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**Research Article**

# PERFORMANCE OF AN UNRELIABLE RETRIAL QUEUE WITH TWO TYPES OF CUSTOMER ARRIVALS AND SERVICE ORBIT

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**Abstract:** This research provides a comprehensive analysis of a complex retrial queue, specifically a $M_1,M_2/G_1,G_2/1$ model. The unique characteristic of this model is its consideration of customer impatience, which can manifest as either persistent or impatient behavior. The study explores the intricate dynamics of the system, including the interplay between customer impatience and the retrial, service, repair, and reserved processes. To enhance realistic modeling, the study introduces a service orbit and repair services that are activated when the server breaks down. The Chapman-Kolmogorov equations are established, and the supplementary variables method is used to present the steady-state solutions. We provide the necessary and sufficient condition for the system to be stable,

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along with several specific cases. Explicit closed-form expressions for various performance measures are provided, which are then used to construct an expected total cost function. Numerical results are also presented to demonstrate how system parameters affect performance measures and the total cost function.

**Keywords:** Unreliable retrial queue, reserved times, two types of orbit, performance measures, cost function.

**MSC:** 60K25, 90B22.

## 1. INTRODUCTION

Over recent years, there has been a significant growth in the literature on retrial queues. The concept of retrial has been widely used as mathematical models in various real-world communication systems, such as cognitive radio networks, communications networks, call centers, cellular mobile networks, and IP networks. For an overview of main results and methods, see Falin [1], Falin and Templeton [2], Templeton [3], Artalejo and Gómez-Corral [4], and Yang and Templeton [5].

Customers highly prioritize their waiting time and are inclined to discontinue their engagement with a system when service delays become excessive. These incidents can occur in queueing systems when server reliability is not high. The service interruptions caused by breakdowns and subsequent repairs contribute to an increase in customer waiting times and provide an additional opportunity for incoming customers to join the retrial group. Retrial queues with server breakdowns and repairs have been addressed by many authors [6, 7, 8, 9, 10, 11, 12, 13, 14].

Impatience is a critical factor affecting the dynamics and performance of queueing systems, as traditional models often overlook the complexities introduced by customer dissatisfaction and frustration during long waits. Customers who become impatient may abandon the queue, a behavior that significantly impacts system performance and optimization. To address these issues, researchers have extended traditional queueing theory by including models that account for customer behavior, abandonment rates, and retrial processes. Notable models, such as the finite-source retrial queue, retrial orbits, and impatient customer queues, provide insights into how impatience affects system performance and design [15, 16, 17, 18, 19, 20]. Additionally, server reliability issues and service interruptions further complicate the situation, leading to increased waiting times and affecting customer decisions to rejoin or abandon the queue [6, 7, 8, 9, 10, 11, 12, 13, 14, 21, 22, 23]. Understanding the dual aspects of customer patience-persistent customers who reattempt service and impatient customers who may renege or balk-is essential for analyzing and improving queueing systems.

Single-server retrial queueing models with two or more types of customer arrivals have been attracted significant attention from numerous researchers, including two-way communication [6, 24, 25, 26, 27], two classes of customers [28, 29, 30], batch arrivals [8, 20, 31, 32], two classes of batch arrivals [7, 8, 9], Markovian arrival process [28], [33], [34], priority customers [17, 35, 36, 37] negative customers [12, 13, 38, 39]. Other researchers have investigated retrial queueing models that account for both persistent and impatient customers. Taleb and Aissani [40] studied an unreliable retrial queue that ac-

commodates both persistent and impatient customers. In this system, if a primary customer finds the server available, they are served immediately. However, if the server is busy, persistent customers enter into an orbit and continue to attempt to receive service until the server becomes available. In contrast, impatient customers leave the system without being served. The study also incorporated corrective and preventive maintenance strategies, which allowed for calculating performance metrics and analyzing the time until the server's first failure. Recently, Aissani et al. [16] conducted a study on an unreliable retrial queueing system with a linear retrial policy, accommodating both persistent and impatient customers under different service distributions. Persistent customers, if blocked by a busy server, enter a retrial orbit and continue to attempt access until they succeed, whereas impatient customers leave the system if blocked. The server is subject to both passive and non-conservative active breakdowns. The authors used the supplementary variable method to derive the stationary probability distribution and various performance metrics. Additionally, they applied the embedded Markov chain method to determine the stationary distribution of customers in the retrial orbit at the end of service epochs.
In this study, we investigate an unreliable retrial queueing system denoted as $M_1, M_2/G_1, G_2/1$, featuring a repairable server and incorporating elements such as balking, reneging, a service orbit, and reservation mechanisms. The service orbit is designed only for impatient customers who face service disruptions due to server breakdowns. Instead of choosing to leave the system permanently, these customers opt to enter a service orbit, where they wait for the issue to be resolved and for their service to be completed. This model accommodates either persistent or impatient customers. Although these customer types play a critical role, their impact has often been overlooked in queueing literature. It is essential to understand how the distinct behaviors and needs of persistent versus impatient customers affect queue dynamics, as their influence can significantly alter system performance.

**Motivation of this study**:

The motivation for this study arises from the growing need to analyze the performance of complex retrial queueing systems in various real-world applications. Queueing systems are prevalent across many domains, including telecommunications, transportation, manufacturing, and service industries. Retrial queueing systems are particularly noteworthy because they account for customer impatience, which can manifest as either persistent or impatient behavior.

– Focus of the study: This study specifically examines a $M_1, M_2/G_1, G_2/1$ retrial queue with two types of customer arrivals: persistent and impatient. It also considers service times, retrial times, repair times, customer balking behavior, and reserved times for impatient customers only. This focus reflects the complexity of real-world scenarios. Understanding and analyzing such intricate systems can lead to improvements in customer satisfaction, resource utilization, and operational efficiency.

– Theoretical foundation: By addressing the ergodicity requirement for system stability and deriving analytical findings for the stationary distribution. This study aims to provide a solid theoretical foundation for understanding the behavior of the retrial queueing system under various conditions. Additionally, the development of diverse performance metrics allows for a comprehensive evaluation of the system's efficiency and effectiveness.

– Cost analysis: A thorough cost analysis is crucial for optimizing practical applications. In this study, we have developed a detailed cost model for the proposed queueing system and calculated the total expected cost per unit of time. This approach enables decision-makers to make well-informed choices that enhance system performance while balancing various trade-offs effectively.

– Overall contribution: This study aims to provide valuable insights and methodologies for analyzing the performance of complex retrial queueing systems. By addressing the needs and challenges encountered in modern applications, the study contributes to various fields, including communication networks, cognitive radio networks, call centers, transportation, manufacturing, and service industries.

**Scope and novelty of this study**:

– The scope of this work is focused on addressing a significant research gap in queueing theory. Specifically, it examines the unreliable retrial queue $M_1, M_2/G_1, G_2/1$ with a repairable server, two types of customer arrivals, service orbit, and reserved times-an area not yet explored in the existing literature. By considering customers' impatience, which can manifest as either persistent or impatient behavior. This study aims to provide a comprehensive understanding of the dynamics and performance of such complex queueing systems.

– To achieve this objective, the study employs the supplementary variable technique and the generating function method to evaluate the behavior of the considered retrial queue. These techniques facilitate a thorough analysis of system dynamics, including the impact of both persistent and impatient customer behavior on system performance.

– Furthermore, the study extends beyond theoretical analysis by proposing a novel structure for the queueing system and presenting numerical examples to illustrate its practical implications. By using graphs, the study provides insights into the behavior of the proposed retrial queue under various scenarios and parameter settings.

– In addition to theoretical and numerical analysis, the study introduces a comprehensive cost function to quantify system performance. More precisely, we analyze how these performance measures impact the expected total cost of the system. We also explore how variations in service times, repair times, and customer behaviors influence overall costs. This analysis provides insights into cost-effective strategies for managing the queue and repair processes.

– Overall, the novelty of this work lies in its holistic approach to addressing a previously unexplored research gap in queueing theory. By combining analytical and numerical techniques, the study offers valuable insights into the sensitivity analysis of the performance of complex queueing systems with customers, who may exhibit either persistent or impatient behavior. This contributes to both theoretical advancements and practical applications across various domains.

The primary goal of this investigation is to assess the queue length distribution and the orbit size, which will aid in the development and evaluation of new metrics for measuring system behavior. Additionally, the managerial implications demonstrate how the model's results can be applied to enhance operational efficiency, customer satisfaction, and profitability in various service environments.

The paper is organized as follows. Section 2 presents a mathematical description of the model with a practical motivation. Section 3 offers an exhaustive analysis of the considered model. System performance measures and some reliability indices of interest are developed in Section 4. Illustrative numerical examples for sensitivity analysis are provided in Section 5. A cost analysis and the key managerial implications are discussed in Section 6. Finally, Section 7 concludes the paper.

## 2. DESCRIPTION OF THE MODEL

We consider an unreliable retrial queue $M_1, M_2/G_1, G_2/1$ with a repairable server, including balking and reneging, a service orbit, and reservations that accommodate either persistent or impatient customers. The dynamics of the retrial queueing model are illustrated in Figure 1.

This model is described as follows:

- Persistent (respectively, impatient) customers arrive at the system according to a Poisson process with rate $\lambda_1 > 0$ (resp. $\lambda_2 > 0$).

