Yugoslav Journal of Operations Research
28 (2018), Number 1, 59–77
DOI: https://doi.org/10.2298/YJOR170212021A

NEW CONDITIONS FOR THE EXISTENCE OF EQUILIBRIUM PRICES

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Received: February 2017 / Accepted: October 2017

Abstract: We study the existence of equilibrium price vector in a supply-demand model taking into account the transaction costs associated with the sale of products. In this model, the demand function is the solution to the problem of maximizing the utility function under budget constraints. The supply function is the solution to the problem of maximizing the profit (with given transaction losses) on the technology set. We establish sufficient conditions for the existence of the equilibrium price vector, which are consequences of some theorems in the theory of covering mappings.

Keywords: Economic Equilibrium, Demand Function, Supply Function, Transaction Costs, Coincidence Points, Covering Mappings.

MSC: 91B50, 91B52, 91B55.

1. INTRODUCTION

The existence of equilibrium is still one of the most important questions in the study of economic models.

The concept of economic equilibrium was first formulated by Leon Walras [1]. He established a law, that implies the equality of the number of equilibrium prices and the number of equations to which they satisfy. However, he did not prove the existence of equilibrium due to lack of mathematical apparatus, which appeared later on, and included the Brouwer fixed-point theorem, Kakutani fixed-point theorem, Gale-Nikaido-Debreu lemma, Ky Fan inequality, etc. This made it possible to investigate the conditions of the existence of equilibrium in some economic models.

One of the first results about the conditions of the existence of the competitive equilibrium was obtained in 1954 by Arrow and Debreu [2]. Further development of the theory, taking into account the role of the credit and financial instrument, allowed to investigate the conditions of the existence of the equilibrium in models of economic dynamics; see, e.g., Aliprantis *e*t al. [3] and Hildenbrand *e*t al. [4].

However, these results are not applicable for studying models with transaction costs, where the price a consumer pays exceeds the price a producer pays. Transaction costs may have different origins: inflation of production costs, tax on sales, racket, etc. Adapting the concept of the economic equilibrium for models with the transaction costs has led to modification of the basic constructions of mathematical economics and to developement of the theory of non-classical equilibriums; see, e.g., Pospelov [5] and Petrov *e*t al. [6].

In [7], [8] the existence of non-classical equilibriums has been reduced to the existence of solutions of special variational inequalities. However, such inequalities do not satisfy the Walras law, whence the standard approach to proving the existence of their solutions (based on the Gale–Nikaido–Debreu lemma) is not applicable. According to our knowledge, none of well-known papers contain any sufficient conditions of the existence of equilibriums in economic models with transaction costs. This is related to the fact that the existing mathematical apparatus is not sufficient for solving this problem.

In this paper, we present a solution to this problem based on [7] – [9], devoted to the existence of coincidence points of mappings in metric spaces. Using the results and methods developed in these papers, we obtain constructive sufficient conditions for the existence of equilibriums in economic models with transaction costs. Here we consider the vector of equilibrium prices as a coincidence point of two mappings: the demand and supply functions. These functions are considered as mappings $X \to Y$, where X, Y are subsets of finite-dimensional Euclidean spaces, whence X, Y are metric spaces with metrics induced by the ambient Euclidean spaces. The methods we use are based on the theory of coincidence points for Lipschitz continuous and covering mappings, developed in a series of recent papers, see, e.g., [7] – [9].

2. CONSUMER AND PRODUCER BEHAVIOR MODEL THE SUPPLY AND DEMAND FUNCTIONS

Suppose that there are $n \in \mathbb{N}$ goods with the price vectors

$$p = (p_1, p_2, \ldots, p_n), \quad \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n), \quad p_i > 0, \ \tilde{p}_i > 0,$$

where p_i is the price of the ith good for the consumer and $\tilde{p}_i$ is the price of the ith good for the producer. Assume that $\tilde{p} = Ap$, where $A = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $0 < \alpha_i < 1$. This relation holds true, for instance, if the producer pays sales tax, or in some economic models with high inflation; see e.g., Petrov et al. [6].

Production capacity of the producer is described by the technology set

$$T = \left\{y = (y_+; y_-) \,|\, \varphi(y_+; y_-) \le 0, y_+, y_- \in \mathbb{R}^n_+\right\} \subset \mathbb{R}^{2n}_+,$$

where $\varphi : \mathbb{R}^{2n}_+ \to \mathbb{R}$ is a strongly convex C^2-smooth function and

$$y_- = (y_{-1}, y_{-2}, \ldots, y_{-n}), \quad y_+ = (y_{+1}, y_{+2}, \ldots, y_{+n})$$

are the vector of inputs and the vector of outputs, respectively. Here $y_{+i} \in \mathbb{R}_+$ is the gross output of the ith good, $y_{-i} \in \mathbb{R}_+$ is the quantity of the ith good needed for y_+.

The producer's goal is to maximize his profit, that is, to solve the following extremum problem:

$$\langle Ap, y_+\rangle - \langle p, y_-\rangle \to \max, \quad (y_+; y_-) \in T, \tag{1}$$

where the triangle brackets denote the standard dot product in $\mathbb{R}^n$. Then, taking into account the assumptions above, the extremum problem (1) can be written in the form

$$\begin{cases} \langle (Ap; -p), (y_+; y_-)\rangle \to \max, \\ \varphi(y_+; y_-) \le 0, \\ (y_+; y_-) \ge 0. \end{cases} \tag{2}$$

Here and below, to write $y \ge 0$ means that all components of the vector y are non-negative.

Let $y^* = (y^*_+, y^*_-)$ be a solution of the extremum problem (2). Then the producer's supply is presented by the mapping

$$S : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad S(p) = y^*_+ - y^*_-.$$

Assume that the consumer possesses the budget $I(p)$, and his preferences are presented by the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, which is C^2-smooth, strictly concave (i.e., the function $-u$ is strictly convex) and does not have maximums. Moreover, assume that the budget function $I : \mathbb{R}^n_+ \to \mathbb{R}_+$ is C^1-smooth, positively homogeneous of degree one, and there exists $C > 0$ such that $I(p) \ge C|p|$ for all $p \in \mathbb{R}^n_+$.

