{m-1}(u))(u - \frac{C_{m-1}^i}{C_m^i}) + 2\lambda B_i^m(u) + \\ & m(m-1)(B_{i-2}^{m-2}(u) - 2B_{i-1}^{m-2}(u) + B_i^{m-2}(u))\Lambda_i^m(u),\end{aligned} \quad (22)$$

$$\begin{aligned}(b_j^{n,\lambda}(v))_{vv} =& 4\lambda n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j}) + 2\lambda B_j^n(v) + n(n-1) \\ & (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))\Lambda_j^n(v).\end{aligned} \quad (23)$$

We denote the partial derivatives of a λ-Bernstein Bézier surface $\mathbf{x}(u,v)$ (given by eq. (17)) of degree $m \times n$ with respect to the surface parameters u and v as $\mathbf{x}_u(u,v)$, $\mathbf{x}_v(u,v)$, $\mathbf{x}_{uu}(u,v)$ and $\mathbf{x}_{vv}(u,v)$, respectively. It turns out that

$$\begin{aligned}\mathbf{x}_u(u,v) =& \sum_{i=0}^{m}\sum_{j=0}^{n}[m(B_{i-1}^{m-1}(u) - B_i^{m-1}(u))\Lambda_i^m(u) + 2\lambda B_i^m(u) \\ & (u - \frac{C_{m-1}^i}{C_m^i})]b_j^{n,\lambda}(v)\, P_{ij},\end{aligned} \quad (24)$$

$$\begin{aligned}\mathbf{x}_v(u,v) =& \sum_{i,j=0}^{m,n}[n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))\Lambda_j^n(v) + 2\lambda B_j^n(v) \\ & (v - \frac{C_{n-1}^j}{C_n^j})]b_i^{m,\lambda}(u) P_{ij},\end{aligned} \quad (25)$$

$$\mathbf{x}_{uu}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}[4\lambda m(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})+2\lambda B_{i}^{m}(u)+ \\ m(m-1)(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))\Lambda_{i}^{m}(u)]b_{j}^{n,\lambda}(v)\,P_{ij}, \tag{26}$$

$$\mathbf{x}_{uv}(u,v)=\sum_{i,j=0}^{m,n}[mn\{B_{i-1}^{m-1}(u)B_{j-1}^{n-1}(v)-B_{i-1}^{m-1}(u)B_{j}^{n-1}(v)-B_{i}^{m-1}(u) \\ B_{j-1}^{n-1}(v)+B_{i}^{m-1}(u)B_{j}^{n-1}(v)\}\Lambda_{i}^{m}(u)\Lambda_{j}^{n}(v)+2\lambda m\{B_{i-1}^{m-1}(u) \\ B_{j}^{n}(v)-B_{i}^{m-1}(u)B_{j}^{n}(v)\}(v-\frac{C_{n-1}^{j}}{C_{n}^{j}})\Lambda_{i}^{m}(u)+2\lambda n \\ \{B_{i}^{m}(u)B_{j-1}^{n-1}(v)-B_{i}^{m}(u)B_{j}^{n-1}(v)\}(u-\frac{C_{m-1}^{i}}{C_{m}^{i}}) \\ \Lambda_{j}^{n}(v)+4\lambda^{2}B_{i}^{m}(u)B_{j}^{n}(v)(u-\frac{C_{m-1}^{i}}{C_{m}^{i}}) \\ (v-\frac{C_{n-1}^{j}}{C_{n}^{j}})]P_{ij}, \tag{27}$$

$$\mathbf{x}_{vv}(u,v)=\sum_{i,j=0}^{m,n}[4\lambda n(B_{j-1}^{n-1}(v)-B_{j}^{n-1}(v))(v-\frac{C_{n-1}^{j}}{C_{n}^{j}})+2\lambda B_{j}^{n}(v)+n\times \\ (n-1)(B_{j-2}^{n-2}(v)-2B_{j-1}^{n-2}(v)+B_{j}^{n-2}(v)\Lambda_{j}^{n}(v))]b_{i}^{m,\lambda}(u)P_{ij}. \tag{28}$$

In the upcoming section, we explore the geometric properties of λ-Bernstein Bézier surfaces, focusing particularly on illustrating biquadratic and bicubic cases.

3. GEOMETRIC CHARACTERISTICS

In this section, we determine the fundamental coefficients, the unit normal, mean curvature (H), Gaussian curvature (K), shape operator coefficients, Christoffel symbols, and Gauss-Weingarten equations for the λ-Bernstein Bézier surface, $\mathbf{x}(u,v)$, as defined in eq. (17). The computations and graphical illustrations of these geometric characteristics deepen our understanding, exemplified by the biquadratic and bicubic cases of these surfaces. The fundamental coefficients of a surface $\mathbf{x}(u,v)$ are defined by

$$E=\langle\mathbf{x}_u,\mathbf{x}_u\rangle,F=\langle\mathbf{x}_u,\mathbf{x}_v\rangle,G=\langle\mathbf{x}_v,\mathbf{x}_v\rangle, \\ e=\langle\mathbf{x}_{uu},N\rangle,f=\langle\mathbf{x}_{uv},N\rangle,g=\langle\mathbf{x}_{vv},N\rangle, \tag{29}$$

with respective partial derivatives defined by

$$E_u=2\langle\mathbf{x}_u,\mathbf{x}_{uu}\rangle,\quad F_u=\langle\mathbf{x}_v,\mathbf{x}_{uu}\rangle+\langle\mathbf{x}_u,\mathbf{x}_{uv}\rangle,\quad G_u=2\langle\mathbf{x}_v,\mathbf{x}_{uv}\rangle, \\ E_v=2\langle\mathbf{x}_u,\mathbf{x}_{uv}\rangle,\quad F_v=\langle\mathbf{x}_v,\mathbf{x}_{uv}\rangle+\langle\mathbf{x}_u,\mathbf{x}_{vv}\rangle,\quad G_v=2\langle\mathbf{x}_v,\mathbf{x}_{vv}\rangle. \tag{30}$$

the Gauss coefficients,

$$\begin{aligned}
\Gamma_{11}^{1} &= \frac{GE_u + FE_v - 2FF_u}{2\left(EG - F^2\right)}, \quad & \Gamma_{12}^{1} = \Gamma_{21}^{1} &= \frac{GE_v - FG_u}{2\left(EG - F^2\right)}, \\
\Gamma_{22}^{1} &= \frac{2GF_v - GG_u - FG_v}{2\left(EG - F^2\right)}, \quad & \Gamma_{11}^{2} &= \frac{2EF_u - FE_u - EE_v}{2\left(EG - F^2\right)}, \\
\Gamma_{12}^{2} = \Gamma_{21}^{2} &= \frac{EG_u - FE_v}{2\left(EG - F^2\right)}, \quad & \Gamma_{22}^{2} &= \frac{EG_v + FG_u - 2FF_v}{2\left(EG - F^2\right)},
\end{aligned} \tag{31}$$

and the Weingarten coefficients,

$$\sigma_1^1 = \frac{Ff - Ge}{EG - F^2}, \sigma_2^1 = \frac{Fe - Ef}{EG - F^2}, \sigma_1^2 = \frac{Fg - Gf}{EG - F^2}, \sigma_2^2 = \frac{Ff - Eg}{EG - F^2}. \tag{32}$$

Gauss-Weingarten equations can be expressed as the linear combinations of $\mathbf{x}_u$, $\mathbf{x}_v$, and the unit normal $\mathbf{N}$ to the surface. The coefficients can be expressed in terms of the first and second fundamental coefficients. These equations describe the intrinsic geometry of a surface. The Gauss-Weingarten equations are:

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^{1}\mathbf{x}_u + \Gamma_{11}^{2}\mathbf{x}_v + e\mathbf{N}, \\
\mathbf{x}_{uv} &= \Gamma_{12}^{1}\mathbf{x}_u + \Gamma_{12}^{2}\mathbf{x}_v + f\mathbf{N}, \\
\mathbf{x}_{vv} &= \Gamma_{22}^{1}\mathbf{x}_u + \Gamma_{22}^{2}\mathbf{x}_v + g\mathbf{N}, \\
\mathbf{N}_u &= \sigma_1^1\mathbf{x}_u + \sigma_2^1\mathbf{x}_v, \\
\mathbf{N}_v &= \sigma_1^2\mathbf{x}_u + \sigma_2^2\mathbf{x}_v,
\end{aligned} \tag{33}$$

where $\mathbf{x}_u$, $\mathbf{x}_v$, and $\mathbf{N}$ are linearly independent and the coefficients Γ_{jk}^{i} and σ_k^j are given by eqs (31) and (32), respectively. The shape operator matrix V for the λ-Bernstein-Bézier surface can be expressed as,

$$V = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix}. \tag{34}$$

Thus, the coefficients of the shape operator matrix can be written as:

$$\begin{aligned}
s_{11} &= (Ge - Ff)/(EG - F^2), \\
s_{12} &= (Gf - Fg)/(EG - F^2) = s_{21}, \\
s_{22} &= (Ef - Fe)/(EG - F^2).
\end{aligned}$$

We start by finding the fundamental coefficients E, F, G, and e, f, g for the λ-Bernstein Bézier surface, using eq.(17) or eq.(18). This helps us calculate the geometric properties like the Gauss curvature K, mean curvature H, and the shape operator matrix coefficients σ_1^1, σ_1^2, σ_2^1, and σ_2^2. We also determine the Christoffel symbols needed for the Gauss-Weingarten equations of the λ-Bernstein Bézier surfaces $\mathbf{x}(u, v)$. The first fundamental coefficients E, F, and G, as defined in eq. (29) for the λ-Bernstein surfaces (17),

$$\begin{aligned}
E &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_u b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_u b_j^{n,\lambda}(v)P_{ij}\rangle,\\
F &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_u b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_v b_j^{n,\lambda}(v)P_{ij}\rangle, \qquad (35)\\
G &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_v b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_v b_j^{n,\lambda}(v)P_{ij}\rangle,
\end{aligned}$$

whereas the partial derivatives of these fundamental coefficients E, F, and G with respect to the surface parameters u and v can be expressed as

$$\begin{aligned}
E_u &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_{uu} b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_{uu} b_j^{n,\lambda}(v)P_{ij}\rangle,\\
F_u &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_{uu} b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_{uv} b_j^{n,\lambda}(v)P_{ij}\rangle, \qquad (36)\\
G_u &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_{uv} b_j^{n,\lambda}(v)P_{ij}, (b_i^{m,\lambda}(u))_{uv} b_j^{n,\lambda}(v)P_{ij}\rangle,
\end{aligned}$$

and

$$\begin{aligned}
E_v &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_u (b_j^{n,\lambda}(v))_v P_{ij}, (b_i^{m,\lambda}(u))_u (b_j^{n,\lambda}(v))_v P_{ij}\rangle,\\
F_v &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_u (b_j^{n,\lambda}(v))_v P_{ij}, (b_i^{m,\lambda}(u))_v (b_j^{n,\lambda}(v))_v P_{ij}\rangle, \qquad (37)\\
G_v &= \sum_{i=0}^{m}\sum_{j=0}^{n} \langle (b_i^{m,\lambda}(u))_v (b_j^{n,\lambda}(v))_v P_{ij}, (b_i^{m,\lambda}(u))_v (b_j^{n,\lambda}(v))_v P_{ij}\rangle.
\end{aligned}$$

The mean curvature can be expressed as

$$H = \frac{1}{2}\frac{Eg + Ge - 2Ff}{EG - F^2}, \qquad (38)$$

and the Gaussian curvature can be written as

$$K = \frac{ge - f^2}{EG - F^2}. \qquad (39)$$

For the geometric characteristics of the λ-Bernstein Bézier surfaces provided above, we now explore these properties in the biquadratic and bicubic cases of these surfaces. We illustrate these geometric constructs and provide graphical representations of relevant quantities.

Example 1. ***Bi-quadratic λ-Bernstein Bézier Surfaces***

To illustrate the geometric properties of the λ-Bernstein-Bézier surface, let us consider the specific case where $\mathbf{x}(u,v)$ (17) for $m=2$ and $n=2$ results in the biquadratic λ-Bernstein-Bézier surface, given by,

$$\mathbf{x}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}b_i^{m,\lambda}(u)\,b_j^{n,\lambda}(v)P_{ij}. \tag{40}$$

To determine the shape operator and the geometric properties of the biquadratic λ-Bernstein-Bézier surface, we first parameterize eq.(40) using the known control points for the shape parameter λ,

$$\begin{aligned}\mathbf{x}(u,v)=&\big(2u(2\lambda u^2-3\lambda u+\lambda+1),2v(2\lambda v^2-3\lambda v+\lambda+1),\\&(4\lambda u^4-8\lambda u^3+8\lambda u^2+4u^2-4\lambda u-4u+1)\\&(4\lambda v^4-8\lambda v^3+8\lambda v^2+4v^2-4\lambda v-4v+1)\big)\end{aligned} \tag{41}$$

Next, we need to calculate the partial derivatives of the biquadratic λ-Bernstein-Bézier surface with respect to u and v to find the fundamental coefficients and the corresponding shape operator. The surface (40), along with its partial derivatives with respect to the surface parameters and the corresponding fundamental coefficients, are given by,

$$\begin{aligned}\mathbf{x}(u,v)=&\big(2u(2-3u+2u^2),2v(2-3v+2v^2),(1-8u+12u^2-\\&8u^3+4u^4)(1-8v+12v^2-8v^3+4v^4)\big),\end{aligned} \tag{42}$$

$$\begin{aligned}\mathbf{x}_u(u,v)=&\big(4(1-3u+3u^2),0,8(-1+3u-3u^2+2u^3)(1-8v+\\&12v^2-8v^3+4v^4)\big),\end{aligned} \tag{43}$$

$$\begin{aligned}\mathbf{x}_v(u,v)=&\big(0,4(1-3v+3v^2),8(1-8u+12u^2-8u^3+4u^4)(-1+\\&3v-3v^2+2v^3)\big),\end{aligned} \tag{44}$$

$$\begin{aligned}\mathbf{x}_{uu}(u,v)=&\big(-12+24u,0,24(1-2u+2u^2)(1-8v+12v^2-8v^3+\\&4v^4)\big),\end{aligned} \tag{45}$$

$$\mathbf{x}_{uv}(u,v)=\big(0,0,64(-1+3u-3u^2+2u^3)(-1+3v-3v^2+2v^3)\big), \tag{46}$$

$$\begin{aligned}\mathbf{x}_{vv}(u,v)=&\big(0,-12+24v,24(1-8u+12u^2-8u^3+4u^4)(1-2v+\\&2v^2)\big).\end{aligned} \tag{47}$$

$$\begin{aligned}&E(u,v)=\\&16(16u^6(4v^4-8v^3+12v^2-8v+1)^2-48u^5(4v^4-8v^3+12v^2-8v+1)^2+\\&3u^4(448v^8-1792v^7+4480v^6-7168v^5+7840v^4-5824v^3+2464v^2-\\&448v+31)-2u^3(704v^8-2816v^7+7040v^6-11264v^5+12320v^4-9152\\&v^3+3872v^2-704v+53)+15u^2(64v^8-256v^7+640v^6-1024v^5+1120\\&v^4-832v^3+352v^2-64v+5)-6u(64v^8-256v^7+640v^6-1024v^5+1120\\&v^4-832v^3+352v^2-64v+5)+64v^8-256v^7+640v^6-1024v^5+1120\\&v^4-832v^3+352v^2-64v+5).\end{aligned} \tag{48}$$

$$\begin{aligned}&F(u,v)=\\&64(8u^7-28u^6+60u^5-80u^4+70u^3-39u^2+11u-1)(8v^7-28v^6+60v^5-\\&80v^4+70v^3-39v^2+11v-1),\end{aligned} \tag{49}$$

$$G(u,v) = 16(5-30v+75v^2-106v^3+93v^4-48v^5+16v^6-32u(2+11u-26u^2+35u^3-32u^4+20u^5-8u^6+2u^7)(-1+2v)^2(1-v+v^2)^2) \tag{50}$$

The graphs of the mean curvature and Gaussian curvature of a biquadratic λ-Bernstein Bézier surface are shown in Figures 3 and 4.