- If the server is idle upon arrival, then the service of the incoming customer (persistent or impatient) begins immediately. If a persistent customer arrives and finds the server blocked, the customer leaves the service area and enters the orbit. On the other hand, if an impatient customer arrives and finds the server blocked, the customer decides either to join the orbit with probability $m$, or to balk the system with probability $\overline{m} = 1 - m$.

- The retrial time of persistent (resp. impatient) customer follows an arbitrary distribution characterized by its probability distribution function (pdf) $A_1(w)$ (resp. $A_2(w)$) and density function $(df)$ $a_1(w)$ (resp. $a_2(w)$). Let $L_{A_1}[s]$ (resp. $L_{A_2}[s]$) denote the Laplace-Stieltjes Transform $(LST)$ of the distribution $A_1(w)$ (resp. $A_2(w)$), and $\alpha_{1j} = (-1)^j L^j_{A_1}[0]$ (resp. $\alpha_{2j} = (-1)^j L^j_{A_2}[0]$) be the $j$th moment of this distribution for persistent (resp. impatient) customers.

- We assume that the service policy follows the $FCFS$ discipline. This means that if the primary customer (a customer arriving from outside) arrives first, it will cancel any access attempts by the secondary customer (a customer coming from the orbit) to the server. In this case, the secondary customer will either return to their position in the orbit with probability $q$ or renege from the system with probability $\overline{q} = 1 - q$.

- The service times of both types of customers are assumed to be independent and identically distributed (iid). The service times for persistent (resp. impatient) customers follow a general distribution with $pdf$ $B_1(x)$ (resp. $B_2(x)$), and $b_1(x)$ (resp. $b_2(x)$) denotes the $df$ of the service time. Let $L_{B_1}[s]$ (resp. $L_{B_2}[s]$) be the $LST$ of the service time distribution for persistent (resp. impatient) customers, and $\beta_{1j} = (-1)^j L^j_{B_1}[0]$ (resp. $\beta_{2j} = (-1)^j L^j_{B_2}[0]$) represents the $j$th moment of the service time of a persistent (resp. impatient) customer.

- The server is subject to active breakdowns and fails after an exponentially distributed time with mean $1/\theta$. Upon breakdown, it undergoes corrective and immediate repair, aimed at quickly addressing any faults or failures, thereby maintaining system performance and reliability, and the repair time is a random variable characterized by an unknown with pdf $C(y)$, with df $c(y)$, LST $L_C[s]$, and the first two moments denoted by $\gamma_1$ and $\gamma_2$, where $\gamma_j = (-1)^j L_C^j[0]$ for $j = 1,2$.

- When an impatient customer's service is interrupted, it has two options: either it stays in the service area with a probability $r$, or it enters a service orbit with a complementary probability $\overline{r} = 1-r$. In contrast, a persistent customer permanently chooses to stay in front of the server. If an impatient customer enters the service orbit due to a server failure, the server must wait for the customer to return after the server repair. This waiting period is referred to as the reserved time. The reserved time is governed by an arbitrary distribution with a pdf $D(v)$, a df $d(v)$, a LST $L_D[s]$, and the first two moments denoted by $\delta_1$ and $\delta_2$, where $\delta_j = (-1)^j L_D^j[0]$ for $j = 1,2$.

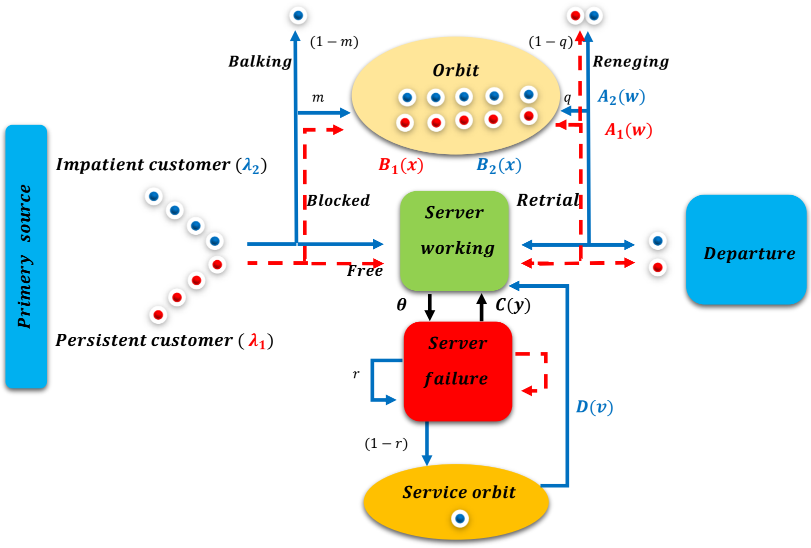


**Figure 1:** Dynamics of the retrial queueing model

All the introduced variables are mutually independent of each other.

Conditional completion rates: for retrial time of a persistent (resp. an impatient) customer is $\alpha_1(w) = \frac{a_1(w)}{1-A_1(w)}$ (resp. $\alpha_2(w) = \frac{a_2(w)}{1-A_2(w)}$), for service time of a persistent (resp. an impatient) customer is $\beta_1(x) = \frac{b_1(x)}{1-B_1(x)}$ (resp. $\beta_2(x) = \frac{b_2(x)}{1-B_2(x)}$), for repair time (resp. reserved time) is $\gamma(y) = \frac{c(y)}{1-C(y)}$ (resp. $\delta(v) = \frac{d(v)}{1-D(v)}$).

Now, we define the probability distribution function (pdf) $F(x)$, and its complement $\overline{F}(x) = 1-F(x)$ over the interval $[0,1]$. The *LST* of the *pdf*, denoted as $L_F[s]$, is given by: $L_F[s] = \int_0^\infty e^{-sx} dF(x)$. The complementary *LST*, $\overline{L}_F[s]$, is defined as: $\overline{L}_F[s] = \int_0^\infty e^{-sx}(1-F(x))dx$. This complementary *LST* can be related to $L_F[s]$ by the formula: $\overline{L}_F[s] = \frac{1-L_F[s]}{s}$.

We use $\hbar$ as a parameter to differentiate between the service periods offered to impatient and persistent customers within the system, where:

$$\hbar = \begin{cases} 0, & \text{indicates service to impatient customers,} \\ 1, & \text{indicates service to persistent customers.} \end{cases}$$

### 2.1. Practical justification for the proposed queueing model in call centers

**Example 1.** *Customer service plays a crucial role in telecom companies such as Ooredoo, Orange, and T-Mobile, which offer a wide range of communication services. These companies assign two-way phones in their service centers to effectively handle customer requests, complaints, and issues.*

*In this system, the server represents the radio channel that customers use to make their calls. Customers can be classified into two categories based on their calling behavior:*

*Persistent callers: These customers persistently attempt to make a call, even when faced with difficulties or delays. They demonstrate determination and patience in their pursuit of accessing the service.*

*Impatient callers: These customers are more likely to give up or balk if they encounter obstacles or delays while trying to access the service. They are prone to abandoning their call attempts under unfavorable circumstances.*

*Impatient callers exhibit different behaviors when they encounter an unavailable communication channel:*

*Probability $m$: The primary impatient caller continues to wait for the communication channel to become free, remaining in the retrial orbit. They are willing to wait for their turn and keep retrying.*

*Probability $\overline{m}$: The primary impatient caller leaves the current channel and seeks an alternative option by balking.*

*Probability $q$: The secondary impatient caller makes another call attempt after a random interval while in the retrial orbit.*

*Probability $\overline{q}$: The secondary impatient caller leaves the system by reneging, deciding not to retry, and abandoning their intention to use the service.*

*The communication channel can handle only one call at a time due to its limited capacity. Factors such as network congestion or technical issues can impact the channel's reliability. During maintenance or downtime, callers are unable to access the service until the connection is restored.*

*When the channel fails and an impatient caller is using it, this caller has two options:*

**i-** *With probability $r$, it can remain in front of the server to complete their service,*

**ii-** *With probability $\overline{r}$, it can enter the service orbit.*

*Once the channel is repaired and becomes available again, it must wait for the impatient customer in the service orbit to return. Customers in the service orbit are given the highest priority to regain access. This prioritization ensures uninterrupted service and enhances user satisfaction.*

## 3. QUEUEING MODEL ANALYSIS

In this section, we establish the stability condition and the steady-state distributions using Markov process techniques (supplementary variables method and generating functions). Figure 2 shows the rate transition diagram.

The Markov process can be represented by:

$$X(t)_{t\geq 0}=\{\varpi(t),\varphi(t),\kappa(t),\xi_{01}(t),\xi_{02}(t),\xi_1(t),\xi_2(t),\xi_3(t),\xi_4(t)\}_{t\geq 0},$$

in which,

- $\varpi(t)$ indicates the server state:

$$\varpi(t)=\begin{cases} 0, \text{ if the server is free,}\\ 1 \text{ (resp. 2), if the persistent (resp. impatient) customer occupies}\\ \quad\text{the server,}\\ 3, \text{ if the server is being repaired after an active breakdown,}\\ 4, \text{ if the server is reserved.}\end{cases}$$

- $\varphi(t)$ denotes the state of a customer in service after an active breakdown:

$$\varphi(t)=\begin{cases} 0 \text{ (resp. 1), if the persistent (resp. impatient) customer in service}\\ \quad\text{remains in front of the server,}\\ 2, \text{ the impatient customer in service enters into the service orbit.}\end{cases}$$

- $\kappa(t)$ indicates the number of customers in the orbit at time $t$.

