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Research Article

PYTHAGOREAN FUZZY EOQ MODEL WITH STOCK- AND HYBRID PRICE-DEPENDENT DEMAND UNDER PRESERVATION TECHNOLOGY AND ADVANCE PAYMENT

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Abstract: This research implements the impact of an advance payment policy on inventory management while incorporating preservation technology to mitigate product deterioration. In a real scenario, the sum of the membership and non-membership degrees of uncertain parameters is greater than one. Hence, the primary objective is to optimize

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cycle time, selling price and maximum profit by utilizing an advanced payment policy, hybrid price-dependent demand rate under interval-valued Pythagorean fuzzy numbers to handle imprecise parameters. Two inventory models are developed: one incorporating preservation technology and another without it. Then, the corresponding fuzzy models are obtained under interval-valued Pythagorean fuzzy environment. A novel ranking method is employed to defuzzify the models and then the defuzzified models are solved by using analytic solution method of maximization problem. The models are validated through numerical examples by assuming hypothetical data, and the results are compared across crisp, Pythagorean fuzzy, and intuitionistic fuzzy frameworks. Key findings indicate that adopting preservation technology significantly improves profitability and reduces deterioration losses. Moreover, the Pythagorean fuzzy approach proves to be more effective in capturing uncertainty compared to intuitionistic fuzzy sets. These findings suggest that businesses can enhance inventory decision-making by leveraging advanced fuzzy techniques to optimize financial and operational outcomes.

Keywords: Inventory model, hybrid price- and stock-dependent demand, preservation technology, advance payment, discount facility, interval-valued Pythagorean fuzzy number.

MSC: 90B05; 03E72.

1. INTRODUCTION

A large collection of items exhibited in a supermarket often creates huge sales due to its variety. For these causes, demand can be considered stock-dependent demand in the inventory model. In the real market, the demand doesn't only depend on the price of the item but also depends on the stock in the inventory systems. Sometimes, demand is linearly dependent on selling price for a season, and again, it is dependent on selling price in non-linear form for another season (Mishra et al. [1]). Thus, many researchers or decision makers (DMs) take the demand rate as a hybrid form of price. Hence, it is considered that demand depends on a hybrid form of selling price and stock level of products in this proposed model.

Nowadays, with the development of the global economy, deteriorating products are performing a vital role. When retailers sell perishable products, then retailers can incur lower profits due to lost sales of deteriorated products. Therefore, retailers need an effective inventory strategy that handles the decay of products to obtain maximum profit. To reduce the deterioration rate of perishable products, the retailers often use preservation technology (PT).

While PT is applied to reduce the loss of perishable products, the investment in PT will also result in additional costs. So the application of PT is also an important factor in the model. Das et al. [2] developed an economic order quantity (EOQ) model for deteriorating products, and also they utilized the effect of PT investment to reduce the deterioration rate. Barman et al. [3] developed a production model for deteriorating products with PT investment to decline the deterioration rate. Hence, PT investment for reducing the deterioration rate is applied in this proposed model.

In this present market scenario, the policy of advance payment is observed as a prominent practice. The manufacturers or suppliers may ask the retailer to pay a portion of the entire purchase price in advance. Both sides gain from this early payment plan in different ways. By accepting prepayments, the manufacturers or suppliers lower their chance of having to cancel orders. Conversely, the retailer expeditiously makes the advance payment to guarantee prompt delivery. To encourage business, suppliers and manufacturers may provide various price breaks, credit options, or other types of facilities in exchange for the advance payment (Khan et al. [4]). It is seen that a wholesaler of some merchandises asks retailers for some payments in part or in full before delivery of the ordered quantity. It is also noticed that the wholesaler allows a certain percentage of discount on the total purchase cost for that particular order. Further, there are situations in which if a retailer gives an extra advance payment, he (or she) may get some price discount during the final payment. On the other hand, the retailer loses the interest on the amount of money that is paid as an advance payment for a period of time from the placing of the order till the supply of the order. Thus, we see that the advance payment is a real-life phenomenon, and the decision regarding the amount of advance payment to be made has a crucial impact on the total profit and inventory decisions. Paul et al. [5] developed a production model with multiple prepayment strategies. Furthermore, delivery lead time between a manufacturer and a retailer is not always constant. It may be a variable parameter. Hence, an advanced payment scheme along with an uncertain delivery lead time is implemented in the proposed work.

In the competitive market, the EOQ models get more challenges from various aspects for which the deterministic models are unable to solve the complexities of the problem. Generally, these complexities occur due to the imprecise nature of inventory parameters, the lack of pieces of evidence, the existence of some restricted information, etc. For such reasons, many researchers have developed various uncertain environments such as fuzzy, stochastic, interval fuzzy, randomness, roughness, and robustness (as discussed by Goli [6]) and Goli et al. [7], etc. Furthermore, Goli [8] discussed the robustness analysis for making product portfolio design in a closed-loop supply chain management. Bellman and Zadeh [9] introduced a decision-making problem using fuzzy set (FS) theory to tackle a non-random uncertainty situation. After a generalization of FS, Atanassov [10] defined an intuitionistic fuzzy set (IFS) in which the uncertainty was handled by a membership degree and non-membership degree, and the sum of membership and non-membership degrees has to be in the interval $[0,1]$. But, when people independently assign the membership and non-membership degrees to the same parameter in the actual decision-making process, then sometimes the sum of membership and non-membership degrees is greater than the number 1, while the sum of their squares does not exceed the number 1. Hence, Yager [11] extended the IFS into a Pythagorean fuzzy set (PFS) so that DMs could characterize fuzzy information. more effectively. Recently, Yang and Chang [12] introduced an interval-valued Pythagorean fuzzy (IVPF) set for solving multi-attribute decision-making problems. In an inventory system, due to a lack of evidence, demand rate, deterioration rate, holding cost, delivery lead-time, the discount rate on purchased cost, etc., are imprecise. Hence, in this proposed study, the inventory parameters are taken under an interval-valued triangular Pythagorean fuzzy (IVTPF) environment to make more accurate decisions for maximizing the average profit.

Therefore, each item described above has an important role to formulate the proposed model. The motivation of this study is to see how hybrid-price-stock-dependent demand, advance payment schemes, and PT investment affect an EOQ problem under an IVTPF environment. The objective is to find an optimum order quantity, cycle time, PT investment and number of advance payment instalments by maximizing the average profit per unit time, and to discuss a comparative study between crisp and IVTPF models.

According to our best knowledge, only a few researchers have combined advanced payment policies, hybrid price-dependent demand, and PT to develop EOQ models of perishable products. However, none of the researchers who examined inventory models in the literature did so by taking into account IVTPF inventory parameters with hybrid price- and stock-dependent demand using PT and an advance payment scheme. Hence, the research objectives and aims of this study are to find the answers to the following research questions:

- How advanced payment policy is incorporated in an inventory management and how it affects the profit of a retailer?
- How do retailers utilize a PT, and what is the effect of PT on increasing retailers' profit?
- How hybrid-price and stock-dependent demand rate affects EOQ models?
- What is the effect of IVTPE in an EOQ model to handle uncertain parameters?

Hence, this study develops two EOQ models that integrate hybrid price-stock dependent demand with an advanced payment policy under an IVTPF environment to enhance decision-making and maximize profitability in uncertain and dynamic market conditions. Also, PT is applied to reduce the deterioration rate of products. The proposed EOQ models under the IVTPF environment are converted into two deterministic models by using the linear ranking index method defined in Definition 6. Thereafter, the deterministic models are solved by the analytical definition of the non-linear maximization problem. The overall research methodology for this study is demonstrated in Figure 2.

The outline of the remaining part of this study is depicted as follows: A literature review, research gaps, and contributions of this study are described in Section 2. Section 3 reflects some preliminaries about FS, IFS, PFS, and IVTPF numbers together with some basic operators. Section 4 contains notations, assumptions, and the formulations of crisp and IVTPF models under two subcases. Solution methodology for solving the corresponding formulated models is provided in Section 5. Some application examples for validating the proposed models are investigated in Section 6, whereas the results of these application examples are discussed in Section 7. Sensitivity analysis along with some managerial insights for some practitioners is explained in Section 8. Finally, the paper comes to an end with a conclusion along with some future research directions which are referred to in Section 9.

2. LITERATURE REVIEW

Herein, a brief literature is reviewed towards four main directions: (i) Inventory problem with variable demand rate, (ii) Inventory problem with PT for perishable products, (iii) Inventory problem with different payment policies, and (iv) Inventory problem under uncertain circumstance.

2.1. Inventory problem with variable demand rate

Because of product's variety, the massive display of products in a supermarket frequently generates a lot of sales. When a retailer has a high stock of a product, demand for that retailer rises; when there is a low stock of products, demand falls. For these reasons, in an inventory model, demand may be correlated with product stock levels. Furthermore, in the real market, demand doesn't only depend on product's stock level, it is also highly affected by product's selling price in an inventory system. Das et al. [13] and Khedlekar et al. [14] investigated an inventory model for deteriorating products under price-dependent demand function. They introduced PT investment in their work to reduce product's deterioration. Pervin et al. [15] developed inventory models by taking price- and stock-dependent demand rates. Meisam et al. [16] integrated price sensitive demands and disposal decision in a disassembly EOQ problem to maximize the profit of disassembly systems. Abdul Halim et al. [17] studied a production inventory model whose goal was to discuss the model for deteriorating items under linear stock and non-linear price-dependent demand consideration. They also investigated continuous and overtime production systems in their model. A two-warehouse inventory model was developed by Tiwari et al. [18] where stock-dependent demand was assumed under inflationary conditions. In the real situation, demand rate is a hybrid form of the selling price, i.e., it is changing both linearly and non-linearly with the selling price. A few researchers such as Mishra et al.[1] investigated a hybrid-price and stock dependent demand in their proposed work. Gharaei et al. [19] and Dadashi [20] analyzed an economic production quantity model by considering stochastic demand rate and stochastic procurement cost.

2.2. Inventory problem with PT for perishable products

In the development of inventory management problems for perishable products, deterioration is a vital issue for getting optimal decisions. Products' deterioration is affected by various common factors like as, time, temperature, humidity, impreciseness, etc. Paul et al.[21] examined the effect of price-sensitive dependent demand and default risk on optimal trade credit period for a deteriorating inventory model in which constant deterioration rate was considered. Over the past decades, many researchers recommended to use PT for reducing the reckless deterioration rate. Mashud et al.[22] used PT to curb the deterioration rate for a sustainable inventory model. Shah et al.[23] analysed inventory control system through preservation technology. A mathematical inventory model for decaying items has been presented by Kumar et al. [24] under PT with the present COVID-19 epidemic environment.