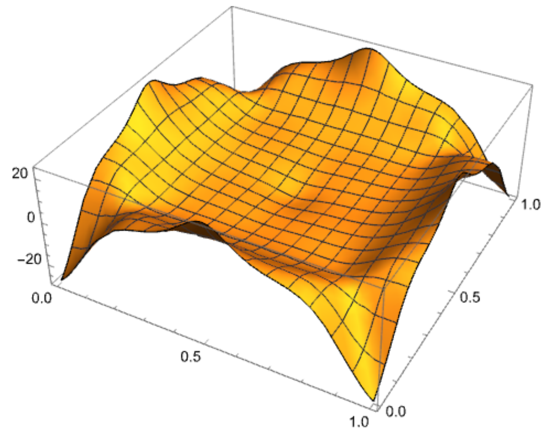


Figure 3: The mean curvature of biquadratic λ-Bernstein Bézier surface

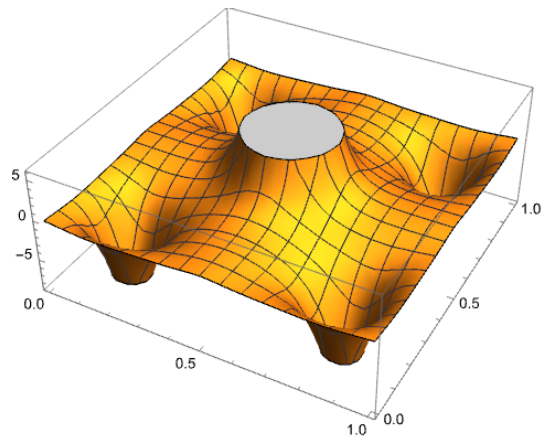


Figure 4: The Gaussian curvature of biquadratic λ-Bernstein Bézier surface

The metric coefficients of the biquadratic λ-Bernstein-Bézier surface at the point $(u,v)=(0,0)$ are,

$$E = 80, \qquad F = 64, \qquad G = 80, \tag{51}$$

and their partial derivatives are,

$$\begin{array}{lll} E_u = -480, & E_v = -1024, & F_u = -704, \\ F_v = -704,, & G_u = -1024, & G_v = -480. \end{array} \tag{52}$$

Therefore, the corresponding metric of the biquadratic λ-Bernstein-Bézier surface is,

$$ds^2 = E\ du^2 + 2F\ du\ dv + G\ dv^2 = 80\ du^2 + 64\ du\ dv + 80\ dv^2. \tag{53}$$

From equations (43) to (47), the first and second-order partial derivatives of the biquadratic λ-Bernstein-Bézier surface $\mathbf{s}(u, v)$ are,

$$\begin{array}{l} \mathbf{x}_u(0,0) = (4,0,-8), \mathbf{x}_v(0,0) = (0,4,-8), \\ \mathbf{x}_{uu}(0,0) = (-12,0,24), \mathbf{x}_{uv}(0,0) = (0,0,64), \\ \mathbf{x}_{vv}(0,0) = (0,-12,24). \end{array} \tag{54}$$

Thus, the unit normal $\mathbf{N}$ to the biquadratic λ-Bernstein-Bézier surface $\mathbf{s}(u, v)$ at the point $(u, v) = (0, 0)$ can be computed and it is,

$$\mathbf{N}(u,v)|_{(u,v)\ =\ (0,0)} = \mathbf{N}(0,0) = \frac{\mathbf{x}_u(0,0)\ \wedge \mathbf{x}_u(0,0)}{\|\mathbf{x}_u(0,0)\ \wedge \mathbf{x}_u(0,0)\|} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right). \tag{55}$$

The fundamental coefficients e, f, g given by eq. (29) for the biquadratic λ-Bernstein-Bézier surface are,

$$e = 0, \qquad f = \frac{64}{3}, \qquad g = 0. \tag{56}$$

Substituting the values of the fundamental coefficients from eq. (51) into $\det(\omega) = EG - F^2$ and from eq. (56) into $\det(b) = eg - f^2$, it follows that

$$\det(\omega) = 2304, \qquad \det(b) = -\frac{4096}{9}. \tag{57}$$

Now, the coefficients s_{11}, s_{12}, s_{21}, and s_{22} of the matrix V corresponding to the shape operator of the biquadratic λ-Bernstein-Bézier surface are given by eqs (34), (51) and (56),

$$s_{11} = \frac{5}{144}, \quad s_{12} = -\frac{1}{36} = s_{21}, \quad s_{22} = \frac{5}{144}. \tag{58}$$

The expressions for the mean curvature (38) and the Gaussian curvature (39) provide fundamental geometric quantities that characterize the biquadratic λ-Bernstein-Bézier surface and in this case they are,

$$K = -\frac{16}{81}, \qquad H = -\frac{256}{9}. \tag{59}$$

The Christoffel symbols of the second kind Γ^i_{jk} (as given by eq. (31)) depending on the first fundamental coefficients and their derivatives as found in eq. (51) and eq. (52) are

$$\Gamma^1_{11} = -3 = \Gamma^2_{22}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{32}{9} = \Gamma^2_{12} = \Gamma^2_{21}, \qquad \Gamma^1_{22} = 0 = \Gamma^2_{11}, \tag{60}$$

The Gauss-Weingarten coefficients, as determined by equation (32), depend on the first and second fundamental coefficients. Specifically, for the bi-quadratic case, they are:

$$\sigma^1_1 = \frac{16}{27}, \sigma^2_1 = -\frac{20}{27}, \sigma^1_2 = -\frac{20}{27}, \sigma^2_2 = \frac{16}{27}. \tag{61}$$

Example 2. ***Bi-cubic λ-Bernstein Bézier Surfaces for different values of λ***

Following the discussion of the biquadratic case, we now turn to the bicubic λ-Bernstein-Bézier surface as another illustration of its geometric characteristics. Here, $\mathbf{x}(u, v)$ (17) for $m = 3$ and $n = 3$ reduces to,

$$\mathbf{x}\left(u, v\right) = \sum_{i=0}^{3} \sum_{j=0}^{3} b_i^{m,\lambda}\left(u\right) b_j^{n,\lambda}\left(v\right) P_{ij}. \tag{62}$$

To find the shape operator and geometric characteristics of the bi-cubic λ-Bernstein-Bézier surface, we express eq.(62) in its parameterized form with known control points and the shape parameter λ.

$$\begin{aligned}\mathbf{x}(u, v) =& (u(2\lambda + 4\lambda u^2 - 6\lambda u + 3), v(2\lambda + 4\lambda v^2 - 6\lambda v + 3), (2u - 1)(2v - 1)\\ &(4\lambda u^4 - 8\lambda u^3 + 8\lambda u^2 + 4u^2 - 4\lambda u - 4u + 1)(4\lambda v^4 - 8\lambda v^3 + 8\lambda v^2 +\\ &4v^2 - 4\lambda v - 4v + 1))\end{aligned} \tag{63}$$

The surface (63), along with its partial derivatives with respect to the surface parameters and the corresponding fundamental coefficients, are as follows,

$$\begin{aligned}\mathbf{x}(u, v) =& (u(4u^2 - 6u + 5), v(4v^2 - 6v + 5), (2u - 1)(2v - 1)\\ &(4u^4 - 8u^3 + 12u^2 - 8u + 1)(4v^4 - 8v^3 + 12v^2 - 8v + 1)), \qquad (64)\\ \mathbf{x}_u(u, v) =& (12u^2 - 12u + 5, 0, 2(20u^4 - 40u^3 + 48u^2 - 28u + 5)\\ &(2v - 1)(4v^4 - 8v^3 + 12v^2 - 8v + 1)), \qquad (65)\\ \mathbf{x}_v(u, v) =& (0, 12v^2 - 12v + 5, 2(2u - 1)(4u^4 - 8u^3 + 12u^2 - 8u + 1)\\ &(20v^4 - 40v^3 + 48v^2 - 28v + 5)), \qquad (66)\\ \mathbf{x}_{uu}(u, v) =& (12(-1 + 2u), 0, 8(-1 + 2u)(7 - 10u + 10u^2)(-1 + 2v)\\ &(1 - 8v + 12v^2 - 8v^3 + 4v^4)), \qquad (67)\\ \mathbf{x}_{uv}(u, v) =& (0, 0, 4(5 - 28u + 48u^2 - 40u^3 + 20u^4)(5 - 28v + 48v^2 -\\ &40v^3 + 20v^4)), \qquad (68)\\ \mathbf{x}_{vv}(u, v) =& (0, 12(2v - 1), 8(1 - 2u)(1 - 2v)(1 - 8u + 12u^2 - 8u^3 +\\ &4u^4)(7 - 10v + 10v^2)), \qquad (69)\end{aligned}$$

$$\begin{aligned}&E(u,v)=\\&25(5-80v+624v^2-2496v^3+5856v^4-8832v^5+9216v^6-6912v^7+\\&3648v^8-1280v^9+256v^{10})-40u(31-560v+4368v^2-17472v^3+\\&40992v^4-61824v^5+64512v^6-48384v^7+25536v^8-8960v^9+1792\\&v^{10})+32u^3(395-7720v+60216v^2-240864v^3+565104v^4-852288\\&v^5+889344v^6-667008v^7+352032v^8-123520v^9+24704v^{10})+8u^2\\&(665-12640v+98592v^2-394368v^3+925248v^4-1395456v^5+\\&1456128v^6-1092096v^7+576384v^8-202240v^9+40448v^{10})-16u^4\\&(1195-23720v+185016v^2-740064v^3+1736304v^4-2618688v^5+\\&2732544v^6-2049408v^7+1081632v^8-379520v^9+75904v^{10})-320\\&u^5(62-44u+20u^2-5u^3)(1-2v)^2(1-8v+12v^2-8v^3+4v^4)^2\end{aligned} \tag{70}$$

$$\begin{aligned}&F(u,v)=\\&4(160u^9-720u^8+1824u^7-3024u^6+3456u^5-2760u^4+1464u^3-\\&468u^2+78u-5)(160v^9-720v^8+1824v^7-3024v^6+3456v^5-\\&2760v^4+1464v^3-468v^2+78v-5),\end{aligned} \tag{71}$$

$$\begin{aligned}&G(u,v)=\\&16(-1+u)u(5-14u+16u^2-10u^3+4u^4)(1-4u+10u^2-6u^3+4u^4)\\&(5-28v+48v^2-40v^3+20v^4)^2+5(25-248v+1064v^2-2528v^3+\\&3824v^4-3968v^5+2816v^6-1280v^7+320v^8).\end{aligned} \tag{72}$$

The graphs of the mean curvature and Gaussian curvature of biquadratic λ-Bernstein Bézier surface are shown in Figures 5 and 6.

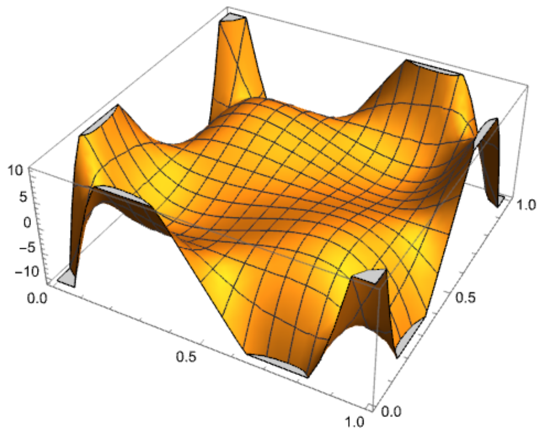


Figure 5: The mean curvature of bicubic λ-Bernstein Bézier surface

The metric coefficients of the bi-cubic λ-Bernstein-Bézier surface at the point $(u,v)=(0,0)$ are,

$$E=125, \qquad F=100, \qquad G=125, \tag{73}$$

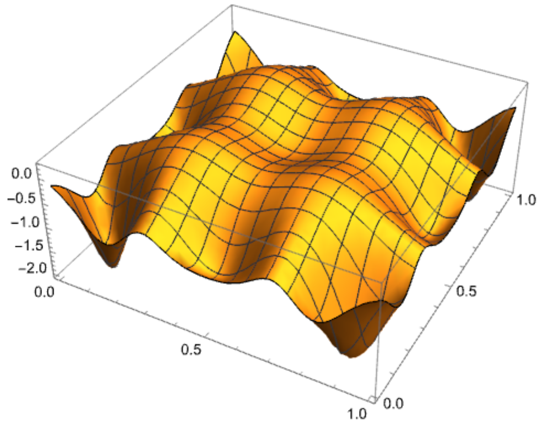


Figure 6: The Gaussian curvature of bicubic λ-Bernstein Bézier surface

and their partial derivatives are

$$\begin{array}{lll} E_u = -1240, & E_v = -2000, & F_u = -1560, \\ F_v = -1560,, & G_u = -2000, & G_v = -1240. \end{array} \tag{74}$$

The corresponding metric of the bi-cubic λ-Bernstein-Bézier surface is,

$$ds^2 = E\ du^2 + 2F\ du\ dv + G\ dv^2 = 125\ du^2 + 200\ du\ dv + 125\ dv^2. \tag{75}$$

Now, from the eqs. (65) to (69), first and second-order partial derivatives of the bi-cubic λ-Bernstein-Bézier surface $\mathbf{s}(u, v)$ are

$$\begin{array}{l} \mathbf{x}_u(0,0) = (5, 0, -10), \mathbf{x}_v(0,0) = (0, 5, -10), \mathbf{x}_{uu}(0,0) = (-12, 0, 56), \\ \mathbf{x}_{uv}(0,0) = (0, 0, 100), \mathbf{x}_{vv}(0,0) = (0, -12, 56). \end{array} \tag{76}$$