$\xi_{01}(t)$ (resp. $\xi_{02}(t)$): Elapsed retrial time of a persistent (resp. an impatient) customer,

$\xi_1(t)$ (resp. $\xi_2(t)$): Elapsed service time of a persistent (resp. an impatient) customer,

$\xi_3(t)$: Elapsed repair time,

$\xi_4(t)$: Elapsed reserved time of an impatient customer.

We establish the corresponding probabilities for the scenario where there are $n$ customers in the system at time $t$:

- $P^{(1)}_{0,0}(t)=P^{(1)}(\varpi(t)=0,\kappa(t)=0)$ (resp. $P^{(2)}_{0,0}(t)=P^{(2)}(\varpi(t)=0,\kappa(t)=0)$): Probability that the system is empty when the server is serving persistent (resp. impatient) customers.
- $P^{(1)}_{0,n}(t,w)\partial w=P^{(1)}(\varpi(t)=0,\kappa(t)=n,w\leq\xi_{01}(t)<w+\partial w)$ (resp. $P^{(2)}_{0,n}(t,w)\partial w=P^{(2)}(\varpi(t)=0,\kappa(t)=n,w\leq\xi_{02}(t)<w+\partial w)$, $n\geq 1$: Probability that the server is free during the retrial period of persistent (resp. impatient) customers.
- $P_{1,n}(t,x)\partial x$=$P(\varpi(t)=1,\kappa(t)=n,x\leq\xi_1(t)<x+\partial x)$ (resp. $P_{2,n}(t,x)\partial x=P(\varpi(t)=2,\kappa(t)=n,x\leq\xi_2(t)<x+\partial x)$): Probability that the server will be occupied by a persistent (resp. an impatient) customer during the retrial period.

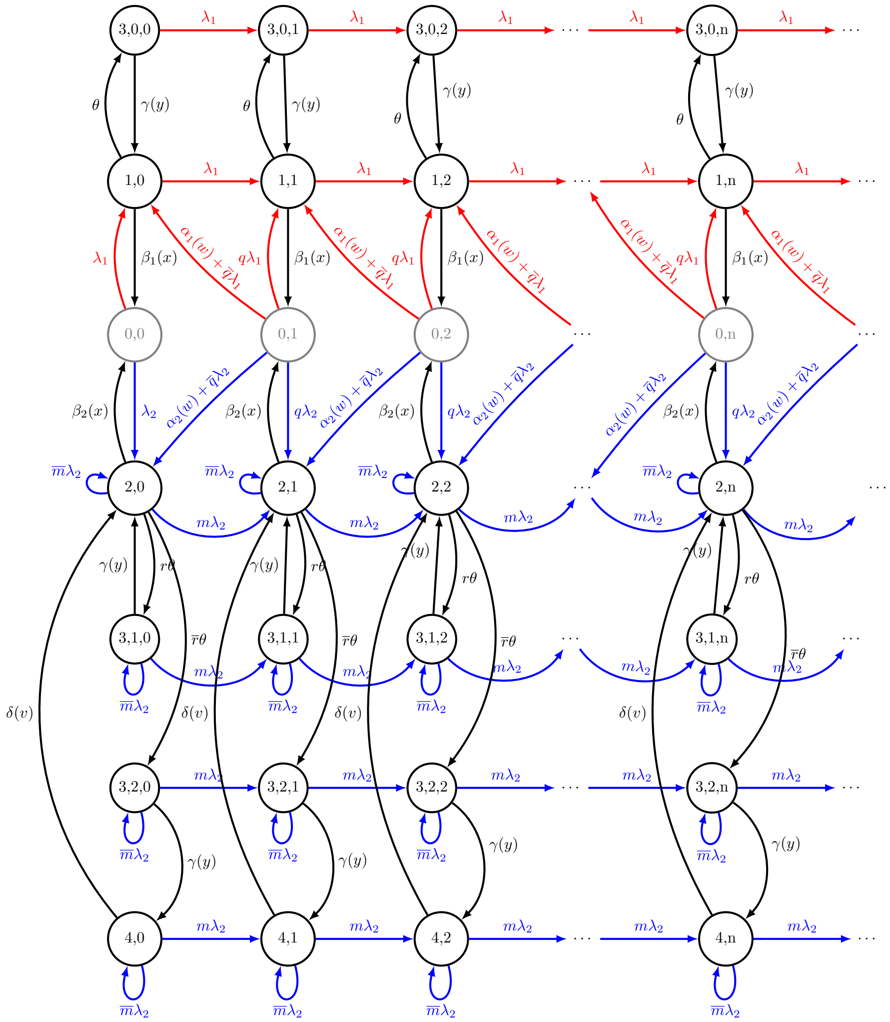


**Figure 2:** Rate transition diagram of the model

- $P_{3,0,n}(t,x,y)\partial x\partial y$=$P(\varpi(t)=3,\varphi(t)=0,\kappa(t)=n,x\leq\xi_1(t)<x+\partial x,y\leq\xi_3(t)<y+\partial y)$ (resp. $P_{3,1,n}(t,x,y)\partial x\partial y$ =$P(\varpi(t)=3,\varphi(t)=1,\kappa(t)=n,x\leq\xi_2(t)<x+\partial x,y\leq\xi_3(t)<y+\partial y)$): Probability that the persistent (resp. impatient) customer remains in front of the server when the server is under repair.
- $P_{3,2,n}(t,x,y)\partial x\partial y$=$P(\varpi(t)=3,\varphi(t)=2,\kappa(t)=n,x\leq\xi_2(t)<x+\partial x,y\leq\xi_3(t)<y+\partial y)$: Probability that the impatient customer enters into a service orbit when the server is under repair.
- $P_{4,n}(t,x,v)\partial x\partial v$=$P(\varpi(t)=4,\kappa(t)=n,x\leq\xi_2(t)<x+\partial x,v\leq\xi_4(t)<v+\partial v)$: Probability that the impatient customer reserved the server during the retrial period.

### 3.1. The steady-state distribution

Now, we define the limiting probabilities associated with the process $X(t)_{t\geq 0}$:

$$\lim_{t\to\infty} P^{(k)}_{0,0}(t) = P^{(k)}_{0,0},\ \lim_{t\to\infty} P^{(k)}_{0,n}(t,w) = P^{(k)}_{0,n}(w),\ \lim_{t\to\infty} P_{k,n}(t,x) = P_{k,n}(x),\ k=1,2,$$
$$\lim_{t\to\infty} P_{3,l,n}(t,x,y) = P_{3,l,n}(x,y),\ l=0,1,2,\ \lim_{t\to\infty} P_{4,n}(t,x,v) = P_{4,n}(x,v).$$

Let us assume that for $x=y=v=0$; $P_{k,-1}(x)=0$, for $k=1,2$; $P_{3,l,-1}(x,y)=0$, for $l=0,1,2$; and $P_{4,-1}(x,v)=0$.

### 3.2. The generating functions

Next, we introduce the following probability-generating functions:

$$P^{(k)}_0(z,w) = \sum_{n=1}^{\infty} P^{(k)}_{0,n}(w)z^n,\ \ P_k(z,x) = \sum_{n=0}^{\infty} P_{k,n}(x)z^n,\ k=1,2,$$
$$P_{3,l}(z,x,y) = \sum_{n=0}^{\infty} P_{3,l,n}(x,y)z^n,\ l=0,1,2,\ \ P_4(z,x,v) = \sum_{n=0}^{\infty} P_{4,n}(x,v)z^n,$$

which are convergent for each $w\geq 0$, $x\geq 0$, $y\geq 0$, $v\geq 0$, and for all $|z|\leq 1$.