2.3. Inventory problem with different payment policies

In addition, the classical EOQ model was developed by ignoring the various types of payment policies. But, there are several payment options such as advance payment, payment in delay, etc. between a supplier and a retailer. Khan et al. [25] considered an advance payment scheme which was made by equal n instalments before receiving the products. Also, Shaikh et al. [26] incorporated a two-warehouse inventory system under an advance payment policy where it was assumed that the retailer's ordered is placed by providing a fixed portion of the total purchased cost in advance. A sustainable inventory model was developed by Mashud et al. [27] with advanced payment policy. They also considered price dependent demand and concluded that advanced payment policy is a more effective policy to run a business. Thereafter, Mashud et al. [28] established a resilient hybrid payment supply chain inventory model for post Covid-19 recovery by composing multiple prepayments and a delay in payment options. Recently, Mondal et al. [29] discussed pricing strategies and advance payment-based inventory model with partially backlogged shortages under interval uncertainty.

2.4. Inventory problem under uncertain circumstance

In classical EOQ inventory models, all related parameters have been considered as crisp values. But, the values are not always crisp in nature. Furthermore, in real-life situations, those parameters may have slight deviations from their exact values which may not follow any probability distribution. In such situations, if those parameters are taken as fuzzy parameters in a model, then the model becomes more realistic. Henceforth, recently, the concept of fuzzy parameters has been introduced in the inventory problems by several researchers. Shekarian et al.[30] analyzed an EOQ inventory model by assuming all input parameters as triangular fuzzy numbers and applied Graded Mean Integration Value for defuzzification the fuzzy parameters. Later, Barman et al. [31] formulated an economic production quantity model for deteriorating products in which demand rate, holding cost and deterioration rate were considered as triangular cloudy fuzzy numbers. Poswal et al. [32] developed an inventory model with weibull deteriorating rate and variable holding cost under fuzzy environment. Again, some researchers have paid their attention to the application of PFS in inventory management problems. Kaur and Priya [33] introduced PFS to solve an inventory problem under an uncertain situation. De and Mahata [34] proposed a fuzzy lock leadership game approach for solving an imperfect quality EOQ model. After then, Nayak et al. [35] proposed an EOQ model for defective items under the Pythagorean fuzzy (PF) environment. They analyzed score functions with the help of (α,β) cuts of fuzzy parameters. Rahaman et al. [36] derived a lock fuzzy EPQ model with selling price-dependent demand rate using PT investment. Mukherjee et al. [37] discussed a sustainable inventory model under uncertain environment. Furthermore, Tirkolaee et al. [38] discussed a supply chain model for sustainable development under interval-valued neutrosophic fuzzy set. Previous studies related to this study are marked in Table 1.

2.5. Research gap and our contribution

Various retailers of perishable products such as fruits, vegetables, milk products, etc. in a super market use suitable PT to handle products' freshness. It is seen that products' demands depend on selling price and stock level. In the existing literature, it is observed that a lot of research works have been done by considering the price- and stock-dependent demand rates for deteriorating products. Some researchers have formulated their works using PT investment and advance payment scheme. But, in the real market, sometimes demand decreases linearly with respect to the selling price and sometimes demand also decreases non-linearly with the increasing of selling price. So, demand depends on the selling price in hybrid form. In a little few of the existing literature, hybrid price-dependent demand has been considered. For the highly demandable products, the manufacturers or suppliers sometimes request the retailer to pay a portion of the total purchase cost in advance. In this advance payment scheme, both parties are benefited in different ways. In this way, the suppliers or manufacturers reduce the risk of cancelling the orders by receiving prepayments. On the other hand, retailer pays the advance payment with alacrity in order to ensure on-time delivery. In response to the advance payment, suppliers or manufacturers offer different price discounts or some credit facilities or some other kinds of facilities in order to promote their business. In the literature, most of the researchers developed inventory models by assuming crisp inventory parameters, but practically these parameters are imprecise and ambiguous. Hence, researchers have considered the inventory parameters in fuzzy or IF environments. But, in real-life, before starting a business, a retailer should know that what the demand rate of a product is. Hence, the retailer asks to an expert what is his or her opinion about the statement "demand rate of a product is 90 units per month". Suppose, an expert declares that the possibility of the statement is true is $\frac{\sqrt{3}}{2}$ and the possibility of the statement is false is $\frac{1}{2}$ between $((0,1)$. Thus, there occurs a vagueness in the nature of demand rate and other several parameters. In this type of uncertainty, the sum of membership and non-membership degrees does not lie in [0,1] whereas the sum of their squares does lie in [0,1]. The general FS and IFS fail to handle such situation. This situation can be handled by introducing PFS, and DMs can take decisions without any modification of provided evidences. A PFS can be defined as an augmented and amplified version of an IFS for measuring the impreciseness of a real-life complication. In particular, IVPF is important for inventory management because it makes data more flexible and realistic and allows membership and non-membership degrees of uncertain parameters to be interval values instead of fixed numbers, which improves the representation of uncertainty. Additionally, by taking into account a greater variety of options than FS and IFS, the IVPF set assists a decision maker in efficiently managing changes in demand, pricing, supplier dependability, etc. To the best of our knowledge, few researchers developed EOQ models by incorporating hybrid price-dependent demand, advanced payment policy and PT for perishable products. But, none of the researchers in literature studied an inventory model by considering IVTPF inventory parameters with hybrid price- and stock-dependent demand using PT and advance payment scheme. Hence, the motivation of this study is to study an EOQ model with hybrid price- and stock-dependent demand using PT and advance payment scheme under IVTPF environment is proposed. The main contributions of this study are listed as:

(1) This research develops two novel crisp EOQ models for products with hybrid selling price- and stock-dependent demand under an advance payment scheme that includes a discount facility and PT investment strategy to mitigate the deterioration rate. The proposed model integrates practical business considerations such as financial incentives and inventory-dependent pricing, offering a more realistic and applicable framework for decision-makers.

(2) The study introduces the concept of IVTPF numbers in EOQ modeling for the first time. This innovation enhances the model's ability to flexibly express and compute uncertainty, overcoming limitations of traditional crisp and fuzzy models. By accommodating a wider degree of vagueness and imprecision, IVTPF numbers provide a more robust approach to inventory management under uncertain demand

conditions.

(3) A linear ranking function is proposed to systematically transform the IVTPF-based EOQ models into the equivalent deterministic models, ensuring computational feasibility and practical usability while preserving the fuzzy system's inherent uncertainty representation.

(4) This study conducts a comparative analysis between the crisp and IVTPF models, demonstrating that the IVTPF approach, as an extension of PFS, offers a broader membership space and superior capability in handling uncertainty and vagueness. The findings highlight that IVTPF models provide greater decision-making flexibility and accuracy compared to traditional fuzzy sets, making them a more effective tool in complex inventory scenarios.

Table 1: An assessment between the proposed model with some previously published works

Authors	Payment's type	Demand pattern	Deterioration	PT	Discount	Transportation cost	Known parameters taken in environment
Mishra et al. [1]	NA	HPD & SD	Yes	NA	NA	NA	Crisp
Das et al. [13]	NA	LPD	Yes	DVPT	NA	NA	Crisp
Shaikh et al. [26]	PAV	LPD	Yes	NA	NA	NA	IV
Barman et al. [31]	NA	TD	Yes	NA	NA	NA	Cloudy Fuzzy
Nayak et al. [35]	NA	LPD	Yes	NA	NA	NA	PF
De and Sana [39]	NA	NLPD	NA	NA	NA	NA	Fuzzy and IF
Garai et al. [40]	NA	SD	NA	NA	NA	NA	IF
Pervin et al. [41]	NA	LPD & SD	Yes	CPT	NA	NA	Crisp
Rahman et al. [42]	NA	IV	Yes	Interval	NA	NA	IV
Ali et al. [43]	NA	Constant	Yes	NA	NA	Yes	Triangular Fuzzy
Bardhan et al. [44]	NA	SD	Yes	DVPT	NA	NA	Crisp
Mashud et al. [45]	PAV	LPD	Yes	NA	Yes	NA	Crisp
This paper	PAV	HPD & SD	Yes	DVPT	Yes	Yes	IVPF

Notes: LPD: Linear price-dependent, NLPD: Non-linear price-dependent, SD: Stock-dependent, PAV: Partial advance payment policy, CPT: PT investment is considered as a known constant parameter, DVPT: PT investment is considered as a decision variable, HPD: Hybrid price-dependent, TD: Time-dependent, PFF: Parabolic-flat Fuzzy, IV: Interval-valued, NA: Not applicable.

3. PRELIMINARIES

At first, Zadeh [46] introduced the FS concept and thereafter, FS theory is evolved as an important tool to tackle uncertainty and vagueness in many research fields including inventory management problems also. Atanassov [10] introduced IFS concept which is a generalization of FS theory. Later, Atanassov and Gargov [47] extended IFS to IVIFS concept. Yager [11] introduced the concept of PFS by extending IFS theory. For convenience, here, definitions of IVIFS, interval-valued intuitionistic fuzzy number (IVIFN), PFS and interval-valued Pythagorean fuzzy number (IPFN) are presented.

Definition 1 (Mondal and Roy [48]). *Let Y be a finite universe of discourse and $Intv([0,1])$ denote set of all subintervals of the interval $[0,1]$. An IVIFS is a set of the form $\widetilde{S^I} = \{\langle y, \Phi_{\widetilde{S^I}}(y), \Psi_{\widetilde{S^I}}(y)\rangle | y \in Y\}$, where $\Phi_{\widetilde{S^I}}(y) : Y \mapsto [0,1], \Psi_{\widetilde{S^I}}(y) : Y \mapsto [0,1]$ are interval-valued membership and non-membership functions, respectively, and $\Phi_{\widetilde{S^I}}(y)$, $\Psi_{\widetilde{S^I}}(y)$ are* *membership and non-membership degrees, respectively, of the element* $y \in Y$ *in* $\widetilde{S^I}$, *given that* $Sup\ \Phi_{\widetilde{S^I}}(y) + Sup\ \Psi_{\widetilde{S^I}}(y) = 1$ *for all* $y \in Y$.