When buying a set of goods $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$, the consumer purchases y_1 units of the first good, y_2 units of the second good, etc. The consumer's goal is to maximize his utility function, that is, to solve the following extremum problem:

$$u(y) \to \max, \quad \langle p, y \rangle \le I(p), \quad y \ge 0. \tag{3}$$

The consumer's demand is presented by the mapping

$$D : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad D(p) = \operatorname{Argmax} \left\{ u(y) | y \in \mathbb{R}^n_+, \ \langle p, y \rangle \le I(p) \right\}.$$

If for a given p there exists i such that $D_i(p) > S_i(p)$, then there is deficit of the ith good at the market. On the contrary, if there exists i such that $D_i(p) < S_i(p)$, then there is surplus of the ith good. Neither of the situations is satisfactory, since consumer's (respectively, producer's) interests are violated. Thus, the best situation is described by the equality $S(p) = D(p)$.

3. THE EXISTENCE OF EQUILIBRIUM PRICES

Consider the model of economic equilibrium described by the following data:

$$\sigma = (\alpha, \varphi, u, I, c_1, c_2),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in (0, 1)$, $i = \overline{1, n}$, are given numbers, $\varphi : \mathbb{R}^{2n}_+ \to \mathbb{R}$ is a strongly convex C^2-smooth function, $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a strongly concave C^2-smooth function, $I : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a C^1-smooth function, positively homogeneous of degree one. Here $c_1 = (c_{11}, \ldots, c_{n1})$, $c_2 = (c_{12}, \ldots, c_{n2}) \in \mathbb{R}^n_+$ are given vectors satisfying the condition $c_{i1} < c_{i2}$, $i = \overline{1, n}$.

The set (α, φ, u, I) uniquely defines the demand functions

$$D : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad D(\cdot) = (D_1(\cdot), \ldots, D_n(\cdot)),$$

and the supply functions

$$S : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad S(\cdot) = (S_1(\cdot), \ldots, S_n(\cdot)).$$

The numbers α_i $\forall i = \overline{1, n}$ characterize the transaction costs of producers. The components of the vectors c_1, c_2 determine natural constrains for the prices of goods: $c_{i1} \le p_i \le c_{i2}$, $i = \overline{1, n}$.

By Σ denote the set of $\sigma = (\alpha, \varphi, u, I, c_1, c_2)$ satisfying the inequalities $c_{i2} > c_{i1}$, $i = \overline{1, n}$. We shall further assume that $\sigma \in \Sigma$.

Definition 1. *If $S(p) = D(p)$, then $p \in \mathbb{R}^n_+$ is called the equilibrium price vector in the model σ.*

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ consider the norms

$$\|x\|_m = 2 \max_{i=\overline{1,n}} \frac{|x_i|}{c_{i2} - c_{i1}}, \quad \|x\|_\infty = \max_{i=\overline{1,n}} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}.$$

For a linear operator $G:\mathbb{R}^n\to\mathbb{R}^n$ define the norms

$$\|G\|_{m,\infty}=\max_{\|\xi\|_m=1}\|G\xi\|_\infty,\quad \|G\|_2=\max_{\|\xi\|_2=1}\|G\xi\|_2.$$

Let $y=y(p)$, $p\in M=\mathop{\times}\limits_{i=1}^{n}[c_{i1};c_{i2}]$, be the solution of the extremum problem

$$\begin{cases} u(y)\to\max,\\ \langle p,y\rangle\le I(p),\\ y\ge 0,\end{cases}\tag{4}$$

$\bar y$ be the solution of the extremum problem (4) with $p=c_1+c_2$, and $(\bar y_+,\bar y_-)$ be the solution of the extremum problem

$$\begin{cases} \langle (A(c_1+c_2);-c_1-c_2),(y_+;y_-)\rangle\to\max,\\ \varphi(y_+;y_-)\le 0,\\ (y_+;y_-)\ge 0.\end{cases}$$

Put

$$\tilde\alpha(\sigma)=(2C_1(u'')C_2(u''))^{-1}\min_{i=\overline{1,n}}\frac{(c_{i2}-c_{i1})^2}{c_{i2}^2}\min_{i=\overline{1,n}}\frac{c_{i1}}{c_{i2}-c_{i1}}\times$$
$$\times\min_{p\in M}\sum_{i=1}^{n}|I'_{p_i}(p)-y_i(p)|-\sqrt{n}C(u')C(\bar\lambda)\max_{i=\overline{1,n}}(c_{i2}-c_{i1})\left(\max_{i=\overline{1,n}}c_{i1}\right)^{-1},$$

$$\tilde\beta(\sigma)=\sqrt{n}\max_{i=\overline{1,n}}(c_{i2}-c_{i1})C(\varphi')C(\lambda)\left(\max_{i=\overline{1,n}}c_{i1}\right)^{-1},$$

$$\tilde\gamma(\sigma)=\max_{i=\overline{1,n}}|\bar y_i-\bar y_{+i}+\bar y_{-i}|,$$

where $\lambda_i(y_+;y_-)$, $i=\overline{1,2n}$, and $\bar\lambda_i(y)$, $i=\overline{1,n}$, are the eigenvalues of the matrices $\varphi''(y_+;y_-)$ and $u''(y)$, respectively, and

$$C(\lambda)=\max_{(y_+;y_-)\in K_S}\max_{i=\overline{1,2n}}(\lambda_i(y_+;y_-))^{-1},\quad C(\bar\lambda)=\max_{y\in K_D}\max_{i=\overline{1,n}}\left|\bar\lambda_i(y)\right|^{-1},$$

$$C(\varphi')=\max_{(y_+;y_-)\in K_S}\|\varphi'(y_+;y_-)\|_\infty,\quad C(u')=\max_{y\in K_D}\|u'(y)\|_\infty$$

$$C_1(u'')=\max_{y\in K_D}\max_{\|\xi\|_\infty=1}\|u''(y)\xi\|_\infty,\quad C_2(u'')=\max_{y\in K_D}\max_{\|\xi\|_\infty=1}\|(u'')^{-1}(y)\xi\|_\infty,$$

$$K_D=\bigcup_{p\in M}\operatorname{Argmax}\left\{u(y)|y\in\mathbb{R}^n_+,\ \langle p,y\rangle\le I(p)\right\},$$

$$K_S=\bigcup_{p\in M}\operatorname{Argmax}\left\{\langle (Ap;-p),(y_+;y_-)\rangle|y\in\mathbb{R}^n_+,\ \varphi(y_+;y_-)\le 0\right\}.$$

Theorem 2. *Suppose that the model $\sigma\in\Sigma$ satisfies the following conditions:*
1) $\tilde\alpha(\sigma)>\tilde\beta(\sigma)$;
2) $\tilde\gamma(\sigma)<\tilde\alpha(\sigma)-\tilde\beta(\sigma)$.
Then the model σ possesses the equilibrium price vector $p=(p_1,\dots,p_n)$ such that $c_{i1}<p_i<c_{i2}$, $i=\overline{1,n}$.