**Theorem 2.** *The inequality* $\hbar\left[\frac{\lambda_1\beta_{11}\left(1+\theta\gamma_1\right)}{1-q(1-L_{A_1}[\lambda_1])}\right]+\bar{\hbar}\left[\frac{m\lambda_2\beta_{21}\left(1+\theta\gamma_1+\bar{r}\theta\delta_1\right)}{1-q(1-L_{A_2}[\lambda_2])}\right]<1$, *is a necessary and sufficient condition for the ergodicity.*

*Proof.* By applying the technique of supplementary variables to the Kolmogorov differential equations and multiplying these equations by $z^n$, considering the sum over $n>0$, we find the following expressions:

$$[\lambda_k+\alpha_k(w)+\frac{\partial}{\partial w}]P^{(k)}_0(z,w) \quad = \quad 0,\ \ k=1,2, \tag{1}$$

$$\begin{aligned}[\theta+\lambda_1+\beta_1(x)+\frac{\partial}{\partial x}]P_1(z,x) \quad &= \quad \int_0^{\infty}\gamma(y)P_{3,0}(z,x,y)\,dy \\ &+ \quad \lambda_1 zP_1(z,x),\end{aligned} \tag{2}$$

$$\begin{aligned}\left[\theta+m\lambda_2+\beta_2(x)+\frac{\partial}{\partial x}\right]P_2(z,x) \quad &= \quad \int_0^{\infty}\gamma(y)P_{3,1}(z,x,y)\,dy+\int_0^{\infty}\delta(v) \\ &\times \quad P_4(z,x,v)\,dv+m\lambda_2 zP_2(z,x),\end{aligned} \tag{3}$$

$$\left[\lambda_1+\gamma(y)+\frac{\partial}{\partial x}\right]P_{3,0}(z,x,y) \quad = \quad \lambda_1 zP_{3,0}(z,x,y), \tag{4}$$

$$\left[m\lambda_2+\gamma(y)+\frac{\partial}{\partial x}\right]P_{3,1}(z,x,y) \quad = \quad m\lambda_2 zP_{3,1}(z,x,y), \tag{5}$$

$$\left[m\lambda_2+\gamma(y)+\frac{\partial}{\partial x}\right]P_{3,2}(z,x,y) = m\lambda_2 zP_{3,2}(z,x,y), \tag{6}$$

$$\left[m\lambda_2+\delta(v)+\frac{\partial}{\partial x}\right]P_4(z,x,v) = m\lambda_2 zP_4(z,x,v), \tag{7}$$

$$P_0^{(k)}(z,0) = \int_0^\infty P_k(z,x)\beta_k(x)\,dx-\lambda_k P_{0,0}^{(k)},\ \ k=1,2, \tag{8}$$

$$\begin{aligned}P_1(z,0) &= \frac{\lambda_1(1-q\bar{z})}{z}\int_0^\infty P_0^{(1)}(z,w)\,dw+\frac{1}{z}\int_0^\infty \alpha_1(w)P_0^{(1)}(z,w)\,dw\\ &+ \lambda_1 P_{0,0}^{(1)},\end{aligned} \tag{9}$$

$$\begin{aligned}P_2(z,0) &= \frac{\lambda_2(1-q\bar{z})}{z}\int_0^\infty P_0^{(2)}(z,w)\,dw+\frac{1}{z}\int_0^\infty \alpha_2(w)P_0^{(2)}(z,w)\,dw\\ &+ \lambda_2 P_{0,0}^{(2)},\end{aligned} \tag{10}$$

$$P_{3,0}(z,x,0) = \theta P_1(z,x), \tag{11}$$

$$P_{3,1}(z,x,0) = r\theta P_2(z,x), \tag{12}$$

$$P_{3,2}(z,x,0) = \bar{r}\theta P_2(z,x), \tag{13}$$

$$P_4(z,x,0) = \int_0^\infty \gamma(y)P_{3,2}(z,x,y)\,dy. \tag{14}$$

The normalization equation is given as follows:

$$\begin{aligned}&\hbar\Bigg[P_{0,0}^{(1)}+\sum_{n=1}^{\infty}\int_0^\infty P_{0,n}^{(1)}(1,w)\,dw+\sum_{n=0}^{\infty}\Bigg(\int_0^\infty P_{1,n}(1,x)\,dx\\ &+\int_0^\infty\int_0^\infty P_{3,0,n}(1,x,y)\,dxdy\Bigg)\Bigg]+\bar{\hbar}\Bigg[P_{0,0}^{(2)}+\sum_{n=1}^{\infty}\int_0^\infty P_{0,n}^{(2)}(1,w)\,dw\\ &+\sum_{n=0}^{\infty}\Bigg(\int_0^\infty P_{2,n}(1,x)\,dx+\int_0^\infty\int_0^\infty P_{3,1,n}(1,x,y)\,dxdy\\ &+\int_0^\infty\int_0^\infty P_{3,2,n}(1,x,y)\,dxdy+\int_0^\infty\int_0^\infty P_{4,n}(1,x,v)\,dxdv\Bigg)\Bigg]=1.\end{aligned} \tag{15}$$

Let,

$$\begin{aligned}N(z) &= \lambda_1\bar{z}+\theta\Big(1-L_C[\lambda_1\bar{z}]\Big),\\ M(z) &= m\lambda_2\bar{z}+\theta-L_C[m\lambda_2\bar{z}]\Big(r\theta+\bar{r}\theta L_D[m\lambda_2\bar{z}]\Big).\end{aligned}$$

By substituting equation (11) into (4), we find

$$P_{3,0}(z,x,y) = \theta P_1(z,x)e^{-\lambda_1\bar{z}y}(1-C(y)). \tag{16}$$

Now, by replacing equation (12) into (5), we get

$$P_{3,1}(z,x,y) \quad = \quad r\theta P_2(z,x)e^{-\bar{z}m\lambda_2 y}(1-C(y)). \tag{17}$$

Substituting equation (13) into (6), gives

$$P_{3,2}(z,x,y) \quad = \quad \bar{r}\theta P_2(z,x)e^{-\bar{z}m\lambda_2 y}(1-C(y)). \tag{18}$$

Replacing equations (14) and (18) into (7), leads to

$$P_4(z,x,v) \quad = \quad \bar{r}\theta L_C[\bar{z}m\lambda_2]P_2(z,x)e^{-\bar{z}m\lambda_2 v}(1-D(v)). \tag{19}$$

By substituting equations (9) and (16) into (2), we find

$$\begin{aligned} P_1(z,x) \quad &= \quad \left[\frac{\lambda_1(1-q\bar{z})}{z}\int_0^\infty P_0^{(1)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_1(w)P_0^{(1)}(z,w)\,dw \right. \\ &+ \quad \left. \lambda_1 P_{0,0}^{(1)}\right] e^{-N(z)x}\overline{B}_1(x). \end{aligned} \tag{20}$$

Substitution of equations (10), (17), and (19) into (3), provides

$$\begin{aligned} P_2(z,x) \quad &= \quad \left[\frac{\lambda_2(1-q\bar{z})}{z}\int_0^\infty P_0^{(2)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_2(w)P_0^{(2)}(z,w)\,dw \right. \\ &+ \quad \left. \lambda_2 P_{0,0}^{(2)}\right] e^{-M(z)x}\overline{B}_2(x). \end{aligned} \tag{21}$$

From equations (16) and (20), we get

$$\begin{aligned} P_{3,0}(z,x,y) \quad &= \quad \theta\left[\frac{\lambda_1(1-q\bar{z})}{z}\int_0^\infty P_0^{(1)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_1(w)P_0^{(1)}(z,w)\,dw \right. \\ &+ \quad \left. \lambda_1 P_{0,0}^{(1)}\right] e^{-N(z)x}\overline{B}_1(x)\; e^{-\lambda_1\bar{z}y}\overline{C}(y). \end{aligned} \tag{22}$$

From equations (17) and (21), we find

$$\begin{aligned} P_{3,1}(z,x,y) \quad &= \quad r\theta\left[\frac{\lambda_2(1-q\bar{z})}{z}\int_0^\infty P_0^{(2)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_2(w)P_0^{(2)}(z,w)\,dw \right. \\ &+ \quad \left. \lambda_2 P_{0,0}^{(2)}\right] e^{-M(z)x}\overline{B}_2(x)e^{-m\lambda_2\bar{z}y}\overline{C}(y). \end{aligned} \tag{23}$$

According to equations (18) and (21), we get

$$\begin{aligned} P_{3,2}(z,x,y) \quad &= \quad \bar{r}\theta\left[\frac{\lambda_2(1-q\bar{z})}{z}\int_0^\infty P_0^{(2)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_2(w)P_0^{(2)}(z,w)\,dw \right. \\ &+ \quad \left. \lambda_2 P_{0,0}^{(2)}\right] e^{-M(z)x}\overline{B}_2(x)e^{-m\lambda_2\bar{z}y}\overline{C}(y). \end{aligned} \tag{24}$$

From equations (19) and (21), we have

$$\begin{aligned} P_4(z,x,v) &= \bar{r}\theta\left[\frac{\lambda_2(1-q\bar{z})}{z}\int_0^\infty P_0^{(2)}(z,w)\,dw + \frac{1}{z}\int_0^\infty \alpha_2(w)P_0^{(2)}(z,w)\,dw \right. \\ &+ \left. \lambda_2 P_{0,0}^{(2)}\right] L_C[m\lambda_2\bar{z}]e^{-M(z)x}\overline{B}_2(x)e^{-m\lambda_2\bar{z}v}\overline{D}(v). \end{aligned} \tag{25}$$

From equations (1) (for $k=1$), (8) (for $k=1$), and (20), we obtain

$$P_0^{(1)}(z,0) = \frac{\lambda_1 z P_{0,0}^{(1)}\Big[1-L_{B_1}[N(z)]\Big]}{L_{B_1}[N(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_1}[\lambda_1]\Big)-z}. \tag{26}$$

From equations (1) (for $k=2$), (8) (for $k=2$), and (21), we find

$$P_0^{(2)}(z,0) = \frac{\lambda_2 z P_{0,0}^{(2)}\Big[1-L_{B_2}[M(z)]\Big]}{L_{B_2}[M(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_2}[\lambda_2]\Big)-z}. \tag{27}$$

We introduce equation (26) in (1) (for $k=1$), we obtain

$$P_0^{(1)}(z,w) = \frac{\lambda_1 z P_{0,0}^{(1)}\Big[1-L_{B_1}[N(z)]\Big]e^{-\lambda_1 w}\overline{A}_1(w)}{L_{B_1}[N(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_1}[\lambda_1]\Big)-z}. \tag{28}$$

We introduce equation (27) in (1) (for $k=2$), we find

$$P_0^{(2)}(z,w) = \frac{\lambda_2 z P_{0,0}^{(2)}\Big[1-L_{B_2}[M(z)]\Big]e^{-\lambda_2 w}\overline{A}_2(w)}{L_{B_2}[M(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_2}[\lambda_2]\Big)-z}. \tag{29}$$

Similarly, by determining $P_0^{(1)}(z,w)$ and $P_0^{(2)}(z,w)$ as in equations (28) and (29), we can derive the expressions for $P_1(z,x)$, $P_2(z,x)$, $P_{30}(z,x,y)$, $P_{31}(z,x,y)$, $P_{32}(z,x,y)$, and $P_4(z,x,v)$, all of which depend on $P_{0,0}^{(1)}$ or $P_{0,0}^{(2)}$.

