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Research Article

A PRODUCTION INVENTORY MODEL WITH IMPERFECT ITEM AND LOW QUALITY MANUFACTURING UNDER α-CUT TYPE-2 FUZZY ENVIRONMENT USING FUZZY H_2 DIFFERENTIATION

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Abstract: This study delves into a fuzzy economic manufacturing model focused on inventory models encountering imperfect and low-quality production processes, inclusive of rework scenarios. The concern arises primarily during epidemics when unsold items accumulate, escalating maintenance costs due to constant deterioration. Uniquely, this research incorporates a special mathematical formulation contrasting discounted low-quality items with non-discounted ones under fuzzy conditions. Enhanced computer techniques founded on fuzzy logic are employed to refine identification, decision-making, and

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optimization. These situations are depicted through a bi-objective framework aiming for a simultaneous minimization of overall cost and emissions, subject to a meticulously devised constraint set. The significance of optimal manufacturing is accentuated by attributing a triangular fuzzy number to economic production quantity. The EPQ model's optimal total cost is discerned in its crisp form, meriting emphasis. Production inventory, varying from raw materials to unfinished products, sometimes includes imperfect items, posing significant challenges like increased production costs and delayed processes. Addressing this involves implementing rigorous quality control measures and possibly adopting lean manufacturing principles aimed at minimizing waste and enhancing production efficiency. These strategies aid in maintaining quality, ensuring customer satisfaction, and sustaining profits by mitigating the challenges posed by inferior quality items within the production inventory.

Keywords: Production inventory, inventory model using fuzzy differentiation, deterioration items, low quality and imperfect items.

MSC: 90B05.

1. INTRODUCTION

In the past, a wide range of Economic Production Quantity (EPQ) models have been scrutinized by numerous scholars. Some of these models have considered the occurrence of defective goods during production. For example, the study by Sana et. al. [1] postulated that the duration until the manufacturing process deviates into a sub optimal condition follows an exponential distribution, featuring flawed quality and changeable production rates. Salameh and Jaber [2] presented a novel approach to inventory management that considered the presence of defective goods. They incorporated the EPQ/Economic Order Quantity (EOQ) formulation into their model to account for defective items. Several other contributions to the field of imperfect production were made by scholars such as Rosenblatt and Lee [3], Ben-Daya and Hariga [4], Hayek and Salameh [5], Goyal and Cardenas-Barron [6], Chung and Hou [7], Goyal, Hung, and Chen (2003), Ghosh-Dey [8], etc. Additionally, Manna, Dey, and Mondal [9] introduced a three-layer supply chain in an imperfect production inventory model with dual storage facilities in a fuzzy-rough setting. In recent research conducted by Manna [10], a model was developed to address imperfections in inventory management. This model took into account promotional demand within a random planning period and utilized a population varying genetic algorithm technique.

Numerous studies have delved deeper into production inventory models incorporating equipment malfunctions, taking into account preventive maintenance and rework as crucial factors. The economic batch sizing implications of machinery failure and corrective maintenance were examined in a study conducted by Groenevelt et al. [11], with building upon their research. Subsequently, Giri et al. [12] put forth an Economic Manufacturing Quantity (EMQ) model featuring equipment failure and general repair time. Preventive maintenance, generally implemented to diminish machine breakdown, has been the focus of numerous research efforts, with significant contributions from Cheung and Hausman [13], Dohi et al. [14], Lin and Gong [15], Halim et al. [16], El-Ferik [17], Lia and others [18], Chiu et al. [19], Chiu and Chang [20], among others.

However, upon reviewing the literature, it appears that few scholars have explored imperfect production inventory models with returned goods, in the context of stochastic machine breakdown and stochastic repair time. Lin and Gong (2006) designed an EPQ model featuring a deteriorating inventory with machine failure and fixed repair time. Introducing imperfect production where the manufacturing system may transition from a 'controlled state' to an 'uncontrolled state' unpredictably. Additionally, it contemplates returned goods from markets, replacing them with flawless new products. Two scenarios are considered: one with stochastic machine repair time and the other with production downtime.

Fuzzy sets, a concept introduced by Lotfi A. Zadeh in 1965, are a mathematical approach to representing uncertainty and imprecision in data. They provide a more flexible and intuitive way [21] to model complex systems and make decisions when dealing with incomplete or vague information. A collection of elements with a continuous membership function, ranging from 0 to 1, where each element has a degree of membership in the set. b. Membership Function: A function that maps elements to their membership values in a fuzzy set. It can be represented by various mathematical forms (e.g., triangular, trapezoidal, Gaussian).

Fuzzy differential equations(FDE) [22] are a type of differential equation where the coefficients and/or initial conditions are fuzzy sets instead of crisp numbers. These categories of equations are employed in circumstances where the factors or initial states are uncertain or imprecise and rendering conventional differential equations are inapplicable. Here are some key points about fuzzy differential equations: Fuzzy differential equations [23] can be utilized to model a diverse array of systems, including biological systems, ecological systems, and financial systems. There are several methods for rectifying fuzzy differential equations, including the interval method [24], the parametric method, and the direct method. The Outcome to a fuzzy differential equation is a fuzzy set [25], which represents the range of possible values for the solution. Fuzzy differential equations can be used to analyze the stability of a system [26] and to make predictions about its behavior over time. Fuzzy differential equations are often used in conjunction with fuzzy logic and fuzzy control systems to model and control complex systems. One of the challenges of working with fuzzy differential equations is that they can be computationally expensive to solve, particularly when dealing with high-dimensional systems.

Hukuhara differentiability [27], also known as pseudo-differentiability or fuzzy differentiability, is a generalization of classical differentiability that applies to functions defined on fuzzy sets. In classic analysis, a function is considered differentiable at a point if its derivative is defined or present at that point. However, in fuzzy calculus, a function can be "partially differentiable" or "fuzzy differentiable" at a point, meaning that its derivative exists only in a fuzzy or uncertain sense. Specifically, a function f defined on a fuzzy set X is mentioned to be Hukuhara differentiable at a point x_0 in X if there exists a fuzzy number μ such that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x in X, the following inequality holds:

$$|f(x) - f(x_0)| \leq \mu(|x - x_0| + \varepsilon) \tag{1}$$

where $|x - x_0|$ denotes the distance between x and x_0. The value of μ represents the degree of differentiability of the function at x_0.

Hukuhara differentiability [27] has applications in fields such as fuzzy control systems, fuzzy optimization, and image processing, among others.

The differentiability of the fuzzy function must first be established in order to create an FDE. A fuzzy function is a type of function that maps elements from a fuzzy set to elements in another fuzzy set. Unlike classical functions, which have precise and well-defined outputs for each input, fuzzy functions allow for more flexible and uncertain mappings. Zadeh and Chang were the ones who initially presented the idea of fuzzy derivative in 1972. Hukuhara differentiability is a concept in mathematics that is related to the notion of differentiability in calculus. The concept was introduced by the Japanese mathematician Kôsaku Yosida in the 1950s.

Hukuhara differentiability [28] is a elaboration on the theory of differentiability that is suitable for functions that are not necessarily continuous. A function is said to be Hukuhara differentiable at a point if it satisfies a certain Lipschitz-type condition in a neighborhood of that point.

Formally, let f be a genuine numerical function established on a subset U of Euclidean space. Then, f is said to be Hukuhara derivable at a point x in U if there exists a non-negative constant L such that for all y in a neighborhood of x,

$$|f(y) - f(x)| \leq L\omega(|y - x|) \tag{2}$$

where ω is a continuity modulus, which is a function that measures the degree of continuity of f. The Lipschitz-type condition implies that the function f is locally Lipschitz, which means that it is Lipschitz continuous in a neighborhood of x. Hukuhara differentiability has applications in a variety of fields, including control theory, signal processing, and optimization. In these fields, Hukuhara differentiability is used to analyze the behavior of functions that may not be continuous, and to develop techniques for optimizing such functions. Hukuhara differentiability is a broadening of the concept of differentiability [29] that is suitable for functions that are not necessarily continuous. It has applications in a variety of fields, and is used to analyze the behavior of functions that may not be well-behaved under traditional calculus methods.

Differentiability of fuzzy functions [30] is an advanced concept within the field of fuzzy mathematics, specifically fuzzy calculus. Fuzzy functions, also known as fuzzy-valued functions or fuzzy mappings, are functions that map elements from one fuzzy set to another. The study of differentiability of fuzzy functions involves understanding how these functions change with respect to their input variables and under what conditions they can be considered differentiable. And Hukuhara differentiability, named after Michio Hukuhara, is a notion of differentiability specifically designed for fuzzy functions. It offers a structure to specify the fuzzy-valued function's derivative, allowing for the analysis of rates of change and local linear approximations in fuzzy calculus. Differentiability of fuzzy functions <**empty citation**> is an advanced concept within the field of fuzzy mathematics, specifically fuzzy calculus. Fuzzy functions, also known as fuzzy-valued functions or fuzzy mappings, are functions that map elements from one fuzzy set to another. The study of differentiability of fuzzy functions involves understanding how these functions change with respect to their input variables and under what conditions they can be considered differentiable. And Hukuhara differentiability, named after Michio

Hukuhara, is a notion of differentiability specifically designed for fuzzy functions. It offers a structure to specify the fuzzy-valued function's derivative, allowing for the analysis of rates of change and local linear approximations in fuzzy calculus.

Hukuhara differentiability [28] is based on the theory of α-cuts, which are crisp sets obtained by fixing a threshold value α and intersecting it with the fuzzy set. The α-cuts capture the extent to which elements in the domain have membership in the fuzzy set. The main idea behind Hukuhara differentiability is to describe the derivative of a fuzzy function in terms of the α-cuts. Here are some key points regarding Hukuhara differentiability:

α-Cuts: In relation to a fuzzy set, the α-cut at a particular level α is determined by intersecting the fuzzy set with a definite set delineated by the α threshold. The consequent definite set signifies the elements having a membership degree that is equal to or higher than α. Hukuhara Difference: The Hukuhara difference among two fuzzy sets A and B is described as the maximum difference between corresponding α-cuts, where the maximum is chosen over all α values between 0 and 1. It measures the extent of overlap or separation among the fuzzy sets. Hukuhara Derivative: Pertaining to a fuzzy function, the Hukuhara derivative at a specific point is delineated in terms of the Hukuhara difference of the α-cuts of the fuzzy function evaluated at that point. It measures the local rate of change or slope of the fuzzy function. Properties and Applications: Hukuhara differentiability has several useful properties, such as linearity, chain rule, and relation to convexity. It has been applied in various areas, incorporating of fuzzy optimization, fuzzy control systems [29], and fuzzy differential equations, providing a mathematical foundation for analyzing fuzzy systems and making decisions in the presence of uncertainty. Hukuhara differentiability offers a valuable tool for extending the notion of differentiability to fuzzy functions, enabling the study of local behavior and approximation of fuzzy systems through linearization techniques.

Type-1 fuzzy differential equations are a type of differential equation where the variables and parameters are fuzzy sets. Fuzzy sets are sets that allow for partial membership of an element in the set, where the degree of membership is represented by a membership function. In type-1 fuzzy differential equations [31], the differential equations involve fuzzy numbers [32] as coefficients or initial/boundary conditions. The solutions to these equations are also fuzzy sets, and the degree of membership of a point in the solution set is determined by the membership functions of the fuzzy sets in the equation.

This study initially presents the concept of Hukuhara discrepancy (H-difference) grounded on defined perfect Type-2 Fuzzy Numbers [T2FNs], and subsequently, by implementing this concept in a production inventory model, it is revealed that the Type-2 Hukuhara discrepancy [33] H_2-difference is essentially the H-difference. This methodology, when applied within a mathematical structure in a production inventory model, is used to identify the most profitable production and inventory strategies for a business engaged in the manufacture and sale of a product. The goal of the model is to minimize the overall production and inventory costs while fulfilling consumer demand. In this context, triangular Perfect Quasi Type-2 Fuzzy Numbers [34] $(TPT2FN)$, a form of fuzzy number frequently employed to represent uncertainty in scenarios with incomplete or imprecise information, are utilized. Finally, an admissibility and a method for resolving T2FDEs [35] are exemplified via illustrations and examples.

2. FUNDAMENTAL IDEAS WITH FUZZY

The utility of Type-1 Fuzzy Numbers (T1FNs) is found in their ability to embody uncertainty when data is either partial or inaccurate. T1FNs are distinguished by their membership function, which accords a membership level to each member within the set.

Each constituent in the scope of discourse within a T1FN is mapped by the membership function to a membership grade ranging from 0 to 1. This degree of association demonstrates the extent to which the constituent belongs to the fuzzy set. Typically, T1FNs are delineated by a triangular or trapezoidal membership function, simplifying both understanding and calculation. As an illustration, a triangular T1FN [36] with parameters (a,b,c) possesses a membership function equating to 1 at $x = b$ and drops progressively to 0 at $x = a$ and $x = c$. In the same vein, a trapezoidal T1FN bearing parameters (a,b,c,d) holds a membership function equal to 1 for x within the range $[b,c]$ and tapers off proportionately to 0 for x within the ranges $[a,b]$ and $[c,d]$.

T1FNs are widely used in diverse areas, involving regulatory mechanisms, decision-making, and pattern recognition [37], among others. They provide a way to handle uncertainty and imprecision in a quantitative manner, making them a powerful tool for modeling real-world problems.

In this article, the symbol $\mathbb{R}$ represents the entirety of real numbers. The collection of Type-1 Fuzzy Numbers (T1FNs) on $\mathbb{R}$ is denoted as E_1, while the set of perfect T2FNs on $\mathbb{R}$ is represented by E_2. The $\alpha-$ cut of a fuzzy set A is indicated as A^{α}.

A Type-1 fuzzy set is the simplest and most basic form of fuzzy set in fuzzy logic. It is distinguished by a membership function that assigns a degree of membership to each element within the domain of discussion. The membership function correlates elements, from the universe of discourse to a value between 0 and 1, expressing the extent to which an element is a member of the fuzzy set.

In a formal context, a Type-1 fuzzy set A within a domain of discourse U is characterized by its membership function $\mu A(x)$, where x represents an element from U. The membership function $\mu A(x)$ quantifies the extent to which x belongs to the fuzzy set A, varying between 0 (indicating no membership) and 1 (representing complete membership). The mathematical form of the membership function can adopt different shapes, such as triangular, trapezoidal, or Gaussian, based on the distinctive features and shape of the fuzzy set.

For example, consider a universe of discourse U representing the heights of people in a certain population. A Type-1 fuzzy set ”Tall” can be defined in this context, where the membership function assigns a degree of membership to each height value in U, indicating the degree to which a person is considered tall. The membership function may be a triangular function centered around a certain height value, such as 180 cm, with membership values decreasing as the height deviates from the center.

That is, for every element of the domain of discussion, the membership function assigns an interval of values between 0 and 1, rather than a single value. This interval can be interpreted as the range of possible degrees of membership of the element in the fuzzy set.

For example, suppose we have a universe of discourse of temperatures and we want to define a fuzzy set ”Hot”. Instead of assigning a single value for each temperature, we assign an interval of possible degrees of membership. So, for a temperature of 30 degrees Celsius, we might assign the interval $[0.7, 0.9]$ to represent the fact that it is very likely to be "hot", but there is some indeterminacy in the membership degree.

When there's a need to manage ambiguous or inexact membership within a fuzzy set, interval-valued fuzzy sets prove beneficial. They introduce a broader way of portraying ambiguity and are applicable in the formation of complicated and erratic systems. These special kinds of fuzzy sets, denoted as IVFS, offer a richer and subtler way of modeling uncertain or vague data. Traditional fuzzy sets provide only a singular value of membership to each set component, however, IVFS allocate a range of values demonstrating the membership degree of every member. The distinguishing attribute of an IVFS is a projection from a discourse universe to a collection of real line intervals, where each interval signifies a member's membership degree from a portion of the set. Several key features of IVFS are their closure under both intersection and union and a monotonically ascending attribute which guarantees that a member's degree of membership won't diminish as the interval's breadth expands. IVFS have a diverse array of applications in fields such as decision-making [38], pattern recognition, and control systems. For example, IVFS can be used to model uncertainty in sensor readings or to represent imprecise knowledge in expert systems. Inference with IVFS can be more complex than with traditional fuzzy sets, as it involves computing the intersection and union of intervals rather than simple arithmetic operations. However, several algorithms have been developed for performing inference with IVFS, including interval arithmetic and interval extension methods. The use of IVFS can provide several advantages over traditional fuzzy sets, including greater expressiveness, more robustness to noisy or imprecise data, and better modeling of uncertainty and imprecision.