Thus, the unit normal $\mathbf{N}$ to the bi-cubic λ-Bernstein-Bézier surface $\mathbf{s}(u, v)$ at the point $(u, v) = (0, 0)$ can be computed and it is,

$$\mathbf{N}(u, v)|_{(u,v)\ =\ (0,0)} = \mathbf{N}(0, 0) = \frac{\mathbf{x}_u(0,0)\ \wedge \mathbf{x}_u(0,0)}{\|\mathbf{x}_u(0,0)\ \wedge \mathbf{x}_u(0,0)\|} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right). \tag{77}$$

The fundamental coefficients e, f, g (29) of the bi-cubic λ-Bernstein-Bézier surface are

$$e = \frac{32}{3}, \qquad f = \frac{100}{3}, \qquad g = \frac{32}{3}. \tag{78}$$

Plugging the values of fundamental coefficients from eq. (73) into $\det(\omega) = EG - F^2$ and from eq. (78) into $\det(b) = eg - f^2$, we find that

$$\det(\omega) = 5625, \qquad \det(b) = -\frac{2992}{3}. \tag{79}$$

The coefficients s_{11}, s_{12}, s_{21}, and s_{22} of the matrix V, which correspond to the shape operator of the bi-cubic λ-Bernstein-Bézier surface, can be found using equations (34), (73), and (78), as

$$s_{11} = \frac{1}{45}, \quad s_{12} = s_{21} = -\frac{4}{225}, \quad s_{22} = \frac{1}{45}. \tag{80}$$

For the bi-cubic, λ-Bernstein-Bézier surface $\mathbf{s}(u, v)$, we can now find the mean curvature (38) and the Gaussian curvature (39)

$$K = -\frac{2992}{16875}, \qquad H = -\frac{80}{3}. \tag{81}$$

The Christoffel symbols (eq. (31)) of the second kind Γ^i_{jk} depending on the first fundamental coefficients and their derivatives as found in eq. (73) and eq. (74) are,

$$\begin{aligned} \Gamma^1_{11} &= -\frac{172}{45}, \Gamma^1_{12} = -\frac{40}{9} = \Gamma^1_{21}, \Gamma^1_{22} = -\frac{64}{45}, \\ \Gamma^2_{11} &= -\frac{64}{45}, \Gamma^2_{12} = -\frac{40}{9} = \Gamma^2_{21}, \Gamma^2_{22} = -\frac{172}{45}. \end{aligned} \tag{82}$$

Eq. (32) defines the Gauss-Weingarten coefficients, which depend on the first and second fundamental coefficients. In the context of the bicubic case, these coefficients are given by,

$$\sigma^1_1 = \frac{16}{45}, \sigma^2_1 = -\frac{124}{225}, \sigma^1_2 = -\frac{124}{225}, \sigma^2_2 = \frac{16}{45}. \tag{83}$$

In the following section, we present the results concerning the extremal conditions for λ-Bernstein Bézier surfaces under the quasi-harmonic functional.

4. QUASI-HARMONIC λ-BERNSTEIN B´EZIER SURFACES

In this section, we investigate λ-Bernstein Bézier surfaces as extremals of the quasi-harmonic functional, providing the constraint integrals. A λ-Bernstein Bézier surface, $\mathbf{x}(u, v)$ (given by the eq.(17)), of degree (m, n), defined by its $(m+1) \times (n+1)$ set of control points $\{P_{ij}\}^{m,n}_{i,j=0}$ $(P_{ij} = (x^a_{ij}), a = 1, 2, 3)$, serves as an extremal of the quasi-harmonic functional with a prescribed border. This can be achieved by finding the vanishing gradient condition for the quasi-harmonic functional $\mu(\mathbf{x})$,

$$\frac{\partial \mu(\mathbf{x})}{\partial x^a_{ij}} = I^{[1]}_{ij} + I^{[2]}_{ij} + I^{[3]}_{ij} + I^{[4]}_{ij} = 0, \tag{84}$$

where $I^{[1]}_{ij}$, $I^{[2]}_{ij}$, $I^{[3]}_{ij}$ and $I^{[4]}_{ij}$ are the integrals obtained from the gradient of the quasi-harmonic functional $\mu(\mathbf{x})$ for the λ-Bernstein Bézier surface, $\mathbf{x}(u, v)$, defined by,

$$\mu(\mathbf{x}) = \frac{1}{2} \int_{\Omega} (\langle \mathbf{x}_{uu}, \mathbf{x}_{uu} \rangle + 2\langle \mathbf{x}_{uu}, \mathbf{x}_{vv} \rangle + \langle \mathbf{x}_{vv}, \mathbf{x}_{vv} \rangle) du dv. \tag{85}$$

Here, $\mathbf{x}(u,v)$ represents the surface parametrization over a domain Ω for $0 \leq u, v \leq 1$, and $\mathbf{x}_{uu}$ and $\mathbf{x}_{vv}$ denote the second partial derivatives of $\mathbf{x}$ with respect to u and v, respectively.

For the λ-Bernstein Bézier surface, $\mathbf{x}(u,v)$ (eq. (17)), of degree (m,n), the gradient of the quasi-harmonic energy functional eq. (85) for the Bézier patch with Bernstein basis function *w.r.t.* the coordinates of inetrior control points, $P_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3\right)$, can be written as

$$\frac{\partial \mu\left(\mathbf{x}\right)}{\partial x_{ij}^a} = \int\limits_{\Omega} \Big(\langle \frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}, \mathbf{x}_{uu}\rangle + \langle \frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}, \mathbf{x}_{vv}\rangle + \langle \mathbf{x}_{uu}, \frac{\partial \mathbf{x}_{vv}}{\partial x_{ij}^a}\rangle + \langle \frac{\partial \mathbf{x}_{vv}}{\partial x_{ij}^a}, \mathbf{x}_{vv}\rangle\Big) du dv. \quad (86)$$

where $a = 1,2,3,\ i = 0,1,2,\ldots,m,\ j = 0,1,2,\ldots,n$. The terms $\mathbf{x}_{uu}$ and $\mathbf{x}_{vv}$ denote the second partial derivatives of $\mathbf{x}$ with respect to u and v respectively. This formulation captures the surface's behavior in terms of its curvature and smoothness characteristics. Eq. (86) can be expressed as the sum of four integrals, which are defined as follows,

$$I_{ij}^{[1]} = \int\limits_{\Omega} \langle \frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}, \mathbf{x}_{uu}\rangle du dv, \quad (87)$$

$$I_{ij}^{[2]} = \int\limits_{\Omega} \langle \frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}, \mathbf{x}_{vv}\rangle du dv, \quad (88)$$

$$I_{ij}^{[3]} = \int\limits_{\Omega} \langle \mathbf{x}_{uu}, \frac{\partial \mathbf{x}_{vv}}{\partial x_{ij}^a}\rangle du dv, \quad (89)$$

$$I_{ij}^{[4]} = \int\limits_{\Omega} \langle \frac{\partial \mathbf{x}_{vv}}{\partial x_{ij}^a}, \mathbf{x}_{vv}\rangle du dv. \quad (90)$$

The integrals can be computed over the domain Ω. To find these integrals, we need to determine $\frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}$ and $\frac{\partial \mathbf{x}_{vv}}{\partial x_{ij}^a}$. For the first integral (87), we start by finding $\frac{\partial \mathbf{x}_{uu}}{\partial x_{ij}^a}$. Using eq. (17), we can express it as follows,

$$\frac{\partial}{\partial x_{ij}^a}(\mathbf{x}_{uu}(u,v)) = \frac{\partial^2}{\partial u^2}\Big(\sum_{p,q=0}^{m,n} b_p^{m,\lambda}(u) b_q^{n,\lambda}(v) \frac{\partial P_{ij}}{\partial x_{ij}^a}\Big) = (b_i^{m,\lambda}(u))_{uu} b_j^{n,\lambda}(v) e^a. \quad (91)$$

Substituting eq.(22) in the above eq. (91), we obtain,

$$\begin{aligned} &\frac{\partial}{\partial x_{ij}^a}(\mathbf{x}_{uu}(u,v)) = \\ &[4\lambda m (B_{i-1}^{m-1}(u) - B_i^{m-1}(u))(u - \frac{C_{m-1}^i}{C_m^i}) + 2\lambda B_i^m(u) + m(m-1) \\ &(B_{i-2}^{m-2}(u) - 2B_{i-1}^{m-2}(u) + B_i^{m-2}(u))(\Lambda_i^m(u))] b_j^{n,\lambda}(v) e^a. \end{aligned} \quad (92)$$

By interchanging the roles of u and v, and m and n in eq. (91), we obtain the expression for $\frac{\partial \mathbf{x}_{vv}}{\partial x^a_{ij}}$, which is given by,

$$\frac{\partial}{\partial x^a_{ij}}(\mathbf{x}_{vv}(u,v)) = \left(b^{n,\lambda}_j(v)\right)_{vv} b^{m,\lambda}_i(u)e^a, \tag{93}$$

and thus, substituting eq. (23) in eq. (93), we find that

$$\begin{aligned}&\frac{\partial}{\partial x^a_{ij}}(\mathbf{x}_{vv}(u,v)) = \\ &\Big[4\lambda n(B^{n-1}_{j-1}(v) - B^{n-1}_j(v))(v - \frac{C^j_{n-1}}{C^j_n}) + 2\lambda B^n_j(v) + n(n-1) \\ &(B^{n-2}_{j-2}(v) - 2B^{n-2}_{j-1}(v) + B^{n-2}_j(v))\Lambda^n_j(v)\Big] b^{m,\lambda}_i(u)e^a.\end{aligned} \tag{94}$$

Now we proceed to find the integrals $I^{[1]}_{ij}, I^{[2]}_{ij}, I^{[3]}_{ij}, I^{[4]}_{ij}$. For the first integral $I^{[1]}_{ij}$, given by the expression (87), we substitute (92) into it to obtain,

$$\begin{aligned}I^{[1]}_{ij} = \int_\Omega &\Big[4\lambda m(B^{m-1}_{i-1}(u) - B^{m-1}_i(u))(u - \frac{C^i_{m-1}}{C^i_m}) + 2\lambda B^m_i(u) + m(m-1) \\ &(B^{m-2}_{i-2}(u) - 2B^{m-2}_{i-1}(u) + B^{m-2}_i(u))\Lambda^m_i(u)\Big] b^{n,\lambda}_j(v)\langle e^a, \mathbf{x}_{uu}\rangle dudv.\end{aligned} \tag{95}$$

Substituting eq. (26) in above eq. (95) to obtain,

$$I^{[1]}_{ij} = \sum_{k,l=0}^{m,n} U^{kl}_{ij}\langle e^a, P_{kl}\rangle, \tag{96}$$

where U^{kl}_{ij} denotes the integral in the above eq. (96),

$$\begin{aligned}&U^{kl}_{ij}(u,v) = \\ &\int_\Omega [4\lambda m(B^{m-1}_{i-1}(u) - B^{m-1}_i(u))(u - \frac{C^i_{m-1}}{C^i_m}) + 2\lambda B^m_i(u) + m \\ &(m-1)(B^{m-2}_{i-2}(u) - 2B^{m-2}_{i-1}(u) + B^{m-2}_i(u))\Lambda^m_i(u)][4\lambda m(B^{m-1}_{k-1}(u) - \\ &B^{m-1}_k(u))(u - \frac{C^k_{m-1}}{C^k_m}) + 2\lambda B^m_k(u) + m(m-1)(B^{m-2}_{k-2}(u) - \\ &2B^{m-2}_{k-1}(u) + B^{m-2}_k(u))\Lambda^m_k(u)] b^{n,\lambda}_j(v)\ b^{n,\lambda}_l(v)dudv.\end{aligned} \tag{97}$$

The product of the two factors within the square brackets in eq. (97) yields the following simplified expression,

$$U_{ij}^{kl}(u,v)= \int_{\Omega} \big[16\lambda^2m^2\{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(B_{k-1}^{m-1}(u)-B_{k}^{m-1}(u))(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})(u-\frac{C_{m-1}^{k}}{C_{m}^{k}})\}+8\lambda^2m \{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))B_{k}^{m}(u)(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})\}+4\lambda m^2(m-1)\{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(B_{k-2}^{m-2}(u) -2B_{k-1}^{m-2}(u)+B_{k}^{m-2}(u))\Lambda_k^m(u)(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})\}+8\lambda^2m\{B_{i}^{m}(u)(B_{k-1}^{m-1}(u)-B_{k}^{m-1}(u))(u-\frac{C_{m-1}^{k}}{C_{m}^{k}})\} +4\lambda^2\{B_{i}^{m}(u)B_{k}^{m}(u)\}+2\lambda m(m-1)\{B_{i}^{m}(u)(B_{k-2}^{m-2}(u)-2B_{k-1}^{m-2}(u)+B_{k}^{m-2}(u))\Lambda_k^m(u)\}+4\lambda m^2 (m-1)\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))(B_{k-1}^{m-1}(u)-B_{k}^{m-1}(u))(u-\frac{C_{m-1}^{k}}{C_{m}^{k}})\Lambda_i^m(u)\}+2\lambda m (m-1)\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))B_{k}^{m}(u)\Lambda_i^m(u)\}+m^2(m-1)^2\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u) +B_{i}^{m-2}(u))(B_{k-2}^{m-2}(u)-2B_{k-1}^{m-2}(u)+B_{k}^{m-2}(u))\Lambda_i^m(u)\Lambda_k^m(u)\}\big]b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)dudv, \tag{98}$$

which can be further simplified as follows,

$$U_{ij}^{kl}(u,v)= \int_{\Omega} \big[16\lambda^2m\{B_{i-1}^{m-1}(u)B_{k-1}^{m-1}(u)-B_{i-1}^{m-1}(u)B_{k}^{m-1}(u)-B_{i}^{m-1}(u)B_{k-1}^{m-1}(u)+B_{i}^{m-1}(u)B_{k}^{m-1}(u)\} (u-\frac{C_{m-1}^{i}}{C_{m}^{i}})(u-\frac{C_{m-1}^{k}}{C_{m}^{k}})+8\lambda^2m\{B_{i-1}^{m-1}(u)B_{k}^{m}(u)-B_{i}^{m-1}(u)B_{k}^{m}(u)\}(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})+4\lambda m^2(m-1) \{B_{i-1}^{m-1}(u)B_{k-2}^{m-2}(u)-2B_{i-1}^{m-1}(u)B_{k-1}^{m-2}(u)+B_{i-1}^{m-1}(u)B_{k}^{m-2}(u)-B_{i}^{m-1}(u)B_{k-2}^{m-2}(u)+2B_{i}^{m-1}(u) B_{k-1}^{m-2}(u)-B_{i}^{m-1}(u)B_{k}^{m-2}(u)\}(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})\Lambda_k^m(u)+8\lambda^2m\{B_{i}^{m}(u)B_{k-1}^{m-1}(u)-B_{i}^{m}(u)B_{k}^{m-1}(u)\} (u-\frac{C_{m-1}^{k}}{C_{m}^{k}})+4\lambda^2\{B_{i}^{m}(u)B_{k}^{m}(u)\}+2\lambda m(m-1)\{B_{i}^{m}(u)B_{k-2}^{m-2}(u)-2B_{i}^{m}(u)B_{k-1}^{m-2}(u)+B_{i}^{m}(u) B_{k}^{m-2}(u)\}\Lambda_k^m(u)+4\lambda m^2(m-1)\{B_{i-2}^{m-2}(u)B_{k-1}^{m-1}(u)-B_{i-2}^{m-2}(u)B_{k}^{m-1}(u)-2B_{i-1}^{m-2}(u)B_{k-1}^{m-1}(u)+ 2B_{i-1}^{m-2}(u)B_{k}^{m-1}(u)+B_{i}^{m-2}(u)B_{k-1}^{m-1}(u)-B_{i}^{m-2}(u)B_{k}^{m-1}(u)\}(u-\frac{C_{m-1}^{k}}{C_{m}^{k}})\Lambda_i^m(u)+2\lambda m(m-1) \{B_{k}^{m}(u)B_{i-2}^{m-2}(u)-2B_{k}^{m}(u)B_{i-1}^{m-2}(u)+B_{k}^{m}(u)B_{i}^{m-2}(u)\}\Lambda_i^m(u)+m^2(m-1)^2\{B_{i-2}^{m-2}(u)B_{k-2}^{m-2}(u) -2B_{i-2}^{m-2}(u)B_{k-1}^{m-2}(u)+B_{i-2}^{m-2}(u)B_{k}^{m-2}(u)-2B_{i-1}^{m-2}(u)B_{k-2}^{m-2}(u)+4B_{i-1}^{m-2}(u)B_{k-1}^{m-2}(u)-2\times B_{i-1}^{m-2}(u)B_{k}^{m-2}(u)+B_{i}^{m-2}(u)B_{k-2}^{m-2}(u)-2B_{i}^{m-2}(u)B_{k-1}^{m-2}(u)+B_{i}^{m-2}(u)B_{k}^{m-2}(u)\}\Lambda_i^m(u)\Lambda_k^m(u)\big] b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)dudv \tag{99}$$