Definition 2. *An interval-valued triangular intuitionistic fuzzy number (IVTIFN) on the real axis is defined by*
$\widetilde{A^{II}} = \left[\left\langle (p^L, q, r^L), \Phi^L_{\widetilde{A^{II}}}(y), \Psi^U_{\widetilde{A^{II}}}(y)\right\rangle; \left\langle (p^U, q, r^U), \Phi^U_{\widetilde{A^{II}}}(y), \Psi^L_{\widetilde{A^{II}}}(y)\right\rangle\right]$,
where $\left[\Phi^L_{\widetilde{A^{II}}}(y), \Phi^U_{\widetilde{A^{II}}}(y)\right]$ *and* $\left[\Psi^L_{\widetilde{A^{II}}}(y), \Psi^U_{\widetilde{A^{II}}}(y)\right]$ *are the intervals for membership and non-membership functions, respectively, of IVTPFN* $\widetilde{A^{II}}$. *In addition,* $p^U < p^L < q < r^L < r^U$. *The functions* $\Phi^L_{\widetilde{A^{II}}}$, $\Phi^U_{\widetilde{A^{II}}}$, $\Psi^L_{\widetilde{A^{II}}}$ *and* $\Psi^U_{\widetilde{A^{II}}}$ *are calculated subsequently:*

$$\Phi^L_{\widetilde{A^{II}}}(y) = \begin{cases} 0, & \text{if } y < p^L \ \& \ y > r^L, \\ \frac{y-p^L}{q-p^L}, & \text{if } p^L \le y \le q, \\ \frac{r^L-y}{r^L-q}, & \text{if } q \le y \le r^L, \end{cases}$$

$$\text{and } \Phi^U_{\widetilde{A^{II}}}(y) = \begin{cases} 0, & \text{if } y < p^U \ \& \ y > r^U, \\ \frac{y-p^U}{q-p^U}, & \text{if } p^U \le y \le q, \\ \frac{r^U-y}{r^U-q}, & \text{if } q \le y \le r^U, \end{cases}$$

$$\Psi^L_{\widetilde{A^{II}}}(y) = \begin{cases} 1, & \text{if } y < p^U \ \& \ y > r^U, \\ \frac{q-y}{q-p^U}, & \text{if } p^U \le y \le q, \\ \frac{y-q}{r^U-q}, & \text{if } q \le y \le r^U \end{cases}$$

$$\text{and } \Psi^U_{\widetilde{A^{II}}}(y) = \begin{cases} 1, & \text{if } y < p^L \ \& \ y > r^L, \\ \frac{q-y}{q-p^L}, & \text{if } p^L \le y \le q, \\ \frac{y-q}{r^L-q}, & \text{if } q \le y \le r^L. \end{cases}$$

Definition 3. *Let* Y *be a finite universe of discourse. A PFS* $\widetilde{S^P}$ *in* Y *is defined as a set of ordered triplet* $\widetilde{S^P} = \{\langle y, \Phi_{\widetilde{S^P}}(y), \Psi_{\widetilde{S^P}}(y)\rangle | y \in Y\}$. *Here, the functions* $\Phi_{\widetilde{S^P}}(y) : Y \mapsto [0,1]$ *and* $\Psi_{\widetilde{S^P}}(y) : Y \mapsto [0,1]$ *satisfy the condition:* $0 \le \Phi^2_{\widetilde{S^P}}(y) + \Psi^2_{\widetilde{S^P}}(y) \le 1$. *The values* $\Phi_{\widetilde{S^P}}(y)$ *and* $\Psi_{\widetilde{S^P}}(y)$ *are membership and non-membership degrees of an element* $y \in Y$ *to the set* $\widetilde{S^P}$, *respectively. Moreover, there is a degree of hesitation between membership and non-membership functions which is defined as:* $\Pi_{\widetilde{S^P}}(y) = \sqrt{1 - (\Phi^2_{\widetilde{S^P}}(y) - \Psi^2_{\widetilde{S^P}}(y))}$.

Definition 4. *A triangular Pythagorean fuzzy number (TPFN) on the real axis is represented as* $\widetilde{A^P} = \{\langle (f, g, h); \Phi_{\widetilde{A^P}}(y), \Psi_{\widetilde{A^P}}(y)\rangle | y \in Y\}$, *where* $\Phi_{\widetilde{A^P}}(y) \ge 0$, $\Psi_{\widetilde{A^P}}(y) \ge 0$, $0 \le \Phi^2_{\widetilde{A^P}}(y) + \Psi^2_{\widetilde{A^P}}(y) \le 1$ *and* $0 \le f \le g \le h$. *The membership function* $\Phi_{\widetilde{A^P}}(y)$ *and the non-membership function* $\Psi_{\widetilde{A^P}}(y)$, *respectively, are determined as follows:*

$$\Phi_{\widetilde{A^P}}(y) = \begin{cases} 0, & \text{if } y < f \ \& \ y > h, \\ \frac{y-f}{g-f}, & \text{if } f \le y \le g, \\ \frac{h-y}{h-g}, & \text{if } g \le y \le h \end{cases}$$

$$\text{and } \Psi_{\widetilde{A^P}}(y) = \begin{cases} 1, & \text{if } y < f \ \& \ y > h, \\ \sqrt{1-(\frac{y-f}{g-f})^2}, & \text{if } f \leq y \leq g, \\ \sqrt{1-(\frac{h-y}{h-g})^2}, & \text{if } g \leq y \leq h. \end{cases}$$

Definition 5. *An interval-valued triangular Pythagorean fuzzy number (IVTPFN) on* $\mathbb{R}$ *is defined by*

$\widetilde{A^{IP}} = \left[\left\langle (f^L, g, h^L), \Phi^L_{\widetilde{A^{IP}}}(y), \Psi^U_{\widetilde{A^{IP}}}(y) \right\rangle; \left\langle (f^U, g, h^U), \Phi^U_{\widetilde{A^{IP}}}(y), \Psi^L_{\widetilde{A^{IP}}}(y) \right\rangle\right]$,

where $\left[\Phi^L_{\widetilde{A^{IP}}}(y), \Phi^U_{\widetilde{A^{IP}}}(y)\right]$ *and* $\left[\Psi^L_{\widetilde{A^{IP}}}(y), \Psi^U_{\widetilde{A^{IP}}}(y)\right]$ *are the intervals for membership and non-membership functions, respectively, of IVTPFN* $\widetilde{A^{IP}}$. *In addition,* $f^U < f^L < g < h^L < h^U$. *The graphical representation of an IVTPFN is shown in Figure 1. The functions* $\Phi^L_{\widetilde{A^{IP}}}$, $\Psi^L_{\widetilde{A^{IP}}}$, $\Phi^U_{\widetilde{A^{IP}}}$ *and* $\Psi^U_{\widetilde{A^{IP}}}$ *are calculated subsequently as:*

$$\Phi^L_{\widetilde{A^{IP}}}(y) = \begin{cases} 0, & \text{if } y < f^L \ \& \ y > h^L, \\ \frac{y-f^L}{g-f^L}, & \text{if } f^L \leq y \leq g, \\ \frac{h^L-y}{h^L-g}, & \text{if } g \leq y \leq h^L, \end{cases}$$

$$\text{and } \Psi^L_{\widetilde{A^{IP}}}(y) = \begin{cases} 1, & \text{if } y < f^U \ \& \ y > h^U, \\ \sqrt{1-(\frac{y-f^U}{g-f^U})^2}, & \text{if } f^U \leq y \leq g, \\ \sqrt{1-(\frac{h^U-y}{h^U-g})^2}, & \text{if } g \leq y \leq h^U \end{cases}$$

$$\Phi^U_{\widetilde{A^{IP}}}(y) = \begin{cases} 0, & \text{if } y < f^U \ \& \ y > h^U, \\ \frac{y-f^U}{g-f^U}, & \text{if } f^U \leq y \leq g, \\ \frac{h^U-y}{h^U-g}, & \text{if } g \leq y \leq h^U, \end{cases}$$

$$\text{and } \Psi^U_{\widetilde{A^{IP}}}(y) = \begin{cases} 1, & \text{if } y < f^L \& y > h^L, \\ \sqrt{1-(\frac{y-f^L}{g-f^L})^2}, & \text{if } f^L \leq y \leq g, \\ \sqrt{1-(\frac{h^L-y}{h^L-g})^2}, & \text{if } g \leq y \leq h^L. \end{cases}$$

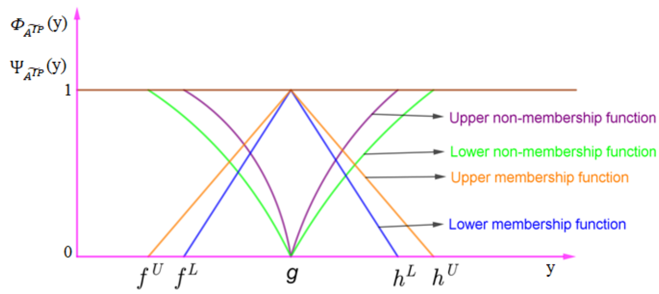

Figure 1: Graphical presentation of membership and non-membership functions of an IVTPFN.

Definition 6. *Let* $\widetilde{A^{IP}} = \left[\left\langle (f_1^L, g_1, h_1^L), \Phi^L_{\widetilde{A^{IP}}}(y), \Psi^U_{\widetilde{A^{IP}}}(y) \right\rangle; \left\langle (f_1^U, g_1, h_1^U), \Phi^U_{\widetilde{A^{IP}}}(y), \Psi^L_{\widetilde{A^{IP}}}(y) \right\rangle\right]$ *and* $\widetilde{B^{IP}} = \left[\left\langle (f_2^L, g_2, h_2^L), \Phi^L_{\widetilde{B^{IP}}}(y), \Psi^U_{\widetilde{B^{IP}}}(y) \right\rangle; \left\langle (f_2^U, g_2, h_2^U), \Phi^U_{\widetilde{B^{IP}}}(y), \Psi^L_{\widetilde{B^{IP}}}(y) \right\rangle\right]$ *be two*

IVTPFNs, $\rho > 0$ be a real number. Then, the basic operational rules of IVTPFN are defined as follows:

I. $\widetilde{A^{IP}} \oplus \widetilde{B^{IP}} = \begin{bmatrix} \left\langle \left(f_1^L + f_2^L, g_1 + g_2, h_1^L + h_2^L\right), \Phi_{\oplus}^L(y), \Psi_{\oplus}^U(y) \right\rangle; \\ \left\langle \left(f_1^U + f_2^U, g_1 + g_2, h_1^U + h_2^U\right), \Phi_{\oplus}^U(y), \Psi_{\oplus}^L(y) \right\rangle \end{bmatrix},$

$\Phi_{\oplus}^{L(U)}(y) = \sqrt{\left(\Phi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^2 + \left(\Phi_{\widetilde{B^{IP}}}^{L(U)}(y)\right)^2 - \left(\Phi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^2 . \left(\Phi_{\widetilde{B^{IP}}}^{L(U)}(y)\right)^2},$

$\Psi_{\oplus}^{L(U)}(y) = \Psi_{\widetilde{A^{IP}}}^{L(U)}(y).\Psi_{\widetilde{B^{IP}}}^{L(U)}(y).$

II. $\widetilde{A^{IP}} \otimes \widetilde{B^{IP}} = \begin{bmatrix} \left\langle \left(f_1^L.f_2^L, g_1.g_2, h_1^L.h_2^L\right), \Phi_{\otimes}^L(y), \Psi_{\otimes}^U(y) \right\rangle; \\ \left\langle \left(f_1^U.f_2^U, g_1.g_2, h_1^U.h_2^U\right), \Phi_{\otimes}^U(y), \Psi_{\otimes}^L(y) \right\rangle \end{bmatrix},$