The "Hukuhara discrepancy" is a notion in fuzzy set theory that is employed to gauge the extent of dissimilarity among two fuzzy sets. In other words, the Hukuhara difference [39] between A and B is the largest possible degree of membership of an element in A that is not a member of B, or vice versa. The Hukuhara difference is a useful measure in fuzzy set theory because it captures the degree of difference between two fuzzy sets [40], taking into account both their overlapping and non-overlapping regions. It is also a continuous function, meaning that minor modifications in the fuzzy sets will result in small changes in their Hukuhara difference.

Definition 1. *Let's suppose that we have $f : (a,b) \to E_1$ and $t_0 \in (a,b) \subset \mathbb{R}$. Now, $f(t)$ will be differentiable at t_0, as per the original representation, provided there's an element $f'(t_0) \in E_1$, that ensures for every $h > 0$, adequately near zero, there exist $f(t_0+h) \frac{H}{f} f(t_0), f(t_0) \frac{H}{f} f(t_0-h)$, along with the associated limit conditions:*

$$\lim_{h\to 0} \frac{f(t_0+h) \frac{H}{f} f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0) \frac{H}{f} f(t_0-h)}{h} = f'(t_0) \tag{3}$$

Alternatively, the function $f(t)$ suppose to be derivable at t_0, in the second format, if there exists an entity $f'(t_0) \in E_1$, such that for all $h > 0$, sufficiently close to zero, there are $f(t_0) \frac{H}{f} (t_0+h), f(t_0-h) \frac{H}{f} f(t_0)$, as well as the subsequent boundaries.

$$\lim_{h\to 0} \frac{f(t_0) \frac{H}{f} f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h) \frac{H}{f} f(t_0)}{-h} = f'(t_0) \tag{4}$$

In the given scenario, the borders are taken into account in the context of the metric space (E_1, d_1), where $d_1(u, v)$ is defined as the supremum of the set of Hausdorff distances, $d_H(u^\alpha, v^\alpha)$, for all α satisfying $0 \leq \alpha \leq 1$ and for any u, v that belong to E_1. Here, d_H signifies the Hausdorff distance.

In the context of type 2 fuzzy logic, the utilization of point-valued representation is a particular approach to denote ambiguity in the realm of type 2 fuzzy sets[41]. Contrasting type 1 fuzzy sets, which allot a unique membership worth to each constituent of the discourse universe, type 2 fuzzy sets introduce increased adaptability. This is made possible by accommodating the portrayal of ambiguity directly in the membership values.

In type 2 fuzzy sets, the membership function is determined with respect to a fuzzy set comprising fuzzy sets. This implies that, for every element x in the universe of discourse, the membership degree is expressed by a type 1 fuzzy set defined within the range of [0,1]. Nevertheless, in certain scenarios, it might be preferable to represent the membership degree of an element in a type 2 fuzzy set using a solitary point instead of a type 1 fuzzy set. This representation is referred to as the point-valued representation.

The point-valued representation in type 2 fuzzy sets involves [42] replacing the type 1 fuzzy sets that define the membership grades for each element with a single point that represents the most likely membership grade for that element. This point is referred to as the ”footprint of uncertainty” and is defined as the centroid of the type 1 fuzzy set that represents the membership grades for that element. The use of the point-valued representation can simplify the computation of operations on type 2 fuzzy sets, as it reduces the complexity of working with sets of sets. However, it also results in some loss of information, as the point-valued representation [42] does not capture the full range of possible membership grades for each element. As such, the choice of whether to use the point-valued representation or not depends on the specific application and the trade-off [43] between accuracy and computational intractability.

This membership function displays the extent to which an element x is a part of a set, represented by a triangular form. The function’s height signifies the level of membership, with a peak value of 1.0 denoting complete membership and a minimum value of 0.0 denoting absence of membership. A vertical cross-section of this membership function for a specific value of x, let’s say x=0.5, would be a line parallel to the y-axis intersecting the membership function at the membership level for x=0.5. In this scenario, the vertical cross-section would be a line intersecting the membership function at a value of 0.5, suggesting that the membership level of x=0.5 in the set represented by the membership function is 0.5.

The illustration of a vertical slice of a membership function is a useful tool for understanding how fuzzy logic works and for visualizing the degree to which an element belongs to a particular set. By examining the shape of the membership function and the degree of membership for different values of the input variable, it is possible to make informed decisions about how to use fuzzy logic to model complex systems.

Definition 2. *The visualization of a vertical section: The membership function serves as a mathematical representation of the degree to which a specific element belongs to a set. In the realm of fuzzy logic, it is utilized to determine the level of membership of an element in a particular set. A vertical section of a membership function refers to a* *graphical depiction that illustrates the membership degree of a specific element in a set while holding the input variable at a constant value. The vertical section of a Type-2 Fuzzy Set (T2FS) at a fixed point $x_0 \in X$ is a Type-1 Fuzzy Set (T1FS) with its domain as the secondary domain or primary membership denoted as J_{x_0}. The membership function of the vertical section is known as the second-tier membership function, $\mu_{\tilde{A}}(x_0)$, and it can be expressed as: $\mu_{\tilde{A}}(x_0,u) = \tilde{A}(x_0) = \{(u,\mu) \mid u \in J_{x_0}, \mu \in [0,1]\}$.*

Each constituent of a discourse universe is linked with a degree of membership in a fuzzy set through a membership function. The depiction of a membership function through a vertical slice representation generates a chart that exhibits the membership degree for every value within the discourse range.

To create a vertical slice representation, you can choose a fixed value for the degree of membership and draw a horizontal line across the membership function. The points where the line crosses or meets the membership function represent the values in the universe of discourse that have that degree of membership.

For example, suppose we have a fuzzy set "tall" that maps people's heights to a degree of membership. We might use a vertical slice representation to show which heights have a degree of membership of 0.5 (i.e., are "somewhat tall"). The graph would show the heights where the membership function is equal to 0.5.

Vertical slice representations can be useful for understanding how fuzzy sets represent uncertainty, as they provide a visual representation of how degrees of membership vary across the universe of discourse [44].

Definition 3. *$S_{\tilde{A}}(x_0 \mid \tilde{\alpha})$ denotes an $\tilde{\alpha}$ - cut the subordinate membership function $\mu_{\tilde{A}}(x_0)$.*

In the domain of fuzzy logic, an $\tilde{\alpha}$-cut of a fuzzy set refers to a clear-cut subset of the universe of discourse that encompasses all the elements having a membership degree equal to or exceeding a specified threshold value $\tilde{\alpha}$.

Likewise, the $\tilde{\alpha}$-cut of the secondary membership function of a fuzzy set, as discussed in [45], refers to a distinct subset of the secondary universe of discourse. This subset includes all the elements whose degree of secondary membership equals or surpasses the specified threshold value $\tilde{\alpha}$. To understand this better, consider a fuzzy set "Tall" that maps people's heights to a degree of membership. The secondary membership function could represent the degree of certainty about the degree of membership, such as how certain we are about the degree of tallness of a person given their height. For example, if the primary membership function for "Tall" gives a degree of membership of 0.7 for someone who is 6 feet tall, the secondary membership function could give a degree of 0.8 for our degree of certainty about that person being tall. Assuming we assign $\tilde{\alpha}$ a value of 0.6 for the secondary membership function, the $\tilde{\alpha}$-cut corresponding to the secondary membership function as per [46], will encapsulate a segment of the secondary discourse universe. This segment will include all values having a secondary membership degree that is 0.6 or higher. This subset signifies the values that offer a relatively assured determination of a person's tallness based on their height.

In the realm of type-2 fuzzy logic, the $\tilde{\alpha}$-slice plane representation is utilized as a tool for depicting a type-2 fuzzy set [47]. This type-2 fuzzy set is a unique fuzzy set where the membership function itself takes the form of a fuzzy set. From another perspective, the membership function for a type-2 fuzzy set can be seen as a collection of membership functions, each signifying a distinct uncertainty level or ambiguity related to the extent of an element's membership in the set. The $\tilde{\alpha}$-cut plane representation can be employed to illustrate a type-2 fuzzy set, showcasing how each element's membership degree fluctuates with changing levels of uncertainty. The $\tilde{\alpha}$-cut plane functions as a two-dimensional chart which maps out each set element's membership degree in relation to the level of ambiguity. By setting a specific $\tilde{\alpha}$ value (which represents the level of ambiguity) and taking a vertical cut through the membership function at that point, the $\tilde{\alpha}$-cut plane is formed. The slice that results is a type-1 fuzzy set that reflects each element's membership degree in the set for that specific level of ambiguity. By showcasing these type-1 fuzzy sets across a spectrum of $\tilde{\alpha}$ values, the $\tilde{\alpha}$-cut plane enables the visualization of the membership degree fluctuation relative to changing ambiguity levels.

The $\tilde{\alpha}$-cut plane representation can be useful for visualizing the level of uncertainty in a type-2 fuzzy set and for understanding how the degree of membership of each element in the set changes as the level of uncertainty changes. It can also be used for performing operations on type-2 fuzzy sets, such as union, intersection, and complement. A triangular second-order fuzzy set [48] is a fuzzy set characterized by a triangular membership function with uncertain levels of membership. In a second-order fuzzy set, both the degree of membership and the degree of uncertainty itself are uncertain. A second-order fuzzy set shaped as a triangle is characterized by a lower membership function (LMF), an upper membership function (UMF), and a principle set (PS) [29]. The secondary membership function is portrayed in the top insert, depicting a vertical cut at the position x_0. The $\tilde{\alpha}$-cut at the position x_0 is designated as $\tilde{\alpha} = 0.7$. The uncertainty footprint of a second-order fuzzy set, referred to as the $\tilde{\alpha}$-plane, is displayed when $\tilde{\alpha} = 0$. The two-dimensional field of a type 2 fuzzy set is comprised of both primary and secondary membership degrees. The primary membership degree signifies an element's membership degree in the fuzzy set, while the secondary membership degree corresponds to the level of uncertainty related to the primary membership degree.

For example, suppose we have a fuzzy set A that symbolizes the concept of "tallness" in humans. The primary membership function for A might be defined using a triangular membership function that assigns a degree of membership to each height value. However, because the concept of tallness is somewhat subjective, there may be a degree of uncertainty associated with each degree of membership. The secondary membership function for A could be defined as a Gaussian distribution that models this uncertainty. Alternatively, it could be argued that there exists a limited area within the T2FS on the x and u axes in the two-dimensional plane, which can be referred to as the uncertainty footprint of $\tilde{A}$. In other words, the uncertainty regarding the primary memberships of the T2FS is contained within this bounded region. The FOU($\tilde{A}$) is the same as $\tilde{A}_0(\tilde{\alpha} = 0$ - Plane) (Figure 1).

Differentiable fuzzy mappings are a type of mathematical function that maps input values to fuzzy output values. Fuzzy mappings are used to represent uncertain or imprecise information in a more realistic way than classical mappings. A differentiable fuzzy mapping is a fuzzy mapping that has a well-defined derivative at each point in its domain. This property is important in many applications, such as control systems, machine learning, and optimization, where it is necessary to compute the gradient or Jacobian of the mapping.For the establishment of a differentiable fuzzy mapping, it is necessary to outline the membership functions that represent the fuzzy sets in both the input and output domains. Thereafter, we can utilize the conventional calculus laws to ascertain the mapping's derivative concerning the input parameters. A wide array of applications, encompassing fuzzy control, fuzzy decision-making, and fuzzy clustering, employ differentiable fuzzy mappings. These mappings serve as a robust instrument for modeling intricate systems grappling with ambiguity and inexactness. The concept of a fuzzy interval is integral within the structure of fuzzy set theory, encapsulating number sets with diverse membership degrees that are contained within a specific interval.

This means that instead of belonging to either 0 or 1 (representing fully membership in a set), a number can have a value that ranges between 0 and 1. This is useful when dealing with uncertain data or when making decisions based on values that have a range of certainty. For example, if a person is asked to assign a confidence level to their answer to a question, they may not be able to assign a numerical value of 0 or 1, but might instead give a value between 0.5 and 0.8, indicating that they are somewhat certain of their answer. Fuzzy intervals can also be used to compare different groups of data and see how they differ in terms of certainty. For any $\alpha \in [0,1]$, we denote

$$F_\alpha(x) = \left[\underline{f}_\alpha(x), \bar{f}_\alpha(x)\right], x \in T$$

Here, for each $\alpha \in [0,1]$, the endpoint functions $\bar{f}\alpha, \underline{f}\alpha : T \to \mathbb{R}$ are referred to as the upper and lower functions of F, respectively.

Following that, we present the concept of derivatives (referred to as gH derivatives) associated with fuzzy functions, which originate from the difference (gH-difference) in fuzzy intervals. In the event that there is an $F'(x_0) \in \mathscr{F}C$ which complies with the mentioned formula, we proclaim that F is differentiable under Hukuhara (concisely, gH-differentiable) at the point x_0.

It should be noted that the concept of gH-differentiability for interval-valued functions has been established based on the gH-difference of intervals, as discussed in [49]. Additionally, alternative definitions have been put forward in [19], utilizing inner-difference, and in [50], employing π-difference, in the instance of interval-valued functions. In this segment, our goal is to investigate the correlation between the gH-derivative of a fuzzy function F and the gH-differentiability of the span of interval functions F_α. Furthermore, we are enthralled to comprehending how the gH-differentiability of a fuzzy function F is linked to the differentiability of its boundary functions $\underline{f}\alpha$ and $\bar{f}\alpha$.