We can further break down the equation (99) into the integrals $U_i^{k[1]}$, $U_i^{k[2]}$, ..., $U_i^{k[9]}$, which can be expressed in terms of these integrals as follows,

$$\begin{aligned}U_{ij}^{kl}(u,v)=&16\lambda^2m^2U_i^{k[1]}+8\lambda^2mU_i^{k[2]}+4m^2(m-1)U_i^{k[3]}+8\lambda^2mU_i^{k[4]}+\\&4\lambda^2U_i^{k[5]}+2\lambda m(m-1)U_i^{k[6]}+4\lambda m^2(m-1)U_i^{k[7]}+\\&2\lambda m(m-1)U_i^{k[8]}+m^2(m-1)^2U_i^{k[9]},\end{aligned} \tag{100}$$

where $U_i^{k[1]}$ to $U_i^{k[9]}$ in the above eq. (100) represent the following integrals,

$$U_i^{k[1]} = \int_\Omega (B_{i-1}^{m-1}(u)B_{k-1}^{m-1}(u) - B_{i-1}^{m-1}(u)B_k^{m-1}(u) - B_i^{m-1}(u)B_{k-1}^{m-1}(u) + B_i^{m-1}(u)B_k^{m-1}(u))(u - \frac{C_{m-1}^i}{C_m^i})(u - \frac{C_{m-1}^k}{C_m^k})b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)dudv, \tag{101}$$

$$U_i^{k[2]} = \int_\Omega (B_{i-1}^{m-1}(u)B_k^m(u) - B_i^{m-1}(u)B_k^m(u))(u - \frac{C_{m-1}^i}{C_m^i}) b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)dudv \tag{102}$$

$$U_i^{k[3]} = \int_\Omega (B_{i-1}^{m-1}(u)B_{k-2}^{m-2}(u) - 2B_{i-1}^{m-1}(u)B_{k-1}^{m-2}(u) + B_{i-1}^{m-1}(u)B_k^{m-2}(u) - B_i^{m-1}(u)B_{k-2}^{m-2}(u) + 2B_i^{m-1}(u)B_{k-1}^{m-2}(u) - B_i^{m-1}(u)B_k^{m-2}(u)) \times (u - \frac{C_{m-1}^i}{C_m^i})\Lambda_k^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{103}$$

$$U_i^{k[4]} = \int_\Omega (B_i^m(u)B_{k-1}^{m-1}(u) - B_i^m(u)B_k^{m-1}(u))(u - \frac{C_{m-1}^k}{C_m^k}) \times b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{104}$$

$$U_i^{k[5]} = \int_\Omega B_i^m(u)B_k^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{105}$$

$$U_i^{k[6]} = \int_\Omega (B_i^m(u)B_{k-2}^{m-2}(u) - 2B_i^m(u)B_{k-1}^{m-2}(u) + B_i^m(u)B_k^{m-2}(u)) \times \Lambda_k^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{106}$$

$$U_i^{k[7]} = \int_\Omega (B_{i-2}^{m-2}(u)B_{k-1}^{m-1}(u) - B_{i-2}^{m-2}(u)B_k^{m-1}(u) - 2B_{i-1}^{m-2}(u)B_{k-1}^{m-1}(u) + 2B_{i-1}^{m-2}(u)B_k^{m-1}(u) + B_i^{m-2}(u)B_{k-1}^{m-1}(u) - B_i^{m-2}(u)B_k^{m-1}(u)) (u - \frac{C_{m-1}^k}{C_m^k})\Lambda_i^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{107}$$

$$U_i^{k[8]} = \int_\Omega (B_{i-2}^{m-2}(u)B_k^m(u) - 2B_{i-1}^{m-2}(u)B_k^m(u) + B_i^{m-2}(u)B_k^m(u)) \times \Lambda_i^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{108}$$

$$U_i^{k[9]} = \int_\Omega (B_{i-2}^{m-2}(u)B_{k-2}^{m-2}(u) - 2B_{i-2}^{m-2}(u)B_{k-1}^{m-2}(u) + B_{i-2}^{m-2}(u)B_k^{m-2}(u) - 2B_{i-1}^{m-2}(u)B_{k-2}^{m-2}(u) + 4B_{i-1}^{m-2}(u)B_{k-1}^{m-2}(u) - 2B_{i-1}^{m-2}(u)B_k^{m-2}(u) + B_i^{m-2}(u)B_{k-2}^{m-2}(u) - 2B_i^{m-2}(u)B_{k-1}^{m-2}(u) + B_i^{m-2}(u)B_k^{m-2}(u)) \times \Lambda_i^m(u)\Lambda_k^m(u)b_j^{n,\lambda}(v)b_l^{n,\lambda}(v)\, dudv \tag{109}$$

The eqs. (101) through (109) can be written in the following simpler form by expressing the product of two Bernstein polynomials in terms of the higher degree Bernstein polynomials, as follows,

$$U_i^{k[1]} = \int\limits_{\Omega} \Big[\frac{\binom{m-1}{i-1}\binom{m-1}{k-1}}{\binom{2m-2}{i+k-2}} B_{i+k-2}^{2m-2}(u) - \frac{\binom{m-1}{i-1}\binom{m-1}{k}}{\binom{2m-2}{i+k-1}} B_{i+k-1}^{2m-2}(u) - \frac{\binom{m-1}{i}\binom{m-1}{k-1}}{\binom{2m-2}{i+k-1}} B_{i+k-1}^{2m-2}(u) + \frac{\binom{m-1}{i}\binom{m-1}{k}}{\binom{2m-2}{i+k}} B_{i+k}^{2m-2}(u)\Big] (u - \frac{C_{m-1}^i}{C_m^i})(u - \frac{C_{m-1}^k}{C_m^k}) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v)\, dudv, \tag{110}$$

$$U_i^{k[2]} = \int\limits_{\Omega} \Big[\frac{\binom{m-1}{i-1}\binom{m}{k}}{\binom{2m-1}{i+k-1}} B_{i+k-1}^{2m-1}(u) - \frac{\binom{m-1}{i}\binom{m}{k}}{\binom{2m-1}{i+k}} B_{i+k}^{2m-1}(u)\Big] (u - \frac{C_{m-1}^i}{C_m^i}) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v)\, dudv, \tag{111}$$

$$U_i^{k[3]} = \int\limits_{\Omega} \Big[\frac{\binom{m-1}{i-1}\binom{m-2}{k-2}}{\binom{2m-3}{i+k-3}} B_{i+k-3}^{2m-3}(u) - 2\frac{\binom{m-1}{i-1}\binom{m-2}{k-1}}{\binom{2m-3}{i+k-2}} B_{i+k-2}^{2m-3}(u) + \frac{\binom{m-1}{i-1}\binom{m-2}{k}}{\binom{2m-3}{i+k-2}} B_{i+k-1}^{2m-3}(u) - \frac{\binom{m-1}{i}\binom{m-2}{k-2}}{\binom{2m-3}{i+k-2}} B_{i+k-2}^{2m-3}(u) - \frac{\binom{m-1}{i}\binom{m-2}{k}}{\binom{2m-3}{i+k}} B_{i+k}^{2m-3}(u) + 2\frac{\binom{m-1}{i}\binom{m-2}{k-1}}{\binom{2m-3}{i+k-1}} B_{i+k-1}^{2m-3}(u)\Big] (u - \frac{C_{m-1}^i}{C_m^i}) \Lambda_k^m(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v)\, dudv \tag{112}$$

$$U_i^{k[4]} = \int\limits_{\Omega} [\frac{\binom{m}{i}\binom{m-1}{k-1}}{\binom{2m-1}{i+k-1}} B_{i+k-1}^{2m-1}(u) - \frac{\binom{m}{i}\binom{m-1}{k}}{\binom{2m-1}{i+k}} B_{i+k}^{2m-1}(u)](u - \frac{C_{m-1}^k}{C_m^k}) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v) dudv, \tag{113}$$

$$U_i^{k[5]} = \int\limits_{\Omega} \frac{\binom{m}{i}\binom{m}{k}}{\binom{2m}{i+k}} B_{i+k}^{2m}(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v) dudv, \tag{114}$$

$$U_i^{k[6]} = \int\limits_{\Omega} [\frac{\binom{m}{i}\binom{m-2}{k-2}}{\binom{2m-2}{i+k-2}} B_{i+k-2}^{2m-2}(u) - 2\frac{\binom{m}{i}\binom{m-2}{k-1}}{\binom{2m-2}{i+k-1}} B_{i+k-1}^{2m-2}(u) + \frac{\binom{m}{i}\binom{m-2}{k}}{\binom{2m-2}{i+k}} B_{i+k}^{2m-2}(u)] \Lambda_k^m(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v) dudv, \tag{115}$$

$$U_i^{k[7]} = \int\limits_{\Omega} \Big[\frac{\binom{m-2}{i-2}\binom{m-1}{k-1}}{\binom{2m-3}{i+k-3}} B_{i+k-3}^{2m-3}(u) - \frac{\binom{m-2}{i-2}\binom{m-1}{k}}{\binom{2m-3}{i+k-2}} B_{i+k-2}^{2m-3}(u)$$
$$- 2\frac{\binom{m-2}{i-1}\binom{m-1}{k-1}}{\binom{2m-3}{i+k-2}} B_{i+k-2}^{2m-3}(u) + 2\frac{\binom{m-2}{i-1}\binom{m-1}{k}}{\binom{2m-3}{i+k-1}} B_{i+k-1}^{2m-3}(u)$$
$$+ \frac{\binom{m-2}{i}\binom{m-1}{k-1}}{\binom{2m-3}{i+k-1}} B_{i+k-1}^{2m-3}(u) - \frac{\binom{m-2}{i}\binom{m-1}{k}}{\binom{2m-3}{i+k}} B_{i+k}^{2m-3}(u)\Big]$$
$$\times \left(u - \frac{C_{m-1}^k}{C_m^k}\right) \Lambda_i^m(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v)\, dudv, \tag{116}$$

$$U_i^{k[8]} = \int\limits_{\Omega} [\frac{\binom{m}{k}\binom{m-2}{i-2}}{\binom{2m-2}{i+k-2}} B_{i+k-2}^{2m-2}(u) - 2\frac{\binom{m}{k}\binom{m-2}{i-1}}{\binom{2m-2}{i+k-1}} B_{i+k-1}^{2m-2}(u) + \frac{\binom{m}{k}\binom{m-2}{i}}{\binom{2m-2}{i+k}} B_{i+k}^{2m-2}(u)]$$
$$\Lambda_i^m(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v) dudv, \tag{117}$$

$$U_i^{k[9]} = \int\limits_{\Omega} \Big[\frac{\binom{m-2}{i-2}\binom{m-2}{k-2}}{\binom{2m-4}{i+k-4}} B_{i+k-4}^{2m-4}(u) - 2\frac{\binom{m-2}{i-2}\binom{m-2}{k-1}}{\binom{2m-4}{i+k-3}} B_{i+k-3}^{2m-4}(u)$$
$$+ \frac{\binom{m-2}{i-2}\binom{m-2}{k}}{\binom{2m-4}{i+k-2}} B_{i+k-2}^{2m-4}(u) - 2\frac{\binom{m-2}{i-1}\binom{m-2}{k-2}}{\binom{2m-4}{i+k-3}} B_{i+k-3}^{2m-4}(u)$$
$$+ 4\frac{\binom{m-2}{i-1}\binom{m-2}{k-1}}{\binom{2m-4}{i+k-2}} B_{i+k-2}^{2m-4}(u) - 2\frac{\binom{m-2}{i-1}\binom{m-2}{k}}{\binom{2m-4}{i+k-1}} B_{i+k-1}^{2m-4}(u)$$
$$+ \frac{\binom{m-2}{i}\binom{m-2}{k-2}}{\binom{2m-4}{i+k-2}} B_{i+k-2}^{2m-4}(u) - 2\frac{\binom{m-2}{i}\binom{m-2}{k-1}}{\binom{2m-4}{i+k-1}} B_{i+k-1}^{2m-4}(u)$$
$$+ \frac{\binom{m-2}{i}\binom{m-2}{k}}{\binom{2m-4}{i+k}} B_{i+k}^{2m-4}(u)\Big] \Lambda_i^m(u) \Lambda_k^m(u) b_j^{n,\lambda}(v) b_l^{n,\lambda}(v)\, dudv. \tag{118}$$