Using the normalization condition (see equation (15)), the probability $P_{0,0}$ can be found as follows: $P_{0,0} = \hbar P_{0,0}^{(1)} + \bar{\hbar} P_{0,0}^{(2)}$.

Thus, after some mathematical manipulations, we find that:

$$\begin{aligned} P_{0,0} = \hbar&\left[\frac{1-q(1-L_{A_1}[\lambda_1])-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)}{\Big[1-q(1-L_{A_1}[\lambda_1])\Big]\Big[1+\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)\Big]-\lambda_1\beta_{11}L_{A_1}[\lambda_1]\Big(1+\theta\gamma_1\Big)}\right] \\ +\bar{\hbar}&\left[\frac{1-q(1-L_{A_2}[\lambda_2])-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}{\Big[1-q(1-L_{A_2}[\lambda_2])\Big]\Big[1+\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)\Big]-m\lambda_2\beta_{21}L_{A_2}[\lambda_2]\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}\right]. \end{aligned} \tag{30}$$

From equation (30), we obtain the expression $\rho = 1 - P_{0,0}$. To ensure the stability of the system, it is necessary and sufficient that $\rho < 1$. This condition guarantees that the probability $P_{0,0}$ (probability that the system is empty) is strictly positive. In other words, the system is stable if and only if:

$$\hbar\left[\frac{\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)}{1-q(1-L_{A_1}[\lambda_1])}\right]+\bar{\hbar}\left[\frac{m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}{1-q(1-L_{A_2}[\lambda_2])}\right]<1.$$

□

## 4. SYSTEM PERFORMANCE MEASURES

The main aim of this section is to derive explicit formulas for the probabilities of server states, as well as for various performance measures.

### 4.1. Probability of the server states:

**Corollary 3.** *If the stability condition is satisfied, the probability generating functions for the server states are as follows:*

1. *When the server is free during the retrial period of persistent (resp. impatient) customers:*

$$P_0^{(1)} = \frac{\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)\Big(1-L_{A_1}[\lambda_1]\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)},$$

$$P_0^{(2)} = \frac{m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)\Big(1-L_{A_2}[\lambda_2]\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

2. *When the server is occupied by a persistent (resp. an impatient) customer:*

$$P_1 = \frac{\lambda_1\beta_{11}\Big(1-q(1-L_{A_1}[\lambda_1])\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)},$$

$$P_2 = \frac{\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

3. *When the server is under repair and the persistent (resp. impatient) customer currently being served must remains in the service area:*

$$P_{3,0} = \frac{\theta\gamma_1\lambda_1\beta_{11}\Big(1-q(1-L_{A_1}[\lambda_1])\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)}.$$

$$P_{3,1} = \frac{r\theta\gamma_1\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

4. *When the server is under repair and the impatient customer currently being served enters into a service orbit:*

$$P_{3,2} = \frac{\bar{r}\theta\gamma_1\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

5. *When the server is reserved by an impatient customer:*

$$P_4 = \frac{\bar{r}\theta\delta_1\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

6. *When the server is blocked by a persistent customer:*

$$P_{B_P} = \frac{\lambda_1\beta_{11}\Big(1-q(1-L_{A_1}[\lambda_1])\Big)\Big(1+\theta\gamma_1\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)}.$$

7. *When the server is blocked by an impatient customer:*

$$P_{B_I} = \frac{\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)\Big(1-q\big(1-L_{A_2}[\lambda_2]\big)\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

8. *When the server is under repair:*

$$P_{Repair} = \frac{\hbar\theta\gamma_1\lambda_1\beta_{11}\Big(1-q(1-L_{A_1}[\lambda_1])\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)} + \frac{\bar{\hbar}\theta\gamma_1\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

*Proof.* The demonstration is based on the following mathematical relationships: $P_0^{(k)} = \int_0^\infty P_0^{(k)}(1,w)\,dw$, $P_k = \int_0^\infty P_k(1,x)\,dx$, for $k=1,2$; $P_{3,j} = \int_0^\infty\int_0^\infty P_{3,j}(1,x,y)\,dxdy$, for $j=0,1,2$; $P_4 = \int_0^\infty\int_0^\infty P_4(1,x,v)\,dxdv$; $P_{B_P} = P_1 + P_{3,0}$; $P_{B_I} = P_2 + P_{3,1} + P_{3,2} + P_4$; and $P_{Repair} = \hbar P_{3,0} + \bar{\hbar}(P_{3,1}+P_{3,2})$. □

### 4.2. Reliability analysis

In this section, we aim to present reliability indices for our retrial queue.

**Corollary 4.** *If the system is in a steady state,*

1. *The active breakdown frequency is defined as:*

$$W_{AB_F} = \frac{\hbar\theta\lambda_1\beta_{11}\Big(1-q(1-L_{A_1}[\lambda_1])\Big)P_{0,0}^{(1)}}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)} + \frac{\bar{\hbar}\theta\lambda_2\beta_{21}\Big(1-q(1-L_{A_2}[\lambda_2])\Big)P_{0,0}^{(2)}}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}.$$

2. *The state system availability $A_v$ is obtained as:*

$$
\begin{aligned}
A_v &= \hbar P_{0,0}^{(1)}\left[1+\frac{\lambda_1\beta_{11}\Big(2+\theta\gamma_1-(1+\theta\gamma_1-q)L_{A_1}[\lambda_1]-q\Big)}{1-q\Big(1-L_{A_1}[\lambda_1]\Big)-\lambda_1\beta_{11}\Big(1+\theta\gamma_1\Big)}\right]+\bar{\hbar}P_{0,0}^{(2)}\Bigg[1+\\
&+ \frac{\lambda_2\beta_{21}\Bigg(1+m(1+\theta\gamma_1+\bar{r}\theta\delta_1)-L_{A_2}[\lambda_2]\Big(m(1+\theta\gamma_1+\bar{r}\theta\delta_1)+q\Big)\Bigg)}{1-q\Big(1-L_{A_2}[\lambda_2]\Big)-m\lambda_2\beta_{21}\Big(1+\theta\gamma_1+\bar{r}\theta\delta_1\Big)}\Bigg].
\end{aligned}
$$

*Proof.* To derive the results for the active breakdown frequency of the server and the steady-state availability of the server. It is easy to see that:

$$
\begin{aligned}
W_{AB_F} &= \theta(\hbar P_1+\bar{\hbar}P_2),\\
A_v &= \hbar(P_{0,0}^{(1)}+P_0^{(1)}+P_1)+\bar{\hbar}(P_{0,0}^{(2)}+P_0^{(2)}+P_2).
\end{aligned}
$$

□

### 4.3. Number of customers in terms of generating functions

**Corollary 5.** *The probability generating functions associated with the number of customers in the retrial orbit ($P_o(z)$) and in the system ($P_s(z)$) are given as follows:*

$$
\begin{aligned}
P_o(z) &= P_{0,0}+\hbar\left(P_0^{(1)}(z)+P_1(z)\Big[1+\theta\overline{L}_C[\lambda_1\bar{z}]\Big]\right)+\bar{\hbar}\Bigg(P_0^{(2)}(z)\\
&+ P_2(z)\Big[1+\theta\overline{L}_C[m\lambda_2\bar{z}]+\bar{r}\theta L_C[m\lambda_2\bar{z}]\overline{L}_D[m\lambda_2\bar{z}]\Big]\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
P_s(z) &= P_{0,0}+\hbar\left(P_0^{(1)}(z)+zP_1(z)\Big[1+\theta\overline{L}_C[\lambda_1\bar{z}]\Big]\right)+\bar{\hbar}\Bigg(P_0^{(2)}(z)\\
&+ zP_2(z)\Big[1+\theta\overline{L}_C[m\lambda_2\bar{z}]+\bar{r}\theta L_C[m\lambda_2\bar{z}]\overline{L}_D[m\lambda_2\bar{z}]\Big]\Bigg),
\end{aligned}
$$