Definition 4. *Suppose that every vertical slice of a T2FS, $\tilde{A}$, has a subordinate level equal to one, denoted as $f_x(u) = 1$. A primary cluster of $\tilde{A}$ is the amalgamation of all points where this condition holds true, and can be described as follows:*

$$\mu_{PS}(x) = \int_{x \in X} u/x \text{ s.t. } f_x(u) = 1 \tag{5}$$

where $\mu_{PS}(x)$ is the membership function of primary cluster of $\tilde{A}$. the α-cut of that $\tilde{\alpha}$ -plane are outlined below

$$\tilde{A}^\alpha_{\tilde{\alpha}} = \left(\underline{A}^\alpha_{\tilde{\alpha}}, \bar{A}^\alpha_{\tilde{\alpha}}\right) \tag{6}$$

where $\underline{A}_{\tilde{\alpha}}^{\alpha}$ *and* $\bar{A}_{\tilde{\alpha}}^{\alpha}$ *are the* α*-cuts of the LMF and UMF of* $\tilde{\alpha}$ *- plane of* $\tilde{A}$ *respectively.*

Theorem 5. *Symbolization of T2FS using* α*-cuts: A T2FS,* $\tilde{A}$*, can be expressed by aggregating all its* α*-cuts in the following manner:*

$$\tilde{A} = \bigcup_{\forall\tilde{\alpha}\in[0,1]} \tilde{\alpha} \bigcup_{\forall\alpha\in[0,1]} \alpha \tilde{A}_{\tilde{\alpha}}^{\alpha} \tag{7}$$

Theorem 6. *The principle of extensional for T2FS* α*-cuts states that if we have a Cartesian product* $X = X_1 \times \ldots \times X_n$ *of universes and T2FSs* $\tilde{A}_1, \ldots, \tilde{A}_n$ *defined in each respective universe, and if we have another universe* Y *with a T2FS* $\tilde{B} \in Y$ *that can be obtained as a result of a monotonic mapping* $f : X \to Y$ *applied to* $\tilde{A}_1, \ldots, \tilde{A}_n$*, then* $\tilde{B}$ *can be represented as the union of adopting the same mapping to all of its decomposed* α*-cuts:*

$$\tilde{B} = f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \bigcup_{\forall\tilde{\alpha}\in[0,1]} \tilde{\alpha} \bigcup_{\forall\alpha\in[0,1]} \alpha f\left(\tilde{A}_{1,\alpha}^{\alpha}, \ldots, \tilde{A}_{n,\alpha}^{\alpha}\right) \tag{8}$$

To establish the differentiability of functions with type-2 fuzzy number values, it is necessary to define a metric space. Huang and Yang [51] have introduced a distance measure for T2FSs, which we adopt in this study.

$$d_2(\tilde{A}, \tilde{B}) = \int_a^b H_f\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right) dx \tag{9}$$

where

$$\begin{aligned} H_f\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right) &= \frac{\int_0^1 \tilde{\alpha}\, d_H\left(S_{\tilde{A}}(x \mid \tilde{\alpha}), S_{\tilde{B}}(x \mid \tilde{\alpha})\right) d\tilde{\alpha}}{\int_0^1 \tilde{\alpha}\, d\tilde{\alpha}} \\ &= 2\int_0^1 \tilde{\alpha}\, d_H\left(S_{\tilde{A}}(x \mid \tilde{\alpha}), S_{\tilde{B}}(x \mid \tilde{\alpha})\right) d\tilde{\alpha}. \end{aligned} \tag{10}$$

In fuzzy set theory, a second type of fuzzy set extends the concept of a first type of fuzzy set by considering uncertainty not only in the membership values but also in the membership grades themselves. To measure the distance between type-2 fuzzy sets, a distance measure in the context of metric spaces can be employed. Here are some key points about the distance measure for type-2 fuzzy sets: Type-2 Fuzzy Sets: A type-2 fuzzy set is defined by a set of type-1 fuzzy sets, where each type-1 fuzzy set represents a membership function over a universe of discourse. The uncertainty in a type-2 fuzzy set arises from the range of possible membership functions that could represent the set. And a metric space is a mathematical construct that defines a distance function, also known as a metric, between elements in a set. The distance function satisfies certain properties such as non-negativity, symmetry, and the triangle inequality. Also the distance measure for type-2 fuzzy sets extends the concept of a metric to handle the uncertainty associated with type-2 fuzzy sets. It quantifies the similarity or dissimilarity between type-2 fuzzy sets based on their membership grades and the uncertainty associated with them.

Theorem 7. *The measurement of distance, denoted as* d_2*, establishes a metric within the realm of type-2 fuzzy sets.*

Proof. Confirmation of the triangle inequality is detailed subsequently, while the other attributes are apparent.

$$\begin{aligned}
& d_2(\tilde{A},\tilde{B}) + d_2(\tilde{B},\tilde{C}) \\
= & \int_a^b H_f\left(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\right) dx + \int_a^b H_f\left(\mu_{\tilde{B}}(x),\mu_{\tilde{C}}(x)\right) dx \\
= & \int_a^b \big(H_f\left(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\right) + H_f\left(\mu_{\tilde{B}}(x),\mu_{\tilde{C}}(x)\right)\big) dx \\
= & 2\int_a^b \int_0^1 \tilde{\alpha}\left\{d_H\left(S_{\tilde{A}}(x\mid\tilde{\alpha}),S_{\tilde{B}}(x\mid\tilde{\alpha})\right) + d_H\left(S_{\tilde{B}}(x\mid\tilde{\alpha}),S_{\tilde{C}}(x\mid\tilde{\alpha})\right)\right\} d\tilde{\alpha}\, dx \\
\geqslant & 2\int_a^b \int_0^1 \tilde{\alpha}\, d_H\left(S_{\tilde{A}}(x\mid\tilde{\alpha}),S_{\tilde{C}}(x\mid\tilde{\alpha})\right) d\tilde{\alpha}\, dx = d_2(\tilde{A},\tilde{C}).
\end{aligned}$$

The ultimate inequality is formulated following the triangular inequality of Hausdorff distance

2.1. Type-2 fuzzy numbers

Type-2 fuzzy numbers (T2FNs) are an extension of Type-1 fuzzy numbers, which incorporate additional uncertainty in their membership function. T2FNs represent the membership degrees of a crisp value being within a fuzzy set, considering both primary and secondary uncertainties. Unlike Type-1 fuzzy numbers that have a single membership function, T2FNs have a primary membership function and a secondary membership function. The primary membership function represents the uncertainty in the degree of membership, while the secondary membership function represents the uncertainty in the support of the T2FN.

Definition 8. *A T2FS, $\tilde{A}$, is referred to as a perfect T2FN when the following requirements are fulfilled: 1. The UMF and LMF of $FOU(\tilde{A})$ are themselves T1FNs,*

2. *The upper membership function (UMF) and lower membership function (LMF) of the PS of $\tilde{A}$ themselves represent T1FNs.*

A comprehensive depiction of a T2FN can be realized by utilizing its FOU and PS, under the condition that each vertical segment comprises T1FNs and the segmented functions possess identical features (for instance, linear). The representation of a triangular perfect QT2FN in a parametric closed form simplifies the discussion in this document, as we examine a collection of such QT2FNs where the LMF and UMF overlap within each PS. The mathematical formula that outlines the membership function of a set is the parametric equation of a triangular fuzzy set. A triangular fuzzy set is a specific category of fuzzy sets characterized by a membership function that is triangular in form.

Let $\tilde{w} \in E_2$ be a triangular ideal QT2FN with the core m. The parametric equation of the triangular ideal QT2FN can be expressed as follows:

$$\begin{aligned}
&\tilde{w}^{\alpha}_{\tilde{\alpha}} = (\underline{w}^{\alpha}_{\tilde{\alpha}}, \bar{w}^{\alpha}_{\tilde{\alpha}}) \\
&\bar{w}^{\alpha}_{\tilde{\alpha}} = \left[\bar{L}^{\alpha}_{w^{\alpha}_{\tilde{\alpha}}}, \bar{R}^{\alpha}_{w_{\tilde{\alpha}}}\right], \\
&\quad \bar{L}^{\alpha}_{w^{\alpha}_{\tilde{\alpha}}} = L^{\alpha}_{w_1} - (1-\tilde{\alpha})\left(L^{\alpha}_{w_1} - \bar{L}^{\alpha}_{w_0}\right), \\
&\quad \bar{R}^{\alpha}_{w_{\tilde{\alpha}}} = R^{\alpha}_{w_1} + (1-\tilde{\alpha})\left(\bar{R}^{\alpha}_{w_0} - R^{\alpha}_{w_1}\right) \\
&\underline{w}^{\alpha}_{\tilde{\alpha}} = \left[\underline{L}^{\alpha}_{w_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{w_{\tilde{\alpha}}}\right], \\
&\quad \underline{L}^{\alpha}_{w_{\tilde{\alpha}}} = L^{\alpha}_{w_1} - (1-\tilde{\alpha})\left(L^{\alpha}_{w_1} - \underline{L}^{\alpha}_{w_0}\right), \\
&\quad \underline{R}^{\alpha}_{w_{\tilde{\alpha}}} = R^{\alpha}_{w_1} + (1-\tilde{\alpha})\left(\underline{R}^{\alpha}_{w_0} - R^{\alpha}_{w_1}\right)
\end{aligned} \tag{11}$$

where, $\bar{L}^{\alpha}_{w_0} \leqslant L^{\alpha}_{w_1} \leqslant \underline{L}^{\alpha}_{w_0} \leqslant \underline{R}^{\alpha}_{w_0} \leqslant R^{\alpha}_{w_1} \leqslant \bar{R}^{\alpha}_{w_0}$. The α - cuts of LMF and UMF of $FOU(\tilde{w})$ are $\underline{w}^{\alpha}_0 = \left[\underline{L}^{\alpha}_{w_0}, \underline{R}^{\alpha}_{w_0}\right], \bar{w}^{\alpha}_0 = \left[\bar{L}^{\alpha}_{w_0}, \bar{R}^{\alpha}_{w_0}\right]$ respectively where $\underline{L}^{\alpha}_{w_0} = m - (1-\alpha)\left(m - \underline{L}_{w_0}\right), \underline{R}^{\alpha}_{w_0} = m + (1-\alpha)\left(\underline{R}_{w_0} - m\right)$ are the left and right endpoints of $\underline{w}^{\alpha}_0$ with the support $\left[\underline{L}_{w_0}, \underline{R}_{w_0}\right], \bar{L}^{\alpha}_{w_0} = m - (1-\alpha)\left(m - \bar{L}_{w_0}\right), \bar{R}^{\alpha}_{w_0} = m + (1-\alpha)\left(\bar{R}_{w_0} - m\right)$ are the left and right endpoints of the $\bar{w}^{\alpha}_0$ with the support $\left[\bar{L}_{w_0}, \bar{R}_{w_0}\right]$ respectively. The $\alpha-$ cut of PS is $w^{\alpha}_1 = \left[L^{\alpha}_{w_1}, R^{\alpha}_{w_1}\right]$ where $L^{\alpha}_{w_1} = m - (1-\alpha)(m - L_{w_1}), R^{\alpha}_{w_1} = m + (1-\alpha)(R_{w_1} - m)$ are the left and right endpoints of w^{α}_1 respectively with the support $[L_{w_1}, R_{w_1}]$.

Consequently, the triangular perfect QT2FN, $\tilde{w}$, can be constructed using the septuple $\tilde{w} = \left(\bar{L}_{w_0}, L_{w_1}, \underline{L}_{w_0}, m, \underline{R}_{w_0}, R_{w_1}, \bar{R}_{w_0}\right)$ where $\bar{L}_{w_0} \leqslant L_{w_1} \leqslant \underline{L}_{w_0} \leqslant m \leqslant \underline{R}_{w_0} \leqslant R_{w_1} \leqslant \bar{R}_{w_0}$ [52].

$$[\tilde{w} \circ \tilde{z}]^{\alpha}_{\tilde{\alpha}} = \left(\left[\underline{w}^{\alpha}_{\tilde{\alpha}} \circ \underline{z}^{\alpha}_{\tilde{\alpha}}\right], [\bar{w}^{\alpha}_{\tilde{\alpha}} \circ \bar{z}^{\alpha}_{\tilde{\alpha}}]\right)$$

where $\tilde{w} \circ \tilde{z}$ means $\tilde{w} + \tilde{z}$ or $\tilde{w} - \tilde{z}$ or $\tilde{w} \times \tilde{z}$ or $\tilde{w} \div \tilde{z}$.

2.2. Differentials of type-2 fuzzy functions

In this section, we demonstrate the differentiability of functions possessing type-2 fuzzy number values, a concept that bears resemblance to the idea of robustly generalized differentiability[53].

Theorem 9. *Given $\tilde{x}, \tilde{y} \in E_2$, the $\tilde{\alpha}$-plane pertaining to the H_2-difference between $\tilde{x}$ and $\tilde{y}$ corresponds to the H-difference between the LMF and UMF of $\tilde{x}$ and $\tilde{y}$.*

Proof. Suppose that the H_2-difference between $\tilde{x}$ and $\tilde{y}$ is $\tilde{z}$. As a result, $\tilde{x} = \tilde{y} + \tilde{z}$. Using the $\tilde{\alpha}-$ plane extension principle as suggested by [52], we derive $\tilde{x}\tilde{\alpha} = \tilde{y}\tilde{\alpha} + \tilde{z}_{\tilde{\alpha}}$. It implies $(\underline{x}_{\tilde{\alpha}}, \bar{x}_{\tilde{\alpha}}) = \left(\underline{y}_{\tilde{\alpha}}, \bar{y}_{\tilde{\alpha}}\right) + \left(\underline{z}_{\tilde{\alpha}}, \bar{z}_{\tilde{\alpha}}\right) = \left(\underline{y}_{\tilde{\alpha}} + \underline{z}_{\tilde{\alpha}}, \bar{y}_{\tilde{\alpha}} + \bar{z}_{\tilde{\alpha}}\right)$, which further leads to $\underline{x}_{\tilde{\alpha}} = \underline{y}_{\tilde{\alpha}} + \underline{z}_{\tilde{\alpha}}$ and $\bar{x}_{\tilde{\alpha}} = \bar{y}_{\tilde{\alpha}} + \bar{z}_{\tilde{\alpha}}$. Then, it can be inferred that $\underline{z}_{\tilde{\alpha}} = \underline{x}_{\tilde{\alpha}} \underline{H} y_{\tilde{\alpha}}$ and $\bar{z}_{\tilde{\alpha}} = \bar{x}_{\tilde{\alpha}} \underline{H} \bar{y}_{\tilde{\alpha}}$

Definition 10. *Suppose $T = (a,b) \subseteq \mathbb{R}$. Hence, $\tilde{f}$ is characterized as a type-2 fuzzy number-valued function on T if $\tilde{f} : T \to E_2$. It is represented as an n-dimensional vector of type-2 fuzzy number-valued functions on T if $\tilde{f} : T \to \underbrace{E_2 \times E_2 \times \ldots \times E_2}\, n = E2^n$.*

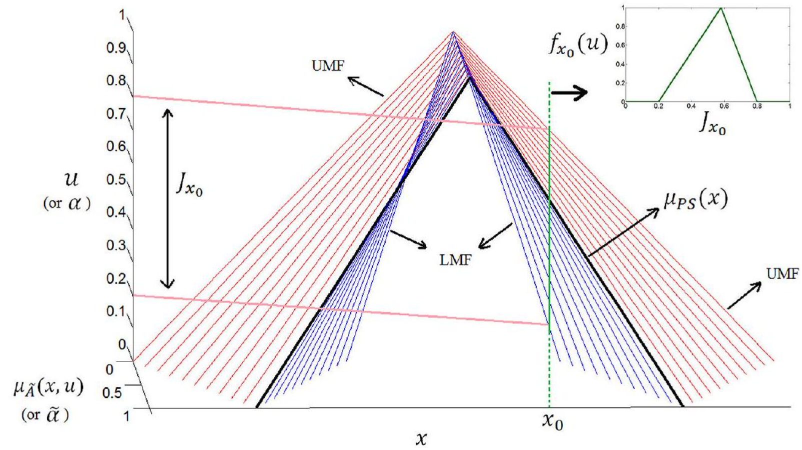


Figure 1: Type-2 fuzzy functions with α-cut illustration

In the given context, the boundaries are acknowledged within the metric space (E_2, d_2). We term $\tilde{f}(t)$ as "the H_2-differentials outlined in its primary form" if it is differentiable as per the Definition, and "the H_2-differentials outlined in the secondary form" if it is indeed differentiable. This is similar to the theorem displayed in the citation [54]. If $\tilde{f}(t)$ is an H_2-differential defined according to the Definition, then $\tilde{f}'(t_0) \in \mathbb{R}$.

Fuzzy H2 differentiability is a concept that extends the traditional notion of differentiability to fuzzy functions. In the context of fuzzy sets and fuzzy logic, differentiability is often ill-defined or not applicable due to the inherent uncertainty and imprecision associated with fuzzy sets. Fuzzy H2 differentiability attempts to address this limitation by providing a framework for characterizing the differentiability of fuzzy functions.

In the research introduced here, type-2 fuzzy sets are identified as vital tools for addressing the numerous complex uncertainties and imprecision inherent in real-world inventory systems, particularly those with imperfect items. Unlike type-1 fuzzy sets that utilize clear-cut membership functions, type-2 fuzzy sets incorporate an additional dimension with fuzzy membership functions. This supplementary element is essential for encompassing the intrinsic ambiguities when assessing item imperfections. For example, in production systems, the evaluation of an item's quality isn't always clear-cut, as there are shades of gray which can be accurately represented by type-2 fuzzy sets. Moreover, in volatile environments where supply chain disruptions, demand volatility, and production quality variation are common, type-2 fuzzy sets provide a more adaptable and resilient modelling approach. By leveraging type-2 fuzzy sets, the decision-making process in the model is significantly improved, resulting in more dependable, sturdy, and insightful outcomes. This is crucial for optimizing inventory management, lowering costs, and enhancing customer satisfaction by maintaining a balance between high-quality and imperfect items. In conclusion, type-2 fuzzy sets are essential for capturing real-world intricacies and promoting informed decision-making in inventory systems.