$\Phi_{\otimes}^{L(U)}(y) = \Phi_{\widetilde{A^{IP}}}^{L(U)}(y).\Phi_{\widetilde{B^{IP}}}^{L(U)}(y),$

$\Psi_{\otimes}^{L(U)}(y) = \sqrt{\left(\Phi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^2 + \left(\Psi_{\widetilde{B^{IP}}}^{L(U)}(y)\right)^2 - \left(\Psi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^2 . \left(\Psi_{\widetilde{B^{IP}}}^{L(U)}(y)\right)^2}.$

III. $\rho.\widetilde{A^{IP}} = \begin{bmatrix} \left\langle \left(\rho.f_1^L, \rho.g_1, \rho.h_1^L\right), \Phi_{\rho.\widetilde{A^{IP}}}^L(y), \Psi_{\rho.\widetilde{A^{IP}}}^U(y) \right\rangle; \\ \left\langle \left(\rho.f_1^U, \rho.g_1, \rho.h_1^U\right), \Phi_{\rho.\widetilde{A^{IP}}}^U(y), \Psi_{\rho.\widetilde{A^{IP}}}^L(y) \right\rangle \end{bmatrix},$

$\Phi_{\rho.\widetilde{A^{IP}}}^{L(U)}(y) = \sqrt{1 - \left(1 - \Phi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^{\rho}}$ *and* $\Psi_{\rho.\widetilde{A^{IP}}}^{L(U)}(y) = \left(\Psi_{\widetilde{A^{IP}}}^{L(U)}(y)\right)^{\rho}.$

Definition 7. ***Ranking Index of an IVTPFN:***
Let $\widetilde{A^{IP}} = \left[\left\langle (f^L, g, h^L), \Phi_{\widetilde{A^{IP}}}^L(y), \Psi_{\widetilde{A^{IP}}}^U(y) \right\rangle; \left\langle (f^U, g, h^U), \Phi_{\widetilde{A^{IP}}}^U(y), \Psi_{\widetilde{A^{IP}}}^L(y) \right\rangle\right]$ *be an IVTPFN and* $\tau \in (0,1)$ *be a fixed parameter. Then the lower and upper ranking indices of* $\widetilde{A^{IP}}$ *are defined subsequently as:*

$$\Re^L(\widetilde{A^{IP}}) = \frac{\int_{f^L}^{h^L} y.\left(\Phi_{\widetilde{A^{IP}}}^L\right)^2(y)dy + (1-\tau)\int_{f^L}^{h^L} y.\left(\Psi_{\widetilde{A^{IP}}}^U\right)^2(y)dy}{\int_{f^L}^{h^L} \Phi_{\widetilde{A^{IP}}}^L(y)dy + (1-\tau)\int_{f^L}^{h^L} \Psi_{\widetilde{A^{IP}}}^U(y)dy}$$
$$= \frac{1}{4}\frac{\tau(f^L + h^L + 2g) + (1-\tau)(5f^L + 5h^L - 2g)}{2-\tau} \quad (1)$$

$$\text{and } \Re^U(\widetilde{A^{IP}}) = \frac{\int_{f^U}^{h^U} y.\left(\Phi_{\widetilde{A^{IP}}}^U\right)^2(y)dy + (1-\tau)\int_{f^U}^{h^U} y.\left(\Psi_{\widetilde{A^{IP}}}^L\right)^2(y)dy}{\int_{f^U}^{h^U} \Phi_{\widetilde{A^{IP}}}^U(y)dy + (1-\tau)\int_{f^U}^{h^U} \Psi_{\widetilde{A^{IP}}}^L(y)dy}$$
$$= \frac{1}{4}\frac{\tau(f^U + h^U + 2g) + (1-\tau)(5f^U + 5h^U - 2g)}{2-\tau}. \quad (2)$$

Therefore, the ranking index interval of $\widetilde{A^{IP}}$ *is* $\Re I(\widetilde{A^{IP}}) = [\Re^L(\widetilde{A^{IP}}), \Re^U(\widetilde{A^{IP}})]$*. Now, the ranking index of an IVTPFN is defined by*

$$\Re(\widetilde{A^{IP}}) = \frac{\Re^L(\widetilde{A^{IP}}) + \Re^U(\widetilde{A^{IP}})}{2}$$

$$= \frac{1}{8}\frac{\tau(f^L + f^U + h^L + h^U + 4g) + (1-\tau)(5f^L + 5f^U + 5h^L + 5h^U - 4g)}{2-\tau}. \quad (3)$$

Here, τ *is an weight that represents DM's preference information.*

Theorem 8. *If* $\widetilde{A^{IP}} = \left[\left\langle \left(f_1^L, g_1, h_1^L\right), \Phi^L_{\widetilde{A^{IP}}}(y), \Psi^U_{\widetilde{A^{IP}}}(y)\right\rangle; \left\langle \left(f_1^U, g_1, h_1^U\right), \Phi^U_{\widetilde{A^{IP}}}(y), \Psi^L_{\widetilde{A^{IP}}}(y)\right\rangle\right]$ *and* $\widetilde{B^{IP}} = \left[\left\langle \left(f_2^L, g_2, h_2^L\right), \Phi^L_{\widetilde{A^{IP}}}(y), \Psi^U_{\widetilde{A^{IP}}}(y)\right\rangle; \left\langle \left(f_2^U, g_2, h_2^U\right), \Phi^U_{\widetilde{A^{IP}}}(y), \Psi^L_{\widetilde{A^{IP}}}(y)\right\rangle\right]$ *be two IVTPFNs, then for two positive real numbers* k_1 *and* k_2, $\Re(k_1.\widetilde{A^{IP}} + k_2.\widetilde{B^{IP}}) = k_1.\Re(\widetilde{A^{IP}}) + k_2.\Re(\widetilde{B^{IP}})$.

Proof. From Definition 6 and Equation (3), it is obtained that

$$
\begin{aligned}
&\Re(k_1.\widetilde{A^{IP}} + k_2.\widetilde{B^{IP}}) \\
&= \frac{1}{8}\frac{\tau\Big(k_1 f_1^L + k_2 f_2^L + k_1 f_1^U + k_2 f_2^U + k_1 h_1^L + k_2 h_1^L + k_1 h_1^U + k_2 h_2^U + 4(k_1 g_1 + k_2 g_2)\Big)}{2-\tau} \\
&+ \frac{1}{8}\frac{(1-\tau)\Big(5(k_1 f_1^L + k_2 f_2^L) + 5(k_1 f_1^U + k_2 f_2^U) + 5(k_1 h_1^L + k_2 h_2^L) + 5(k_1 h_1^U + k_2 h_2^U)\Big)}{2-\tau} \\
&- \frac{1}{8}\frac{(1-\tau)(4k_1 g_1 + 4k_2 g_2)}{2-\tau} \\
&= k_1.\frac{1}{8}\frac{\tau(f_1^L + f_1^U + h_1^L + h_1^U + 4g_1) + (1-\tau)(5f_1^L + 5f_1^U + 5h_1^L + 5h_1^U - 4g_1)}{2-\tau} \\
&+ k_2.\frac{1}{8}\frac{\tau(f_2^L + f_2^U + h_2^L + h_2^U + 4g_2) + (1-\tau)(5f_2^L + 5f_2^U + 5h_2^L + 5h_2^U - 4g_2)}{2-\tau} \\
&= k_1.\Re(\widetilde{A^{IP}}) + k_2.\Re(\widetilde{B^{IP}}). \qquad (4)
\end{aligned}
$$

This completes the proof of the Theorem. □

4. MODEL FORMULATION

Before formulating the proposed models, we define problem description, required notations and assumptions.

4.1. Problem description

In real-world markets, product demand does not always follow a fixed pattern of dependence on selling price; instead, it exhibits hybrid behavior, fluctuating non-linearly with seasonal changes and stock availability. Traditional EOQ models fail to adequately capture these dynamic variations, especially under conditions of uncertainty in deterioration rate, holding cost, and delivery lead-time. Decision-makers struggle to optimize pricing and inventory control due to these imprecise parameters. Hence, this study develops two EOQ models that integrate hybrid price-stock dependent demand with an advanced payment policy under an IVTPF environment to enhance decision-making and maximize profitability in uncertain and dynamic market conditions. Also, PT is applied to reduce deterioration rate of products. The proposed EOQ models under IVTPF environment are converted into two deterministic models by using the linear ranking index method defined in Definition 6. Thereafter, the deterministic models are solved analytical definition of non-linear maximization problem. The overall research methodology for this study is demonstrated in Figure 2.

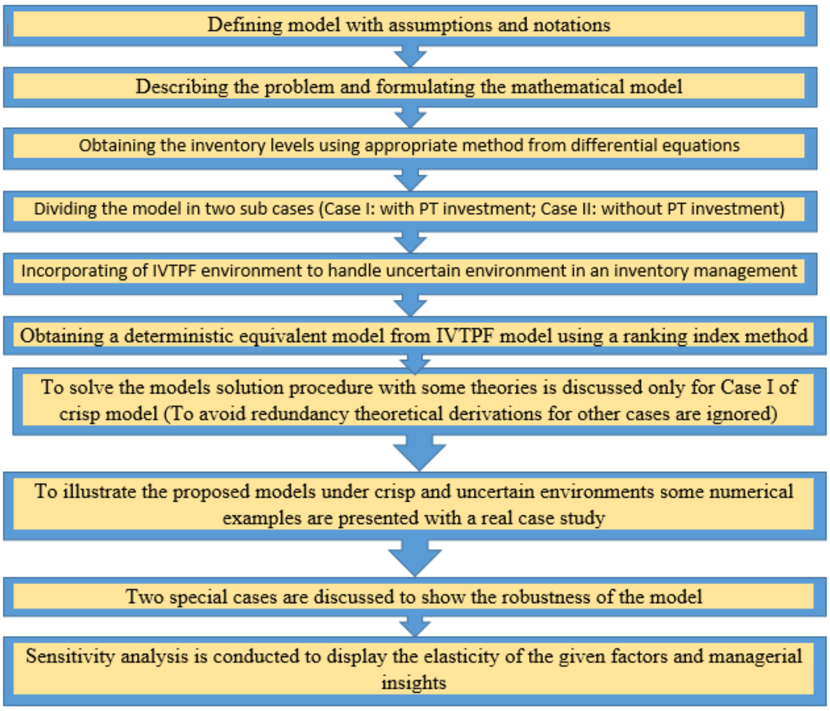

Figure 2: Schematic diagram of the proposed study.

Notations:

The following notations are used to develop this model.