*where,*

$$
P_0^{(1)}(z)=\frac{zP_{0,0}^{(1)}\Big[1-L_{B_1}[N(z)]\Big]\Big[1-L_{A_1}[\lambda_1]\Big]}{L_{B_1}[N(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_1}[\lambda_1]\Big)-z},\ P_1(z)=\frac{\lambda_1P_{0,0}^{(1)}\Big[(1-q\bar{z})+q\bar{z}L_{A_1}[\lambda_1]-z\Big]\overline{L}_{B_1}[N(z)]}{L_{B_1}[N(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_1}[\lambda_1]\Big)-z},
$$

$$
P_0^{(2)}(z)=\frac{zP_{0,0}^{(2)}\Big[1-L_{B_2}[M(z)]\Big]\Big[1-L_{A_2}[\lambda_2]\Big]}{L_{B_2}[M(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_2}[\lambda_2]\Big)-z},\ P_2(z)=\frac{\lambda_2P_{0,0}^{(2)}\Big[(1-q\bar{z})+q\bar{z}L_{A_2}[\lambda_2]-z\Big]\overline{L}_{B_2}[M(z)]}{L_{B_2}[M(z)]\Big((1-q\bar{z})+q\bar{z}L_{A_2}[\lambda_2]\Big)-z}.
$$

*Proof.* The generating functions are given by:
$P_o(z)=\hbar P_o^{(1)}(z)+\bar{\hbar}P_o^{(2)}(z)$, and $P_s(z)=\hbar P_s^{(1)}(z)+\bar{\hbar}P_s^{(2)}(z)$,

with,

$$P_o^{(1)}(z) = P_{0,0}^{(1)} + P_0^{(1)}(z) + P_1(z) + P_{3,0}(z),$$
$$P_o^{(2)}(z) = P_{0,0}^{(2)} + P_0^{(2)}(z) + P_2(z) + P_{3,1}(z) + P_{3,2}(z) + P_4(z),$$
$$P_s^{(1)}(z) = P_{0,0}^{(1)} + P_0^{(1)}(z) + z\Big(P_1(z) + P_{3,0}(z)\Big),$$
$$P_s^{(2)}(z) = P_{0,0}^{(2)} + P_0^{(2)}(z) + z\Big(P_2(z) + P_{3,1}(z) + P_{3,2}(z) + P_4(z)\Big).$$

□

### 4.4. Mean performance measures

**Corollary 6.** • *Mean number of customers in the orbit ($E[N_o]$) and in the system ($E[N_s]$):*

$$E[N_o] = \hbar\Big[\eta_1^{(1)} + \eta_2^{(1)} + \eta_3^{(1)}\Big] + \bar{\hbar}\Big[\eta_1^{(2)} + \eta_2^{(2)} + \eta_3^{(2)}\Big],$$
$$E[N_s] = E[N_o] + \hbar P_1\Big(1 + \theta\gamma_1\Big) + \bar{\hbar}P_2\Big(1 + \theta\gamma_1 + \bar{r}\theta\delta_1\Big).$$

• *Mean waiting time of customers in the orbit ($W_o$) and in the system ($W_s$):*

$$W_o = \frac{E[N_o]}{\hbar\lambda_1 + \bar{\hbar}\lambda_2}, \quad W_s = \frac{E[N_s]}{\hbar\lambda_1 + \bar{\hbar}\lambda_2},$$

*where,*

$$\eta_1^{(1)} = \frac{E_1''(1)E_2'(1) - E_1'(1)E_2''(1)}{2\Big(E_2'(1)\Big)^2}, \quad \eta_1^{(2)} = \frac{G_1''(1)G_2'(1) - G_1'(1)G_2''(1)}{2\Big(G_2'(1)\Big)^2},$$

$$\eta_2^{(1)} = P_1'(1)\Big(1 + \theta\gamma_1\Big), \quad \eta_2^{(2)} = P_2'(1)\Big(1 + \theta\gamma_1 + \bar{r}\theta\delta_1\Big),$$

$$\eta_3^{(1)} = P_1\Big(\frac{\lambda_1\theta\gamma_2}{2}\Big), \quad \eta_3^{(2)} = P_2\Bigg(\frac{m\lambda_2\Big(\bar{r}\theta\Big(2\gamma_1\delta_1 + \delta_2\Big) + \theta\gamma_2\Big)}{2}\Bigg),$$

$$P_1'(1) = \frac{E_3'''(1)E_4''(1) - E_3''(1)E_4'''(1)}{3\Big(E_4''(1)\Big)^2}, \quad P_2'(1) = \frac{G_3'''(1)G_4''(1) - G_3''(1)G_4'''(1)}{3\Big(G_4''(1)\Big)^2},$$

$$E_1'(1) = -\lambda_1\beta_{11}P_{0,0}^{(1)}\Big[1 - L_{A_1}[\lambda_1]\Big]\Big[1 + \theta\gamma_1\Big],$$

$$E_1''(1) = -\lambda_1 P_{0,0}^{(1)}\Big[1 - L_{A_1}[\lambda_1]\Big]\Bigg(4\beta_{11}\Big[1 + \theta\gamma_1\Big] + \lambda_1\beta_{12}\Big[1 + \theta\gamma_1\Big]^2 + \lambda_1\beta_{11}\Big[\theta\gamma_2\Big]\Bigg),$$

$$E_2'(1) = \lambda_1\beta_{11}\Big[1 + \theta\gamma_1\Big] + q\Big[1 - L_{A_1}[\lambda_1]\Big] - 1,$$

$$E_2''(1) = \lambda_1\Bigg(2\beta_{11}q\Big[1 - L_{A_1}[\lambda_1]\Big]\Big[1 + \theta\gamma_1\Big] + \lambda_1\beta_{12}\Big[1 + \theta\gamma_1\Big]^2 + \lambda_1\beta_{11}\Big[\theta\gamma_2\Big]\Bigg),$$

$$E_3''(1) = -2\lambda_1^2\beta_{11}P_{0,0}^{(1)}\Big[q(1 - L_{A_2}[\lambda_2]) - 1\Big]\Big[1 + \theta\gamma_1\Big],$$

$$E_3'''(1) = -3\lambda_1 P_{0,0}^{(1)}\Big[q(1 - L_{A_1}[\lambda_1]) - 1\Big]\Big[\big(N'(1)\big)^2\beta_{12} - N''(1)\beta_{11} - 2\beta_{11}N'(1)\Big],$$

$$E_4''(1) = 2N'(1)E_2'(1), \quad E_4'''(1) = 3\Big[E_2''(1)N'(1) + E_2'(1)N''(1)\Big],$$

$$N'(1) = -\lambda_1\Big[1+\theta\gamma_1\Big], \quad N''(1) = -\lambda_1^2\Big[\theta\gamma_2\Big],$$
$$G_1'(1) = -m\lambda_2\beta_{21}P_{0,0}^{(2)}\Big[1-L_{A_2}[\lambda_2]\Big]\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big],$$
$$G_1''(1) = -m\lambda_2P_{0,0}^{(2)}\Big[1-L_{A_2}[\lambda_2]\Big]\Bigg(4\beta_{21}\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big]+m\lambda_2\beta_{22}\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big]^2+$$
$$m\lambda_2\beta_{21}\Big[\theta\gamma_2+2\bar{r}\theta\gamma_1\delta_1+\bar{r}\theta\delta_2\Big]\Bigg),$$
$$G_2'(1) = m\lambda_2\beta_{21}\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big]+q\Big[1-L_{A_2}[\lambda_2]\Big]-1,$$
$$G_2''(1) = m\lambda_2\Bigg(2\beta_{21}q\Big[1-L_{A_2}[\lambda_2]\Big]\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big]+m\lambda_2\beta_{22}\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big]^2+$$
$$m\lambda_2\beta_{21}\Big[\theta\gamma_2+2\bar{r}\theta\gamma_1\delta_1+\bar{r}\theta\delta_2\Big]\Bigg),$$
$$G_3''(1) = -2m\lambda_2^2\beta_{21}P_{0,0}^{(2)}\Big[q(1-L_{A_2}[\lambda_2])-1\Big]\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big],$$
$$G_3'''(1) = -3\lambda_2P_{0,0}^{(2)}\Big[q(1-L_{A_2}[\lambda_2])-1\Big]\Big[\big(M'(1)\big)^2\beta_{22}-M''(1)\beta_{21}-2\beta_{21}M'(1)\Big],$$
$$G_4''(1) = 2M'(1)G_2'(1), \quad G_4'''(1) = 3\Big[G_2''(1)M'(1)+G_2'(1)M''(1)\Big],$$
$$M'(1) = -m\lambda_2\Big[1+\theta\gamma_1+\bar{r}\theta\delta_1\Big], \textit{ and } M''(1) = -(m\lambda_2)^2\Big[\theta\gamma_2+2\bar{r}\theta\gamma_1\delta_1+\bar{r}\theta\delta_2\Big].$$

*Proof.* By definition, we have

$$E[N_o] = \hbar P_o^{'(1)}(1)+\bar{\hbar}P_o^{'(2)}(1), \; E[N_s] = \hbar P_s^{'(1)}(1)+\bar{\hbar}P_s^{'(2)}(1).$$

The quantities $W_s$ and $W_o$ are calculated by using Little's formula. □

### 4.5. Particular cases

In this subsection, we explore some particular cases that align with the literature, as detailed below:

1. If the parameters $(\lambda_2, \theta) \to (0,0)$, then our model reduces to an $M/G/1$ retrial queue with a reliable server and no customer loss. This means that the findings of our study expand upon the research conducted by Gómez-Corral [41]. In this case, the stability condition is : $\frac{\lambda_1\beta_{11}}{L_{A_1}[\lambda_1]} < 1$.
2. If $\lambda_1 \to 0$, and $D(v) = 1-e^{-\delta v}$, then our model reduces to an $M/G/1$ retrial queue with balking, which aligns with the conclusions reported by Wu et al. [18]. In this case, the stability condition becomes: $\frac{m\lambda_2\beta_{21}(1+\theta\gamma_1+\frac{\bar{r}\theta}{\delta_1})}{1-q(1-L_{A_2}[\lambda_2])} < 1$.
3. In the case where $\lambda_1 \to 0$, our model reduces to an unreliable retrial queue with random reserved time and geometric loss, which aligns with the results obtained by Taleb et al. [19]. In this case, the stability condition of our system becomes: $\frac{m\lambda_2\beta_{21}(1+\theta\gamma_1+\bar{r}\theta\delta_1)}{1-q(1-L_{A_2}[\lambda_2])} < 1$.