The above integrals $U_i^{k[1]}, U_i^{k[2]}, U_i^{k[3]}, U_i^{k[4]}, U_i^{k[5]}, U_i^{k[6]}, U_i^{k[7]}, U_i^{k[8]}, U_i^{k[9]}$ can be computed for $0 \leq u, v \leq 1$ and substituted back in eq. (100) to obtain the eq. (96). To find the integral $I_{ij}^{[2]}$, substitute eq. (91) along with eq. (22) in eq. (88) for $\partial \mathbf{x}_{uu}/\partial x_{ij}^a$ along with $(b_i^{m,\lambda}(u))_{uu}$ and $\mathbf{x}_{vv}$. This gives us,

$$I_{ij}^{[2]} = \int\limits_{\Omega} [4\lambda m(B_{i-1}^{m-1}(u) - B_i^{m-1}(u))(u - \frac{C_{m-1}^i}{C_m^i}) + 2\lambda B_i^m(u) + m(m-1)$$
$$(B_{i-2}^{m-2}(u) - 2B_{i-1}^{m-2}(u) + B_i^{m-2}(u))\Lambda_i^m(u)] b_j^{n,\lambda}(v) \langle e^a, \mathbf{x}_{vv}\rangle dudv. \tag{119}$$

By using eq. (28), we can rewrite it as,

$$I_{ij}^{[2]}=\sum_{k,l=0}^{m,n}\int_{\Omega}[4\lambda m(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})+2\lambda B_{i}^{m}(u)+m(m-1)(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))\Lambda_{i}^{m}(u)][4\lambda n(B_{l-1}^{n-1}(v)-B_{l}^{n-1}(v))(v-\frac{C_{n-1}^{l}}{C_{n}^{l}})+2\lambda B_{l}^{n}(v)+n(n-1)(B_{l-2}^{n-2}(v)-2B_{l-1}^{n-2}(v)+B_{l}^{n-2}(v))\Lambda_{l}^{n}(v)]b_{j}^{n,\lambda}(v)b_{k}^{m,\lambda}(u)\langle e^{a},P_{kl}\rangle dudv. \tag{120}$$

For convenience, this can be expressed as,

$$I_{ij}^{[2]}=\sum_{k,l=0}^{m,n}V_{ij}^{kl}\langle e^{a},P_{kl}\rangle, \tag{121}$$

where V_{ij}^{kl} represents the following integral,

$$V_{ij}^{kl}(u,v)=\int_{\Omega}[4\lambda m(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})+2\lambda B_{i}^{m}(u)+m(m-1)(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))\Lambda_{i}^{m}(u)][4\lambda n(B_{l-1}^{n-1}(v)-B_{l}^{n-1}(v))(v-\frac{C_{n-1}^{l}}{C_{n}^{l}})+2\lambda B_{l}^{n}(v)+n(n-1)(B_{l-2}^{n-2}(v)-2B_{l-1}^{n-2}(v)+B_{l}^{n-2}(v))\Lambda_{l}^{n}(v)]b_{j}^{n,\lambda}(v)b_{k}^{m,\lambda}(u)dudv. \tag{122}$$

The multiplication of the two factors within square brackets reduces eq. (122) to,

$$V_{ij}^{kl}(u,v)=\int_{\Omega}[16\lambda^{2}mn\{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(B_{l-1}^{n-1}(v)-B_{l}^{n-1}(v))(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})(v-\frac{C_{n-1}^{l}}{C_{n}^{l}})\}+8\lambda^{2}n\{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))B_{l}^{n}(v)(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})\}+4\lambda mn(n-1)\{(B_{i-1}^{m-1}(u)-B_{i}^{m-1}(u))(B_{l-2}^{n-2}(v)-2B_{l-1}^{n-2}(v)+B_{l}^{n-2}(v))\Lambda_{l}^{n}(v)\}(u-\frac{C_{m-1}^{i}}{C_{m}^{i}})+8\lambda^{2}n\{B_{i}^{m}(u)(B_{l-1}^{n-1}(v)-B_{l}^{n-1}(v))(v-\frac{C_{n-1}^{l}}{C_{n}^{l}})\}+4\lambda^{2}\{B_{i}^{m}(u)B_{l}^{n}(v)\}+2\lambda n(n-1)\{B_{i}^{m}(u)(B_{l-2}^{n-2}(v)-2B_{l-1}^{n-2}(v)+B_{l}^{n-2}(v))\Lambda_{l}^{n}(v)\}+4\lambda mn(m-1)\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))(B_{l-1}^{n-1}(v)-B_{l}^{n-1}(v))(v-\frac{C_{n-1}^{l}}{C_{n}^{l}})\Lambda_{i}^{m}(u)\}+2\lambda m(m-1)\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))B_{l}^{n}(v)\Lambda_{i}^{m}(u)\}+mn(m-1)(n-1)\{(B_{i-2}^{m-2}(u)-2B_{i-1}^{m-2}(u)+B_{i}^{m-2}(u))(B_{l-2}^{n-2}(v)-2B_{l-1}^{n-2}(v)+B_{l}^{n-2}(v))\Lambda_{i}^{m}(u)\Lambda_{l}^{n}(v)\}]b_{k}^{m,\lambda}(u)b_{j}^{n,\lambda}(v)dudv. \tag{123}$$

This can be further simplified to,

$$V_{ij}^{kl}(u,v) = \int_\Omega \big[16\lambda^2 mn\{B_{i-1}^{m-1}(u)B_{l-1}^{n-1}(v) - B_{i-1}^{m-1}(u)B_l^{n-1}(v) - B_i^{m-1}(u)B_{l-1}^{n-1}(v) + B_i^{m-1}(u)B_l^{n-1}(v)\} (u - \frac{C_{m-1}^i}{C_m^i})(v - \frac{C_{n-1}^l}{C_n^l}) + 8\lambda^2 m\{B_{i-1}^{m-1}(u)B_l^n(v) - B_i^{m-1}(u)B_l^n(v)\}(u - \frac{C_{m-1}^i}{C_m^i}) + 4\lambda mn(n-1) \{B_{i-1}^{m-1}(u)B_{l-2}^{n-2}(v) - 2B_{i-1}^{m-1}(u)B_{l-1}^{n-2}(v) + B_{i-1}^{m-1}(u)B_l^{n-2}(v) - B_i^{m-1}(u)B_{l-2}^{n-2}(v) + 2B_i^{m-1}(u)B_{l-1}^{n-2}(v) - B_i^{m-1}(u)B_l^{n-2}(v)\}\Lambda_l^n(v)(u - \frac{C_{m-1}^i}{C_m^i}) + 8\lambda^2 n\{B_i^m(u)B_{l-1}^{n-1}(v) - B_i^m(u)B_l^{n-1}(v)\}(v - \frac{C_{n-1}^l}{C_n^l}) + 4\lambda^2\{B_i^m(u)B_l^n(v)\}2\lambda n(n-1)\{B_i^m(u)B_{l-2}^{n-2}(v) - 2B_i^m(u)B_{l-1}^{n-2}(v) + B_i^m(u)B_l^{n-2}(v)\} \Lambda_l^n(v) + 4\lambda mn(m-1)\{B_{i-2}^{m-2}(u)B_{l-1}^{n-1}(v) - B_{i-2}^{m-2}(u)B_l^{n-1}(v) - 2B_{i-1}^{m-2}(u)B_{l-1}^{n-1}(v) + 2B_{i-1}^{m-2}(u) B_l^{n-1}(v) + B_i^{m-2}(u)B_{l-1}^{n-1}(v) - B_i^{m-2}(u)B_l^{n-1}(v)\}\Lambda_i^m(u)(v - \frac{C_{n-1}^l}{C_n^l}) + 2\lambda m(m-1)\{B_{i-2}^{m-2}(u) B_l^n(v) - 2B_{i-1}^{m-2}(u)B_l^n(v) + B_i^{m-2}(u)B_l^n(v)\}\Lambda_i^m(u) + mn(m-1)(n-1)\{B_{i-2}^{m-2}(u)B_{l-2}^{n-2}(v) - 2B_{i-2}^{m-2}(u)B_{l-1}^{n-2}(v) + B_{i-2}^{m-2}(u)B_l^{n-2}(v) - 2B_{i-1}^{m-2}(u)B_{l-2}^{n-2}(v) + 4B_{i-1}^{m-2}(u)B_{l-1}^{n-2}(v) - 2B_{i-1}^{m-2}(u) B_l^{n-2}(v) + B_i^{m-2}(u)B_{l-2}^{n-2}(v) - 2B_i^{m-2}(u)B_{l-1}^{n-2}(v) + B_i^{m-2}(u)B_l^{n-2}(v)\}\Lambda_i^m(u)\Lambda_l^n(v)\big] \times b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv. \tag{124}$$

For convenience, we can express the above eq. (124) as follows,

$$\begin{aligned} V_{ij}^{kl}(u,v) =& 16\lambda^2 mn V_i^{l[1]} + 8\lambda^2 m V_i^{l[2]} + 4\lambda mn\,(n-1)\,V_i^{l[3]} + 8\lambda^2 n V_i^{l[4]} + \\ & 4\lambda^2 V_i^{l[5]} + 2\lambda n\,(n-1)\,V_i^{l[6]} + 4\lambda mn\,(m-1)\,V_i^{l[7]} + \\ & 2\lambda m\,(m-1)\,V_i^{l[8]} + mn\,(m-1)\,(n-1)\,V_i^{l[9]}, \end{aligned} \tag{125}$$

where $V_i^{l[1]}$, $V_i^{l[2]}$ ... $V_i^{l[9]}$ represent the following integrals,

$$\begin{aligned} V_i^{l[1]} =& \int_\Omega \big(B_{i-1}^{m-1}(u)\,B_{l-1}^{n-1}(v) - B_{i-1}^{m-1}(u)\,B_l^{n-1}(v) - B_i^{m-1}(u)\,B_{l-1}^{n-1}(v) \\ & +B_i^{m-1}(u)\,B_l^{n-1}(v)\big)\,(u - \frac{C_{m-1}^i}{C_m^i})(v - \frac{C_{n-1}^l}{C_n^l})b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \end{aligned} \tag{126}$$

$$\begin{aligned} V_i^{l[2]} =& \int_\Omega \big(B_{i-1}^{m-1}(u)\,B_l^n(v) - B_i^{m-1}(u)\,B_l^n(v)\big)\,(u - \frac{C_{m-1}^i}{C_m^i}) \\ & b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \end{aligned} \tag{127}$$

$$\begin{aligned} V_i^{l[3]} =& \int_\Omega \big(B_{i-1}^{m-1}(u)\,B_{l-2}^{n-2}(v) - 2B_{i-1}^{m-1}(u)\,B_{l-1}^{n-2}(v) + B_{i-1}^{m-1}(u)\,B_l^{n-2}(v) \\ & - B_i^{m-1}(u)\,B_{l-2}^{n-2}(v) + 2B_i^{m-1}(u)\,B_{l-1}^{n-2}(v) - B_i^{m-1}(u)\,B_l^{n-2}(v)\big) \\ & (u - \frac{C_{m-1}^i}{C_m^i})\Lambda_l^n(v)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \end{aligned} \tag{128}$$

$$V_i^{l[4]} = \int_\Omega \left(B_i^m(u)\,B_{l-1}^{n-1}(v) - B_i^m(u)\,B_l^{n-1}(v)\right)(v - \frac{C_{n-1}^l}{C_n^l}) \\ b_{k,\lambda}^m(u)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{129}$$

$$V_i^{l[5]} = \int_\Omega B_i^m(u)B_l^n(v)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{130}$$

$$V_i^{l[6]} = \int_\Omega \left(B_i^m(u)\,B_{l-2}^{n-2}(v) - 2B_i^m(u)\,B_{l-1}^{n-2}(v) + B_i^m(u)\,B_l^{n-2}(v)\right)\Lambda_l^n(v) \\ b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{131}$$

$$V_i^{l[7]} = \int_\Omega \left(B_{i-2}^{m-2}(u)\,B_{l-1}^{n-1}(v) - B_{i-2}^{m-2}(u)\,B_l^{n-1}(v) - 2B_{i-1}^{m-2}(u)\,B_{l-1}^{n-1}(v) \right. \\ \left. + 2B_{i-1}^{m-2}(u)\,B_l^{n-1}(v) + B_i^{m-2}(u)\,B_{l-1}^{n-1}(v) - B_i^{m-2}(u)\,B_l^{n-1}(v)\right) \\ (v - \frac{C_{n-1}^l}{C_n^l})\Lambda_i^m(u)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{132}$$

$$V_i^{l[8]} = \int_\Omega \left(B_{i-2}^{m-2}(u)\,B_l^n(v) - 2B_{i-1}^{m-2}(u)\,B_l^n(v) + B_i^{m-2}(u)\,B_l^n(v)\right) \\ \Lambda_i^m(u)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{133}$$

$$V_i^{l[9]} = \int_\Omega \left(B_{i-2}^{m-2}(u)\,B_{l-2}^{n-2}(v) - 2B_{i-2}^{m-2}(u)\,B_{l-1}^{n-2}(v) + B_{i-2}^{m-2}(u)\,B_l^{n-2}(v) \right. \\ - 2B_{i-1}^{m-2}(u)\,B_{l-2}^{n-2}(v) + 4B_{i-1}^{m-2}(u)\,B_{l-1}^{n-2}(v) - 2B_{i-1}^{m-2}(u)\,B_l^{n-2}(v) \\ \left. + B_i^{m-2}(u)\,B_{l-2}^{n-2}(v) - 2B_i^{m-2}(u)\,B_{l-1}^{n-2}(v) + B_i^{m-2}(u)\,B_l^{n-2}(v)\right) \\ \Lambda_i^m(u)\,\Lambda_l^n(v)\,b_k^{m,\lambda}(u)b_j^{n,\lambda}(v)dudv \tag{134}$$

The above integrals (126) through (134) can be substituted back in $V_{ij}^{kl}(u,v)$ provided by eq. (125) to obtain eq. (121) and subsequently obtain the second integral (88). To determine the integral $I_{ij}^{[3]}$, substitute eq. (94) along with eqs. (26) and (28) into eq. (89) for $\partial \mathbf{x}_{vv}/\partial x_{ij}^a$, $(b_i^{m,\lambda}(v))_{vv}$ and $\mathbf{x}_{vv}$, we have

$$I_{ij}^{[3]} = \int_\Omega [4\lambda n\left(B_{j-1}^{n-1}(v) - B_j^{n-1}(v)\right)(v - \frac{C_{n-1}^j}{C_n^j}) + 2\lambda B_j^n(v) + n(n-1) \\ (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))\Lambda_j^n(v)]b_i^{m,\lambda}(u)\langle e^a, \mathbf{x}_{uu}\rangle dudv, \tag{135}$$

and substituting eq. (26) in above eq. (135), we find that

$$
\begin{aligned}
I_{ij}^{[3]} = \sum_{k,l=0}^{m,n} \int_{\Omega} & [4\lambda n(B_{j-1}^{n-1}(v) - B_{j}^{n-1}(v))(v - \frac{C_{n-1}^{j}}{C_{n}^{j}}) + 2\lambda B_{j}^{n}(v) + n(n-1) \\
& (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_{j}^{n-2}(v))\Lambda_{j}^{n}(v)][4\lambda m(B_{k-1}^{m-1}(u) - B_{k}^{m-1}(u)) \\
& (u - \frac{C_{m-1}^{k}}{C_{m}^{k}}) + 2\lambda B_{k}^{m}(u) + m(m-1)(B_{k-2}^{m-2}(u) - 2B_{k-1}^{m-2}(u) + \\
& B_{k}^{m-2}(u))\Lambda_{k}^{m}(u)]b_{i}^{n,\lambda}(u)b_{l}^{n,\lambda}(v) \langle e^{a}, P_{kl}\rangle dudv.
\end{aligned}
\tag{136}
$$