Parameters	
A	Ordering cost per cycle
D_o	Original demand rate of customers per unit time
p_c	Unit purchasing cost
d_c	Unit deteriorating cost
λ_0	Deterioration rate before usage of PT
h	Holding cost per product
T_f	Fixed transportation cost
T_v	Variable transportation cost per product
δ	Fraction of purchasing cost to be paid as multiple advance payments
α	Discount rate on purchasing cost offered by the manufacturer
I_e	Interest rate per unit time

Dependent	**variables**
$D(S_p, I(t))$	Demand rate of customers per unit time
$\lambda(G)$	Reduced deterioration rate after applying PT
Q	Order quantity of retailer
$I(t)$	Inventory level of retailer
T_1	Length of time during which the retailer will pay the advance payments
Decision	**variables**
M	Number of equally spaced advance payments to be made before receiving the order
S_p	Unit product's selling price
G	PT investment to reduce deterioration rate
T	Length of cycle time

Assumptions:

1. Similar to Mishra et al. [1], consumers' demand for products depends on selling price in hybrid form and on-hand stock-level of inventory. So, the demand rate for product is defined as: $D(S_p, I(t)) = D(S_p) + cI(t)$, where c $(0 < c < 1)$ is a stock dependent consumers' consumption rate and $D(S_p) = \varphi(D_o - aS_p) + (1-\varphi)bS_p^{-\kappa}$ is the hybrid selling price-dependent demand that depends on the selling price in both linear and non-linear form. For the function $D(S_p)$, the conditions $D_o, b > 0$, $0 \leq \varphi \leq 1$, $\frac{D_o}{a} \geq S_p$ and $\kappa > 1$ must hold. When $\varphi = 1$ then $D(s_p)$ decreases linearly with respect to the selling price S_p. Furthermore, when $0 \leq \varphi < 1$ then $D(S_p)$ decreases exponentially with respect to increasing of S_p.
2. Shortages are not permitted.
3. When an order is placed by a retailer to a manufacturer, the manufacturer asks the retailer to pay δ $(0 < \delta < 1)$ part of the total purchase cost in advance before delivering the order quantity (see Mashud et al. [45]).
4. The retailer pays the advance amount under multiple instalments with an equal amount. Furthermore, due to this advance payment, the manufacturer offers the retailer a discount α $(0 < \alpha < 1)$ on purchasing cost.
5. To reduce the product's deterioration rate the retailer invested capital in PT. The reduced deterioration rate after applying PT is $\lambda(G) = \lambda_0 e^{-\xi G}$ as discussed by Das et al. [13]. The function $\lambda(G)$ fulfils the conditions $\frac{d\lambda(G)}{G} = -\frac{\lambda_0}{\xi} e^{-\xi G} < 0$ and $\frac{d^2\lambda(G)}{dG^2} > 0$. Therefore, the function $\lambda(G)$ is a decreasing function of PT investment G. Here, the deterioration rate λ_0 lies in $(0,1)$ and $\xi > 0$ is the sensitive coefficient of G to the deterioration rate.
6. There is no option to repair or return a deteriorated product.
7. All known parameters are assumed as constant values for crisp model.
8. The lead time, deterioration rate and holding cost are assumed as IVTPFNs for IVTPF models.

4.2. Formulation of the proposed EOQ model

Here, an EOQ model is made in the consideration of the hypothesis mentioned before. Retailers may be asked to pay a portion of the entire purchase price before delivering the products by manufacturers or suppliers. Both sides gain different benefits from this early payment plan. By accepting prepayments, suppliers or manufacturers lower the chance of orders being cancelled. Conversely, the retailer promptly makes the advance payment in order to guarantee on-time delivery. To promote their business, suppliers and manufacturers may provide various price concessions, credit options, or other services in exchange for an advance payment. Furthermore, as inflation is increasing for most of the products, it becomes hard to a retailer for paying the full amount in a single instalment. Keeping all this in the mind, the manufacturer proposes the retailer a scheme of advance payment. Figure 3 refers physical situation of the proposed EOQ model where the retailer makes order Q units of a deteriorating product by prepaying a fraction δ of total purchasing cost $(1-\alpha)p_cQ$ with M instalments at equal intervals $\frac{T_1}{M}$ within the lead time T_1. The retailer receives the ordered quantity by completely paying the rest purchasing cost at time $t = 0$. After receiving the ordered quantity Q, the retailer's inventory level $I(t)$ starts to diminish due to both demand and deterioration. Finally, the inventory level reaches zero at time $t = T$.

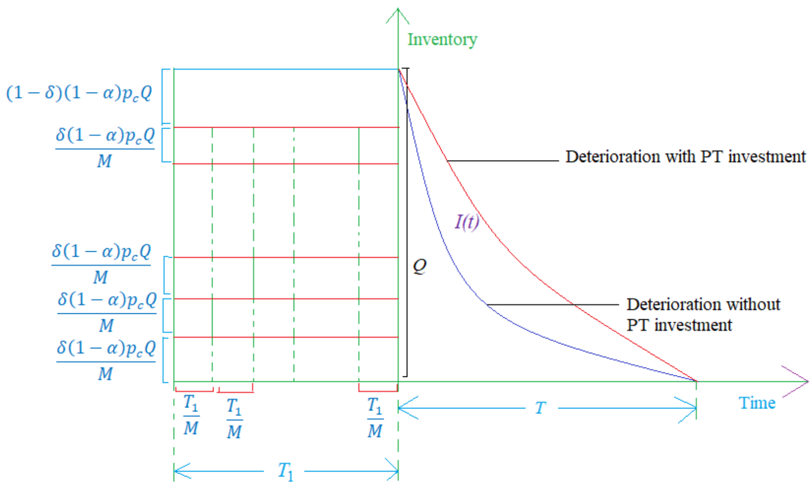

Figure 3: Graphical representation of the proposed model

Throughout the replenishment cycle time [0, T], the inventory level at time t is obtained by solving the subsequent differential equation:

$$\begin{aligned}\frac{dI(t)}{dt} + \lambda(G)I(t) &= -D(S_p, I(t)) \\ &= -(D(S_p) + cI(t)),\end{aligned} \tag{5}$$

with the boundary conditions $I(0) = Q$ and $I(T) = 0$.
By solving Equation (5) under the condition $I(T) = 0$, the inventory level $I(t)$ is obtained as:

$$I(t) = D(S_p)\frac{e^{(\lambda(G)+c)(T-t)} - 1}{\lambda(G)+c}. \tag{6}$$

At $t = 0$, $I(t) = Q$. Therefore, from Equation (6), we get

$$Q = \frac{D(S_p)}{B}\left(e^{BT} - 1\right), \text{where } B = \lambda(G) + c. \tag{7}$$

4.2.1. Model with PT investment

In this model, the retailer has a plan to preserve the perishable products for a long time by using PT. Hence, the retailer needs an extra investment in PT. The components corresponding to the retailer's total profit function per cycle are depicted below:

i. Total ordering cost $(OC) = A$.

ii. Total purchasing cost $(PC) = (1-\alpha)\, p_c Q$.

iii. Total stock holding cost (HC) is calculated as follows:

$$HC = h\int_0^T I(t)dt = h\int_0^T \frac{D(S_p)}{B}\left(e^{B(T-t)} - 1\right)dt$$

$$= h\frac{D(S_p)}{B}\left(-\frac{1}{B} + \frac{e^{BT}}{B} - T\right) = h\frac{D(S_p)}{B^2}\left(e^{BT} - TB - 1\right).$$

iv. Total deteriorating cost is defined as:

$DC = d_c \times$ (Total amount of deteriorated products)

$= d_c\lambda(G)Q = d_c\lambda(G)\frac{D(S_p)}{B}\left(e^{BT} - 1\right).$

v. Total preservation cost $PRC = GT$ for reducing the deterioration rate of products.

vi. **Transportation cost:** Total transportation cost (TC) is expressed as:

$TC =$ Fixed transportation cost $+ Qd_{st} \times$ (Variable transportation cost)

$= T_f + Qd_{st}T_v = T_f + T_v d_{st}\frac{D(S_p)}{B}\left(e^{BT} - 1\right).$

vii. **Capital cost:** The manufacturer gives an offer to retailer due to pay δ part of total purchased cost in advance. The offer is that α fraction discount on purchasing cost p_c per product. The retailer pays the advance amount with M equal instalments within T_1 months from the time of placing order. When the retailer does not have enough fund during the time T_1, he/she may take a loan from bank at an interest I_e per unit money to pay the advance amount. Therefore, the total interest of taking a loan is calculated as:

$$TI = I_e\frac{\delta(1-\alpha)p_cQ}{M}\left(\frac{T_1}{M}\right)M + I_e\frac{\delta(1-\alpha)p_cQ}{M}\left(\frac{T_1}{M}\right)(M-1)$$

$$+ I_e\frac{\delta(1-\alpha)p_cQ}{M}\left(\frac{T_1}{M}\right)(M-2) + \ldots + I_e\frac{\delta(1-\alpha)p_cQ}{M}\left(\frac{T_1}{M}\right)$$

$$(M-(M-1))$$

$$= I_e\frac{\delta(1-\alpha)p_cQ}{M}\left(\frac{T_1}{M}\right)(M + M - 1 + M - 2 + \ldots + 1)$$

$$= I_e\frac{\delta(1-\alpha)p_cQ}{2M}T_1(M+1).$$

viii. **Total sales revenue:** Total sales revenue (TSR) is calculated as:

$$TSR = S_p(Q - \lambda(G)Q) = S_p(1-\lambda(G))\frac{D(S_p)}{B}\left(e^{BT} - 1\right).$$

From the above results, the total relevant profit per cycle for the retailer can be expressed as: $TPF = TSR - OC - PC - HC - DC - PRC - TC - TI$; after doing some simplification, the following average profit function (APF) per unit time is achieved:

$$\begin{aligned}APF &= \frac{1}{T}\Big[S_p\left(1-\lambda(G)\right)\frac{D(S_p)}{B}\left(e^{BT}-1\right) - A - GT - T_f\Big] \\ &- \frac{1}{T}h\frac{D(S_p)}{B^2}\left(e^{BT}-TB-1\right) - \frac{1}{T}\frac{D(S_p)}{B}\left(e^{BT}-1\right)\Big[(1-\alpha)\,p_c + d_c\lambda(G) \\ &+ T_v d_{st} + I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}T_1\,(M+1)\Big] \\ &= S_p\left(1-\lambda(G)\right)D(S_p)\left(1+\frac{BT}{2}\right) - \frac{A+T_f}{T} - hD(S_p)\frac{1}{T} - G \\ &- D(S_p)\left(1+\frac{BT}{2}\right)\left[(1-\alpha)\,p_c + d_c\lambda(G) + T_v d_{st} + I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}T_1\,(M+1)\right]. \qquad (8)\end{aligned}$$

4.2.2. Model without PT investment

If the retailer does not apply any PT then there is no any extra cost for PT and the average profit per unit time is as below.

$$\begin{aligned}APF &= \frac{TSR - OC - PC - HC - DC - TC - TI}{T} \\ &= S_p\left(1-\lambda_0\right)D(S_p)\left(1+\frac{BT}{2}\right) - \frac{A+T_f}{T} - hD(S_p)\frac{1}{T} \\ &- D(S_p)\left(1+\frac{BT}{2}\right)(1-\alpha)\,p_c - D(S_p)\left(1+\frac{BT}{2}\right)[d_c\lambda_0 + T_v d_{st} \\ &+ I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}T_1\,(M+1)\Big], \qquad (9)\end{aligned}$$

where the related components TSR, OC, PC, HC, DC, TC and TI are described in Subsection 4.2.1.