4. AUXILIARY RESULTS

The proof of Theorem 2 is based on some auxiliary results.

Let X and Y be metric spaces with the metrics ρ_X and ρ_Y, respectively. By $B_X(c,r)$ ($B_Y(c,r)$) denote the closed ball in the space X (respectively, Y) with the center c and radius r.

Definition 3 (Arutyunov [9]). *Given $\alpha > 0$, $D : X \to Y$ is called α-covering mapping, if*

$$D(B_X(x,r)) \supseteq B_Y(D(x), \alpha r) \ \ \forall r \geq 0,\ \forall x \in X.$$

Theorem 4 (about coincidence points, Arutyunov [9]). *Let X be a complete metric space, $D : X \to Y$ be a continuous α-covering mapping, $S : X \to Y$ be a Lipschitz continuous mapping with the Lipschitz constant $\beta < \alpha$. Then for any $x_0 \in X$, there exists $\xi = \xi(x_0) \in X$ such that*

$$D(\xi) = S(\xi), \tag{5}$$

$$\rho_X(x_0, \xi) \leq \frac{\rho_Y(D(x_0), S(x_0))}{\alpha - \beta}.$$

The solution ξ of equation (5) is called the coincidence point of the mappings D and S. It is worth observing that the coincidence point is not necessarily unique.

From Theorem 4, it follows (see Arutyunov [9]) the Milyutin theorem about perturbation of covering mappings.

Theorem 5 (about perturbation). *Let X be a complete metric space, Y be a normed vector space, and $D : X \to Y$ be a continuous α-covering mapping. Then for every Lipschitz continuous mapping $S : X \to Y$ with the Lipschitz constant $\beta < \alpha$, the mapping $D + S : X \to Y$ is $(\alpha - \beta)$-covering.*

Let $M \subset X$ be an arbitrary nonempty set.

Definition 6 (Arutyunov *et al.* [8]). *Let $\alpha > 0$. The mapping $D : X \to Y$ is called α-covering on M, if for any $x \in M$, $r > 0$ such that $B_X(x,r) \subseteq M$ the following inclusion holds true:*

$$D(B_X(x,r)) \supseteq B_Y(D(x), \alpha r).$$

By $\operatorname{cov}(D|\, M)$ denote the supremum of $\alpha > 0$ such that the mapping D is α-covering on M. If $M = X$, denote the supremum by $\operatorname{cov}(D)$.

Theorem 7 (Arutyunov *et al.* [8]). *Let X be a complete metric space, $x_0 \in X$, $\alpha > 0$, $R > 0$, and $D : X \to Y$ be a closed mapping, α-covering on $B_X(x_0, R)$. Let $S : B_X(x_0, R) \to Y$ be a Lipschitz continuous mapping with the Lipschitz constant $\beta < \alpha$ such that*

$$\rho_Y(D(x_0), S(x_0)) \leq (\alpha - \beta)R.$$

Then there exists the coincidence point $\xi \in X$ of the mappings D and S, i.e., $D(\xi) = S(\xi)$, such that

$$\rho_X(x_0, \xi) \le \frac{\rho_Y\big(D(x_0), S(x_0)\big)}{\alpha - \beta}.$$

It is worth observing that in the economic model considered above, the equilibrium price vector is the coincidence point of the demand and supply functions.

To establish sufficient conditions for the existence of the equilibrium price vector, consider the following two extremum problems. The first problem is

$$\langle x, y\rangle \to \max, \quad \varphi(y) \le 0. \tag{6}$$

Here the optimization proceeds with respect to the variable $y \in \mathbb{R}^k$, while the variable $x \in \mathbb{R}^k$ plays the role of parameter. Assume that the function $\varphi : \mathbb{R}^k \to \mathbb{R}$, which presents the constrains, is C^2-smooth and strongly convex, i.e., there exists $\varepsilon > 0$ such that the matrix $\varphi''(y) - \varepsilon E$ is positive definite for every y. Moreover, assume that there exists $\bar{y} \in \mathbb{R}^k : \varphi(\bar{y}) < 0$.

Let $\|\cdot\|$ be an arbitrary norm in the space $\mathbb{R}^k$.

Lemma 8. *For every $x \neq 0$, the extremum problem (6) has a unique solution $y = y(x)$, which is C^1-smooth in the domain $x \neq 0$ and*

$$y'(x) = \frac{\|\psi\|}{\|x\|}\left(\Psi^{-1} - \frac{(\Psi^{-1}\psi^*)\left(\Psi^{-1}\psi^*\right)^*}{\langle \Psi^{-1}\psi^*, \psi^*\rangle}\right) = \frac{\|\psi\|}{\|x\|}\left(\Psi^{-1} - \frac{(\Psi^{-1}x^*)\left(\Psi^{-1}x^*\right)^*}{\langle \Psi^{-1}x^*, x^*\rangle}\right). \tag{7}$$

Here

$$\Psi = \varphi''(y), \quad \psi = \varphi'(y), \tag{8}$$

and the symbol "$*$" means the transpose.

Proof. The Lagrange function for the extremum problem (6) reads

$$L(x, y, \lambda) = -\langle x, y\rangle + \lambda\varphi(y).$$

By hypothesis, the set $\{y : \varphi(y) \le 0\}$ is nonempty, compact and strictly convex. A linear function attains a unique maximum on a strictly convex compact. Therefore, for every $x \neq 0$, the extremum problem (6) has a unique solution $y = g(x)$.

Since the function φ is strongly convex, the following regularity condition holds:

$$\varphi'(y) \neq 0 \quad \forall y : \ \varphi(y) = 0. \tag{9}$$

Therefore, taking into account the Lagrange principe, we conclude that there exist $\lambda \ge 0$ such that

$$-x + \lambda\varphi'(y) = 0, \quad \varphi(y) = 0. \tag{10}$$

Let us introduce mapping $F:\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}\to\mathbb{R}^k\times\mathbb{R}$ by the formula

$$F(x,y,\lambda)=(x-\lambda\varphi'(y),\varphi(y))\,.$$

Then the relations (10) are equivalent to the equation

$$F(x,y,\lambda)=0. \tag{11}$$

with the parameter x and the unknowns (y,λ). Let us show that this equation satisfies the conditions of the implicit function theorem with respect by (y,λ), that is, the matrix $\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)$ is non-degenerate.