For convenience, we can write the eq. (136) in the form,

$$
I_{ij}^{[3]} = \sum_{k,l=0}^{m,n} W_{ij}^{kl} \langle e^{a}, P_{kl}\rangle,
\tag{137}
$$

where W_{ij}^{kl} denotes the following integral,

$$
\begin{aligned}
& W_{ij}^{kl}(u,v) = \\
& \int_{\Omega} [4\lambda n(B_{j-1}^{n-1}(v) - B_{j}^{n-1}(v))(v - \frac{C_{n-1}^{j}}{C_{n}^{j}}) + 2\lambda B_{j}^{n}(v) + n(n-1)(B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + \\
& B_{j}^{n-2}(v))\Lambda_{j}^{n}(v)]\ [4\lambda m(B_{k-1}^{m-1}(u) - B_{k}^{m-1}(u))(u - \frac{C_{m-1}^{k}}{C_{m}^{k}}) + 2\lambda B_{k}^{m}(u) + m(m-1)\times \\
& (B_{k-2}^{m-2}(u) - 2B_{k-1}^{m-2}(u) + B_{k}^{m-2}(u))\Lambda_{k}^{m}(u)]b_{i}^{n,\lambda}(u)b_{l}^{n,\lambda}(v)\, dudv.
\end{aligned}
\tag{138}
$$

Combining the factors within the square brackets in eq. (138) results in,

$$
\begin{aligned}
& W_{ij}^{kl}(u,v) = \\
& \int_{\Omega} [16\lambda^{2} mn\{B_{k-1}^{m-1}(u)B_{j-1}^{n-1}(v) - B_{k-1}^{m-1}(u)B_{j}^{n-1}(v) - B_{k}^{m-1}(u)B_{j-1}^{n-1}(v) + B_{k}^{m-1}(u)B_{j}^{n-1}(v)\} \\
& (u - \frac{C_{m-1}^{k}}{C_{m}^{k}})(v - \frac{C_{n-1}^{j}}{C_{n}^{j}}) + 8\lambda^{2} m\{B_{k-1}^{m-1}(u)B_{j}^{n}(v) - B_{k}^{m-1}(u)B_{j}^{n}(v)\}(u - \frac{C_{m-1}^{k}}{C_{m}^{k}}) + 4\lambda mn(n-1) \\
& \{B_{k-1}^{m-1}(u)B_{j-2}^{n-2}(v) - 2B_{k-1}^{m-1}(u)B_{j-1}^{n-2}(v) + B_{k-1}^{m-1}(u)B_{j}^{n-2}(v) - B_{k}^{m-1}(u)B_{j-2}^{n-2}(v) + 2B_{k}^{m-1}(u)B_{j-1}^{n-2}(v) \\
& - B_{k}^{m-1}(u)\, B_{j}^{n-2}(v)\}\Lambda_{j}^{n}(v)(u - \frac{C_{m-1}^{k}}{C_{m}^{k}}) + 8\lambda^{2} n\{B_{k}^{m}(u)B_{j-1}^{n-1}(v) - B_{k}^{m}(u)B_{j}^{n-1}(v)\}(v - \frac{C_{n-1}^{j}}{C_{n}^{j}}) \\
& + 4\lambda^{2}\{B_{k}^{m}(u)B_{j}^{n}(v)\} + 2\lambda n(n-1)\{B_{k}^{m}(u)B_{j-2}^{n-2}(v) - 2B_{k}^{m}(u)B_{j-1}^{n-2}(v) + B_{k}^{m}(u)B_{j}^{n-2}(v)\} \\
& \Lambda_{j}^{n}(v) + 4\lambda mn(m-1)\{B_{k-2}^{m-2}(u)B_{j-1}^{n-1}(v) - B_{k-2}^{m-2}(u)B_{j}^{n-1}(v) - 2B_{k-1}^{m-2}(u)B_{j-1}^{n-1}(v) + 2B_{k-1}^{m-2}(u) \\
& B_{j}^{n-1}(v) + B_{k}^{m-2}(u)B_{j-1}^{n-1}(v) - B_{k}^{m-2}(u)B_{j}^{n-1}(v)\}\Lambda_{k}^{m}(u)(v - \frac{C_{n-1}^{j}}{C_{n}^{j}}) + 2\lambda m(m-1)\{B_{k-2}^{m-2}(u) \\
& B_{j}^{n}(v) - 2B_{k-1}^{m-2}(u)B_{j}^{n}(v) + B_{k}^{m-2}(u)B_{j}^{n}(v)\}\Lambda_{k}^{m}(u) + mn(m-1)(n-1)\{B_{k-2}^{m-2}(u)B_{j-2}^{n-2}(v) - \\
& 2B_{k-2}^{m-2}(u)B_{j-1}^{n-2}(v) + B_{k-2}^{m-2}(u)B_{j}^{n-2}(v) - 2B_{k-1}^{m-2}(u)B_{j-2}^{n-2}(v) + 4B_{k-1}^{m-2}(u)B_{j-1}^{n-2}(v) - 2B_{k-1}^{m-2}(u) \\
& B_{j}^{n-2}(v) + B_{k}^{m-2}(u)B_{j-2}^{n-2}(v) - 2B_{k}^{m-2}(u)B_{j-1}^{n-2}(v) + B_{k}^{m-2}(u)B_{j}^{n-2}(v)\}\Lambda_{k}^{m}(u)\Lambda_{j}^{n}(v)] \\
& b_{i}^{m,\lambda}(u)b_{l}^{n,\lambda}(v)dudv.
\end{aligned}
\tag{139}
$$

For the sake of convenience, we can write the above expression as follows,

$$\begin{aligned} W_{ij}^{kl}(u,v) =& 16\lambda^2 mn W_k^{j[1]} + 8\lambda^2 m W_k^{j[2]} + 4\lambda mn(n-1) W_k^{j[3]} + \\ & 8\lambda^2 n W_k^{j[4]} + 4\lambda^2 W_k^{j[5]} + 2\lambda n(n-1) W_k^{j[6]} + 4\lambda mn \\ & (m-1) W_k^{j[7]} + 2\lambda m(m-1) W_k^{j[8]} + \\ & mn(m-1)(n-1) W_k^{j[9]}, \end{aligned} \tag{140}$$

where $W_k^{j[1]}$ through $W_k^{j[9]}$ represent the integrals as given in following eqs. (141) to eq. (149),

$$\begin{aligned} W_k^{j[1]} =& \int_\Omega (B_{k-1}^{m-1}(u)B_{j-1}^{n-1}(v) - B_{k-1}^{m-1}(u)B_j^{n-1}(v) - B_k^{m-1}(u)B_{j-1}^{n-1}(v) + \\ & B_k^{m-1}(u)B_j^{n-1}(v))(u - \frac{C_{m-1}^k}{C_m^k})(v - \frac{C_{n-1}^j}{C_n^j}) b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{141}$$

$$\begin{aligned} W_k^{j[2]} =& \int_\Omega \left(B_{k-1}^{m-1}(u) B_j^n(v) - B_k^{m-1}(u) B_j^n(v)\right)(u - \frac{C_{m-1}^k}{C_m^k}) \\ & b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{142}$$

$$\begin{aligned} W_k^{j[3]} =& \int_\Omega (B_{k-1}^{m-1}(u)B_{j-2}^{n-2}(v) - 2B_{k-1}^{m-1}(u)B_{j-1}^{n-2}(v) + B_{k-1}^{m-1}(u)B_j^{n-2}(v) - \\ & B_k^{m-1}(u)B_{j-2}^{n-2}(v) + 2B_k^{m-1}(u)B_{j-1}^{n-2}(v) - B_k^{m-1}(u)B_j^{n-2}(v)) \\ & (u - \frac{C_{m-1}^k}{C_m^k})\Lambda_j^n(v) b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{143}$$

$$\begin{aligned} W_k^{j[4]} =& \int_\Omega \left(B_k^m(u) B_{j-1}^{n-1}(v) - B_k^m(u) B_j^{n-1}(v)\right)(v - \frac{C_{n-1}^j}{C_n^j}) b_{i,\lambda}^m(u) b_i^{m,\lambda}(u) \\ & b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{144}$$

$$W_k^{j[5]} = \int_\Omega B_k^m(u) B_j^n(v) b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \tag{145}$$

$$\begin{aligned} W_k^{j[6]} =& \int_\Omega \left(B_k^m(u) B_{j-2}^{n-2}(v) - 2B_k^m(u) B_{j-1}^{n-2}(v) + B_k^m(u) B_j^{n-2}(v)\right) \\ & \Lambda_j^n(v) b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{146}$$

$$\begin{aligned} W_k^{j[7]} =& \int_\Omega \left(B_{k-2}^{m-2}(u) B_{j-1}^{n-1}(v) - B_{k-2}^{m-2}(u) B_j^{n-1}(v) - 2B_{k-1}^{m-2}(u) B_{j-1}^{n-1}(v)\right. \\ & \left. + 2B_{k-1}^{m-2}(u) B_j^{n-1}(v) + B_k^{m-2}(u) B_{j-1}^{n-1}(v) - B_k^{m-2}(u) B_j^{n-1}(v)\right) \\ & (v - \frac{C_{n-1}^j}{C_n^j})\Lambda_k^m(u) b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) dudv, \end{aligned} \tag{147}$$

$$W_k^{j[8]} = \int\limits_{\Omega} \left(B_{k-2}^{m-2}(u)\, B_j^n(v) - 2B_{k-1}^{m-2}(u)\, B_j^n(v) + B_k^{m-2}(u)\, B_j^n(v)\right) \Lambda_k^m(u)\, b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) du dv, \tag{148}$$

$$W_k^{j[9]} = \int\limits_{\Omega} \left(B_{k-2}^{m-2}(u)\, B_{j-2}^{n-2}(v) - 2B_{k-2}^{m-2}(u)\, B_{j-1}^{n-2}(v) + B_{k-2}^{m-2}(u)\, B_j^{n-2}(v) - 2B_{k-1}^{m-2}(u)\, B_{j-2}^{n-2}(v) + 4B_{k-1}^{m-2}(u)\, B_{j-1}^{n-2}(v) - 2B_{k-1}^{m-2}(u)\, B_j^{n-2}(v) + B_k^{m-2}(u)\, B_{j-2}^{n-2}(v) - 2B_k^{m-2}(u)\, B_{j-1}^{n-2}(v) + B_k^{m-2}(u)\, B_j^{n-2}(v)\right) \Lambda_k^m(u)\, \Lambda_j^n(v)\, b_i^{m,\lambda}(u) b_l^{n,\lambda}(v) du dv. \tag{149}$$

By inserting the integrals (141) through (149) in the eq. (140) for $W_{ij}^{kl}(u,v)$ and then the final expression in the eq. (137) to obtain third integral (89). To find the integral $I_{ij}^{[4]}$, substitute eq. (94) in eq. (90) for $\partial \mathbf{x}_{vv}/\partial x_{ij}^a$ along with $(b_i^{m,\lambda}(v))_{vv}$ and $\mathbf{x}_{vv}$, we obtain

$$I_{ij}^{[4]} = \int\limits_{\Omega} [4\lambda n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j}) + 2\lambda B_j^n(v) + n(n-1) (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))\Lambda_j^n(v)] b_i^{m,\lambda}(u) \langle e^a, \mathbf{x}_{vv}\rangle du dv. \tag{150}$$

Now substitute eq. (28) in above eq. (150) to obtain,

$$I_{ij}^{[4]} = \int\limits_{\Omega} [4\lambda n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j}) + 2\lambda B_j^n(v) + n(n-1) (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))\Lambda_j^n(v))][4\lambda n(B_{l-1}^{n-1}(v) - B_l^{n-1}(v)) (v - \frac{C_{n-1}^l}{C_n^l}) + 2\lambda B_l^n(v) + n(n-1)(B_{l-2}^{n-2}(v) - 2B_{l-1}^{n-2}(v) + B_l^{n-2}(v))\Lambda_l^n(v)] b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) \langle e^a, P_{kl}\rangle du dv. \tag{151}$$

For the sake of convenience, the integral (151) can be expressed,

$$I_{ij}^{[4]} = \sum_{k,l=0}^{m,n} T_{ij}^{kl} \langle e^a, P_{kl}\rangle, \tag{152}$$

where T_{ij}^{kl} represent the following integral,

$$T_{ij}^{kl} = \int\limits_{\Omega} [4\lambda n(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j}) + 2\lambda B_j^n(v) + n(n-1) (B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))\Lambda_j^n(v))][4\lambda n(B_{l-1}^{n-1}(v) - B_l^{n-1}(v)) (v - \frac{C_{n-1}^l}{C_n^l}) + 2\lambda B_l^n(v) + n(n-1)(B_{l-2}^{n-2}(v) - 2B_{l-1}^{n-2}(v) + B_l^{n-2}(v)) \Lambda_l^n(v)] b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv. \tag{153}$$

The product of the factors within the square brackets in eq. (153) yields the following expression,

$$T_{ij}^{kl}(u,v) = \int_\Omega [16\lambda^2 n^2\{(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(B_{l-1}^{n-1}(v) - B_l^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j})(v - \frac{C_{n-1}^l}{C_n^l})\} + 8\lambda^2 n \{(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))B_l^n(v)(v - \frac{C_{n-1}^j}{C_n^j})\} + 4\lambda n^2(n-1)\{(B_{j-1}^{n-1}(v) - B_j^{n-1}(v))(B_{l-2}^{n-2}(v) - 2B_{l-1}^{n-2}(v) + B_l^{n-2}(v))\Lambda_l^n(v)(v - \frac{C_{n-1}^j}{C_n^j})\} + 8\lambda^2 n\,\{B_j^n(v)\ (B_{l-1}^{n-1}(v) - B_l^{n-1}(v))(v - \frac{C_{n-1}^l}{C_n^l}) + 4\lambda^2 B_j^n(v)B_l^n(v) + 2\lambda n(n-1)\{B_j^n(v)(B_{l-2}^{n-2}(v) - 2B_{l-1}^{n-2}(v) + B_l^{n-2}(v))\Lambda_l^n(v)\} + 4\lambda n^2(n-1) \{(B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))(B_{l-1}^{n-1}(v) - B_l^{n-1}(v))(v - \frac{C_{n-1}^l}{C_n^l})\Lambda_l^n(v) + 2\lambda n(n-1)\} \{(B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))B_l^n(v)\Lambda_l^n(v) + n^2(n-1)^2\}\{(B_{j-2}^{n-2}(v) - 2B_{j-1}^{n-2}(v) + B_j^{n-2}(v))(B_{l-2}^{n-2}(v) - 2B_{l-1}^{n-2}(v) + B_l^{n-2}(v))\Lambda_j^n(u)\Lambda_l^n(v)\}]b_{i,\lambda}^m(u)b_{k,\lambda}^n(u)dudv. \tag{154}$$