3. MATHEMATICAL MODEL

This part showcases multiple examples and situations involving type-2 fuzzy differential equations, coupled with references to their real-world applications. It's worth noting that, for the sake of clarity, our attention is primarily directed towards the triangular perfect QT2FNs in the given examples.

Table 1: Notations for Parameters

Symbol	Name of Variable
$\theta(t)$	inventory status at time t (unit)
P	consistent manufacturing pace (unit/time)
R	steady reprocessing pace (unit/time)
A_p	expense for setup for each cycle in the manufacturing procedure ($)
A_r	expense for setup for every cycle in the reprocessing procedure ($)
$\widetilde{A}$	uncertain inventory expense (($)/item year)
c_p	cost per unit per manufacturing cycle ($)
c_d	cost per unit for degradation per manufacturing duration ($)
c_r	cost per unit of reprocessing per round ($)
c_s	cost of waste per unit per cycle ($)
h_p	cost per unit for inventory per manufacturing cycle ($)
$\widetilde{Q}_{\text{P}}$	Uncertain production volume (unit)
η	fraction of inferior quality items in the manufacturing procedure
ξ	proportion of faulty items in the production procedure
h_r	carrying cost of inventory per unit per round in the reprocessing procedure ($)
φ	depreciation rate of fixed stock
ξ_0	proportion of faulty items manufactured in the previous production procedure
ξ_r	percentage of defective goods in the reprocessing procedure
ψ	reduction rate in defect rate post quality enhancement investment
ω	cost of missed opportunities subsequent to quality improvement investment

In a manufacturing process, the raw materials inventory level will decrease as materials are used in production. As production progresses, the work in progress inventory level will increase as partially finished products accumulate. Finally, as finished goods are produced, the finished goods inventory level will increase until the products are sold and shipped out of the facility, at which point the inventory level will decrease. If $\tilde{\theta}_1(t)$

Table 2: Expressions and Functions of Variables

Symbol	Name of Variable
$\theta(t)$	available inventory level (unit)
T	manufacturing duration (unit time)
S	selling price of items (\$)
TP	retailer's total profit (\$)

reflects the inventory quantity at that point from $t = 0$ to $t = t_1$

$$\frac{d\tilde{\theta}_1(t)}{dt} = (1-\xi)P - D - \varphi\tilde{\theta}_1(t), \quad \tilde{\theta}(t) \in E_1, \quad \tilde{\theta}(t_1) = 0 \tag{12}$$

$$\frac{d\tilde{\theta}_2(t)}{dt} = (1-\eta-\xi)P - D - \varphi\tilde{\theta}_2(t), \quad \tilde{\theta}(t) \in E_2, \quad \tilde{\theta}_2(t_1) = \tilde{\theta}_1(t_1) \tag{13}$$

$$\frac{d\tilde{\theta}_3(t)}{dt} = -D - \varphi\tilde{\theta}_3(t), \quad \tilde{\theta}(t) \in E_3, \quad \tilde{\theta}(t_3) = 0 \tag{14}$$

Equation for low quality items

$$\frac{d\tilde{\theta}_4(t)}{dt} = \eta P - D_1 - \varphi\tilde{\theta}_4(t), \quad \tilde{\theta}(0) \in E_2 \tag{15}$$

$$\frac{d\tilde{\theta}_5(t)}{dt} = -D_1 - \varphi\tilde{\theta}_5(t), \quad \tilde{\theta}(0) \in E_3 \tag{16}$$

Equation for imperfect items

$$\frac{d\tilde{\theta}_6(t)}{dt} = \xi P - D_2 - \varphi\tilde{\theta}_6(t), \quad \tilde{\theta}(0) \in E_2 \tag{17}$$

$$\frac{d\tilde{\theta}_7(t)}{dt} = -D_2 - \varphi\tilde{\theta}_7(t), \quad \tilde{\theta}(0) \in E_2 \tag{18}$$

An inventory with high quality and low quality items refers to a stock of goods that includes both products of superior and inferior quality. This type of inventory is common in many industries, including manufacturing, retail, and distribution. The high quality items are products that meet or exceed customer expectations and provide superior value. The items are typically more expensive than low quality items and are often associated with a premium brand or reputation. High quality items tend to have longer lifespans, better performance, and are less likely to fail or need repairs. On the other hand, low quality items are products that do not meet customer expectations and provide lower value. These items are often associated with lower cost brands and are typically cheaper than high quality items. Low quality items tend to have shorter lifespans, lower performance, and are more likely to fail or need repairs.

Managing the inventory with high quality and low quality items can be challenging. Companies need to balance the cost of inventory management with the value of the products they are stocking. They need to ensure that they have enough high quality items to meet demand while minimizing the amount of low quality items in stock to avoid negative customer experiences. The value of the proportion of low quality items η in the

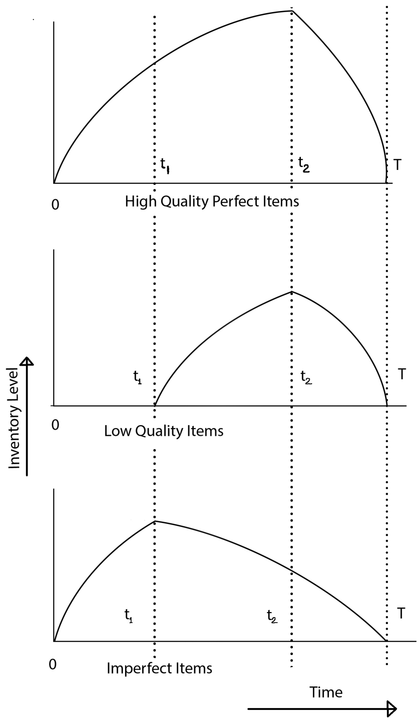


Figure 2: Inventory behavior during a production cycle

production process can vary depending on various factors such as the type of product being manufactured, the complexity of the manufacturing process, and the quality control measures in place. In general, it is desirable to have a low proportion of low quality items in the production process, as this can result in lower costs, higher customer satisfaction, and a better overall reputation for the company.

Based on the H_2-differentiability in the second form, the T2FDE can be considered analogous to the following system of type-1 fuzzy differential equations:

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_1}(t) = P - D - \varphi\overline{\theta}_{s\tilde{\alpha}_1} \\ \overline{\theta}_{\tilde{\alpha}_1}(0) \in E_1 \\ \dot{\underline{\theta}}_{\tilde{\alpha}_1}(t) = P - D - \varphi\underline{\theta}_{s\tilde{\alpha}_1} \\ \underline{\theta}_{\tilde{\alpha}_1}(0) \in E_1 \end{cases} \tag{19}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_2}(t) = P - P\eta - \xi P - D - \varphi\overline{\theta}_{s\tilde{\alpha}_2}, \\ \overline{\theta}_{\tilde{\alpha}_2}(0) \in E_2 \\ \dot{\underline{\theta}}_{\tilde{\alpha}_2}(t) = P - P\eta - \xi P - D - \varphi\underline{\theta}_{s\tilde{\alpha}_2}, \\ \underline{\theta}_{\tilde{\alpha}_2}(0) \in E_2 \end{cases} \tag{20}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_3}(t) = -D - \varphi \overline{\theta}_{s\tilde{\alpha}_3}, \\ \bar{\theta}_{\tilde{\alpha}_3}(0) \in E_3 \\ \underline{\dot{\theta}}_{\tilde{\alpha}_3}(t) = -D - \varphi \underline{\theta}_{s\tilde{\alpha}_3}, \\ \underline{\theta}_{\tilde{\alpha}_3}(0) \in E_3 \end{cases} \tag{21}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_4}(t) = \eta P - D_1 - \varphi \overline{\theta}_{s\tilde{\alpha}_4}, \\ \bar{\theta}_{\tilde{\alpha}_4}(0) \in E_2 \\ \underline{\dot{\theta}}_{\tilde{\alpha}_4}(t) = \eta P - D_1 - \varphi \underline{\theta}_{s\tilde{\alpha}_4}, \\ \underline{\theta}_{\tilde{\alpha}_4}(0) \in E_2 \end{cases} \tag{22}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_5}(t) = -D_1 - \varphi \overline{\theta}_{s\tilde{\alpha}_5}, \\ \bar{\theta}_{\tilde{\alpha}_5}(0) \in E_3 \\ \underline{\dot{\theta}}_{\tilde{\alpha}_5}(t) = \eta P - D_1 - \varphi \underline{\theta}_{s\tilde{\alpha}_5}, \\ \underline{\theta}_{\tilde{\alpha}_5}(0) \in E_3 \end{cases} \tag{23}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_6}(t) = \xi P - D_2 - \varphi \overline{\theta}_{s\tilde{\alpha}_6}, \\ \bar{\theta}_{\tilde{\alpha}_6}(0) \in E_2 \\ \underline{\dot{\theta}}_{\tilde{\alpha}_6}(t) = \eta P - D_2 - \varphi \underline{\theta}_{s\tilde{\alpha}_6}, \\ \underline{\theta}_{\tilde{\alpha}_6}(0) \in E_2 \end{cases} \tag{24}$$

$$\begin{cases} \dot{\overline{\theta}}_{\tilde{\alpha}_7}(t) = -D_2 - \varphi \overline{\theta}_{s\tilde{\alpha}_7}, \\ \bar{\theta}_{\tilde{\alpha}_7}(0) \in E_3 \\ \underline{\dot{\theta}}_{\tilde{\alpha}_7}(t) = \eta P - D_2 - \varphi \underline{\theta}_{s\tilde{\alpha}_7}, \\ \underline{\theta}_{\tilde{\alpha}_7}(0) \in E_3 \end{cases} \tag{25}$$

where $\dot{\overline{\theta}}_{\tilde{\alpha}}(t) = \overline{\left[\frac{d\tilde{\theta}(t)}{dt}\right]}, \underline{\dot{\theta}}^{\tilde{\alpha}}_{\tilde{\alpha}}(t) = \underline{\left[\frac{d\tilde{\theta}(t)}{dt}\right]_{\tilde{\alpha}}}$. Using the $\alpha - cut$, as discussed above in the literature we have two Ordinary Differential Equations [ODEs] systems:

$$\begin{cases} \dot{\bar{L}}^{\alpha}_{\theta_{\tilde{\alpha}}}(t) = -k\bar{L}^{\alpha}_{\theta_{\tilde{\alpha}}}(t) + k\bar{R}^{\alpha}_{\theta^{\alpha}_{s\tilde{\alpha}}} \\ \bar{L}^{\alpha}_{\theta^{\alpha}_{\tilde{\alpha}}}(0) = L^{\alpha}_{\theta_1}(0) - (1-\tilde{\alpha})\left(L^{\alpha}_{\theta_1}(0) - \bar{L}^{\alpha}_{\theta_0}(0)\right) \\ \dot{\bar{R}}^{\alpha}_{\theta_{\tilde{\alpha}}}(t) = -k\bar{R}^{\alpha}_{\theta_{\tilde{\alpha}}}(t) + k\bar{L}^{\alpha}_{\theta^{\alpha}_{s\tilde{\alpha}}} \\ \bar{R}^{\alpha}_{\theta_{\tilde{\alpha}}}(0) = R^{\alpha}_{\theta_1}(0) + (1-\tilde{\alpha})\left(\bar{R}^{\alpha}_{\theta_0}(0) - R^{\alpha}_{\theta_1}(0)\right) \end{cases}$$

Solving equation

$$\theta_1(t) = -\frac{1}{\eta\varphi}(D + (\xi - 1)P)e^{\eta(-t)\varphi}\left(e^{\eta t\varphi} - 1\right) \tag{26}$$

$$\theta_2(t) = \frac{1}{\varphi}e^{t_1(-\varphi)}\left(-\left(e^{t\varphi} - e^{t_1\varphi}\right)(D + P(\eta + \xi - 1)) + \theta_1(t_1)\varphi e^{t\varphi}\right) \tag{27}$$

$$\theta_3(t) = \frac{1}{\varphi}De^{-t\varphi}\left(e^{t_2\varphi} - e^{t\varphi}\right) \tag{28}$$

$$\theta_4(t) = \frac{1}{\varphi}e^{-t\varphi}(\eta P - D_1)\left(e^{t\varphi} - e^{t_1\varphi}\right) \tag{29}$$

$$\tilde{\theta}_5(t) = \frac{1}{\varphi}e^{-t\varphi}\left(D_1\left(e^{t_2\varphi} - e^{t\varphi}\right) + \eta P\left(e^{t\varphi} - e^{t_2\varphi}\right) + \theta_4(t_1)\varphi e^{t_2\varphi}\right) \tag{30}$$

$$\theta_6(t) = \frac{1}{\varphi}e^{-t\varphi}(\xi P - D_2)\left(e^{t\varphi} - e^{t_2\varphi}\right) \tag{31}$$

$$\theta_7(t) = \frac{1}{\varphi}D_2\left(e^{\varphi(t_2-t)} - 1\right) \tag{32}$$

where $\bar{\theta}_{\tilde{\alpha}}^{\alpha}(t) = \left[\bar{L}_{\theta_{\tilde{\alpha}}}^{\alpha}(t), \bar{R}_{\theta_{\tilde{\alpha}}}^{\alpha}(t)\right], \underline{\theta}_{\tilde{\alpha}}^{\alpha}(t) = \left[\underline{L}_{\theta_{\tilde{\alpha}}}^{\alpha}(t), \underline{R}_{\theta_{\tilde{\alpha}}}^{\alpha}(t)\right], \bar{\theta}_{s\tilde{\alpha}}^{\alpha} = \left[\bar{L}_{\theta_{s\bar{x}}}^{\alpha}, \bar{R}_{\theta_{s\bar{z}}}^{\alpha}\right]$ and $\underline{\theta}_{s\tilde{\alpha}}^{\alpha} = \left[\underline{L}_{\theta_{s\bar{z}}}^{\alpha}, \underline{R}_{\theta_{s\bar{s}}}^{\alpha}\right]$.

$$\begin{cases} \bar{\theta}_{\tilde{\alpha}}^{\alpha}(t) = \left[\left(L_{\theta_1}^{\alpha}(0) - (1-\tilde{\alpha})\left(L_{\theta_1}^{\alpha}(0) - \bar{L}_{\theta_0}^{\alpha}(0)\right)\right)e^{-kt} + \bar{R}_{\theta_{sj\alpha}}^{\alpha}\left(1 - e^{-kt}\right),\right. \\ \left.\left(R_{\theta_1}^{\alpha}(0) + (1-\tilde{\alpha})\left(\bar{R}_{\theta_0}^{\alpha}(0) - R_{\theta_1}^{\alpha}(0)\right)\right)e^{-kt} + \bar{L}_{\theta_{si}}^{\alpha}\left(1 - e^{-kt}\right)\right] \\ \underline{\theta}_{\tilde{\alpha}}^{\alpha}(t) = \left[\left(L_{\theta_1}^{\alpha}(0) - (1-\tilde{\alpha})\left(L_{\theta_1}^{\alpha}(0) - \underline{L}_{\theta_0}^{\alpha}(0)\right)\right)e^{-kt} + \underline{R}_{\theta_{si\alpha}}^{\alpha}\left(1 - e^{-kt}\right),\right. \\ \left.\left(R_{\theta_1}^{\alpha}(0) + (1-\tilde{\alpha})\left(\underline{R}_{\theta_0}^{\alpha}(0) - R_{\theta_1}^{\alpha}(0)\right)\right)e^{-kt} + \underline{L}_{\theta_{si}}^{\alpha}\left(1 - e^{-kt}\right)\right] \\ \alpha, \tilde{\alpha} \in [0,1] \end{cases}$$

Based on the T2FS α-cut principle of mathematical extension, the explanation for $0 \leqslant t \leqslant 10, k = 0.05$ is

$$\tilde{\theta}(t) = \bigcup_{\forall\tilde{\alpha}\in[0,1]} \tilde{\alpha} \bigcup_{\forall\alpha\in[0,1]} \alpha\left(\underline{\theta}_{\tilde{\alpha}}^{\alpha}(t), \bar{\theta}_{\tilde{\alpha}}^{\alpha}(t)\right)$$

that can be de-fuzzified using a technique like the centroid approach. The FOU($\tilde{\theta}(t)$) for $0 \leqslant t \leqslant 10$ and $k = 0.05$ is shown by its LMF and UMF. The assumption is made that the membership functions obtained from the professor's experience are more reliable and trustworthy compared to those obtained from the students, and hence are considered as the primary source of information at $\tilde{\alpha} = 1$, while the membership functions obtained from the students are considered at lower levels of $\tilde{\alpha}$, i.e., $0 \leqslant \tilde{\alpha} < 1$.