Indeed, using the notations (8) and taking into account (10), for all $x\in\mathbb{R}^k$, $y\in\mathbb{R}^k$, $\lambda\in\mathbb{R}$ we have

$$\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)=\begin{pmatrix}-\lambda\varphi''(y) & -(\varphi'(y))^*\\ \varphi'(y) & 0\end{pmatrix}=\begin{pmatrix}-\lambda\Psi & -\psi^*\\ \psi & 0\end{pmatrix}.$$

It suffices to prove that for every w,w_{k+1} the linear system

$$\begin{pmatrix}-\lambda\Psi & -\psi^*\\ \psi & 0\end{pmatrix}\begin{pmatrix}v\\ v_{k+1}\end{pmatrix}=\begin{pmatrix}w\\ w_{k+1}\end{pmatrix} \tag{12}$$

has a solution v,v_{k+1}. By the assumptions made above $\lambda>0$, the matrix Ψ is positive definite (and consequently, nondegenerate) and $\psi\neq 0$.

The system (12) can be written in the form

$$-\lambda\Psi v-\psi^*v_{k+1}=w,\quad \langle\psi^*,v\rangle=w_{k+1}. \tag{13}$$

Therefore, we have $\lambda\Psi v=-w-\psi^*v_{k+1}$, and consequently,

$$v=-(\lambda\Psi)^{-1}w-(\lambda\Psi)^{-1}\psi^*v_{k+1}. \tag{14}$$

Substituting the obtained expression for v in the second equality in (13), we get

$$-\left\langle\psi^*,(\lambda\Psi)^{-1}w\right\rangle-\left\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\right\rangle v_{k+1}=w_{k+1}.$$

In view of what we have said above, $\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\rangle>0$. Hence this yields

$$v_{k+1}=-\frac{\left\langle\psi^*,(\lambda\Psi)^{-1}w\right\rangle}{\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\rangle}-\frac{w_{k+1}}{\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\rangle}.$$

Substituting the obtained expression for v_{k+1} in (14), we get

$$v=-(\lambda\Psi)^{-1}w+\frac{\left\langle\psi^*,(\lambda\Psi)^{-1}w\right\rangle(\lambda\Psi)^{-1}a^*}{\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\rangle}+\frac{w_{k+1}(\lambda\Psi)^{-1}\psi^*}{\langle\psi^*,(\lambda\Psi)^{-1}\psi^*\rangle}.$$

Thus, the obtained pair v,v_{k+1} is a solution of (12), and the matrix $\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)$ is non-degenerate.

From what we have said above, it follows that the solution v, v_{k+1} can be represented in the form

$$\begin{pmatrix} v \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} -(\lambda\Psi)^{-1} + bc^*c & bc^* \\ -bc & -b \end{pmatrix} \begin{pmatrix} w \\ w_{k+1} \end{pmatrix},$$

where $b = \left\langle \psi^*, (\lambda\Psi)^{-1}\psi^* \right\rangle^{-1}$, $c = \left((\lambda\Psi)^{-1}\psi^*\right)^*$. This gives the following formula for the inverse matrix:

$$\left(\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)\right)^{-1} = \begin{pmatrix} -(\lambda\Psi)^{-1} + bc^*c & bc^* \\ -bc & -b \end{pmatrix}. \tag{15}$$

Consequently, for every $x \neq 0$, equation (11) has the solution $(y,\lambda)(x)$. Moreover, this solution is unique, since in the maximization problem (6) the Lagrange principle is sufficient condition for maximum, the maximum point is unique, and (in light of the regularity condition (9)) the corresponding Lagrange multiplier λ is uniquely defined. By the implicit function theorem, the mapping $(y,\lambda)(\cdot)$ is C^1-smooth in the domain $x \neq 0$.

Differentiating the identity $F(x,(y,\lambda)(x)) \equiv 0$ by x, we get

$$\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)\frac{\partial(y,\lambda)}{\partial x} + \frac{\partial F}{\partial x}(x,y,\lambda) = 0.$$

Then

$$\frac{\partial(y,\lambda)}{\partial x} = -\left(\frac{\partial F}{\partial(y,\lambda)}(x,y,\lambda)\right)^{-1}\frac{\partial F}{\partial x}(x,y,\lambda).$$

Taking into account (15) and the obvious identity

$$\frac{\partial F}{\partial x}(x,y,\lambda) \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

we have

$$\frac{\partial(y,\lambda)}{\partial x} = \begin{pmatrix} (\lambda\Psi)^{-1} - bc^*c \\ bc \end{pmatrix}.$$

Then

$$\frac{dy}{dx} = (\lambda\Psi)^{-1} - bc^*c = (\lambda\Psi)^{-1} - \left\langle \psi^*, (\lambda\Psi)^{-1}\psi^* \right\rangle^{-1} \left((\lambda\Psi)^{-1}\psi^*\right)\left((\lambda\Psi)^{-1}\psi^*\right)^*.$$

Taking into account the equality $\psi = x/\lambda$ (which follows from (8) and (10), we finally obtain (7). $\square$

Consider the second extremum problem

$$u(y) \to \max, \quad \langle x, y\rangle \le I(x), \quad y \ge 0. \tag{16}$$

Here the optimization proceeds with respect to the variable $y \in \mathbb{R}^k_+$, while the variable

$$x \in X_c = \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k : x_i \ge c_i,\ i = \overline{1,k}\},$$

plays the role of parameter, $c_i > 0$ are given numbers.

Assume that the function $I : \mathbb{R}^k_+ \to \mathbb{R}_+$ is C^1-smooth, positively homogeneous of degree one, and there exists $C > 0$ such that $I(x) \ge C|x|$ for all $x \in \mathbb{R}^n_+$. Assume that the function $u : \mathbb{R}^k \to \mathbb{R}$ is strongly concave C^2-smooth function (i.e., the matrix $u''(y)$ is negative definite for all y) and does not have maximums. Hence, the set of admissible points $\{y \in \mathbb{R}^k_+ : \langle x, y\rangle \le I(x)\}$ is closed, convex and bounded uniformly with respect to $x \in X_c$. Extremum problem (16) consists of maximization of a strictly concave function on a convex compact, hence, for any x, it has a unique solution $y = f(x)$. Taking into account further applications of extremum problem (16), we assume that the function u satisfies the conditions such that $f(x) > 0$ and $\langle x, f(x)\rangle = I(x)$ for every $x \in X_c$.

Lemma 9. *For $x \in X_c$, problem (16) has a unique solution $y = f(x)$, where the function f is C^1-smooth and*

$$\frac{df}{dx}(x) = \frac{\|\bar\psi\|}{\|x\|}\left(\bar\Psi^{-1} - \frac{\left(\bar\Psi^{-1}x^*\right)\left(\bar\Psi^{-1}x^*\right)^*}{\langle \bar\Psi^{-1}x^*, x^*\rangle}\right) - \frac{\bar\Psi^{-1}x^*(y - I'(x))}{\langle \bar\Psi^{-1}x^*, x^*\rangle}, \tag{17}$$

where $\bar\Psi = u''(y)$, $\bar\psi = u'(y)$.