## 5. NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

### 5.1. Sensitivity analysis

In this section, we conduct a sensitivity analysis to examine how different system parameters influence key performance metrics. By exploring various numerical scenarios and adjusting system parameters, we aim to understand their impact on the system's behavior. The following assumptions are made for this analysis:

- The service times for both customer types (persistent and impatient) follow a Gamma distribution. The probability density functions are $b_1(x;0.8,\beta_1) = \frac{\beta_1^{0.8}}{\Gamma(0.8)} x^{-0.2} e^{-\beta_1 x}$ (for persistent customers) and $b_2(x;0.4,\beta_2) = \frac{\beta_2^{0.4}}{\Gamma(0.4)} x^{-0.6} \times e^{-\beta_2 x}$ (for impatient customers).
- The retrial times for both customer types follow a two-step Erlang distribution ($E_2$). The probability density functions are $a_1(w) = \alpha_1^2 w e^{-\alpha_1 w}$ (for persistent customers) and $a_2(w) = \alpha_2^2 w e^{-\alpha_2 w}$ (for impatient customers).
- The repair times are Gamma distributed with the probability density function $c(y;1.5,\gamma) = \frac{2\gamma^{1.5}}{\sqrt{\pi}} y^{0.5} e^{-\gamma y}$.
- The reservation times are Gamma distributed with the probability density function $d(v;2.5,\delta) = \frac{4\delta^{2.5}}{3\sqrt{\pi}} v^{1.5} e^{-\delta v}$.

The procedure for calculating $P_{0,0}$, $W_{AB_F}$, $E[N_o]$, and $E[N_s]$ is outlined in Algorithm 1.

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**Algorithm 1** : Algorithm for Computing $P_{0,0}$, $W_{AB_F}$, $E[N_o]$ and $E[N_s]$

**Begin**
**Input:** $n$, $\hbar$, $\lambda_1$, $\lambda_2$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $m$, $q$, $r$, $\theta$, $\gamma$, and $\delta$.

1. **For** $i = 1$ **to** $n$
   (a) Compute $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$, $\alpha_{22}$, $\beta_{11}$, $\beta_{12}$, $\beta_{21}$, $\beta_{22}$, $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ from Section 5.1.
   (b) Compute $P_{0,0}$.
   (c) **If** $\rho < 1$ **then**
      i. Compute steady-state probabilities from Section 4.1.
      ii. Compute $W_{AB_F}$ from Section 4.2.
      iii. Compute $E[N_o]$ from Section 4.4.
      iv. Compute $E[N_s]$ from Section 4.4.
   (d) **Output:** $P_{0,0}$, $W_{AB_F}$, $E[N_o]$, and $E[N_s]$.
   (e) **End If**
2. **End For**

**End**

---

### 5.2. Results and discussion

The following values are chosen to satisfy the stability condition: $\hbar = 0$, $\lambda_1 = 6$, $\lambda_2 = 6$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 5$, $\beta_2 = 5$, $m = q = r = 0.5$, $\theta = 2$, $\gamma = 6$, and $\delta = 5$.

– As illustrated in Figures 3 and 4, an increase in the arrival rate ($\lambda_2$) leads to greater congestion within the system, which consequently results in a decrease in the probability $P_{0,0}$. As the system becomes busier, the strain on resources intensifies, raising the likelihood of faults or failures. Consequently, a higher $\lambda_2$ correlates with more frequent active breakdowns ($W_{AB_F}$).

Higher values of $\beta_2$, $\gamma$, and $\delta$ suggest that the system can handle issues and service customers more efficiently. This efficiency increases the probability $P_{0,0}$. Specifically, an increase in $\beta_2$ enhances the system's reliability and stability, which decreases the $W_{AB_F}$. However, while higher $\gamma$ and $\delta$ values indicate better recovery and service capabilities, they may also reflect more frequent interruptions in the normal service process, potentially leading to a rise in the $W_{AB_F}$.

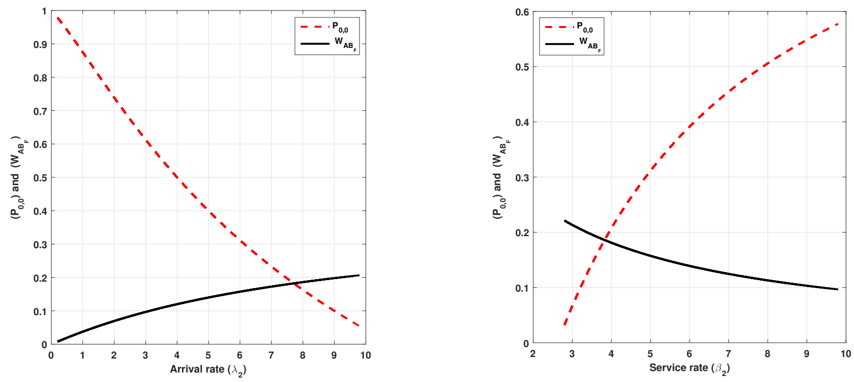


**Figure 3:** Effect of $\lambda_2$ and $\beta_2$ on $P_{0,0}$ and $W_{AB_F}$

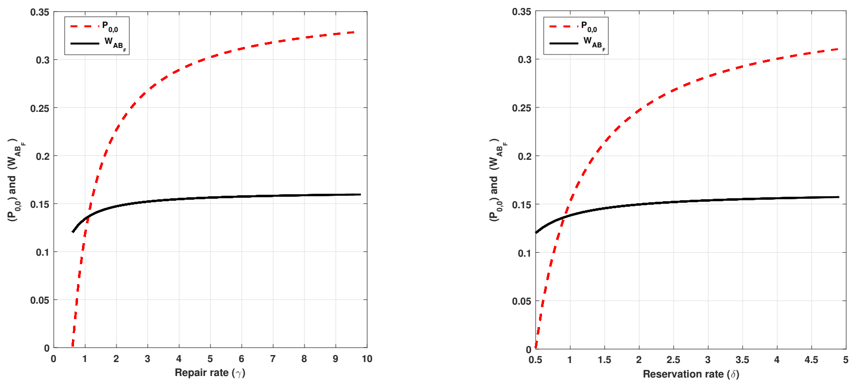


**Figure 4:** Effect of $\gamma$ and $\delta$ on $P_{0,0}$ and $W_{AB_F}$

– Figures 5 and 6 show that increasing the parameters $\beta_2$ and $r$ is associated with a reduction in the mean number of customers in orbit ($E[N_o]$) and the mean number of customers in the system ($E[N_s]$). This indicates that improving service quality and

reducing waiting times, particularly when customers choose to remain in front of the server until repairs are completed, significantly enhances service efficiency. As a result, the system becomes more effective at handling customer demand, thereby decreasing both $E[N_o]$ and $E[N_s]$.
Conversely, an increase in the parameters $\lambda_2$ and $\theta$ leads to higher values of $E[N_o]$ and $E[N_s]$. This trend is driven by the increased demand and strain on the servers due to higher customer arrival rates. Additionally, higher rates of active breakdowns contribute to a growing backlog of services, causing more customers to accumulate within the system. Consequently, both $E[N_o]$ and $E[N_s]$ increase.