The more reliable an experience is in comparison to others, the larger the $\tilde{\alpha}$-plane that includes their membership functions. In accordance with the guidance of a knowledgeable source and others with a less competent level than that of the influence, the starting population can be depicted. According to H_2-differentiability at the starting point, the T2FDE can be of similar significance to T1FDEs system, $\widetilde{A} = (a_1, a_2, a_3, a_4)$, $\widetilde{P} = (p_1, p_2, p_3, p_4)$ and $\widetilde{T} = (A_{p_1}, A_{p_2}, A_{p_3}, A_{p_4})$ be trapezoidal fuzzy numbers. Also let Fuzzy demand rate $\widetilde{R} = (r_1, r_2, r_3, r_4)$.

The fuzzy combined production inventory amount $\widetilde{\theta}_1$ is first obtained as

$$\begin{aligned} \theta_1 = \; & \frac{A_{p_1}d_1}{Q_P} + \frac{a_1Q_P}{2}\left(1 - \frac{r_4}{p_1}\right), \frac{A_{p_2}d_2}{Q_P} + \frac{a_2Q_P}{2}\left(1 - \frac{r_3}{p_2}\right), \\ & \frac{A_{p_3}d_3}{Q_P} + \frac{a_3Q_P}{2}\left(1 - \frac{r_2}{p_3}\right), \frac{A_{p_4}d_4}{Q_P} + \frac{a_4Q_P}{2}\left(1 - \frac{r_1}{p_4}\right)\Bigg]. \end{aligned} \tag{33}$$

The outcome is as follows:

$$P(\theta_1) = \frac{1}{6}\left[\frac{A_{\mathrm{p}_1} d_1}{Q_{\mathrm{P}}} + \frac{a_1 Q_{\mathrm{P}}}{2}\left(1 - \frac{r_4}{p_1}\right) + \frac{2A_{\mathrm{p}_2} d_2}{Q_{\mathrm{P}}} + \frac{2a_2 Q_{\mathrm{P}}}{2}\left(1 - \frac{r_3}{p_2}\right) + \frac{2A_{\mathrm{p}_3} d_3}{Q_{\mathrm{P}}} + \frac{2a_3 Q_{\mathrm{P}}}{2}\left(1 - \frac{r_2}{p_3}\right) + \frac{A_{\mathrm{p}_4} d_4}{Q_{\mathrm{P}}} + \frac{a_4 Q_{\mathrm{P}}}{2}\left(1 - \frac{r_1}{p_4}\right)\right]. \tag{34}$$

$$\frac{\partial P(\theta_1)}{\partial Q_{\mathrm{P}}} = \frac{1}{6}\left[-\frac{1}{Q_{\mathrm{P}}^2}\left(A_{\mathrm{p}_1} d_1 + 2A_{\mathrm{p}_2} d_2 + 2A_{\mathrm{p}_3} d_3 + A_{\mathrm{p}_4} d_4\right)\right] + \frac{1}{2}\left[a_1\left(1 - \frac{r_4}{p_1}\right) + 2a_2\left(1 - \frac{r_3}{p_2}\right) + 2a_3\left(1 - \frac{r_2}{p_3}\right) + a_4\left(1 - \frac{r_1}{p_4}\right)\right]. \tag{35}$$

Let $(\partial P(\theta_1))/\partial Q_{\mathrm{P}} = 0$, then to determine the ideal manufacturing capacity. Q_{P}^*

$$Q_{\mathrm{P}}^* = \sqrt{\frac{2\left(A_{\mathrm{p}_1} d_1 + 2A_{\mathrm{p}_2} d_2 + 2A_{\mathrm{p}_3} d_3 + A_{\mathrm{p}_4} d_4\right)}{a_1(1-(\eta_4/p_1)) + 2a_2(1-(\eta_3/p_2)) + 2a_3(1-(\eta_2/p_3)) + a_4(1-(\eta_1/p_4))}} \tag{36}$$

$$\theta_2 = \left[\frac{A_{\mathrm{p}_1} d_1}{q_{\mathrm{P}_4}} + \frac{a_1 q_{\mathrm{P}_1}}{2}\left(1 - \frac{\eta_4}{p_1}\right), \frac{A_{\mathrm{p}_2} d_2}{q_{\mathrm{P}_3}} + \frac{a_2 q_{\mathrm{P}_2}}{2}\left(1 - \frac{\eta_3}{p_2}\right), \frac{A_{\mathrm{p}_3} d_3}{q_{\mathrm{P}_2}} + \frac{a_3 q_{\mathrm{P}_3}}{2}\left(1 - \frac{\eta_2}{p_3}\right), \frac{A_{\mathrm{p}_4} d_4}{q_{\mathrm{P}_1}} + \frac{a_4 q_{\mathrm{P}_4}}{2}\left(1 - \frac{\eta_1}{p_4}\right)\right]. \tag{37}$$

Additionally, we can derive the representation of $\widetilde{\theta}_2$ as:

$$P(\theta_2) = \frac{1}{6}\left[\frac{A_{\mathrm{p}_1} d_1}{q_{\mathrm{P}_4}} + \frac{a_1 q_{\mathrm{P}_1}}{2}\left(1 - \frac{r_4}{p_1}\right), \frac{2A_{\mathrm{p}_2} d_2}{q_{\mathrm{P}_3}} + \frac{2a_2 q_{\mathrm{P}_2}}{2}\left(1 - \frac{r_3}{p_2}\right), \frac{2A_{\mathrm{p}_3} d_3}{q_{\mathrm{P}_2}} + \frac{2a_3 q_{\mathrm{P}_3}}{2}\left(1 - \frac{r_2}{p_3}\right), \frac{A_{\mathrm{p}_4} d_4}{q_{\mathrm{P}_1}} + \frac{a_4 q_{\mathrm{P}_4}}{2}\left(1 - \frac{r_1}{p_4}\right)\right] \tag{38}$$

with $0 < q_{\mathrm{P}_1} \leqslant q_{\mathrm{P}_2} \leqslant q_{\mathrm{P}_3} \leqslant q_{\mathrm{P}_4}$.
If we replace the inequality criteria, the formula's meaning will not change. And

$$q_{\mathrm{P}_2} - q_{\mathrm{P}_1} \geqslant 0, \quad q_{\mathrm{P}_3} - q_{\mathrm{P}_2} \geqslant 0, \quad q_{\mathrm{P}_4} - q_{\mathrm{P}_3} \geqslant 0 \quad \text{and } q_{\mathrm{P}_1} > 0.$$

The next Phases we apply the extension of the Lagrangian approach. The Lagrangian approach is a mathematical technique used in optimization problems, particularly in the field of calculus of variations and mathematical physics. It is named after the Italian mathematician Joseph-Louis Lagrange, who introduced the method in the late 18th century.

Step 1: We are to solve the unconstrained problem,

$$\text{Minimize } P(\theta_2) = \frac{1}{6}\left[\frac{A_{\mathrm{p}_1} d_1}{q_{\mathrm{P}_4}} + \frac{a_1 q_{\mathrm{P}_1}}{2}\left(1 - \frac{r_4}{p_1}\right), \frac{2A_{\mathrm{p}_2} d_2}{q_{\mathrm{P}_3}} + \frac{2a_2 q_{\mathrm{P}_2}}{2}\left(1 - \frac{r_3}{p_2}\right) + \frac{2A_{\mathrm{p}_3} d_3}{q_{\mathrm{P}_2}} + \frac{2a_3 q_{\mathrm{P}_3}}{2}\left(1 - \frac{r_2}{p_3}\right) + \frac{A_{\mathrm{p}_4} d_4}{q_{\mathrm{P}_1}} + \frac{a_4 q_{\mathrm{P}_4}}{2}\left(1 - \frac{r_1}{p_4}\right)\right]. \tag{39}$$

$$\frac{\partial P}{\partial q_{\mathrm{P}_1}} = \frac{a_1}{2}\left(1 - \frac{r_4}{p_1}\right) - \frac{A_{\mathrm{p}_4} d_4}{q_{\mathrm{P}_1}^2}, \quad \frac{\partial P}{\partial q_{\mathrm{P}_2}} = \frac{2a_2}{2}\left(1 - \frac{r_3}{p_2}\right) - \frac{2A_{\mathrm{p}_3} d_3}{q_{\mathrm{P}_2}^2},$$
$$\frac{\partial P}{\partial q_{\mathrm{P}_3}} = \frac{2a_3}{2}\left(1 - \frac{r_2}{p_3}\right) - \frac{2A_{\mathrm{p}_2} d_2}{q_{\mathrm{P}_3}^2}$$
$$\frac{\partial P}{\partial q_{\mathrm{P}_4}} = \frac{a_4}{2}\left(1 - \frac{r_2}{p_4}\right) - \frac{A_{\mathrm{p}_1} d_1}{q_{\mathrm{P}_4}^2}. \tag{40}$$

After that, set zero to all partial derivatives of the obtained outcomes. Those are

$$\frac{\partial P}{\partial q_{\mathrm{P}_1}} = 0, \text{ then } q_{\mathrm{P}_1} = \sqrt{\frac{2A_{\mathrm{P}_4} d_4}{a_1(1-(r_4/p_1))}}, \quad \frac{\partial P}{\partial q_{\mathrm{P}_2}} = 0, \tag{41}$$

$$\text{then } q_{\mathrm{P}_2} = \sqrt{\frac{4A_{\mathrm{P}_3} d_3}{2a_2(1-(r_3/p_2))}}, \tag{42}$$

$$\frac{\partial P}{\partial q_{\mathrm{P}_3}} = 0, \text{ then } q_{\mathrm{P}_3} = \sqrt{\frac{4A_{\mathrm{P}_2} d_2}{2a_3(1-(r_2/p_3))}}, \quad \frac{\partial P}{\partial q_{\mathrm{P}_4}} = 0, \tag{43}$$

$$\text{then } q_{\mathrm{P}_4} = \sqrt{\frac{2A_{\mathrm{P}_1} d_1}{a(1-(r/p))}}. \tag{44}$$

$$L\left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda\right) = P\left(\widetilde{\theta}_2\right) - \lambda\left(q_{\mathrm{P}_2} - q_{\mathrm{P}_1}\right). \tag{45}$$

Step2: In the second step, we proceed to take the partial derivative of the following expression: $L\left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda\right)$ in connection with to find the minimization of $L\left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda\right)$.

Then we set all partial derivative value is set to zero. Following that, we attain

$$q_{\mathrm{P}_1} = q_{\mathrm{P}_2} = \sqrt{\frac{2\left(2A_{\mathrm{P}_3} d_3 + A_{\mathrm{P}_4} d_4\right)}{a_1\left(1-\frac{r_4}{p_1}\right) + 2a_2(1-(r_3/p_2))}},$$

$$q_{\mathrm{P}_3} = \sqrt{\frac{4A_{\mathrm{P}_2} d_2}{2a_3(1-(r_2/p_3))}}, \quad \text{and} \quad q_{\mathrm{P}_4} = \sqrt{\frac{2A_{\mathrm{P}_1} d_1}{a_4(1-(r_1/p_4))}}. \tag{46}$$

Step 3: The Lagrangian function is then

$$L\left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda_1, \lambda_2\right) = P(C_2) - \lambda_1\left(q_{\mathrm{P}_2} - q_{\mathrm{P}_1}\right) - \lambda_2\left(q_{\mathrm{P}_3} - q_{\mathrm{P}_2}\right)$$

$$\begin{aligned} q_{\mathrm{P}_1} &= q_{\mathrm{P}_2} = q_{\mathrm{P}_3} = \sqrt{\frac{2\left(2A_{\mathrm{P}_2} d_2 + 2A_{\mathrm{P}_3} d_3 + A_{\mathrm{P}_4} d_4\right)}{a_1(1-(r_4/p_1)) + 2a_2(1-(r_3/p_2)) + 2a_3(1-(r_2/p_3))}} \text{ and} \\ q_{\mathrm{P}_4} &= \sqrt{\frac{2A_{\mathrm{P}_1} d_1}{a_4(1-(r_1/p_4))}}. \end{aligned} \tag{47}$$

Step 4: The Lagrangian equation is expressed as follows:

$$\begin{aligned} L\left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda_1, \lambda_2, \lambda_3\right) =& P(\theta_2) - \lambda_1\left(q_{\mathrm{P}_2} - q_{\mathrm{P}_1}\right) - \lambda_2\left(q_{\mathrm{P}_3} - q_{\mathrm{P}_2}\right) \\ & - \lambda_3\left(q_{\mathrm{P}_4} - q_{\mathrm{P}_3}\right). \end{aligned}$$

To find the minimization, consider the Lagrangian:

$$L(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda_1, \lambda_2, \lambda_3).$$

We compute the partial derivatives of

$$L(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda_1, \lambda_2, \lambda_3)$$

with respect to $q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}, \lambda_1, \lambda_2, \lambda_3$, and set each partial derivative to zero to solve for $q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}$.

$$q_{\mathrm{P}_1} = q_{\mathrm{P}_2} = q_{\mathrm{P}_3} = q_{\mathrm{P}_4} \tag{48}$$

And this is equal to

$$q_{\mathrm{P}}^{*} = \sqrt{\frac{2\left(A_{\mathrm{p}_1} d_1 + 2A_{\mathrm{p}_2} d_2 + 2A_{\mathrm{p}_3} d_3 + A_{\mathrm{p}_4} d_4\right)}{a_1\left(1-(r_4/p_1)\right) + 2a_2\left(1-(r_3/p_2)\right) + 2a_3\left(1-(r_2/p_3)\right) + a_4\left(1-(r_1/p_4)\right)}} \quad (49)$$

Due to the previously described curative solution $\widetilde{Q}_{\mathrm{P}} = \left(q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4}\right)$ If each inequality requirement is satisfied, the procedure concludes with $\widetilde{Q}_{\mathrm{P}}$ as the locally ideal answer to the problem.

As there is no other local best answer to the preceding formula, the inventory model with fuzzy production quantity has the best possible answer using the Extensions of the Lagrangian approach. Now using the fuzzy technique the optimum solution will be: $\widetilde{Q}_{\mathrm{P}}^{*} = (q_{\mathrm{P}}^{*}, q_{\mathrm{P}}^{*}, q_{\mathrm{P}}^{*}, q_{\mathrm{P}}^{*})$.

Finally, to relate to a numerical example, we employ trapezoidal fuzzy numbers. Four parameters, a, b, c, and d, make up a trapezoidal fuzzy number, a particular kind of fuzzy number. The fuzzy number appears trapezoid-like form is determined by these characteristics. A membership function that gives a level of membership to each value in the universe of discourse defines the trapezoidal fuzzy number. The size of the trapezium and the separation of the value from its edges define this degree of membership.

4. NUMERICAL EXAMPLE

Within a clothing production site, substandard pieces can be identified through issues like inadequate seams, hue inconsistencies, or incorrect sizes. Pieces showing minor defects may exhibit loose threads, while flawless ones are deemed as premium. Likewise, in an electronics manufacturing site, suboptimal goods might have defective components or shoddy construction, leading them to break down prematurely. Goods with trivial cosmetic blemishes are labeled as blemished, whereas those operating without hitches are considered flawless and function without hitches. The primary goal of any manufacturing site is to boost the production of high-quality goods and reduce the yield of inferior and slightly defective pieces.