Proof. The Lagrange function of the problem (16) reads

$$L(x, y, \lambda) = -u(y) + \lambda(\langle x, y\rangle - I(x)).$$

By the Lagrange principle, there exists the Lagrange multiplier $\lambda \ge 0$ such that

$$-u'(y) + \lambda x = 0, \ \langle x, y\rangle = I(x). \tag{18}$$

Let us introduce mapping $\Phi : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k \times \mathbb{R}$ by the formula

$$\Phi(x, y, \lambda) = (u'(y) - \lambda x, \langle x, y\rangle - I(x))\,.$$

Then the relations (18) are equivalent to

$$\Phi(x, y, \lambda) = 0, \tag{19}$$

with unknowns (y, λ) and parameter x.

Let us show that equation (19) satisfies the conditions of the implicit function theorem, that is, the matrix $\frac{\partial\Phi}{\partial(y,\lambda)}(x, y, \lambda)$ is non-degenerate. Remark that for every $x \in \mathbb{R}^k$, $y \in \mathbb{R}^k$ and $\lambda \in \mathbb{R}$, from (18) it follows

$$\frac{\partial\Phi}{\partial(y, \lambda)}(x, y, \lambda) = \begin{pmatrix} \bar\Psi & -x^* \\ x & 0 \end{pmatrix}.$$

Let us prove that for every w, w_{k+1}, the linear system

$$\begin{pmatrix} \bar{\Psi} & -x^* \\ x & 0 \end{pmatrix} \begin{pmatrix} v \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} w \\ w_{k+1} \end{pmatrix} \tag{20}$$

has solution v, v_{k+1}. From what we have said above, it follows that the matrix $\bar{\Psi}$ is negative definite (consequently, non-degenerate) and $x \neq 0$.

Reasoning analogously to what has gone in Lemma 9, one can write the linear system in the form

$$\bar{\Psi}v - x^* v_{k+1} = w, \quad \langle x^*, v \rangle = w_{k+1}. \tag{21}$$

Hence, we get $\bar{\Psi}v = w + x^* v_{k+1}$. This yields

$$v = \bar{\Psi}^{-1}w + \bar{\Psi}^{-1}x^* v_{k+1}. \tag{22}$$

Substituting the obtained expression for v in the second equality in (21), we get

$$\left\langle x^*, \bar{\Psi}^{-1}w \right\rangle + \left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle v_{k+1} = w_{k+1}.$$

In view of what we have said above, $\langle x^*, \bar{\Psi}^{-1}x^* \rangle < 0$. Hence, this yields

$$v_{k+1} = -\frac{\left\langle x^*, \bar{\Psi}^{-1}w \right\rangle}{\left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle} + \frac{w_{k+1}}{\left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle}.$$

Substituting the obtained expression for v_{k+1} in (22), we get

$$v = \bar{\Psi}^{-1}w - \frac{\bar{\Psi}^{-1}x^* \left\langle x^*, \bar{\Psi}^{-1}w \right\rangle}{\left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle} + \frac{\bar{\Psi}^{-1}x^* w_{k+1}}{\left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle}.$$

Thus, the obtained pair v, v_{k+1} is a solution of (20), and the matrix $\frac{\partial \Phi}{\partial (y,\lambda)}(x, y, \lambda)$ is non-degenerate.

From what we have said above, it follows that the solution v, v_{k+1} can be represented in the form

$$\begin{pmatrix} v \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} (\bar{\Psi}^{-1} - \bar{b}\bar{c}^*\bar{c}) & \bar{b}\bar{c}^* \\ -\bar{b}\bar{c} & \bar{b} \end{pmatrix} \begin{pmatrix} w \\ w_{k+1} \end{pmatrix},$$

where $\bar{b} = \left\langle x^*, \bar{\Psi}^{-1}x^* \right\rangle^{-1}$, $\bar{c} = \left(\bar{\Psi}^{-1}x^*\right)^*$. This yields the formula for the inverse matrix:

$$\left(\frac{\partial \Phi}{\partial (y, \lambda)}(x, y, \lambda)\right)^{-1} = \begin{pmatrix} (\bar{\Psi}^{-1} - \bar{b}\bar{c}^*\bar{c}) & \bar{b}\bar{c}^* \\ -\bar{b}\bar{c} & \bar{b} \end{pmatrix}. \tag{23}$$

Consequently, for every $x \neq 0$, equation (19) has the solution $(y, \lambda)(x)$. Moreover, this solution is unique, since in the maximization problem (16), the Lagrange principle is a sufficient condition for maximum, the maximum point is unique, and (in light of the regularity condition (9)) the corresponding Lagrange multiplier λ

is uniquely defined. By the implicit function theorem, the mapping $(y, \lambda)(\cdot)$ is C^1-smooth in the domain $x \neq 0$.

Differentiating the identity $\Phi(x, (y, \lambda)(x)) \equiv 0$ by x, we get

$$\frac{\partial \Phi}{\partial (y,\lambda)}(x, y, \lambda)\frac{\partial (y,\lambda)}{\partial x} + \frac{\partial \Phi}{\partial x}(x, y, \lambda) = 0.$$

Then

$$\frac{\partial (y,\lambda)}{\partial x} = -\left(\frac{\partial \Phi}{\partial (y,\lambda)}(x, y, \lambda)\right)^{-1}\frac{\partial \Phi}{\partial x}(x, y, \lambda).$$

Taking into account (23), the equality

$$\frac{\partial \Phi}{\partial x}(x, y, \lambda) = \begin{pmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda \\ y_1 - \frac{\partial I}{\partial x_1}(x) & y_2 - \frac{\partial I}{\partial x_2}(x) & \cdots & y_k - \frac{\partial I}{\partial x_k}(x) \end{pmatrix},$$

and the equality $\lambda = \|u'(y)\| \cdot \|x\|^{-1}$ (which follows from (18)), we finally obtain (17). □

For the proof of the main theorem, we need the following lemma:

Lemma 10. *Let H be a symmetric positive or negative definite matrix of the order k with the eigenvalues λ_i, $i = \overline{1,k}$. Given non-zero vector $h = (h_1, h_2, \dots, h_k) \in \mathbb{R}^k$, define the linear operator $\mathcal{H} : \mathbb{R}^k \to \mathbb{R}^k$ by the formula*

$$\mathcal{H}\xi = \left(H - \frac{Hh^*\,(Hh^*)^*}{\langle Hh^*, h^*\rangle}\right)\xi, \quad \forall \xi \in \mathbb{R}^k. \tag{24}$$

Then

$$\|\mathcal{H}\|_{m,\infty} \leq \sqrt{k} \max_{i=\overline{1,k}}(c_{i2} - c_{i1}) \max_{i=\overline{1,k}} |\lambda_i|. \tag{25}$$

We wish to emphasize that the estimation (25) does not depend on h.