4.3. Implication of IVTPF environment in inventory management

Due to the rising complexity of the socio-economic environment and natural intuitive thinking, various restrictions such as inadequate knowledge, information fission and inaccurate data sets may occur in decision making process of an inventory management system. Hence, some inventory parameters such as holding cost, deterioration rate, delivery lead-time, etc. are not always crisp and are sometimes exposed to ambiguity in the real-life situations due to unreliability of the data collection. A classical EOQ model in the crisp environment does not handle this type of uncertain real-life issue appropriately. As a result, it is very much essential to use the best membership function and a suitable de-fuzzification method to convert the real-world problems into an appropriate mathematical model. The basic flexibility of a parameter is taken into account in the broad fuzzy idea without any opposition. Although the IFS opposes the general perspective of fuzzy notions, but it is limited in its definition. For example, the membership degree β and non-membership degree γ of an IF number are connected as $\beta \geq \gamma$ and $\beta + \gamma \leq 1$, $\forall\ \beta, \gamma \in [0,1]$. Suppose, $\beta = 0.7$ and $\gamma = 0.5$ satisfying $\beta \geq \gamma$, but $\beta + \gamma \geq 1$. In an inventory management problem, this situation may arise where all the information is not known. Products' deterioration rate, holding cost, delivery lead-time vary from time to time and delivery lead-time for different manufacturers may also be different. This concept is more realistic than an existing IFS and the best fitted fuzzy relaxation could be accomplished with an IVTPFS.

4.4. Model under IVTPF environment

The formulation of the EOQ model where the delivery lead-time T_1 for advance payment, the deterioration rate λ_0 and holding cost h are expressed by IVTPFNs seems to be viable and gives more flexibility to the retailer. Now, the IVTPF parameters of the proposed model are defined as:

$$\widetilde{T_1^{IP}} = \left[\left\langle (T_{11}^L, T_{12}, T_{13}^L), \Phi^L_{\widetilde{T_1^{IP}}}(y), \Psi^U_{\widetilde{T_1^{IP}}}(y)\right\rangle ; \left\langle (T_{11}^U, T_{12}, T_{13}^U), \Phi^U_{\widetilde{T_1^{IP}}}(y), \Psi^L_{\widetilde{T_1^{IP}}}(y)\right\rangle\right],$$

$$\widetilde{\lambda_0^{IP}} = \left[\left\langle (\lambda_{01}^L, \lambda_{02}, \lambda_{03}^L), \Phi^L_{\widetilde{\lambda_0^{IP}}}(y), \Psi^U_{\widetilde{\lambda_0^{IP}}}(y)\right\rangle ; \left\langle (\lambda_{01}^U, \lambda_{02}, \lambda_{03}^U), \Phi^U_{\widetilde{\lambda_0^{IP}}}(y), \Psi^L_{\widetilde{\lambda_0^{IP}}}(y)\right\rangle\right],$$

$$\widetilde{h^{IP}} = \left[\left\langle (h_1^L, h_2, h_3^L), \Phi^L_{\widetilde{h^{IP}}}(y), \Psi^U_{\widetilde{h^{IP}}}(y)\right\rangle ; \left\langle (h_1^U, h_2, h_3^U), \Phi^U_{\widetilde{h^{IP}}}(y), \Psi^L_{\widetilde{h^{IP}}}(y)\right\rangle\right].$$

The required intervals for membership and non-membership functions of the respective IVTPFNs $\widetilde{T_1^{IP}}$, $\widetilde{\lambda_0^{IP}}$ and $\widetilde{h^{IP}}$ are calculated accordingly using the Definition 5. Henceforth, the models in IVTPF environment are defined below.

4.4.1. *IVTPF Model with PT*

The IVTPF model for deteriorating products under PT is obtained by putting IVTPF parameters $\widetilde{\lambda_0^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{T_1^{IP}}$ instead of λ_0, h and T_1, respectively, in Equation (8) and the corresponding IVTPF average profit is presented as:

$$\begin{aligned}\widetilde{APF^{IP}} = {} & S_p\left(1-\widetilde{\lambda_0^{IP}}e^{-\xi G}\right)D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)-\frac{A+T_f}{T}-\widetilde{h^{IP}}D(S_p)\frac{1}{T}\\ & -G-D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)\left[(1-\alpha)\,p_c+d_c\widetilde{\lambda_0^{IP}}e^{-\xi G}+T_v d_{st}\right]\\ & -D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)\left[I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}\widetilde{T_1^{IP}}\,(M+1)\right], \qquad (10)\end{aligned}$$

where $\widetilde{B^{IP}} = \widetilde{\lambda_0^{IP}}e^{-\xi G} + c$.

4.4.2. *IVTPF Model without PT*

The IVTPF model for deteriorating products without PT investment is obtained by putting $\widetilde{\lambda_0^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{T_1^{IP}}$ instead of λ_0, h and T_1, respectively, in Equation (9) and the corresponding IVTPF average profit is stated as:

$$\widetilde{APF^{IP}} = S_p\left(1-\widetilde{\lambda_0^{IP}}e^{-\xi G}\right)D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)-\frac{A+T_f}{T}-\widetilde{h^{IP}}D(S_p)\frac{1}{T}$$
$$-D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)\left[(1-\alpha)\,p_c+d_c\widetilde{\lambda_0^{IP}}e^{-\xi G}+T_v d_{st}\right]$$
$$-D(S_p)\left(1+\frac{\widetilde{B^{IP}}T}{2}\right)\left[+I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}\widetilde{T_1^{IP}}\,(M+1)\right], \tag{11}$$

where $\widetilde{B^{IP}} = \widetilde{\lambda_0^{IP}}+c$.

4.5. Equivalent deterministic model of IVTPF model

We cannot solve directly the IVTPF models. So, we apply a ranking index (defined in Definition 6) and use Theorem 8 to transform the IVTPF models into equivalent deterministic models.

4.5.1. Deterministic model with PT

By taking the ranking indices of $\widetilde{\lambda_0^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{T_1^{IP}}$ using Definition 6, and putting these indices in Equation (10) the deterministic model with PT is obtained as:

$$\mathfrak{R}\left(\widetilde{APF^{IP}}\right)$$
$$= S_p\left(1-\mathfrak{R}\left(\widetilde{\lambda_0^{IP}}\right)e^{-\xi G}\right)D(S_p)\left(1+\frac{\mathfrak{R}\left(\widetilde{B^{IP}}\right)T}{2}\right)-\frac{A+T_f}{T}-G$$
$$-\mathfrak{R}\left(\widetilde{h^{IP}}\right)D(S_p)\frac{1}{T}-D(S_p)\left(1+\frac{\mathfrak{R}\left(\widetilde{B^{IP}}\right)T}{2}\right)\left[(1-\alpha)\,p_c+d_c\mathfrak{R}\left(\widetilde{\lambda_0^{IP}}\right)e^{-\xi G}\right]$$
$$-D(S_p)\left(1+\frac{\mathfrak{R}\left(\widetilde{B^{IP}}\right)T}{2}\right)\left[T_v d_{st}+I_e\frac{\delta\,(1-\alpha)\,p_c}{2M}\mathfrak{R}\left(\widetilde{T_1^{IP}}\right)(M+1)\right], \tag{12}$$

where $\mathfrak{R}\left(\widetilde{B^{IP}}\right) = \mathfrak{R}\left(\widetilde{\lambda_0^{IP}}\right)e^{-\xi G}+c$.

4.5.2. Deterministic model without PT

By taking the ranking indices of $\widetilde{\lambda_0^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{T_1^{IP}}$ using Definition 6, and putting these indices in Equation (11) the deterministic model without PT is obtained as:

$$\mathfrak{R}\left(\widetilde{APF^{IP}}\right)$$
$$= S_p\left(1-\mathfrak{R}\left(\tilde{\lambda}_0{}^{IP}\right)\right)D(S_p)\left(1+\frac{\mathfrak{R}\left(\widetilde{B^{IP}}\right)T}{2}\right)-\frac{A+T_f}{T}$$
$$-\mathfrak{R}\left(\widetilde{h^{IP}}\right)D(S_p)\frac{1}{T}-D(S_p)\left(1+\frac{\mathfrak{R}\left(\widetilde{B^{IP}}\right)T}{2}\right)\left[(1-\alpha)\,p_c+d_c\mathfrak{R}\left(\widetilde{\lambda_0^{IP}}\right)\right]$$

$$-D(S_p)\left(1+\frac{\Re\left(\widetilde{B^{IP}}\right)T}{2}\right)\left[T_v d_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}\Re\left(\widetilde{T_1^{IP}}\right)(M+1)\right], \tag{13}$$

where $\Re\left(\widetilde{B^{IP}}\right)=\Re\left(\widetilde{\lambda_0^{IP}}\right)+c$.

5. SOLUTION METHODOLOGY

To reduce the length of the paper, in this section, the analytical method is discussed only for solving the above presented crisp model.

5.1. Solution procedure for Model 4.2.1

In this described model, there are four decision variables T, S_p, G and M. Among these M is a positive integer. So analytic methods are not applicable to solve the proposed models. The necessary and sufficient conditions can be used for proving unique optimum solutions T, S_p and G by considering M as constant and then using Mathematica software, the solution for mixed integer problem is obtained.