We can further break down the equation (154) into the integrals $T_j^{l[1]}$, $T_j^{l[2]}$, ..., $T_j^{l[9]}$, that appear in the curly brackets in the above expression for $T_{ij}^{kl}(u,v)$, which can be expressed in terms of these integrals as follows,

$$\begin{aligned} T_{ij}^{kl}(u,v) =& 16\lambda^2 n^2 T_j^{l[1]} + 8\lambda^2 n T_j^{l[2]} + 4n^2(n-1)T_j^{l[3]} + 8\lambda^2 n T_j^{l[4]} + \\ & 4\lambda^2 T_j^{l[5]} + 2\lambda n(n-1)T_j^{l[6]} + 4\lambda n^2(n-1)T_j^{l[7]} + \\ & 2\lambda n(n-1)T_j^{l[8]} + n^2(n-1)^2 T_j^{l[9]}, \end{aligned} \tag{155}$$

where $T_j^{l[1]}$ to $T_j^{l[9]}$ in the above eq. (155) represent the following integrals,

$$\begin{aligned} T_j^{l[1]} =& \int_\Omega (B_{j-1}^{n-1}(v)B_{l-1}^{n-1}(v) - B_{j-1}^{n-1}(v)B_l^{n-1}(v) - B_j^{n-1}(v)B_{l-1}^{n-1}(v) + \\ & B_j^{n-1}(v)B_l^{n-1}(v))(v - \frac{C_{n-1}^j}{C_n^j})(v - \frac{C_{n-1}^l}{C_n^l})b_i^{m,\lambda}(u)b_k^{m,\lambda}(u)dudv, \end{aligned} \tag{156}$$

$$T_j^{l[2]} = \int_\Omega \left(B_{j-1}^{n-1}(v)\, B_l^n(v)\ - B_j^{n-1}(v)\, B_l^n(v)\right)(v - \frac{C_{n-1}^j}{C_n^j})b_i^{m,\lambda}(u)b_k^{m,\lambda}(u)dudv, \tag{157}$$

$$\begin{aligned} T_j^{l[3]} =& \int_\Omega \left(B_{j-1}^{n-1}(v)\, B_{l-2}^{n-2}(v)\ - 2B_{j-1}^{n-1}(v)\, B_{l-1}^{n-2}(v) + B_{j-1}^{n-1}(v)\, B_l^{n-2}(v) - \right. \\ & \left. B_j^{n-1}(v)\, B_{l-2}^{n-2}(v) + 2B_i^{n-1}(v)\ B_{l-1}^{n-2}(v) - B_j^{n-1}(v)\, B_l^{n-2}(v)\right) \\ & (v - \frac{C_{n-1}^j}{C_n^j})\Lambda_l^n(v)\, b_i^{m,\lambda}(u)b_k^{m,\lambda}(u)dudv, \end{aligned} \tag{158}$$

$$T_j^{l[4]} = \int_\Omega \left(B_j^n(v)\, B_{l-1}^{n-1}(v)\ - B_j^n(v)\, B_l^{n-1}(v)\right)(v - \frac{C_{n-1}^l}{C_n^l})b_i^{m,\lambda}(u)b_k^{m,\lambda}(u)dudv, \tag{159}$$

$$T_j^{l[5]} = \int_\Omega B_j^n(v) B_l^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{160}$$

$$T_j^{l[6]} = \int_\Omega \left(B_j^n(v) B_{l-2}^{n-2}(v) - 2B_j^n(v) B_{l-1}^{n-2}(v) + B_j^n(v) B_l^{n-2}(v)\right) \Lambda_l^n(v) b_l^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{161}$$

$$T_j^{l[7]} = \int_\Omega \left(B_{j-2}^{n-2}(v) B_{l-1}^{n-1}(v) - B_{j-2}^{n-2}(v) B_l^{n-1}(v) - 2B_{j-1}^{n-2}(v) B_{l-1}^{n-1}(v) + 2B_{j-1}^{n-2}(v) B_l^{n-1}(v) + B_j^{n-2}(v) B_{l-1}^{n-1}(v) - B_j^{n-2}(v) B_l^{n-1}(v)\right) (v - \frac{C_{n-1}^l}{C_n^l}) \Lambda_j^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{162}$$

$$T_j^{l[8]} = \int_\Omega \left(B_{j-2}^{n-2}(v) B_l^n(v) - 2B_{j-1}^{n-2}(v) B_l^n(v) + B_j^{n-2}(v) B_l^n(v)\right) \Lambda_j^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{163}$$

$$T_j^{l[9]} = \int_\Omega \left(B_{j-2}^{n-2}(v) B_{l-2}^{n-2}(v) - 2B_{j-2}^{n-2}(v) B_{l-1}^{n-2}(v) + B_{j-2}^{n-2}(v) B_l^{n-2}(v) - 2B_{j-1}^{n-2}(v) B_{l-2}^{n-2}(v) + 4B_{j-1}^{n-2}(v) B_{l-1}^{n-2}(v) - 2B_{j-1}^{n-2}(v) B_l^{n-2}(v) + B_j^{n-2}(v) B_{l-2}^{n-2}(v) - 2B_j^{n-2}(v) B_{l-1}^{n-2}(v) + B_j^{n-2}(v) B_l^{n-2}(v)\right) \Lambda_j^n(v) \Lambda_l^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv. \tag{164}$$

Now using the result for the product of two classical Bernstein polynomials with same independent variable, we have

$$T_j^{l[1]} = \int_\Omega \left[\frac{\binom{n-1}{j-1}\binom{n-1}{l-1}}{\binom{2n-2}{j+l-2}} B_{j+l-2}^{2n-2}(v) - \frac{\binom{n-1}{j-1}\binom{n-1}{l}}{\binom{2n-2}{j+l-1}} B_{j+l-1}^{2n-2}(v) - \frac{\binom{n-1}{j}\binom{n-1}{l-1}}{\binom{2n-2}{j+l-1}} B_{j+l-1}^{2n-2}(v) + \frac{\binom{n-1}{j}\binom{n-1}{l}}{\binom{2n-2}{j+l}} B_{j+l}^{2n-2}(v) \right] \times \left(v - \frac{C_{n-1}^j}{C_n^j}\right) \left(v - \frac{C_{n-1}^l}{C_n^l}\right) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u)\, du dv, \tag{165}$$

$$T_j^{l[2]} = \int_\Omega \left[\frac{\binom{n-1}{j-1}\binom{n}{l}}{\binom{2n-1}{j+l-1}} B_{j+l-1}^{2n-1}(v) - \frac{\binom{n-1}{j}\binom{n}{l}}{\binom{2n-1}{j+l}} B_{j+l}^{2n-1}(v) \right] (v - \frac{C_{n-1}^j}{C_n^j}) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{166}$$

$$T_j^{l[3]} = \int_\Omega \Bigg[\frac{\binom{n-1}{j-1}\binom{n-2}{l-2}}{\binom{2n-3}{j+l-3}} B_{j+l-3}^{2n-3}(v) - 2\frac{\binom{n-1}{j-1}\binom{n-2}{l-1}}{\binom{2n-3}{j+l-2}} B_{j+l-2}^{2n-3}(v)$$
$$+ \frac{\binom{n-1}{j-1}\binom{n-2}{l}}{\binom{2n-3}{j+l-1}} B_{j+l-1}^{2n-3}(v) - \frac{\binom{n-1}{j}\binom{n-2}{l-2}}{\binom{2n-3}{j+l-2}} B_{j+l-2}^{2n-3}(v)$$
$$- \frac{\binom{n-1}{j}\binom{n-2}{l}}{\binom{2n-3}{j+l}} B_{j+l}^{2n-3}(v) + 2\frac{\binom{n-1}{j}\binom{n-2}{l-1}}{\binom{2n-3}{j+l-1}} B_{j+l-1}^{2n-3}(v) \Bigg] \times$$
$$\left(v - \frac{C_{n-1}^j}{C_n^j} \right) \Lambda_l^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{167}$$

$$T_j^{l[4]} = \int_\Omega \Bigg[\frac{\binom{n}{j}\binom{n-1}{l-1}}{\binom{2n-1}{j+l-1}} B_{j+l-1}^{2n-1}(v) - \frac{\binom{n}{j}\binom{n-1}{l}}{\binom{2n-1}{j+l}} B_{j+l}^{2n-1}(v) \Bigg] (v - \frac{C_{n-1}^l}{C_n^l})$$
$$b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{168}$$

$$T_j^{l[5]} = \int_\Omega \frac{\binom{n}{j}\binom{n}{l}}{\binom{2n}{j+l}} B_{j+l}^{2n}(v)\, b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv, \tag{169}$$

$$T_j^{l[6]} = \int_\Omega \Bigg[\frac{\binom{n}{j}\binom{n-2}{l-2}}{\binom{2n-2}{j+l-2}} B_{j+l-2}^{2n-2}(v) - 2\frac{\binom{n}{j}\binom{n-2}{l-1}}{\binom{2n-2}{j+l-1}} B_{j+l-1}^{2n-2}(v)$$
$$+ \frac{\binom{n}{j}\binom{n-2}{l}}{\binom{2n-2}{j+l}} B_{j+l}^{2n-2}(v) \Bigg] \Lambda_l^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u)\, du dv, \tag{170}$$

$$T_j^{l[7]} = \int_\Omega \Big[\frac{\binom{n-2}{j-2}\binom{n-1}{l-1}}{\binom{2n-3}{j+l-3}} B_{j+l-3}^{2n-3}(v) - \frac{\binom{n-2}{j-2}\binom{n-1}{l}}{\binom{2n-3}{j+l-2}} B_{j+l-2}^{2n-3}(v)$$
$$- 2\frac{\binom{n-2}{j-1}\binom{n-1}{l-1}}{\binom{2n-3}{j+l-2}} B_{j+l-2}^{2n-3}(v) + 2\frac{\binom{n-2}{j-1}\binom{n-1}{l}}{\binom{2n-3}{j+l-1}} B_{j+l-1}^{2n-3}(v)$$
$$+ \frac{\binom{n-2}{j}\binom{n-1}{l-1}}{\binom{2n-3}{j+l-1}} B_{j+l-1}^{2n-3}(v) - \frac{\binom{n-2}{j}\binom{n-1}{l}}{\binom{2n-3}{j+l}} B_{j+l}^{2n-3}(v) \Big]$$
$$\times \left(v - \frac{C_{n-1}^l}{C_n^l} \right) \Lambda_j^n(v) b_i^{m,\lambda}(u) b_k^{m,\lambda}(u)\, du dv, \tag{171}$$

$$T_j^{l[8]} = \int_\Omega \Big[\frac{\binom{n}{l}\binom{n-2}{j-2}}{\binom{2n-2}{j+l-2}} B_{j+l-2}^{2n-2}(v) - 2\frac{\binom{n}{l}\binom{n-2}{j-1}}{\binom{2n-2}{j+l-1}} B_{j+l-1}^{2n-2}(v)$$
$$+ \frac{\binom{n}{l}\binom{n-2}{j}}{\binom{2n-2}{j+l}} B_{j+l}^{2n-2}(v) \Big] \Lambda_j^n(v)\, b_i^{m,\lambda}(u)\, b_k^{m,\lambda}(u)\, du dv, \tag{172}$$

$$T_j^{l[9]} = \int\limits_{\Omega} \left[\frac{\binom{n-2}{j-2}\binom{n-2}{l-2}}{\binom{2n-4}{j+l-4}} B_{j+l-4}^{2n-4}(v) - 2\frac{\binom{n-2}{j-2}\binom{n-2}{l-1}}{\binom{2n-4}{j+l-3}} B_{j+l-3}^{2n-4}(v) + \right.$$
$$\frac{\binom{n-2}{j-2}\binom{n-2}{l}}{\binom{2n-4}{j+l-2}} B_{j+l-2}^{2n-4}(v) - 2\frac{\binom{n-2}{j-1}\binom{n-2}{l-2}}{\binom{2n-4}{j+l-3}} B_{j+l-3}^{2n-4}(v) +$$
$$4\frac{\binom{n-2}{j-1}\binom{n-2}{l-1}}{\binom{2n-4}{j+l-2}} B_{j+l-2}^{2n-4}(v) - 2\frac{\binom{n-2}{j-1}\binom{n-2}{l}}{\binom{2n-4}{j+l-1}} B_{j+l-1}^{2n-4}(v) +$$
$$\frac{\binom{n-2}{j}\binom{n-2}{l-2}}{\binom{2n-4}{j+l-2}} B_{j+l-2}^{2n-4}(v) - 2\frac{\binom{n-2}{j}\binom{n-2}{l-1}}{\binom{2n-4}{j+l-1}} B_{j+l-1}^{2n-4}(v)$$
$$\left. \frac{\binom{n-2}{j}\binom{n-2}{l}}{\binom{2n-4}{j+l}} B_{j+l}^{2n-4}(v) \right] \Lambda_j^n(v)\, \Lambda_l^n(v)\, b_i^{m,\lambda}(u) b_k^{m,\lambda}(u) du dv. \tag{173}$$

By plugging these integrals in $T_{ij}^{kl}(u,v)$ provided by eq. (155) provides us the expression for the fourth integral (90) for the integral (152). The vanishing condition (84) for the gradient of the quasi-harmonic functional (85) can now be readily obtained by summing up the constituent integrals (96), (121), (137), and (152). Setting the resulting expression to zero allows us to solve for the interior control points in terms of the boundary control points for the λ-Bernstein Bézier surface representing the quasi-harmonic surface. The proposed λ-Bernstein Bézier surfaces provide enhanced control over surface smoothness and computational efficiency; however, there are some limitations to consider. The computational complexity increases with higher polynomial degrees and more complex surface configurations, which may affect real-time applications [60]. Additionally, the method assumes that boundary control points are known, which may not always be the case in practical design scenarios, as traditional path planning methods can involve unexpected changes or uncertainties in the environment [61]. Future work could explore more robust approaches that incorporate dynamic adjustments to boundary conditions, enabling real-time responsiveness and enhanced performance in scenarios with both static and moving obstacles.

5. CONCLUSION

In this paper, we have presented a framework for constructing λ-Bernstein Bézier surfaces and the biquadratic and bicubic cases as the illustration of their geometric characteristics. We have provided a structure of integrals that appear in terms of Bernstein polynomials, which can be solved for given values of m and n. Utilizing these integrals in equations (100) to obtain equation (96), in equation (125) to obtain equation (121), in equation (140) to obtain equation (137), and in equation (155) to obtain equation (152), gives us the vanishing condition for the gradient of the quasi-harmonic functional from equation (84) as $\frac{\partial \mu(\mathbf{x})}{\partial x_{ij}^a} = 0$, which can then be solved for the interior control points that appear as the constraints depending on the known boundary control points, giving us the λ-Bernstein Bézier surfaces as the extremal of the quasi-harmonic functional (85). These surfaces may offer better smooth surfaces with additional control via the shape parameter λ, utilizing computational techniques for further exploration. Their potential applications span across computer graphics, computer science, computational geometry, and engineering, facilitating surface design and computational modeling for quasi-harmonic surfaces, contributing to advancements across various scientific disciplines.