Proof. Since a symmetric matrix can be brought to the diagonal form by using an orthogonal transformation, there exists an orthogonal matrix T such that $H = T^{-1}ZT$, where $Z = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Using the equalities $H = H^* = T^{-1}ZT$ and $T^{-1} = T^*$ (orthogonality), one can transform the matrix from the right-hand side of (24) in the following way:

$$H - \frac{Hh^*\,(Hh^*)^*}{\langle Hh^*, h^*\rangle} = H - \frac{Hh^*hH}{hHh^*} = T^*ZT - \frac{T^*Z(Th^*)(hT^*)ZT}{(hT^*)Z(Th^*)} =$$

$$T^*ZT - \frac{T^*Zz^*(Zz^*)^*T}{zZz^*} = T^{-1}\left(Z - \frac{Zz^*\,(Zz^*)^*}{\langle Zz^*, z^*\rangle}\right)T, \tag{26}$$

where $z^* = Th^*$.

Let us estimate the norm $\|\mathcal{H}\|_2$. Since the Euclidean norm of a vector (and consequently, the corresponding norm of a linear operator) does not change, from (26) it follows that it suffices to prove the estimation (25) for the operator $\mathcal{H}$, given by the formula (24) with

$$H = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}.$$

It is convenient to represent the operator $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

where

$$\mathcal{H}_1\xi = H\xi, \quad \mathcal{H}_2\xi = \frac{Hh^*\,(Hh^*)^*}{\langle Hh^*, h^*\rangle}\xi, \quad \forall \xi \in \mathbb{R}^k.$$

Obviously,

$$\|\mathcal{H}_1\|_2 = \max_{\|\xi\|_2=1} \|H\xi\|_2 = \max_{i=\overline{1,k}} |\lambda_i|.$$

Now let us calculate $\|\mathcal{H}_2\|_2$. Put $Hh^* = d$ and remark that

$$\max_{\|\xi\|_2=1} \langle dd^*\xi, dd^*\xi\rangle = \max_{\|\xi\|_2=1} \langle d,\xi\rangle^2 \langle d,d\rangle = \left\langle d, \frac{d}{\|d\|_2}\right\rangle^2 \langle d,d\rangle = \|d\|_2^4.$$

Then, taking into account that the matrix H is positive or negative definite, and consequently, its eigenvalues have the same sign, we have

$$\|\mathcal{H}_2\|_2 = \left(\sum_{i=1}^k |\lambda_i| h_i^2\right)^{-1} \sum_{i=1}^k \lambda_i^2 h_i^2 \le \left(\sum_{i=1}^k |\lambda_i| h_i^2\right)^{-1} \max_{i=\overline{1,k}} |\lambda_i| \sum_{i=1}^k |\lambda_i| h_i^2 = \max_{i=\overline{1,k}} |\lambda_i|.$$

Consequently,

$$\|\mathcal{H}\|_2 \le \|\mathcal{H}_1\|_2 + \|\mathcal{H}_2\|_2 \le 2 \max_{i=\overline{1,k}} |\lambda_i|.$$

Finally, remark that $\|\mathcal{H}\|_\infty \le \sqrt{k}\|\mathcal{H}\|_2$ and

$$\|\mathcal{H}\|_{m,\infty} = \max_{\xi\neq 0} \frac{\|\mathcal{H}\xi\|_\infty}{\|\xi\|_m} = \max_{\xi\neq 0} \frac{\|\mathcal{H}\xi\|_\infty}{2\max_{i=\overline{1,k}} \frac{|\xi_i|}{c_{i2}-c_{i1}}} \le \frac{1}{2} \max_{i=\overline{1,k}} (c_{i2}-c_{i1}) \max_{\xi\neq 0} \frac{\|\mathcal{H}\xi\|_\infty}{\|\xi\|_\infty} \le \frac{\sqrt{k}}{2} \max_{i=\overline{1,k}} (c_{i2}-c_{i1}) \|\mathcal{H}\|_2.$$

This yields (25). □

5. THE PROOF OF THEOREM 2

Consider the metric spaces (X, ρ_X) and (Y, ρ_Y), where $X = \mathbb{R}^n_+$, $Y = \mathbb{R}^n_+$, the metrics ρ_X and ρ_Y are defined by the norm $\|\cdot\|_m$ and $\|\cdot\|_\infty$, respectively.

Put

$$\tilde{c} = \frac{c_1 + c_2}{2}, \quad M = B_X(\tilde{c}, 1).$$

Let us estimate the Lipschitz constant of the mapping S. By $\operatorname{lip}(S|\,M)$ denote the infinum of the numbers $\beta \geq 0$ such that the restriction of S to the set M is Lipschitz continuous with the Lipschitz constant β. Then

$$\operatorname{lip}(S|\,M) = \sup_{p \in \operatorname{int} M} \left\| \frac{\partial S}{\partial p}(p) \right\|_{m,\infty}.$$

For every $p \in \operatorname{int} M$, we have

$$\frac{\partial S}{\partial p}(p) = \frac{\partial(y_+ - y_-)}{\partial(y_+; y_-)}(p) \frac{\partial(y_+; y_-)}{\partial(Ap; -p)}(p) \frac{\partial(Ap; -p)}{\partial p}(p).$$

Therefore, by Lemma 8, for every $p \in \operatorname{int} M$, we have

$$\frac{\partial S}{\partial p}(p) = \frac{\|\psi\|_\infty}{\|(Ap; -p)\|_\infty} B \left(\Psi^{-1} - \frac{\left(\Psi^{-1}\psi^*\right)\left(\Psi^{-1}\psi^*\right)^*}{\langle \Psi^{-1}\psi^*, \psi^* \rangle} \right) C,$$

where $\Psi = \varphi''(y_+; y_-)$, $\psi = \varphi'(y_+; y_-)$, and the matrices

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & -1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \\ -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}.$$

Consequently, by Lemma 10, we have

$$\operatorname{lip}(S|\,M) \leq \sqrt{n} \max_{i=\overline{1,n}} (c_{i2} - c_{i1}) C(\varphi') C(\lambda) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1} = \tilde{\beta}(\sigma).$$