5.1.1. Necessary conditions for optimum solutions

Theorem 9. *When S_p, M and G are constant, the average profit APF is concave with respect to cycle time T.*

Proof. The first and second order partial derivatives with respect to T of the average profit APF (given in Equation (8)) are

$$\frac{\partial APF}{\partial T}=\frac{1}{2T^2}\Big[2\Big(A+T_f+hD(S_p)\Big)+S_p\Big(1-\lambda(G)\Big)D(S_p)BT^2\Big]$$
$$-D(S_p)B\Big[(1-\alpha)p_c+d_c\lambda(G)+T_v d_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big], \tag{14}$$

$$\frac{\partial^2 APF}{\partial T^2}=-\frac{2(A+T_f+hD(S_p))}{T^3}. \tag{15}$$

Therefore, for optimum solution, $\frac{\partial APF}{\partial T}=0$ which gives

$$\frac{\partial APF}{\partial T}=\frac{1}{2T^2}\Big[2\Big(A+T_f+hD(S_p)\Big)+S_p\Big(1-\lambda(G)\Big)D(S_p)BT^2\Big]$$
$$-D(S_p)B\Big[(1-\alpha)p_c+d_c\lambda(G)+T_v d_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big]=0 \tag{16}$$

and $\frac{\partial^2 APF}{\partial T^2}<0$ as $D(S_p)>0$. Hence, APF is concave with respect to T. □

Theorem 10. *When T, M and G are constant, the average profit APF is concave with respect to cycle time S_p.*

Proof. The first and second order partial derivatives with respect to S_p of the average profit APF (given in Equation (8)) are

$$\frac{\partial APF}{\partial S_p}=\Big(1-\lambda(G)\Big)\Big(1+\frac{BT}{2}\Big)\Big(D(S_p)+S_pD'(S_p)\Big)-D'(S_p)\Big(1+\frac{BT}{2}\Big)(1-\alpha)p_c$$

$$-D'(S_p)\Big[d_c\lambda(G)+T_vd_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big]\Big(1+\frac{BT}{2}\Big), \tag{17}$$

$$\frac{\partial^2 APF}{\partial {S_p}^2}=-\Big[\Big(1-\lambda(G)\Big)\Big(1+\frac{BT}{2}\Big)(2D(S_p)+S_pD''(S_p))+D''(S_p)\Big(1+\frac{BT}{2}\Big)$$
$$\times(1-\alpha)p_c+D''(S_p)\Big[d_c\lambda(G)+T_vd_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big]$$
$$\Big(1+\frac{BT}{2}\Big)\Big]. \tag{18}$$

Therefore, for optimum solution, $\frac{\partial APF}{\partial S_p}=0$ and $\frac{\partial^2 APF}{\partial {S_p}^2}<0$ as $D(S_p), D'(S_p)$ and $D''(S_p)>0$. Hence, APF is concave with respect to S_p. □

Theorem 11. *When T, M and S_p are constant, the average profit APF is concave with respect to cycle time G.*

Proof. The first and second order partial derivatives with respect to G of the average profit APF (given in Equation (8)) are

$$\frac{\partial APF}{\partial G}=\xi\lambda_0D(S_p)\Big(1+\frac{BT}{2}\Big)\Big(S_p+d_c\Big)e^{-\xi G}-1, \tag{19}$$

$$\frac{\partial^2 APF}{\partial G^2}=-\xi^2\lambda_0D(S_p)\Big(1+\frac{BT}{2}\Big)\Big(S_p+d_c\Big)e^{-\xi G}. \tag{20}$$

Therefore, for optimum solution, $\frac{\partial APF}{\partial G}=0$ and $\frac{\partial^2 APF}{\partial G^2}<0$ as $D(S_p)>0$. Hence, APF is concave with respect to G. □

5.1.2. Sufficient conditions for optimum solutions

The concavity of the average profit APF (given in Equation (8)) is shown with respect to T, G and S_p. In order to satisfy the necessary conditions for maximizing the average profit APF, the first order partial derivatives of APF with respect to T, G and S_p are equating to zero. After taking the partial derivatives and rearranging the terms, we get:

$$\frac{\partial APF}{\partial T}=\frac{1}{2T^2}\Big[2\Big(A+T_f+hD(S_p)\Big)+S_p\Big(1-\lambda(G)\Big)D(S_p)BT^2\Big]$$
$$-D(S_p)B\Big[(1-\alpha)p_c+d_c\lambda(G)+T_vd_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big]=0, \tag{21}$$

$$\frac{\partial APF}{\partial G}=\xi\lambda_0D(S_p)\Big(1+\frac{BT}{2}\Big)\Big(S_p+d_c\Big)e^{-\xi G}-1=0, \tag{22}$$

$$\frac{\partial APF}{\partial S_p}=\Big(1-\lambda(G)\Big)\Big(1+\frac{BT}{2}\Big)\Big(D(S_p)+S_pD'(S_p)\Big)-D'(S_p)\Big(1+\frac{BT}{2}\Big)(1-\alpha)p_c$$
$$-D'(S_p)\Big[d_c\lambda(G)+T_vd_{st}+I_e\frac{\delta(1-\alpha)p_c}{2M}T_1(M+1)\Big]\Big(1+\frac{BT}{2}\Big)=0. \tag{23}$$

On solving Equations (21), (22) and (23) simultaneously, the optimal values of T, G and S_p (say $T_{(M)}$, $G_{(M)}, S_{p(M)}$) are obtained for a given value of M. Solution of the simultaneous Equations (21), (22) and (23) are difficult as the expressions are highly non-linear. So, the optimal solutions are obtained numerically by using Mathematica software in Section 6.

Theorem 12. *The average profit APF presented in Equation* (8) *is concave with respect to T, G and S_p for a given value of M. Hence, the average profit APF attains the global maximum value at the point ($T_{(M)}$, $G_{(M)}, S_{p(M)}$), and the point ($T_{(M)}$, $G_{(M)}, S_{p(M)}$) is unique.*

Proof. The theorem is proved by examining the first principle minor $H_{1\times1}$ of the Hessian matrix H of APF is negative while, the second principle minor $H_{2\times2}$ of H is positive and the third principle minor $H_{3\times3}$ of H is negative. The Hessian matrix H of APF (Equation (8)) is as follows:

$$H = \begin{pmatrix} \frac{\partial^2 APF}{\partial T^2} & \frac{\partial^2 APF}{\partial T \partial G} & \frac{\partial^2 APF}{\partial T \partial S_p} \\ \frac{\partial^2 APF}{\partial G \partial T} & \frac{\partial^2 APF}{\partial G^2} & \frac{\partial^2 APF}{\partial G \partial S_p} \\ \frac{\partial^2 APF}{\partial S_p \partial T} & \frac{\partial^2 APF}{\partial S_p \partial G} & \frac{\partial^2 APF}{\partial S_p^2} \end{pmatrix}.$$

Now, the first principle minor $H_{1\times1} = \frac{\partial^2 APF}{\partial T^2} = -\frac{2}{T^3}\left(A + T_f + hD(S_p)\right) < 0$ at $\Big(T_{(M)}$, $G_{(M)}, S_{p(M)}\Big)$, for all parameters because concerning all parameters are positive. The expression of second principle minor $H_{2\times2} = \left(\frac{\partial^2 APF}{\partial T^2}\right)\left(\frac{\partial^2 APF}{\partial G^2}\right) - \left(\frac{\partial^A PF}{\partial T \partial G}\right)^2$ and the expression of third principle minor $H_{3\times3}$ i.e., the determinant of the matrix H are highly non-linear. Due to these complicated expressions, positivity of $H_{2\times2}$ and negativity of $H_{3\times3}$ at $\Big(T_{(M)}$, $G_{(M)}, S_{p(M)}\Big)$ are verified numerically by using Mathematica software in Section 6. Hence, the average profit APF (shown in Equation (8)) is concave at the point $\Big(T_{(M)}$, $G_{(M)}, S_{p(M)}\Big)$, and the point is unique. This completes the proof. □

Next, the following Algorithm 1 is developed to determine the optimal solutions of the model presented in Section 4.2.1.

Algorithm 1	
Step 1:	Set the number of advance payment instalment $M = 1$.
Step 2:	Determine the values of $T_{(M)}$, $G_{(M)}$ and $S_{p(M)}$ by solving Equation (21), (22) and (23), simultaneously.
Step 3:	Substitute $T_{(M)}$, $G_{(M)}$ and $S_{p(M)}$ into the function APF (given in Equation (8) to obtain $APF\Big(T_{(M)}, G_{(M)}, S_{p(M)}\Big)$.
Step 4:	Set $M = M+1$ and repeat Steps 2 and 3 to obtain $APF\Big(T_{(M+1)}$, $G_{(M+1)}, S_{p(M+1)}\Big)$.

Step 5: If $APF\left(T_{(M+1)}, G_{(M+1)}, S_{p(M+1)}\right)$ less than or equal to $APF\left(T_{(M)}, G_{(M)}, S_{p(M)}\right)$ and $APF\left(T_{(M-1)}, G_{(M-1)}, S_{p(M-1)}\right)$ less than or equal to $APF\left(T_{(M)}, G_{(M)}, S_{p(M)}\right)$, then the optimum average profit $APF\left(T^*, G^*, S_p^*\right) = APF\left(T_{(M)}, G_{(M)}, S_{p(M)}\right)$ and hence, the optimal solution $\left(T^*, G^*, S_p^*\right) = \left(T_{(M)}, G_{(M)}, S_{p(M)}\right)$. Otherwise, go to Step 4.

Step 6: Calculate optimal Q^* by putting the optimal values T^*, G^* and S_p^* in Equation (7) and then stop.

The solution procedure for Models 4.2.2, 4.5.1 and 4.5.2 are same for Model 4.2.1 (discussed in subsection 5.1).

6. NUMERICAL EXAMPLES

The proposed work applies to a case of a perishable product retailer. Each retailer of perishable products preserves the products using a suitable PT and tries to accept advanced payment policy to overcome financial problem at the beginning. In March 2022, we visited a most popular tourist spot Nagpur, Maharashtra, India. We have observed that for a particular fruit shop the tourists (customers) were departing and arriving frequently in the business hours. It is also seen that few of them have purchased items and some others came back with empty hands based on orange's price. The reverse logistics have also been viewed for another season as per forecasted yearly records. This kind of phenomenon is viewed due to lack of information, information fusion over the motive of the continuous flow of newly arrived customers for that place. A DM has no prior knowledge of the number of actual customers who are going to purchase particular items on a particular day. Such kind of uncertainty may be considered as non-standard IF system. For that time being, the grade of membership and non membership becomes high or low and hence their sum becomes greater than one or less than one, respectively. IVPF considers such situations rigorously to overcome the complexities in decision making problems. Furthermore, it was known that the fruit shop purchased oranges from a particular firm by paying some prices in advanced and that's why he got some discounts on purchasing cost. In addition, he was preserved the products in a well set up refrigerator to keep fresh the products. Therefore, we use the IVPF rule to solve the problem with PT and advanced payment scheme. A case study has been demonstrated through some numerical examples to access the optimal results of the proposed models. Both advance payment policy and PT have been applied. Due to unavailability of real-life data, we consider the parameters' values based on real-life situation.

In order to justify the appropriateness of the suggested models, four examples are examined for each crisp and IVTPF models. The respective examples are solved by Mathematica software. In the following examples, the units of the business cycle and delivery lead time are taken in month; the units of initial stock, market demand are taken in quintal. Furthermore, units of all costs and profit are assumed in dollar.

Example 13. Crisp model (considering PT and advance payment scheme):
Here, values of all known parameters are assumed as constant. The hypothetical values of known parameters are listed as: $A = 500$, $D_o = 90$, $\varphi = 0.2$, $a = 0.5$, $b = 90$, $c = 0.05$, $\kappa = 2$, $\delta = 0.7$, $\alpha = 0.3$, $\xi = 0.2$, $I_e = 0.12$, $p_c = 10$, $T_f = 0.2$, $T_v = 0.02$, $d_{st} = 100$,$h = 3.5$, $\lambda_0 = 0.08$ and $T_1 = 0.2$. Table 2 is referred to see the obtained optimal solutions for this model. The sufficient conditions of the maximization problem are proved numerically as: $|H_{1\times1}| = -37.0519 < 0$, $|H_{2\times2}| = 72.44 > 0$ and $|H_{3\times3}| = -15.727 < 0$. In addition, the concavity of average profit for this example is shown in Figure 4.