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REFERENCES

[1] R. Osserman, *A survey of Minimal Surfaces.* Dover Publications Inc., 1986.
[2] J. C. C. Nitsche, *Lectures on Minimal Surfaces.* Cambridge University Press, 1989.
[3] F. C. Marques, "Minimal surfaces - variational theory and applications," 2014.
[4] C. Zhu, P. Gu, D. Liu, X. Hu, and Y. Wu, "Evaluation of surface topography of SiCp/Al composite in grinding," *The International Journal of Advanced Manufacturing Technology*, vol. 102, pp. 2807–2821, 2019.
[5] J. Zhang and J. Wang, "Research on the effect path of institutional environment to internal control," in *7th International Conference on Social Science and Higher Education (ICSSHE 2021).* Atlantis Press, 2021, pp. 664–668.
[6] P. Gu, J. Chen, W. Huang, Z. Shi, X. Zhang, and L. Zhu, "Evaluation of surface quality and error compensation for optical aspherical surface grinding," *Journal of Materials Processing Technology*, vol. 327, p. 118363, 2024. doi: 10.1016/j.jmatprotec.2024.118363
[7] A. Goetz, *Introduction to Differential Geometry.* Addison Wesley Publishing Company, 1970.
[8] M. P. do Carmo, *Differential Geometry of Curves and Surfaces.* Prentice Hall, 1976.
[9] F. C. Park, "Optimal Robot Design and Differential Geometry," *Journal of Mechanical Design*, vol. 117, no. B, pp. 87–92, 06 1995. doi: 10.1115/1.2836475
[10] N. Jaquier and T. Asfour, "Riemannian geometry as a unifying theory for robot motion learning and control," in *Robotics Research*, A. Billard, T. Asfour, and O. Khatib, Eds. Cham: Springer Nature Switzerland, 2023. ISBN 978-3-031-25555-7 pp. 395–403.
[11] S. Radhakrishnan and W. Gueaieb, "A state-of-the-art review on topology and differential geometry-based robotic path planning-part I: planning under static constraints," *International Journal of Intelligent Robotics and Applications*, 03 2024. doi: 10.1007/s41315-024-00330-5
[12] A. Ai, M. Ifrim, and D. Tataru, "The time-like minimal surface equation in Minkowski space: low regularity solutions," *Inventiones mathematicae*, vol. 235, no. 3, pp. 745–891, March 2024. doi: 10.1007/s00222-023-01231-3
[13] J. Monterde and H. Ugail, "On harmonic and biharmonic Bézier surfaces," *Computer Aided Geometric Design*, vol. 21, no. 7, pp. 697 – 715, 2004. doi: 10.1016/j.cagd.2004.07.003
[14] D. Ahmad and B. Masud, "A Coons patch spanning a finite number of curves tested for variationally minimizing its area," *Abstract and Applied Analysis*, vol. 2013, 2013. doi: 10.1155/2013/645368
[15] G.Xu, R. Timon, G. Erhan, W. Qing, K. Hui, and W. Guozhao, "Quasi-harmonic Bézier approximation of minimal surfaces for finding forms of structural membranes," *Comput. Struct.*, vol. 161, no. C, pp. 55–63, Dec. 2015. doi: 10.1016/j.compstruc.2015.09.002
[16] D. Ahmad and B. Masud, "Variational minimization on string-rearrangement surfaces, illustrated by an analysis of the bilinear interpolation," *Applied Mathematics and Computation*, vol. 233, pp. 72 – 84, 2014. doi: 10.1016/j.amc.2014.01.172
[17] D. Ahmad and S. Naeem, "Quasi-harmonic constraints for toric Bézier surfaces," *Sigma Journal of Engineering and Natural Sciences*, vol. 36, pp. 325–340, 2018.
[18] D. Ahmad, S. Naeem, A. Haseeb, and M. Mahmood, "A computational approach to a quasi- minimal Bézier surface for computer graphics," *VFAST Transactions on Software Engineering*, vol. 9, no. 4, pp. 150–159, 2021. doi: 10.21015/vtse.v9i4.929

[19] D. Ahmad and B. Masud, "Near-stability of a quasi-minimal surface indicated through a tested curvature algorithm," *Computers & Mathematics with Applications*, vol. 69, no. 10, pp. 1242 – 1262, 2015. doi: 10.1016/j.camwa.2015.03.015
[20] J. Hoschek and D. Lasser, *Fundamentals of Computer Aided Geometric Design.* A K Peters/CRC Press, 2003.
[21] D. Marsh, *Applied Geometry for Computer Graphics and CAD.* Springer Science & Business Media, 2005.
[22] G. Farin, *Curves and Surfaces for Computer-Aided Geometric Design: A Practical Guide*, ser. Computer science and scientific computing. Elsevier Science, 2014. ISBN 9781483296999
[23] X. Jiao, D. Wang, and H. Zha, "Simple and effective variational optimization of surface and volume triangulations," *Engineering with Computers*, vol. 27, pp. 81–94, 2011.
[24] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, ser. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003. ISBN 9781139435024
[25] D. Ahmad and M. Ziad, "Homothetic motions of spherically symmetric space-times," *Journal of Mathematical Physics*, vol. 38, no. 5, pp. 2547–2552, 1997. doi: 10.1063/1.531994
[26] D. Ahmad and K. Habib, "Homotheties of a class of spherically symmetric space-time admitting G_3 as maximal isometry group," *Advances in Mathematical Physics*, vol. 2018, p. 8195208, Jan 2018. doi: 10.1155/2018/8195208
[27] U. Camci and K. Saifullah, "Conformal symmetries of the energy–momentum tensor of spherically symmetric static spacetimes," *Symmetry*, vol. 14, no. 4, p. 647, Mar 2022. doi: 10.3390/sym14040647
[28] K. S. Nisar, J. Ali, M. K. Mahmood, D. Ahmad, and S. Ali, "Hybrid evolutionary Padé approximation approach for numerical treatment of nonlinear partial differential equations," *Alexandria Engineering Journal*, vol. 60, no. 5, pp. 4411–4421, 2021. doi: 10.1016/j.aej.2021.03.030
[29] A. Zafar, M. Ijaz, A. Qaisar, D. Ahmad, and A. Bekir, "On assorted soliton wave solutions with the higher-order fractional Boussinesq-Burgers system," *International Journal of Modern Physics B*, vol. 0, no. 0, p. 2350287, 0. doi: 10.1142/S0217979223502879
[30] M. Saqib, D. Ahmad, A. N. Al-Kenani, and T. Allahviranloo, "Fourth- and fifth-order iterative schemes for nonlinear equations in coupled systems: A novel adomian decomposition approach," *Alexandria Engineering Journal*, vol. 74, pp. 751–760, 2023. doi: 10.1016/j.aej.2023.05.047
[31] C. Cosín and J. Monterde, *Bézier surfaces of minimal area.* Berlin, Heidelberg: Springer Berlin Heidelberg, 2002, pp. 72–81. ISBN 978-3-540-46080-0
[32] J. Monterde, "Bézier surfaces of minimal area: The Dirichlet approach," *Computer Aided Geometric Design*, vol. 21, no. 2, pp. 117 – 136, 2004. doi: 10.1016/j.cagd.2003.07.009
[33] J. Monterde and H. Ugail, "A general 4th-order {PDE} method to generate Bézier surfaces from the boundary," *Computer Aided Geometric Design*, vol. 23, no. 2, pp. 208 – 225, 2006. doi: 10.1016/j.cagd.2005.09.001
[34] L.-Y. Sun and C.-G. Z. (2016), "Approximation of minimal toric Bézier patch," *Advances in Mechanical Engineering*, vol. 8, p. (6).
[35] D. Ahmad, K. Hassan, M. K. Mahmood, J. Ali, I. Khan, and M. Fayz-Al-Asad, "Variationally improved Bézier surfaces with shifted knots," *Advances in Mathematical Physics*, vol. 2021, no. 5, p. 14, 2021. doi: 10.1155/2021/9978633
[36] Y. X. Hao and T. Li, "Construction of quasi-Bézier surfaces from boundary conditions," *Graphical Models*, vol. 123, p. 101159, 2022. doi: 10.1016/j.gmod.2022.101159
[37] X. Li, Y. Zhu, H. Wu, J. Xu, R. Ling, X. Wu, and G. Xu, "Construction of Bézier surfaces with energy-minimizing diagonal curves from given boundary," *Journal of Computational and Applied Mathematics*, vol. 413, p. 114382, 2022. doi: 10.1016/j.cam.2022.114382
[38] D. Ahmad, M. K. Mahmood, Q. Xin, F. M. O. Tawfiq, S. Bashir, and A. Khalid, "A computational model for q-Bernstein Quasi-minimal Bézier surface," *Journal of Mathematics*, vol. 2022, p. 8994112, Sep 2022.
[39] D. Ahmad, K. Naz, M. E. Buttar, P. C. Darab, and M. Sallah, "Extremal solutions for surface energy minimization: Bicubically blended Coons patches," *Symmetry*, vol. 15, no. 6,

2023. doi: 10.3390/sym15061237

[40] S. Bashir and D. Ahmad, "Geometric analysis of non-degenerate shifted-knots Bézier surfaces in minkowski space," *PLOS ONE*, vol. 19, no. 1, pp. 1–28, 01 2024. doi: 10.1371/journal.pone.0296365

[41] S. Bernstein, "Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités," *Comm. Soc. Math. Kharkov*, vol. 13, pp. 1–2, 1912. [Online]. Available: http://mi.mathnet.ru/khmo107

[42] K. Khan and D. Lobiyal, "Bézier curves based on Lupaş (p, q)-analogue of Bernstein functions in CAGD," *Journal of Computational and Applied Mathematics*, vol. 317, pp. 458–477, 2017. doi: 10.1016/j.cam.2016.12.016

[43] G. M. Phillips, "A generalization of the Bernstein polynomials based on the q-integers," *The ANZIAM Journal*, vol. 42, no. 1, pp. 79–86, 07 2000. doi: 10.1017/S1446181100011615

[44] H. Oruc and G. M. Phillips, "q-Bernstein polynomials and Bézier curves," *Journal of Computational and Applied Mathematics*, vol. 151, no. 1, pp. 1 – 12, 2003. doi: 10.1016/S0377-0427(02)00733-1

[45] D. Kim, L.-C. Jang, Y. Kim, and J. Choi, "On p-adic analogue of q-Bernstein polynomials and related integrals," *Discrete Dynamics in Nature and Society*, vol. 2010, 09 2010. doi: 10.1155/2010/179430

[46] T. Kim, "A note on q-Bernstein polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 1, p. 73, 2011. doi: 10.1134/S1061920811010080

[47] S. Araci, M. Acikgoz, and E. Şen, "On the extended Kims p-adic q-deformed fermionic integrals in the p-adic integer ring," *Journal of Number Theory*, vol. 133, no. 10, pp. 3348 – 3361, 2013. doi: 10.1016/j.jnt.2013.04.007

[48] S. Araci and M. Acikgoz, "A note on the values of the weighted q-Bernstein polynomials and weighted q-Genocchi numbers," *Advances in Difference Equations*, 01 2015. doi: 10.1186/s13662-015-0369-y

[49] D. S. K. Lee-Chae Jang, Taekyun Kim and D. V. Dolgy, "On p-adic fermionic integrals of q-Bernstein polynomials associated with q-Euler numbers and polynomials," *Journal of Number Theory*, vol. 1, no. 10, pp. 1–9, 2018. doi: 10.1016/j.jnt.2013.04.007

[50] K. Khan and D. Lobiyal, "Bézier curves based on lupaş (p, q)-analogue of Bernstein functions in CAGD," *Journal of Computational and Applied Mathematics*, vol. 317, pp. 458–477, 2017. doi: 10.1016/j.cam.2016.12.016

[51] t. Ağyüz and t. Açikgöz, "A note on Bernstein polynomials based on (p, q)-calculus," *AIP Conference Proceedings*, vol. 1978, no. 1, p. 040006, 2018. doi: 10.1063/1.5043690

[52] Q.-B. Cai, B.-Y. Lian, and G. Zhou, "Approximation properties of λ-Bernstein operators," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 61, 2018. doi: 10.1186/s13660-018-1653-7

[53] M. Ayman-Mursaleen, M. Nasiruzzaman, N. Rao, M. Dilshad, and K. S. Nisar, "Approximation by the modified λ-Bernstein-polynomial in terms of basis function," *AIMS Mathematics*, vol. 9, no. 2, pp. 4409–4426, 2024. doi: 10.3934/math.2024217

[54] J. Delgado and J. Peña, "Geometric properties and algorithms for rational q-Bézier curves and surfaces," *Mathematics*, vol. 8, no. 4, p. 541, 2020. doi: 10.3390/math8040541

[55] G. Zhou and Q.-B. Cai, "Bivariate λ-Bernstein operators on triangular domain," *AIMS Mathematics*, vol. 9, no. 6, pp. 14 405–14 424, 2024. doi: 10.3934/math.2024700

[56] N. Turhan, F. Özger, and M. Mursaleen, "Kantorovich-Stancu type (α, λ, s) - Bernstein operators and their approximation properties," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 30, no. 1, pp. 228–265, 2024. doi: 10.1080/13873954.2024.2335382

[57] Z.-P. Lin, G. Torun, E. Kangal, Ü. D. Kantar, and Q.-B. Cai, "On the properties of the modified λ-Bernstein-Stancu operators," *Symmetry*, vol. 16, no. 10, p. 1276, 2024. doi: 10.3390/sym16101276

[58] L. Yan and J. Liang, "An extension of the Bézier model," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2863–2879, 2011. doi: 10.1016/j.amc.2011.08.030

[59] G. Hu, J. Wu, and X. Qin, "A novel extension of the Bézier model and its applications to surface modeling," *Advances in Engineering Software*, vol. 125, pp. 27–54, 2018. doi: 10.1016/j.advengsoft.2018.09.002

[60] V. Bulut, "Path planning of mobile robots in dynamic environment based on analytic geom-

etry and cubic bézier curve with three shape parameters," *Expert Systems with Applications*, vol. 233, p. 120942, 2023. doi: 10.1016/j.eswa.2023.120942

[61] M. Elhoseny, A. Tharwat, and A. E. Hassanien, "Bézier curve based path planning in a dynamic field using modified genetic algorithm," *Journal of Computational Science*, vol. 25, pp. 339–350, 2018. doi: 10.1016/j.jocs.2017.08.004