Now let us estimate $\operatorname{cov}(D|\,M)$. By Theorem 4 ([Arutyunov *e*t al. [8]]), we have

$$\operatorname{cov}(D|\,M) = \inf_{p \in \operatorname{int} M} \operatorname{cov}(D|\,p) = \inf_{p \in \operatorname{int} M} \operatorname{cov}\left(\frac{\partial D}{\partial p}(p) \right). \tag{27}$$

By Lemma 9, for $p \in \mathrm{int} M$, we have

$$\frac{\partial D}{\partial p}(p) = \bar{\mathcal{A}}(p) + \bar{\bar{\mathcal{A}}}(p),$$

where $\bar{\mathcal{A}}(p),\ \bar{\bar{\mathcal{A}}}(p) : \mathbb{R}^n \to \mathbb{R}^n$ are linear operators given by the following formulae

$$\bar{\mathcal{A}}(p)\xi = \frac{\bar{\Psi}^{-1} p^* (I'(p) - y)}{\langle \bar{\Psi}^{-1} p^*, p^* \rangle} \xi \quad \forall \xi \in \mathbb{R}^n,$$

$$\bar{\bar{\mathcal{A}}}(p)\xi = \frac{\|\bar{\psi}\|_\infty}{\|p\|_\infty} \left(\bar{\Psi}^{-1} - \frac{\left(\bar{\Psi}^{-1} p^*\right) \left(\bar{\Psi}^{-1} p^*\right)^*}{\langle \bar{\Psi}^{-1} p^*, p^* \rangle} \right) \xi \quad \forall \xi \in \mathbb{R}^n,$$

$\bar{\Psi} = u''(y)$, $\bar{\psi} = u'(y)$. Consequently, by Theorem 5, we obtain the relation

$$\mathrm{cov}\left(\frac{\partial D}{\partial p}(p) \right) = \mathrm{cov}\left(\bar{\mathcal{A}}(p) \right) - \mathrm{lip}\left(\bar{\bar{\mathcal{A}}}(p) \right).$$

Remark that

$$\mathrm{cov}\left(p^* (I'(p) - y) \right) = \|p\|_m \sum_{i=1}^{n} |I'_{p_i}(p) - y_i(p)|,$$

since the linear operator with the matrix $p^*(I'(p) - y)$ acting from $\mathbb{R}^n$ into $\mathbb{R}^n$ sends the unit ball centered at the origin to the segment with the endpoints

$$\sum_{i=1}^{n} |I'_{p_i}(p) - y_i(p)| p \quad \text{and} \quad -\sum_{i=1}^{n} |I'_{p_i}(p) - y_i(p)| p.$$

Therefore,

$$\mathrm{cov}\left(\bar{\mathcal{A}}(p) \right) \geq$$
$$(2C_1(u'')C_2(u''))^{-1} \min_{i=\overline{1,n}} \frac{(c_{i2} - c_{i1})^2}{c_{i2}^2} \min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2} - c_{i1}} \min_{p \in M} \sum_{i=1}^{n} |I'_{p_i}(p) - y_i(p)|.$$

By Lemma 10, we have

$$\mathrm{lip}\left(\bar{\bar{\mathcal{A}}}(p) \right) \leq \sqrt{n} C(u') C(\bar{\lambda}) \max_{i=\overline{1,n}} (c_{i2} - c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1}.$$

Consequently, taking into account (27), we get

$$\mathrm{cov}(D|\, M) \geq$$
$$(2C_1(u'')C_2(u''))^{-1} \min_{i=\overline{1,n}} \frac{(c_{i2} - c_{i1})^2}{c_{i2}^2} \min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2} - c_{i1}} \min_{p \in M} \sum_{i=1}^{n} |I'_{p_i}(p) - y_i(p)|| -$$
$$\sqrt{n} C(u') C(\bar{\lambda}) \max_{i=\overline{1,n}} (c_{i2} - c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1} = \tilde{\alpha}(\sigma).$$

From the by assumption of the theorem and the inequalities $\operatorname{cov}(D|\,M) \geq \tilde{\alpha}(\sigma)$, $\operatorname{lip}(S|\,M) \leq \tilde{\beta}(\sigma)$, it follows that there exist positive numbers $\bar{\alpha}$ and $\bar{\beta}$ such that $\tilde{\beta}(\sigma) < \bar{\beta} < \bar{\alpha} < \tilde{\alpha}(\sigma)$, $\tilde{\gamma}(\sigma) < \bar{\alpha} - \bar{\beta}$, the mapping D is $\bar{\alpha}$-covering on the set M, and the mapping S is Lipschitz continuous on M with the Lipschitz constant $\bar{\beta}$.

Since $\rho_Y(D(\tilde{c}), S(\tilde{c})) = \tilde{\gamma}(\sigma)$, from the second condition of Theorem 2, it follows that $\rho_Y(D(\tilde{c}), S(\tilde{c})) \leq \bar{\alpha} - \bar{\beta}$. Thus, by Theorem 1 ([Arutyunov et al. [8]]), there exists $p \in X$ such that $D(p) = S(p)$ and

$$\rho_X(p, \tilde{c}) \leq \frac{1}{\tilde{\alpha} - \tilde{\beta}} \rho_Y(D(\tilde{c}), S(\tilde{c})).$$

From the last inequality, it follows that $p \in \operatorname{int} M$. Therefore, $c_{i1} < p_i < c_{i2}$ for all $i = \overline{1, n}$. The proof is complete.

6. EXAMPLE

To illustrate the obtained results, let us consider the following example.

Assume that the production capacity of the producer is described by the technology set

$$T = \left\{ y = (y_+; y_-) \;\middle|\; \beta^2 \sum_{i=1}^{n} y_{+i}^2 + \sum_{i=1}^{n} \left(y_{-i} - \frac{1}{\sqrt{n}} \right)^2 \leq 1 \;,\; y_+, y_- \in \mathbb{R}_+^n \right\},$$

where $0 < \beta < 1$, consumer's budget is described by the function

$$I(p) = \sum_{i=1}^{n} \gamma_i p_i, \quad \gamma_i > 0, \quad \forall i = \overline{1, n},$$

and consumer's preferences are presented by the utility function

$$u(y) = \sum_{i=1}^{n} \ln y_i.$$

The set $(\alpha, \beta, \gamma, a)$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $a = (a_1, a_2, \ldots, a_n)$, uniquely defines the demand and supply functions $D : \mathbb{R}_+^n \to \mathbb{R}_+^n$, $S : \mathbb{R}_+^n \to \mathbb{R}_+^n$.