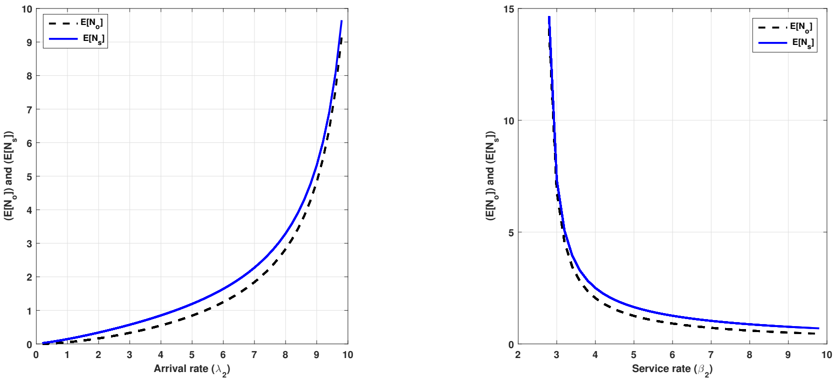


**Figure 5:** Influence of the parameters $\lambda_2$ and $\beta_2$ on $E[N_o]$ and $E[N_s]$

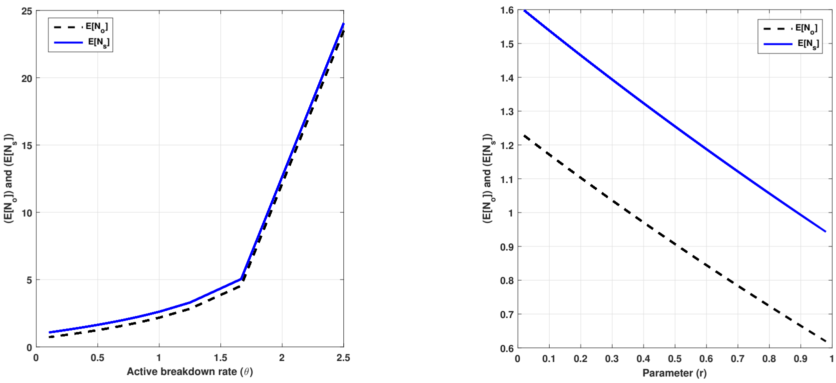


**Figure 6:** Influence of the parameters $\theta$ and $r$ on $E[N_o]$ and $E[N_s]$

## 6. ECONOMIC ANALYSIS OF SYSTEM PERFORMANCE METRICS

In this section, we develop a cost function to enhance the system's cost-effectiveness and present numerical examples to illustrate its impact on performance measures. We also discuss key managerial implications derived from the retrial queueing model.

### 6.1. Cost analysis

This analysis focuses on evaluating the total cost, which comprises two main components: the service cost, which is related to service capacity, and the waiting cost, which is associated with customer waiting. From a quality-of-service perspective, the goal is to understand how system parameters such as the probability $P_{0,0}$, the mean number of customers in orbit ($E[N_o]$), the arrival rate ($\lambda_2$), the reservation rate ($\delta$), and the parameters $m$, $q$, and $r$ affect the total cost ($C_T$).

The total cost $C_T$ per unit of time is given by the following formula:

$$C_T = C_f P_{0,0} + C_b(1-P_{0,0}) + C_c(\hbar\lambda_1 + \bar{\hbar}\lambda_2) + C_o E[N_o],$$

where,

$C_f$ $\equiv$ the cost of server preparation work when the system is empty,

$C_b$ $\equiv$ the cost for server operation when the server is blocked,

$C_c$ $\equiv$ the preparation cost per occupancy cycle,

$C_o$ $\equiv$ the cost of waiting for customers in orbit.

### 6.2. Results and discussion

Now, we fix the appropriate values of parameters $C_o = 80$, $C_f = 250$, $C_b = 940$, and $C_c = 310$, and we choose the other parameters ($\hbar = 0$, $\lambda_1 = 6$, $\lambda_2 = 6$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 5$, $\beta_2 = 5$, $m = q = r = 0.5$, $\theta = 2$, $\gamma = 6$, and $\delta = 5$) to ensure model stability.

As observed in Figures 7 and 8, the total cost ($C_T$) increases with higher values of the parameters $m$, $q$, $r$, $\lambda_2$, and $E[N_o]$. This increase is primarily due to the additional strain these factors place on the system. For example, a higher customer arrival rate ($\lambda_2$) or increases in $m$, $q$, and $r$ lead to a rise in the mean number of customers in orbit ($E[N_o]$), which causes congestion and longer waiting times.

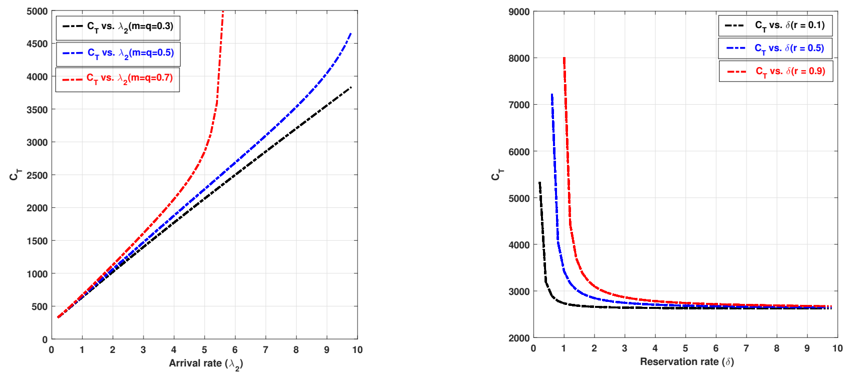


**Figure 7:** Impact of the parameters $\lambda_2$, $m$, $q$, $r$, and $\delta$ on $C_T$

Since $C_T$ accounts for both service costs and costs associated with customer waiting, any factor that contributes to congestion or delays will inevitably lead to a higher total cost. Conversely, increasing the parameter $\delta$ and the probability $P_{0,0}$ leads to a reduction in the total cost. An increase in $\delta$ enhances the system's ability to efficiently manage reserved customers, thereby reducing their waiting time and expediting service. This improvement not only speeds up the service process but also lowers associated waiting costs, resulting in reduced operational costs and, consequently, a lower total cost. Additionally, a higher $P_{0,0}$ indicates that the system operates efficiently during idle periods, with minimal congestion or delays. This optimal performance during downtime further reduces total costs by ensuring effective resource utilization and minimizing service interruptions.

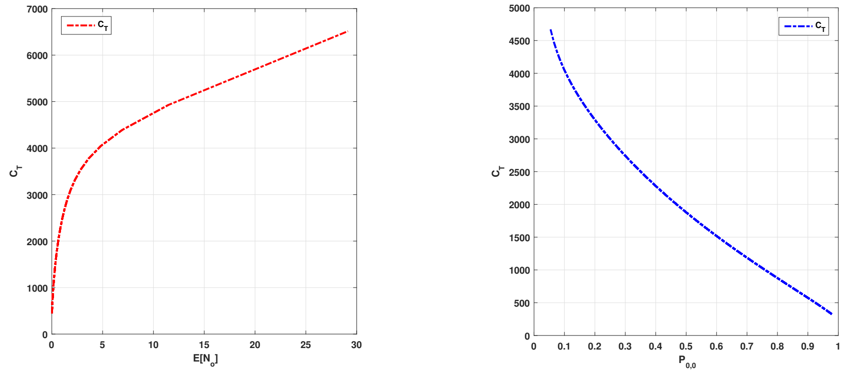


**Figure 8:** Impact of the quantities $E[N_o]$ and $P_{0,0}$ on $C_T$

### 6.3. Managerial implications of the queueing model

This section highlights the key managerial implications derived from our retrial queueing model, demonstrating how its results can be utilized to enhance operational efficiency, customer satisfaction, and profitability across various service environments.

1. **Optimal resource management:**
   - Resource allocation: The model can assist managers in optimally allocating resources based on customer behavior (persistent vs. impatient) and repair times. By adjusting service and repair rates, businesses can minimize waiting times and improve service efficiency.
   - Maintenance planning: The model helps in planning maintenance and repairs more effectively, reducing server downtime and increasing service availability.
2. **Improving customer satisfaction:**
   - Strategies for handling impatient customers: By understanding how impatient customers react to service interruptions, managers can develop strategies to enhance their experience. For instance, optimizing retry rates or offering compensations can reduce abandonment and improve satisfaction.
   - Reducing abandonment rates: Adjusting mechanisms for managing waiting customers, such as refund policies or incentives, can decrease abandonment rates and keep customers engaged.
3. **Cost analysis:**
   - Cost-benefit analysis: The model quantifies costs related to waiting times, service interruptions, and repairs. This information allows managers to perform cost-benefit analyses to make informed investment decisions regarding infrastructure or repair technologies.

- Reducing operational costs: By optimizing model parameters, businesses can lower operational costs associated with repair periods and extended waiting times, including managing customer withdrawals and server maintenance.

4. **Strategic planning:**
- Demand forecasting: The model helps forecast service demand based on customer behavior and repair times. Managers can use these forecasts to adjust staffing levels and service schedules, thereby enhancing operational efficiency.
- Adaptation to changes: Sensitivity analyses demonstrate how changes in system parameters affect performance measures, enabling managers to quickly adapt to fluctuations in demand or variations in service and repair times.

5. **Service system design:**
- Queue system design: Model results can guide the design of service systems by determining the best configuration for handling different customer behaviors. This may involve designing more efficient queue systems and optimizing mechanisms for managing retries and service resumption.

## 7. CONCLUSION

Through this investigation, we explored an unreliable retrial queue $M_1, M_2/G_1, G_2/1$ with a repairable server, including balking, reneging, a service orbit, and reservations that accommodate either persistent or impatient customers. The required condition for the system to be stable is verified. The supplementary variables method was applied to establish the steady-state distributions of the server state, such as the generating function of the number of customers in the system and in the orbit. We acquired some significant system performance measures. Several specific cases were also addressed in this research work, and numerical examples were evaluated to illustrate our findings.

In the future, we suggest further developments to be conducted on the mathematical analysis of an unreliable retrial queue with two types of batch arrivals, one persistent and the other impatient, as well as with repairs and reservations.

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