Hence, manufacturing enterprises should emphasize and focus on elements such as Efficiency, Quality Guarantee, Cost-Effective Strategies, Flexibility, and Consistency. Moreover, batch processing might assist in final product adjustments. These enterprises forecast their output and enhance their methods through various tactics. Considerations include managing capacity, optimizing manufacturing, ensuring quality, overseeing inventory, and advancing improvement strategies.

As an example, let's say the company's predicted need for a single cycle, utilizing fuzzy logic, approximates 11,000 units. The anticipated setup costs amount to 300, and the inventory expense is estimated at 0.5 for every unit. After initiating production, they can produce a maximum of 80 units daily. Additionally, there's a consistent daily requirement of around 80 units during the active manufacturing period.

To sum up, we convert the verbal data expressions "around the value of X" or "near the mark of X" into trapezoidal fuzzy numbers as detailed later. Both expressions "around the value of X" and "near the mark of X" can be denoted as $(1.00, 1.05, 1.1, 1.15)$ respectively.

Table 3: Compare between parameters

ξ_0	η_0	P (unit/time)	Total Cost ($)	θ^* (Units)	$\widetilde{A^*}$ ($/item year)
0.0012	0.0012	2400	12223.44	193	16,672
0.0018	0.0018	2400	12437.49	195	16,973
0.0025	0.0018	2500	12593.71	204	17,176
0.0021	0.0018	2500	12654.18	218	17,378
0.0022	0.0018	2600	12635.25	236	17,471
0.0027	0.0018	2700	12707.45	260	17,474
0.0028	0.0018	2700	12782.06	291	17,675
0.0029	0.0018	2800	12881.69	328	17,777
0.0030	0.0018	2900	12950.74	359	17,872
0.0032	0.0018	3000	13054.67	362	17,979
0.0033	0.0018	3000	13054.67	362	18,170

Finally, the fuzzy traits in this situation can be re-positioned according to the rule stated above. Fuzzy traits refer to properties or aspects of objects or concepts that are vague, hard to quantify, or measure.

It is assumed that the yearly demand which is fuzzy in nature = "greater or lower than 14,000"

$$\widetilde{D} = (d_1, d_2, d_3\ , d_4) = (9500, 10,000, 10,500, 11,000)$$

Also the setup cost in fuzzy = "greater or lower than $100" $= (A_{p_1}, A_{p_2}, A_{p_3}, A_{p_4})$=(100, 105, 105, 115).

And the cost of the inventory in fuzzy expression = "about $450" $= \widetilde{a} = (\mathrm{a}_1, a_2, a_3, a_4)$ = (400$, 455$, 500$, 550$).

The fuzzy production daily rate = "greater on less 80" units daily $= \widetilde{P} = (p_1, p_2, p_3, p_4) = (70, 77, 85, 89)$.

Demand flow rate in fuzzy = "about 60 """ $= \widetilde{R} = (r_1, r_2, r_3, r_4) = (57, 60, 60, 63)$.

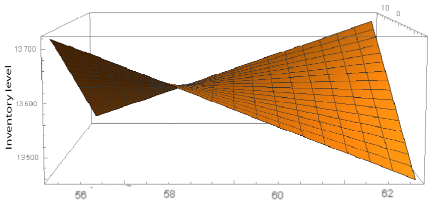
And, the value after manufacturing process in fuzzy $\widetilde{Q}_{\mathrm{P}} = (q_{\mathrm{P}_1}, q_{\mathrm{P}_2}, q_{\mathrm{P}_3}, q_{\mathrm{P}_4})$ with $0 < q_{\mathrm{P}_1} \leqslant q_{\mathrm{P}_2} \leqslant q_{\mathrm{P}_3} \leqslant q_{\mathrm{P}_4}$.

We obtain the ideal fuzzy production quantity by substituting the aforementioned fuzzy parameter values into formula given in equation 37,

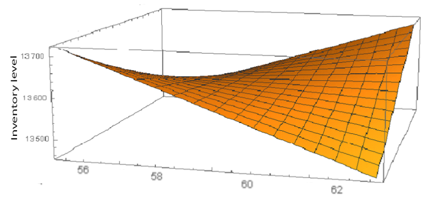
$\widetilde{Q}^*_{\mathrm{P}} = (4028.77, 4028.77, 4028.77, 4028.77)$.

Table 3 highlights the influence of the imperfect rate on the optimal lot size θ and optimal quantity. It is clear that as the percentage of the defective rate increases, the lot size also increases. Likewise, there is a significant variation in the total cost, which increases with higher levels of imperfection in the manufacturing process. However, the impact on the optimal quantity level is relatively humble, with a slight increase observed up to a 15 % defective rate, followed by a decrease with further increases in defective products.

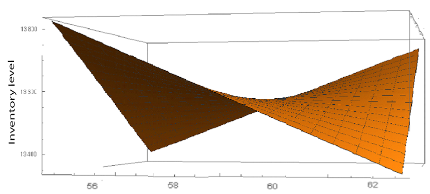
The findings presented in Table 3 reveal that the total cost and optimal lot size exhibit a substantial increase with higher defective rates, particularly in scenarios involving higher demand and production rates. As the number of defective products produced rises, the



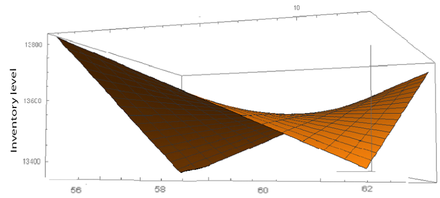
(a) Impact of increasing imperfect production on inventory



(b) Impact of decreasing imperfect production on inventory



(c) Impact of increasing low quality production on inventory



(d) Impact of decreasing low quality production on inventory

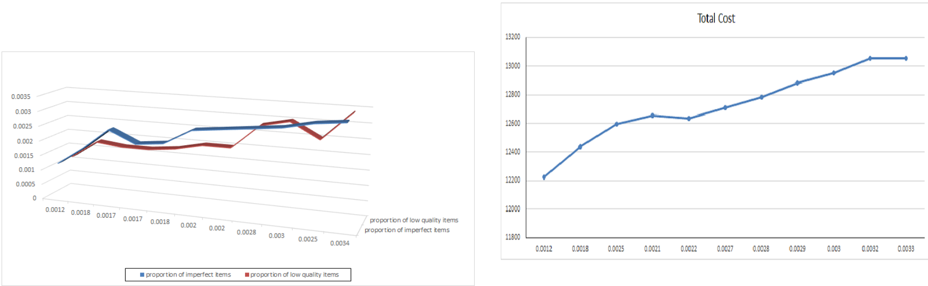
Figure 3: Inventory level in the industrial production process and the impact of imperfect and low quality items parameters

corresponding lot size to be ordered also increases. However, the impact on the number of units is less pronounced for higher production and demand rates, especially when the holding cost per unit is relatively low.

The connection or correlation between the alteration in the inspection rate and the optimal lot size is demonstrated below, utilizing data from Example 1 for further analysis. It is evident that as the frequency of inspections rises, the optimal lot size decreases. In this analysis, the inspection rate is varied from 24,000 units (on par with the rate of production, the frequency of inspections also increases) to 42,000 units per year (one and a half times higher than the production rate), while keeping the defective product rate fixed at 20%. All other data remain unchanged. The combined expenses and quantity level are demonstrated in Table 6. Notably, the quantity level remains relatively stable, exhibiting insignificant variation in response to significant changes in the frequency of inspections, assuming a consistent level of defective products being manufactured.

The optimal lot size and quantity undergo changes corresponding to variations in the defective production rate.

If the examination frequency and flawed goods manufacturing rate are altered concurrently, it can have different effects depending on the specific changes made. For example, if the inspection rate is increased while the defective products production rate remains the same, it is likely that the number of detected defects will increase, resulting in a decrease in the quantity of faulty goods dispatched to clients. This data is shown on the table 4. On the other hand, if the production rate of defective products is increased while the inspection rate remains the same, there will likely be an increase in the number of defective products that are shipped to customers, potentially leading to a decrease in customer satisfaction and increased costs due to returns, repairs, and replacements.



(a) proportion produced items and imperfect items in production process

(b) proportion produced items and low quality items in production process

Figure 4: Images Compare between parameters of imperfect and low quality items production

Table 4: Changes in parameters of various functions

P (Units produced annually)	θ_1	Total Cost ($/Year)	Q^* (Units)	$\widetilde{A}^*$ ($/item year)
23,000	13,300	17,783.27	2,289	16,877
25,000	13,350	17,793.67	2,243	16,971
28,000	13,375	17,803.15	2,202	17,072
30,000	13,400	17,811.81	2,166	17,143
32,000	13,425	17,819.77	2,134	17,192
34,000	13,475	17,827.10	2,106	17,362
36,000	13,550	17,833.88	2,080	17,472
38,000	13,600	17,840.17	2,057	17,553
40,000	13,650	17,846.02	2,035	17,886
42,000	13,725	17,851.47	2,016	17,982

Finally, the simultaneous variation of the inspection rate and defective production rate is examined, and their impact or influence on the size of the lot, quantity level, and sum total of expenses is assessed. The corresponding data is presented in Table 6. It is evident that the total cost increases as both the inspection rate and defective rate increase. Additionally, it is noticeable that when the evaluation rate surpasses the production rate ($M > p$), and optimal conditions in Table 6 reveal that the optimal batch size deduced from our proposed method comes close to the batch size calculated by the economic order quantity model. The variations in the optimal batch size and quantity, considering the changes in the proportion of defects and the rate of inspection, are depicted here.

5. SENSITIVITY ANALYSIS

Changes in holding and ordering costs can affect the inventory structure and the total cost of inventory management. A higher holding cost may result in smaller order

Table 5: Relative differences among various parameters of the desired functions

η	φ	t	ξ	D(Units)	P	$\widetilde{Q}_{\mathrm{P}}$(Units)
0.002	0.085	0.433	1.125	30000	24000	1428
0.002	0.085	0.453	1.125	30000	24000	1428
0.002	0.085	0.453	1.130	30000	24000	1500
0.002	0.085	0.453	1.135	30000	24000	1505
0.002	0.095	0.453	1.140	35000	24000	1737
0.002	0.095	0.453	1.145	35000	24000	1748
0.002	0.095	0.453	1.150	34000	24000	1703
0.005	0.095	0.453	1.150	34000	24000	1703
0.008	0.095	0.453	1.150	34000	24000	1702
0.010	0.095	0.453	1.155	34000	24000	1708
0.010	0.095	0.453	1.155	34000	25000	1715
0.010	0.095	0.453	1.155	32000	22000	1603
0.010	0.095	0.453	1.160	33000	23000	1661
0.010	0.110	0.453	1.160	33000	23000	1661
0.010	0.110	0.453	1.160	33000	23000	1661

quantities, while a higher ordering cost may result in larger order quantities. If any of these parameters change, it can have a significant impact on the inventory model and its outcomes. For example, if the demand rate increases, the reorder point may need to be adjusted to ensure that enough inventory is available to meet the increased demand. If the lead time increases, It might be necessary to raise the safety stock level in order to accommodate the extended waiting period. In inventory management, there are several parameters that can be adjusted to optimize the inventory level and reduce costs. These parameters include Order Quantity, setup cost per cycle in the production process A_p, production rate P etc. Table 5 gives an tabulated form of the parameter change. In a production inventory model, increasing the value of the inventory item generally results in decreasing the total cost of production. This is because higher value items typically have a lower cost per unit of production, due to economies of scale. When the value of an inventory item is increased, the cost of producing each unit of the item generally decreases. This is because many fixed costs associated with production, such as setup costs or equipment maintenance costs, are spread over a larger number of units. As a result, sum total of expenses of producing the inventory item decreases. However, it is important to note that this relationship may not hold true in all cases. Other factors such as demand variability, lead times, and holding costs can also impact the total cost of production. Therefore, it is important to consider all relevant factors when making decisions related to production and inventory management.

Now we divide the production process in phases 1, 2, 3 and 4. In a production inventory model, the production process can be divided into four phases:

Stages 1 Procurement: At the outset of the manufacturing cycle, the enterprise procures essential components or raw materials. This step encompasses determining the

Table 6: The value in this table indicates how well the related models' predictions were made.

Attribute	Adjustments (%)	Attribute Value	θ (Units)	$\widetilde{A^*}$	Total Cost ($/Year)
Cost for setup (A_p)	−35%	60	13471	37	17,671.65
	−25%	90	13801	45	17,759.68
	+25%	150	14325	58	17,899.26
	+35%	180	14547	63	17,958.36
Cost for holding (h_p)	−35%	0.3	12910	37	17,675.81
	−25%	0.45	13389	45	17,762.20
	+25%	0.75	13870	58	17,896.04
	+35%	0.9	13916	63	17,951.37
Production rate (P)	−35%	12,000	13720	47	17,949.81
	−25%	18,000	13915	50	17,881.72
	+25%	30,000	14229	53	18,696.91
	+35%	36,000	14866	54	18,766.95
Imperfect items (ξ)	−35%	18,000	13479	53	17,744.79
	−25%	27,000	13222	52	17,798.52
	+25%	45,000	14910	51	18,858.99
	+35%	54,000	14927	51	18,977.76
Low quality items (η)	−35%	0.1	12020	52	16,410.33
	−25%	0.15	13352	52	17,121.43
	+25%	0.25	14103	51	18,547.80
	+35%	0.3	14720	51	19,263.32
Deteriorating rate (φ)	−35%	3.5	14123	102	17,822.63
	−25%	4.0	14594	69	17,830.06
	+25%	4.5	15171	42	18,836.21
	+35%	5.0	15465	35	18,937.78

material volume, gauging the order's time frame, and finalizing the acquisition with the vendor.

Stages 2 Acquisition: Post the procurement, the company acquires the ordered components or raw materials. This stage involves assessing these materials against quality benchmarks, authenticating the volume secured, and revising stock logs.

Stages 3 Manufacturing: Subsequent to the material validation, the fabrication process kicks off. It encompasses transforming the sourced materials into end products following the enterprise's established procedures. This stage involves orchestrating fabrication timelines, overseeing the crafting process, and appraising the completed items to ensure they match quality benchmarks.

Stages 4 Dispatch: Post the product completion, these are either delivered to clientele or reserved in stock for upcoming demands. This dispatch stage entails organizing goods for delivery, ensuring proper packaging, and revising stock ledgers to denote the dispatched items.

Then using table 6 the changes in the inventory is shown in the following table. There are several factors to implementing production inventory in phases, one of them is reduced risk. Implementing production inventory in phases allows for a more controlled and manageable process. By implementing the inventory system in phases, you can identify any

Table 7: Changes in Phase.

	Phase			
	1	2	3	4
1	97(7.98%)	38(7.20%)	98(11.77%)	37(3.05%)
2	13(11.07%)	88(7.24%)	105(16.38%)	15(8.32%)
3	61(5.03%)	135(17.04%)	110(15.14%)	119(9.79%)

Table 8: Steadily introducing production inventory with changes in Phases.

Joint	Phases 1 and 2	Phases 2 and 3	Joint phases	Phases 1 and 2	Phases 2 and 4
(1,1)	7(0.58%)	0(0.00%)	(3,1)	1(0.08%)	7(0.58%)
(1,2)	3(0.25%)	3(0.25%)	(3,2)	4(0.33%)	82(6.75%)
(1,3)	46(3.79%)	6(0.49%)	(3,3)	25(2.05%)	71(5.84%)
(1,4)	41(3.37%)	4(0.33%)	(3,4)	113(23.05%)	39(3.21%)
(2,1)	4(0.33%)	23(1.89%)	(4,1)	1(0.08%)	31(2.55%)
(2,2)	73(6.01%)	18(1.48%)	(4,2)	8(0.66%)	48(61.56%)
(2,3)	124(7.45%)	22(1.81%)	(4,3)	4(0.33%)	85(7.00%)
(2,4)	337(17.63%)	25(2.06%)	(4,4)	24(1.97%)	51(4.20%)

issues or challenges early on and make adjustments before the system is fully deployed. This can help to reduce the risk of costly errors or failures. Also faster implementation production inventory in phases can also help to speed up the implementation process. By breaking the process down into smaller, more manageable phases, you can focus on specific areas of the inventory system and get them up and running quickly. This can help to reduce the overall time it takes to implement the system. And improved efficiency in production inventory in phases allows you to focus on specific areas of the inventory system and optimize them for efficiency. By taking a phased approach, you can identify areas where improvements can be made and implement changes before moving on to the next phase. This can help to improve overall efficiency and productivity. Implementing production inventory in phases allows you to manage your resources more effectively. By focusing on specific areas of the inventory system, you can allocate resources more efficiently and ensure that you have the right people and tools in place to get the job done. Implementing production inventory in phases can help to reduce risk, speed up implementation, improve efficiency, and manage resources more effectively. It can also help to ensure that the inventory system is optimized for your specific needs and requirements.