Then, in the considered model we have

$$\tilde{\alpha}(\sigma) = \left(2 \max_{y \in K_D} \max_{i=\overline{1,n}} (y_i - a_i)^{-2} \max_{y \in K_D} \max_{i=\overline{1,n}} (y_i - a_i)^2 \right)^{-1} \min_{i=\overline{1,n}} \frac{(c_{i2} - c_{i1})^2}{c_{i2}^2} \times$$
$$\min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2} - c_{i1}} \min_{y \in K_D} \sum_{i=1}^{n} |\gamma_i - y_i| -$$
$$-\sqrt{n} \max_{y \in K_D} \max_{i=\overline{1,n}} (y_i - a_i)^{-1} \max_{y \in K_D} \max_{i=\overline{1,n}} (y_i - a_i)^2 \max_{i=\overline{1,n}} (c_{i2} - c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1}.$$

This yields the estimation

$$\tilde{\alpha}(\sigma) \geq \frac{1}{2} \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-2} \min_{i=\overline{1,n}} \frac{(c_{i2}-c_{i1})^2}{c_{i2}^2} \times \min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2}-c_{i1}} \min_{k,l=1,2} \sum_{i=1}^{n} |\gamma_i - a_i - \delta_{ikl}| - \sqrt{n} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-1} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1},$$

where

$$\delta_{ikl} = (nc_{ik})^{-1} \sum_{j=1}^{n} c_{jl}(\gamma_j - a_j).$$

Moreover,

$$\tilde{\beta}(\sigma) \leq 4\sqrt{n} \max\left\{\beta; \frac{1}{\sqrt{n}}\right\} \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1},$$

$$\bar{y}_i = a_i + (n(c_{i1}+c_{i2}))^{-1} \sum_{j=1}^{n} (c_{j1}+c_{j2})(\gamma_j - a_j), \quad \forall i = \overline{1,n},$$

$$\bar{y}_{+i} = \frac{\alpha_i(c_{i1}+c_{i2})}{\beta} \left(\sum_{j=1}^{n} \left(\alpha_j^2 + \beta^2\right) (c_{j1}+c_{j2})^2 \right)^{-1/2}, \quad \forall i = \overline{1,n},$$

$$\bar{y}_{-i} = \frac{1}{\sqrt{n}} - \beta(c_{i1}+c_{i2}) \left(\sum_{j=1}^{n} \left(\alpha_j^2 + \beta^2\right) (c_{j1}+c_{j2})^2 \right)^{-1/2}, \quad \forall i = \overline{1,n}.$$

Then, for the considered model, the condition 1) in Theorem 1 is equivalent to

$$8\sqrt{n} \max\left\{\beta; \frac{1}{\sqrt{n}}\right\} \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1} < \\ < \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-2} \times \min_{i=\overline{1,n}} \frac{(c_{i2}-c_{i1})^2}{c_{i2}^2} \min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2}-c_{i1}} \min_{k,l=1,2} \sum_{i=1}^{n} |\gamma_i - a_i - \delta_{ikl}| - \\ - \sqrt{n} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-1} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1} \right)^{-1} \quad (28)$$

and the condition 2) is equivalent to

$$\max_{i=\overline{1,n}} \Bigg| a_i + (n(c_{i1}+c_{i2}))^{-1} \sum_{j=1}^{n} (c_{j1}+c_{j2})(\gamma_j - a_j) -$$

$$\frac{1}{\beta}\left(\alpha_i+\beta^2\right)(c_{i1}+c_{i2})\left(\sum_{j=1}^{n}\left(\alpha_j^2+\beta^2\right)(c_{j1}+c_{j2})^2\right)^{-1/2} + \frac{1}{\sqrt{n}}\Bigg| <$$

$$\frac{1}{2} \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \min_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-2} \min_{i=\overline{1,n}} \frac{(c_{i2}-c_{i1})^2}{c_{i2}^2} \times$$

$$\min_{i=\overline{1,n}} \frac{c_{i1}}{c_{i2}-c_{i1}} \min_{k,l=1,2} \sum_{i=1}^{n} |\gamma_i - a_i - \delta_{ikl}| -$$

$$\sqrt{n} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^{-1} \max_{\substack{i=\overline{1,n}\\ k,l=1,2}} \delta_{ikl}^2 \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1}\right)^{-1} -$$

$$4\sqrt{n} \max\left\{\beta; \frac{1}{\sqrt{n}}\right\} \max_{i=\overline{1,n}} (c_{i2}-c_{i1}) \left(\max_{i=\overline{1,n}} c_{i1}\right)^{-1}. \quad (29)$$

By Theorem 2, if the parameters of the model satisfy the conditions (28) and (29), then there exists a equilibrium price vector $p = (p_1, \ldots, p_n)$ such that $c_{i1} < p_i < c_{i2}$, $i = \overline{1,n}$.

In Table 1, we present several sets of parameters (for $n = 2$, $c_{i1} = 0.1$, $c_{i2} = 1$, $i = 1, 2$) satisfying the conditions (28), (29), and the corresponding equilibrium prices calculated numerically with precision of 0.001.

	$\alpha_1 = \alpha_2 = 0.1$ $\beta = 0.1$		$\alpha_1 = \alpha_2 = 0.9$ $\beta = 0.1$	
	$a_1 = 0.1$ $a_2 = 0.1$	$a_1 = 0.1$ $a_2 = 0.5$	$a_1 = 0.1$ $a_2 = 0.1$	$a_1 = 0.1$ $a_2 = 0.5$
$\gamma_1 = 10$ $\gamma_2 = 10$	$p_1 = 0.219$ $p_2 = 0.219$	$p_1 = 0.615$ $p_2 = 0.636$	$p_1 = 0.337$ $p_2 = 0.337$	$p_1 = 0.926$ $p_2 = 0.952$
$\gamma_1 = 10$ $\gamma_2 = 20$	$p_1 = 0.159$ $p_2 = 0.156$	$p_1 = 0.546$ $p_2 = 0.570$	$p_1 = 0.189$ $p_2 = 0.186$	$p_1 = 0.657$ $p_2 = 0.675$

Table 1: Numerically calculated equilibrium prices

Acknowledgement. The paper was supported by the Ministry of Education and Science of the Russian Federation, project no. 1.962.2017/4.6, by the "RUDN University Program 5-100" and by RFBR grant (project no. 17-01-00849). The results of Section 4 are due to the first author, who was supported by a grant from the Russian Science Foundation (project no. 17-11-01168).

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