5.1. Experiment results

When we accept $a_{i,t}\left(X_i^G\right)$ is the total production operation and S_i^v for phase i at time t fulfils the conditions $\left(S_i^{\tilde{v}}, t\right) = (S_i^v, t)$, additionally $a_{i,t}\left(X_i^G\right)$ is $0; a'_{i,t}\left(X_i^G\right)$ is one if the phase S_i^v for Phase i at time t holds the relation $(S_i{}^v, t) = (S_i^v, t)$, and $a'_{i,t}\left(X_i^G\right)$ has reduced value. Also $V_i^G = \left(v_{p,1}^G, v_i^G, \cdots, v_i^G\right)$ and the X_i^G is derived by checking the appropriate phases from the mutant individual generation phases as explained.

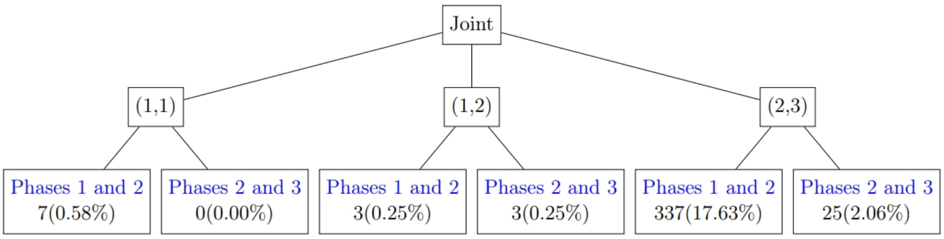


Figure 5: Depicting Steadily production inventory with changes in Phases

Phase 1: Let $\eta_l \geqslant 0$, be a likelihood that the i-th approach in the candidate pool will be chosen, and populate each η_l as $1/K$, where K is the number of tactics in the pool as a whole, meet the requirement $\sum_{l=1}^{K} \eta_l = 1$.

Phase 2: Give the likelihood range $\left[\sum_{s=0}^{l-1} p_s, \sum_{s=0}^{l} p_s\right], l = 1, 2, \ldots, K$ to the i-th phase, satisfying $p_0 = 0$.

Phase 3: Now the $l' - th$ phase to generate V_p^G for the locate X_p^G, whereas R_{rand} is a uniformly generated random number at random from the range $[0, 1]$, and the preferred R_{rand} meets $R_{\text{rand}} \in \left[\sum_{s=0}^{l'-1} p_s, \sum_{s=0}^{l'} p_s\right)$.

Step 4: After finishing up the crossover and selection processes at the current generation G , note how many test subjects produced by the $i - th$ approach were competent enough to enter the following generation, Num_i^G, then steadily add these figures. Num_i^G in a predetermined number of subsequent cycles G' to obtain Num_l, where Num_l is the whole population of trial subjects produced by the $i - th$ method that made the transition into the future over G' generations.

Phase 5: Next phase the value η_l of the $i - th$ strategy with $\eta_l = \frac{\text{Num}_l}{\sum_{s=1}^{K} \text{Num}_s}$ satisfying the condition $G \bmod G' = 0$.

The experimental phase $U_p^G = \left(u_{i,1}^G, u_{i,2}^G, \cdots, u_i^G\right)$ is modified with

$$u_i^G = \begin{cases} v_i^G, & \text{if } q(0,1) \leqslant q \text{ or } q = q_{\text{r}} \\ x_i^G, & \text{otherwise} \end{cases}$$

where $\text{r}_q(0, 1) \in [0, 1]$ is a uniformly distributed random value, with determined probability $CR \in [0, 1]$ is a constant, and q_{r} is a integer taken between 1 to D.

We select the execution which is used to decide if the X_i^G or U_i^G the new group arrives. The phase i of decision process is written as

$$X_i^{G+1} = \begin{cases} U_i^G, & \text{if } f_i\left(U_i^G\right) \geqslant f_i\left(X_i^G\right) \\ X_p^G, & \text{otherwise} \end{cases}$$

When there is the most growth X_{max} is reached, the process is revoked, and the ideal condition is identified. If not, the procedure moves on to the next step.

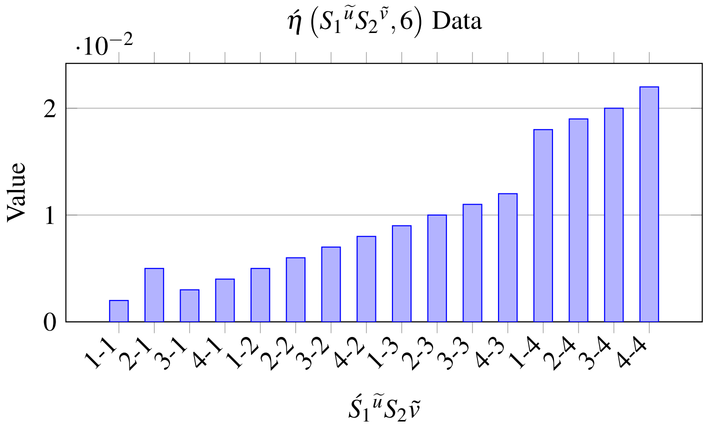


Figure 6: Depicting Data Obtained in Production process in Phases

From the manufacturing process, we derive Four Stages $(n = 4)$, as presented in table 7 and table 8. Around 10% of the components in this segment result in below-standard, flawed items, while the remaining 90% are directly used as complete items for fulfilling orders at the manufacturing site. Even though 90% of the items are finalized by the end of Stage 3, these three Stages can be analyzed to meet the objectives of the suggested production and inventory project.

During regular operational production, $1350\,(T_2 = 1350)$ entries from an inventory data-set were collected, as detailed in the subsequent table.

Considering Stage i, denote $\eta_{i,t}$ and $\eta'_{i,t}$ as the current and forecast fractions of inferior quality products in the production sequence (picking consistent numbers that align with inventory steadiness) at moment t. The difference between these two parameters is given by $\Delta\eta_{i,t} = \theta_{i,t} - \theta'_{i,t}$. Based on this difference, the inventory discrepancy stage S_i^u is segmented into four classifications, namely $i \in \{1,2,3,4\}$: a negative variation state S_i^1, a tolerable state S_i^2, an average variation state S_i^3, and a condition of high fluctuation S_i^4. These categories align with the following duration: $\Delta\eta_i^0 < \Delta\eta_{i,t} \leqslant \Delta\eta_i^1, \Delta\eta_i^1 < \Delta\eta_{i,t} \leqslant \Delta\eta_i^2$, $\Delta\eta_i^2 < \Delta\eta_{i,t} \leqslant \Delta\eta_i^3, \Delta\eta_{i,t} > \Delta\eta_i^3$, where $\Delta\eta_i^0, \Delta\eta_i^1, \Delta\eta_i^2$, and $\Delta\eta_i^3$ are defining parameters with $\Delta\eta_i^0 < \Delta\eta_i^1 < 0$ and $\Delta\eta_i^3 > \Delta\eta_i^2 > 0$. For this study, the values are: $-1500, -1000, 1000$ and 1500 for Stage 1; $-3500, -1000, 1000, 1500$ for Stage 2; $-5000, -3000, 1000, 4000$ for Stage 3; and $-2500, -1000, 1000, 2500$ for Stage 4.

In the subsequent data processing steps, each sample of data is turned into a set of inventory fluctuation phases $\theta_i^u, u \in \{1,2,3,4\}$ and $i \in \{1,2,3,4\}$. These are the phases $\theta_i^u\theta_{i+1}^v, u, v \in \{1,2,3,4\}$. Because, θ_i^u and $\theta_i^u, \theta_{i+1}^v$ further expressed as $\theta_i^u = u$ and $\theta_i^u, \theta_{i+1}^v = (u, v)$. Table 9 and Table 10 describe for every single Component and any mixture of double Phases, the frequency distribution of phases, respectively, based on the practise data. It is noticed that for (single) Stages 1, 2, and 3, the prevailing condition is θ_1^2, θ_2^4 and θ_3^2, representing $75.20\%, 77.31\%$ and 73.04% of all the instances of the stages, respectively.

Table 9: Different Phases in Production process

Parameter value												para ^1_g				η_g^2			
DP	V_D	F	DR	$H_{\max}$	P'	K	NP	H_1	G_1	CK_1	$I_{\max}$	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
37	4	0.4	1.7	75	25	10	2.5	4	0.002	0.7	260	$[0,5]$	$[0,5]$	Integer in $[0,10]$	$[0.5,1]$	$[0,5]$	$[0,5]$	Integer in $[0,10]$	$[0,1]$

Table 10: Data Obtained in Production process in Phases

$\acute{S}_1{}^u S_2 \tilde{v}$	$\acute{\eta}\left(S_1{}^u S_2{}^{\tilde{v}},6\right)$	η_g^1	$P(\widetilde{u},\widetilde{v} \mid 1,2,4,2,3)$	$(S_1{}^u S_2 \widetilde{v},6)$	$\left(S_2{}^{\tilde{v}},6\right)(S_2\tilde{v})$	$P\left(S_2\tilde{v},6\right)\left(S_2{}^{\tilde{v}}\right)$	S_2^v
(1,1)	0.002	13.280	0.025	0.006	0.011 (1)	0.011 (1)	3
(2,1)	0.005	13.280	0.000	0.005	0.011 (1)	0.011(1)	3
(3,1)	0.003	13.280	0.000	0.000	0.011 (1)	0.011(1)	3
(4,1)	0.004	13.280	0.000	0.000	0.011(1)	0.011(1)	3
(1,2)	0.005	13.280	0.000	0.002	0.067(2)	0.071(2)	3
(2,2)	0.006	13.280	0.038	0.060	0.067 (2)	0.071 (2)	3
(3,2)	0.007	13.280	0.000	0.002	0.067 (2)	0.071 (2)	3
(4,2)	0.008	13.280	0.013	0.003	0.067(2)	0.071 (2)	3
(1,3)	0.009	13.280	0.013	0.051	0.491(3)	0.165(3)	3
(2,3)	0.010	13.280	0.150	0.395	0.491 (3)	0.165(3)	3
(3,3)	0.011	13.280	0.000	0.038	0.491 (3)	0.165(3)	3
(4,3)	0.012	13.280	0.000	0.007	0.491 (3)	0.165(3)	3
(1,4)	0.018	13.280	0.025	0.011	0.431 (4)	0.753 (4)	3
(2,4)	0.019	0.133	0.698	0.348	0.431 (4)	0.753(4)	3
(3,4)	0.020	13.280	0.025	0.064	0.431(4)	0.753 (4)	3
(4,4)	0.022	13.280	0.013	0.008	0.431(4)	0.753(4)	3

Table of spreadsheet 9 and 10 present the estimate conclusions for Phase 2 and Phase 3 for normal time periods respectively. The modified result of $\theta_i \tilde{u} S_{i+1}^{\tilde{v}}$ (in the modification), and $(\theta_{i+1}\tilde{v}$, t) together with $S_{i+1}\tilde{v}$ (i.e., the predicted state). For resemblance, the 7th column the relative likelihood is described $P(S_{i+1}\tilde{v},t)$ along with the matching projected state $(S_{i+1}\tilde{v})$ gained from $P(S_i{}^u S_{i+1}\tilde{v},t)$ (i.e., the anticipated outcome without given threshold only on Markov models). The final column displays the current condition θ_{i+1}^{v} for Phase $i+1$ on time t.

The first term's worth is far less than the cost of the reelection campaign, this results in the likelihood of the starting procedure $\widehat{a}_{k(u_1-1)+v_1,k(\widetilde{u}-1)+\widetilde{v}}^{i+1}$ from $(4,2)$ to $(4,2)$ reducing while from $(4,2)$ to $(3,3)$ rising. The outcome is $\left(S_2{}^u S_3{}^{\tilde{v}}, 11\right)$ reaches its highest point (0.588) a joint phases $(3,3)$, higher than its value (0.110) at $(4,2)$. Consequently, $\left(S_3{}^{\tilde{v}}, 11\right)$ has the real-world example that is most likely $S_3^v = 3$, the scenario is as presumed adopting the earlier method. In overview, the exact phase prediction in Table 11 demonstrates that the parameter's adjustment $\eta_g^i P(\widetilde{u},\widetilde{v} \mid c, u_c, v_c, u_r, v_r)$ can enhance non-dominant state prediction greatly in phase(s).

Markov models are widely employed as probabilistic models for predicting future events based on the analysis of past events. These models possess the Markov property, which signifies that the prediction of a future event relies solely on the current state of the system and not its historical context. The predictive accuracy of Markov models depends on various factors, including the model's order, the size and quality of the dataset, and the nature of the events being predicted. Numerous studies have investigated the impact of increasing the model's complexity on the accuracy of its predictions. It has been observed that raising the model's order generally enhances its accuracy up to a certain point. However, beyond that point, further increases in the model's order do not yield significant improvements in prediction accuracy. This phenomenon can be attributed to over-fitting, where the model becomes overly complex and starts capturing noise in the data rather than the underlying pattern.

The prediction accuracy of Markov models can be influenced by various factors, one of which is the size and quality of the dataset. For accurate predictions, Markov models require a substantial amount of data. If the dataset is insufficient in size or of low quality, the model may fail to capture the underlying pattern effectively. Moreover, if the dataset is biased or not representative, the predictions made by the model may be inaccurate. Additionally, the type of events being predicted can impact the prediction accuracy of Markov models. These models perform best when predicting discrete events with a limited number of possible outcomes. Conversely, when dealing with continuous events or those with a large number of potential outcomes, Markov models may not be the most suitable choice. In such cases, alternative methods like neural networks or regression models might be more appropriate. To achieve accurate predictions using Markov models, it is crucial to consider several factors, including the model's order, the size and quality of the dataset, and the nature of the events being predicted. By comprehending these factors and selecting appropriate parameters and techniques, Markov models can be effectively utilized to make accurate predictions across a wide range of applications.

Additionally, the anticipated stages acquired from the initial and the fresh anticipated outcomes are identical, suggesting that the estimated outcomes for inventory phases at

Table 11: Changes made to the variable, combined with its corresponding parameter, and conducting sensitivity analysis by varying different parameters

$(S^{\tilde{v}},t)$ $(S_2{}^{\tilde{v}})$	ξ decreases 10%				η changes 10%			
	Original	New	Change	Degree of change (%)	Original	New	Change	Degree of change (%)
$(S_2^1,1)$ (S_2^1)	0.0022	0.0023	0.0001	4.55	0.0022	0.0021	-0.0001	-4.55
$(S_2^2,1)$ (S_2^2)	0.0000	0.0000	0.0000	0.00	0.0000	0.0000	0.0000	0.00
$(S_2^3,1)$ (S_2^3)	0.0081	0.0086	0.0005	6.17	0.0081	0.0076	-0.0005	-6.17
$(S_2^4,1)$ (S_2^4)	0.9897(4)	0.9891(4)	-0.0006	-0.06	0.9897(4)	0.9903(4)	0.0006	0.06
$(S_2^1,2)$ (S_2^1)	0.0030	0.0032	0.0002	6.67	0.0030	0.0028	-0.0002	-6.67
$(S_2^2,2)$ (S_2^2)	0.0482	0.0488	0.0006	1.24	0.0482	0.0476	-0.0006	-1.24
$(S_2^3,2)$ (S_2^3)	0.3461	0.3401	-0.0060	-1.73	0.3461	0.3512	0.0051	1.47
$(S_2^4,2)$ (S_2^4)	0.6027(4)	0.6079(4)	0.0052	0.86	0.6027(4)	0.5983(4)	-0.0044	-0.73
$(S_2^1,3)$ (S_2^1)	0.0058	0.0061	0.0002	3.39	0.0059	0.0058	-0.0001	-1.69
$(S_2^2,3)$ (S_2^2)	0.0912	0.0892	-0.0010	-1.11	0.0902	0.0911	0.0009	1.00
$(S_2^3,3)$ (S_2^3)	0.2540	0.2505	-0.0035	-1.38	0.2540	0.2572	0.0032	1.26
$(S_2^4,3)$ (S_2^4)	0.6498(4)	0.6541(4)	0.0043	0.66	0.6498(4)	0.6459(4)	-0.0039	-0.60
$(S_2^1,4)$ (S_2^1)	0.0002	0.0002	0.0000	0.00	0.0002	0.0002	0.0000	0.00
$(S_2^2,4)$ (S_2^2)	0.0154	0.0155	0.0001	0.65	0.0154	0.0153	-0.0001	-0.65
$(S_2^3,4)$ (S_2^3)	0.9210(3)	0.9198(3)	-0.0012	-0.13	0.9210(3)	0.9221(3)	0.0011	0.12
$(S_2^4,4)$ (S_2^4)	0.0634	0.0646	0.0012	1.89	0.0634	0.0625	-0.0009	-1.42
$(S_2^1,5)$ (S_2^1)	0.0004	0.0004	0.0000	0.00	0.0004	0.0004	0.0000	0.00
$(S_2^4,9)$ (S_2^4)	0.9605(4)	0.9605(4)	0.0000	0.00	0.9605(4)	0.9605(4)	0.0000	0.00
$(S_2^1,10)$ (S_2^1)	0.0095	0.0095	0.0000	0.00	0.0095	0.0094	-0.0001	-1.05
$(S_2^2,10)$ (S_2^2)	0.0730	0.0729	-0.0001	-0.14	0.0730	0.0730	0.0000	0.00
$(S_2^3,10)$ (S_2^3)	0.2125	0.2102	-0.0023	-1.08	0.2125	0.2147	0.0022	1.04
$(S_2^4,10)$ (S_2^4)	0.7051(4)	0.7074(4)	0.0023	0.33	0.7051(4)	0.7029(4)	-0.0022	-0.31
$(S_2^1,11)$ (S_2^1)	0.0107	0.0107	0.0000	0.00	0.0107	0.0107	0.0000	0.00
$(S_2^2,11)$ (S_2^2)	0.0774	0.077	-0.0004	-0.52	0.0774	0.0777	0.0003	0.39
$(S_2^3,11)$ (S_2^3)	0.1753	0.1744	-0.0009	-0.51	0.1753	0.1762	0.0009	0.51
$(S_2^4,11)$ (S_2^4)	0.7366(4)	0.7379(4)	0.0013	0.18	0.7366(4)	0.7353(4)	-0.0013	-0.18
$(S_2^1,12)$ (S_2^1)	0.0096	0.0097	0.0001	1.04	0.0096	0.0095	-0.0001	-1.04
$(S_2^2,12)$ (S_2^2)	0.0796	0.0791	-0.0005	-0.63	0.0796	0.0801	0.0005	0.63
$(S_2^3,12)$ (S_2^3)	0.1887	0.1867	-0.0020	-1.06	0.1887	0.1907	0.0020	1.06
$(S_2^4,12)$ (S_2^4)	0.7220(4)	0.7244(4)	0.0024	0.33	0.7220(4)	0.7197(4)	-0.0023	-0.32
$(S_2^1,13)$ (S_2^1)	0.0034	0.0036	0.0002	5.88	0.0034	0.0032	-0.0002	-5.88
$(S_2^2,13)$ (S_2^2)	0.0236	0.0253	0.0017	7.20	0.0236	0.0222	-0.0014	-5.93
$(S_2^3,13)$ (S_2^3)	0.6491(3)	0.6329(3)	-0.0162	-2.50	0.6491(3)	0.6633(3)	0.0142	2.19
$(S_2^4,13)$ (S_2^4)	0.3238	0.3382	0.0144	4.45	0.3238	0.3113	-0.0125	-3.86

continuing table...

$(S\tilde{v},t)\,(S\tilde{v}^{v})$	ξ changes 10%				η changes 10%			
	Original	New	Change	Degree of change (%)	Original	New	Change	Degree of change (%)
$(S_3^1,1)\,(S_3^1)$	0.0590	0.0590	0.0000	0.00	0.0590	0.0590	0.0000	0.00
$(S_3^2,1)\,(S_3^2)$	0.7082(2)	0.7082(2)	0.0000	0.00	0.7082(2)	0.7082(2)	0.0000	0.00
$(S_3^3,1)\,(S_3^3)$	0.1399	0.1399	0.0000	0.00	0.1399	0.1399	0.0000	0.00
$(S_3^4,1)\,(S_3^4)$	0.0929	0.0929	0.0000	0.00	0.0929	0.0929	0.0000	0.00
$(S_3^1,2)\,(S_3^1)$	0.0587	0.0587	0.0000	0.00	0.0587	0.0588	0.0001	0.17
$(S_3^2,2)\,(S_3^2)$	0.7080(2)	0.7080(2)	0.0000	0.00	0.7080(2)	0.7080(2)	0.0000	0.00
$(S_3^3,2)\,(S_3^3)$	0.1400	0.1400	0.0000	0.00	0.1400	0.1400	0.0000	0.00
$(S_3^4,2)\,(S_3^4)$	0.0933	0.0933	0.0000	0.00	0.0933	0.0933	0.0000	0.00
$(S_3^1,3)\,(S_3^1)$	0.0438	0.0441	0.0003	0.68	0.0438	0.0437	-0.0001	-0.23
$(S_3^2,3)\,(S_3^2)$	0.7030(2)	0.7029(2)	-0.0001	-0.01	0.7030(2)	0.7031(2)	0.0001	0.01
$(S_3^3,3)\,(S_3^3)$	0.1593	0.1590	-0.0003	-0.19	0.1593	0.1595	0.0002	0.13
$(S_3^4,3)\,(S_3^4)$	0.0939	0.0940	0.0001	0.11	0.0939	0.0938	-0.0001	-0.11
$(S_3^1,4)\,(S_3^1)$	0.0458	0.0460	0.0002	0.44	0.0458	0.0457	-0.0001	-0.22
$(S_3^2,4)\,(S_3^2)$	0.6973(2)	0.6975(2)	0.0002	0.03	0.6973(2)	0.6972(2)	-0.0001	-0.01
$(S_3^3,4)\,(S_3^3)$	0.1599	0.1596	-0.0003	-0.19	0.1599	0.1602	0.0003	0.19
$(S_3^4,4)\,(S_3^4)$	0.0969	0.0969	0.0000	0.00	0.0969	0.0969	0.0000	0.00
$(S_3^1,5)\,(S_3^1)$	0.0226	0.0227	0.0001	0.44	0.0226	0.0226	0.0000	0.00
$(S_3^2,5)\,(S_3^2)$	0.7070(2)	0.7070(2)	0.0000	0.00	0.7070(2)	0.7071(2)	0.0001	0.01
$(S_3^3,5)\,(S_3^3)$	0.1834	0.1834	0.0000	0.00	0.1834	0.1835	0.0001	0.05
$(S_3^4,5)\,(S_3^4)$	0.0869	0.0869	0.0000	0.00	0.0869	0.0869	0.0000	0.00
$(S_3^1,6)\,(S_3^1)$	0.0575	0.0575	0.0000	0.00	0.0575	0.0575	0.0000	0.00
$(S_3^4,10)\,(S_3^4)$	0.0932	0.0932	0.0000	0.00	0.0932	0.0932	0.0000	0.00
$(S_3^1,11)\,(S_3^1)$	0.0483	0.0484	0.0001	0.21	0.0483	0.0482	-0.0001	-0.21
$(S_3^2,11)\,(S_3^2)$	0.1537	0.1538	0.0001	0.07	0.1537	0.1537	0.0000	0.00
$(S_3^3,11)\,(S_3^3)$	0.7006(3)	0.7005(3)	-0.0001	-0.01	0.7006(3)	0.7009(3)	0.0003	0.04
$(S_3^4,11)\,(S_3^4)$	0.0974	0.0974	0.0000	0.00	0.0974	0.0973	-0.0001	-0.10
$(S_3^1,12)\,(S_3^1)$	0.0486	0.0487	0.0001	0.21	0.0486	0.0485	-0.0001	-0.21
$(S_3^2,12)\,(S_3^2)$	0.7058(2)	0.7053(2)	-0.0005	-0.07	0.7058(2)	0.7063(2)	0.0005	0.07
$(S_3^3,12)\,(S_3^3)$	0.1494	0.1496	0.0002	0.13	0.1494	0.1492	-0.0002	-0.13
$(S_3^4,12)\,(S_3^4)$	0.0962	0.0964	0.0002	0.21	0.0962	0.0961	-0.0001	-0.10
$(S_3^1,13)\,(S_3^1)$	0.0490	0.0491	0.0001	0.20	0.0490	0.0489	-0.0001	-0.20
$(S_3^4,13)\,(S_3^4)$	0.0980	0.0980	0.0000	0.00	0.0980	0.0980	0.0000	0.00

Table 12: The mean computational duration of each algorithm during a single execution, along with a comparison of their predictive capabilities

Phase	S			Stage	GA		
	Average	Discrepancy	Time (s)		Average	Discrepancy	Time (s)
Phase 2	0.62	2.26E − 4	38.13	Phase 2	0.69	4.02E − 3	39.25
Phase 3	0.72	8.57E − 4	56.67	Phase 3	0.62	$3.03E-3$	45.62

Table 13: Displaying the anticipated outcomes for individual stages

Authentic periods	Projected stages $S_2\tilde{v}$								Projected phases $S_3^{\tilde{v}}$											
	2	1	4	3	1	1	2	3	4	1	3	4	1	2	3	4	1	2	3	4
1	1	0.3	0	0.74	0	0	0	0.74	1	0.74	1	2	2	0.74	0	0	0	0.74	0	0
2	0	0	0	6.66	0	2.96	1.48	2.22	0	62.96	1	2	2	62.22	2.64	2	1	62.96	1	1
3	0	0	0	26.67	0	0	12.59	14.07	0	28.15	2	2	1	25.93	2.22	0	0	23.71	4.44	0
4	0	2	1	62.92	0	2.39	6.51	65.04	0	7.89	1	2	2	7.14	2	1	3	2.84	2	3.47
Total	1	2	2	100	2	5.55	25.48	75.07	0	89	2	2	1	88.05	3.86	1	2	88.22	5.44	6.47

Phase 3 remain consistent despite alterations in the ηg^2 parameter. Furthermore, all variations fall within a range of 0.68%, providing evidence that the outcomes remain unchanged even with modifications in ηg^2.

Markov models are powerful mathematical tools used to analyze and predict the behavior of systems that exhibit temporal dependencies. These models are based on the concept of Markov processes, which are stochastic processes that satisfy the Markov property. According to the Markov principle, the upcoming actions of a system are determined exclusively by its current position and are not affected by its prior positions. This property makes Markov models particularly useful for modeling and analyzing systems with limited memory or where the history of the system is not relevant. One of the key applications of Markov models is in the field of probability theory and statistics. Markov chains, which are a type of Markov model, are extensively used to model various phenomena such as weather patterns, stock market movements, genetic sequences, and even language generation. Markov models are characterized by a collection of states and a series of transition probabilities linking those states. These transformation probabilities determine the likelihood of transitioning from one state to another in a given time step. Markov models can be classified into different types based on the properties of their state space and transition probabilities, such as homogeneous Markov models, non-homogeneous Markov models, and hidden Markov models.

The final manufacturing process or marinating the inventory level any action taken will be immensely affect the changes. Accordingly, the importance of the recommended Markov models is conclusively demonstrated. Here it is illustrated the contrast of the proper Markov models with the proposed models. In this provisions of clause we exclusively emphasize the resemblance of the inventory variety phase estimations while deliberately ignoring the correlation of modeling propagation of variation. This is because cutting-edge models of variation propagation in the relevant literature [55] primarily employ conventional Markov models (i.e., TMM). Moreover, when predicting the stages of inventory fluctuation, this research presents a preliminary data set for modeling inventory variation. As a result, we provide a brief comparison of the phase estimations between our posited model and the test-wide prediction findings for double Phases. Predictive analytic can be a valuable tool for manufacturers to manage their production inventory more effectively. By analyzing historical data and identifying patterns, predictive models can provide insights into future demand, potential supply chain disruptions [56], and optimal inventory levels. Here are some key notes on prediction findings in production inventory: Forecasting demand: Predictive models can analyze historical sales data, market trends, and other factors to forecast future demand for a particular product. This can help manufacturers optimize their inventory levels and avoid stock outs or overstocking. By analyzing data from quality inspections and other sources, predictive models can identify potential quality issues in the production process. This can help manufacturers address these issues before they impact the final product and reduce the amount of inventory that needs to be scrapped. Predictive models can analyze data from the supply chain [57], such as shipping times and supplier reliability, to predict lead times for raw materials and other components. This can help manufacturers avoid stock outs and delays in the production process. Predictive models can analyze data from the production process, such as machine up time and production rates, to optimize production schedules and reduce idle time. This can help manufacturers improve efficiency and reduce the amount of inventory that is in process at any given time. Overall, predictive analytic can help manufacturers make more informed decisions about their production inventory and improve their overall efficiency and profitability. Table 12 presents the outcome for numerous phases. From Table 12 , it is noticed that for Phases 1 and 2, over all the main components (i.e., $S_1{}^u S_2{}^{\widetilde{v}} = (2,4)\big)$,65.93% and 54.24% are correctly predicted by the model; while for non-dominant phases (i.e., $S_1 \tilde{u} S_2 \tilde{v} \neq (2,4))$,0% and 18.35% are precisely recognised.

6. CONCLUSION

During production, when there's a prevalence of substandard or defective products, it's crucial to understand the underlying system using mathematical modeling. Should one find themselves facing the production of such items, a series of measures can be adopted. First, pinpointing the main reason is essential, be it due to inadequate staff training, outdated machinery, or other contributing factors. Upon recognizing these challenges, proactive solutions can be pursued. It's essential to re-evaluate and enhance production methods, possibly through technological upgrades, additional staff training, or the introduction of robust quality assurance practices. Strengthening quality oversight ensures that flaws are spotted and rectified promptly. It might even be beneficial to onboard more personnel dedicated to this task. Continuous communication with all involved parties is of paramount importance to maintain trust and reinforce relationships. Emphasizing quality can greatly influence the overall success of production, aligning the output with the anticipations of both clients and affiliates. The focus of this research is on introducing deterministic stock frameworks, articulated through precise mathematical equations.

Items with flaws are strictly disallowed, and the effects of economic inflation, value decrease, initial stock influence on demand, and substandard products are scrutinized. To enhance a production process that yields substandard or flawed goods, several strategies can be adopted. A pivotal move is pinpointing the primary reason behind these product deficiencies. Recognizing this fundamental issue offers clarity on the necessary modifications for refining the manufacturing workflow. It's also imperative to enforce stringent quality assurance protocols to detect and rectify production anomalies.

To overcome these limitations, future research could consider incorporating stochastic elements into the model to account for uncertainties in demand and supply. Additionally, the model could be extended to include dynamic factors such as market trends and competitor analysis to enable more accurate decision-making. Furthermore, exploring the potential of incorporating advanced technologies such as machine learning or artificial intelligence into the production process could further optimize quality control and minimize the production of low quality and imperfect items. By addressing these limitations and incorporating the outlined strategies for improvement, the production process can be enhanced, resulting in higher quality products that meet or exceed customer expectations.

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