Table 2: Optimal solutions for Example 13

Unknown parameter	Optimal value
Selling price (S_p^*)	\$97.2143
PT investment (G^*)	\$2.58
Ordered quantity (Q^*)	28.5 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	3 months
Average profit (APF^*)	\$576.01

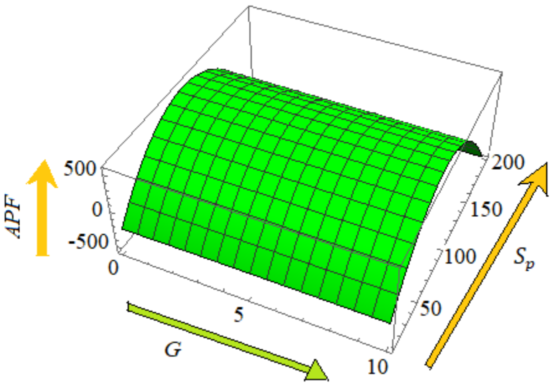

Figure 4: Concavity of APF w.r.t. G and S_p for Example 13

Example 14. Crisp model (considering PT without advance payment scheme)
When an advance payment scheme is not applied then the proposed model is modified with respect to the assumption of PT investment, and the corresponding situation is shown in Figure 5. In addition, there is no discount on purchased cost in this case. To investigate this scenario, the capital cost due to advance payment is omitted from the average profit function. Table 3 represents the optimal solutions of this example. The same Example 13 is solved in this scenario by omitting only the capital cost.

Table 3: Optimal solutions for Example 14

Unknown parameter	Optimal value
Selling price (S_p^*)	\$98.73
PT investment (G^*)	\$5.02
Ordered quantity (Q^*)	27.21 quintals
Cycle length (T^*)	2.99 months
Average profit (APF^*)	\$545.982

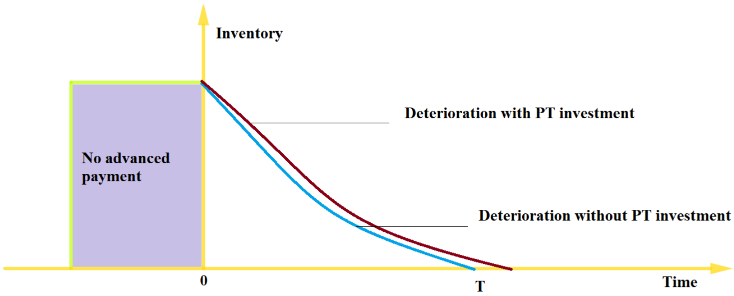

Figure 5: Graphical presentation of the model without advance payment

Example 15. Crisp model (considering no PT and advance payment scheme)
Investment in PT is important for most of the products to control their deteriorations but there are so many products such as books, pens, electronic products, etc., that do not require PT investment to hold their quality for their life period. We modify the proposed model without consideration of PT investment i.e., $G = 0$. The values of remaining parameters are same as Example 13. The optimum solutions are shown in Table 4.

Table 4: Optimal solutions for Example 15

Unknown parameter	Optimal value
Selling price (S_p^*)	\$97.40
Ordered quantity (Q^*)	23.48 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	2.45 months
Average profit (APF^*)	\$465.916

Example 16. Crisp model (considering PT, advance payment scheme and $\varphi = 1$)
In this model, it is assumed that demand is only linearly dependent on the selling price. So, $\varphi = 1$ and the remaining parameters' values are same in Example 13. The optimal solutions are declared in Table 5.

Table 5: Optimal solutions for Example 16

Unknown parameter	Optimal value
Selling price (S_p^*)	\$97.24
PT investment (G^*)	\$5.23
Ordered quantity (Q^*)	144.837 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	3.12 months
Average profit (APF^*)	\$3548.01

Example 17. IVTPF model (considering PT and advance payment scheme)
In this example, deterioration rate λ_0, holding cost h and lead time T_1 are taken in IVTPF environment. The values of the remaining inventory parameters are same as in Example 13. The IVTPF parameters are given as:

$$\widetilde{T_1^{IP}} = \left[\left\langle (0.2, 0.3, 0.4), \Phi^L_{\widetilde{T_1^{IP}}}(y), \Psi^U_{\widetilde{T_1^{IP}}}(y) \right\rangle ; \left\langle (0.1, 0.3, 0.6), \Phi^U_{\widetilde{T_1^{IP}}}(y), \Psi^L_{\widetilde{T_1^{IP}}}(y) \right\rangle \right],$$

$$\widetilde{\lambda_0^{IP}} = \left[\left\langle (0.1, 0.2, 0.25), \Phi^L_{\widetilde{\lambda_0^{IP}}}(y), \Psi^U_{\widetilde{\lambda_0^{IP}}}(y) \right\rangle ; \left\langle (0.05, 0.2, 0.28), \Phi^U_{\widetilde{\lambda_0^{IP}}}(y), \Psi^L_{\widetilde{\lambda_0^{IP}}}(y) \right\rangle \right],$$

$\widetilde{h^{IP}} = \left[\left\langle (1.5, 2, 2.6), \Phi^L_{\widetilde{h^{IP}}}(y), \Psi^U_{\widetilde{h^{IP}}}(y) \right\rangle; \left\langle (1, 2, 3.2), \Phi^U_{\widetilde{h^{IP}}}(y), \Psi^L_{\widetilde{h^{IP}}}(y) \right\rangle\right]$.
Where the corresponding membership and non-membership functions of the IVTPF parameters $\widetilde{\lambda_0^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{T_1^{IP}}$ are calculated by Definition 5. Furthermore, using Definition 6 and taking $\tau = \frac{1}{2}$, the ranking indices of the IVTPF parameters $\widetilde{T_1^{IP}}$, $\widetilde{h^{IP}}$ and $\widetilde{\lambda_0^{IP}}$, respectively, are $\Re(\widetilde{T_1^{IP}}) = 0.325$, $\Re(\widetilde{h^{IP}}) = 2.075$ and $\Re(\widetilde{\lambda_0^{IP}}) = 0.17$. The obtained optimal solutions are displayed in Table 6. The sufficient conditions of the maximization problem are proved numerically as: $|H_{1\times1}| = -37.0519 < 0$, $|H_{2\times2}| = 22.478 > 0$ and $|H_{3\times3}| = -4.744 < 0$. Moreover, the concavity of the average profit APF with respect to S_p and G is presented by Figure 6 and the concavity w.r.t. cycle time T is depicted in Figure 7.

Table 6: Optimal solutions for Example 17

Unknown parameter	Optimal value
Selling price (S_p^*)	\$96.74
PT investment (G^*)	\$3.99
Ordered quantity (Q^*)	25.122 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	2.59 months
Average profit (APF^*)	\$592.18

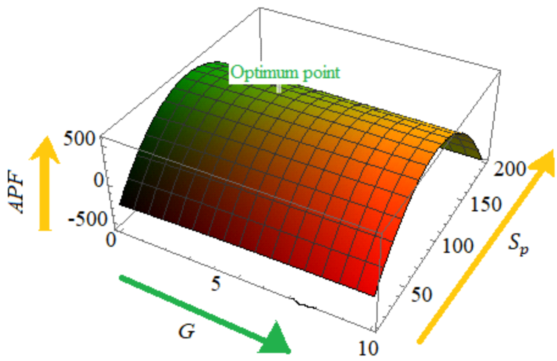

Figure 6: Concavity of APF w.r.t. S_p and G for Example 17

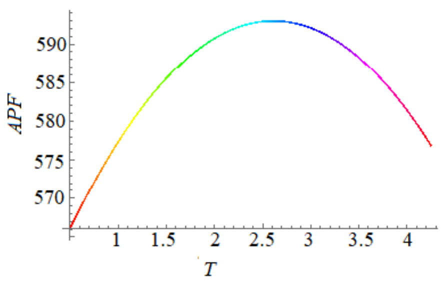

Figure 7: Concavity of APF w.r.t. T for Example 17

Example 18. IVTPF model (considering PT without advance payment scheme)
The same Example 17 is solved in this scenario by omitting only the capital cost from the average profit function APF. The optimal results is presented in Table 7.

Table 7: Optimal solutions for Example 18

Unknown parameter	Optimal value
Selling price (S_p^*)	\$98.19
PT investment (G^*)	\$4.59
Ordered quantity (Q^*)	23.28 quintals
Cycle length (T^*)	2.46 months
Average profit (APF^*)	\$562.978

Example 19. IVTPF model (considering no PT and advance payment scheme)
Here, PT investment is not considered. Hence, there is no need any extra preservation cost. The preservation cost PRC is removed from the average profit function APF and the values of remaining parameters are same as in Example 17. Table 8 shows the optimal solutions for this case.

Table 8: Optimal solutions for Example 19

Unknown parameter	Optimal value
Selling price (S_p^*)	\$92.13
Ordered quantity (Q^*)	24.28 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	1.89 months
Average profit (APF^*)	\$560.978

Example 20. IVTPF model (considering PT, advance payment scheme and $\varphi = 1$)
Here, a linearly selling price dependent demand IVTPF model is studied by applying PT investment and advance payment scheme. Henceforth, $\varphi = 1$ and the remaining parameters' values remain unchanged as in Example 17. The optimal solutions of this example are reflected in Table 9.

Table 9: Optimal solutions for Example 20

Unknown parameter	Optimal value
Selling price (S_p^*)	\$96.57
PT investment (G^*)	\$5.34
Ordered quantity (Q^*)	133.45 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	2.78 months
Average profit (APF^*)	\$3637.65

Example 21. IVTIF model (considering PT investment and advance payment scheme)
Here, the crisp model stated in Example 13 is investigated under IVTIF environment. The deterioration rate λ_0, holding cost h and lead time T_1 are considered as IVTIFNs (see Definition 2). Based on the definition of expected value of an IVTIFN that has been defined by Bharati and Singh [49], the expected crisp values of the IVTIF parameters $\widetilde{T_1^{II}}$, $\widetilde{h^{II}}$ and $\widetilde{\lambda_0^{II}}$ are reflected as: $EV(\widetilde{T_1^{II}}) = 0.3125$, $EV(\widetilde{h^{II}}) = 2.1375$ and $EV(\widetilde{\lambda_0^{II}}) = 0.205$, respectively. The values of the remaining parameters are same as in Example 13. The optimal results for this model is depicted in Table 10.

Table 10: Optimal solutions for Example 21

Unknown parameter	Optimal value
Selling price (S_p^*)	\$96.65
PT investment (G^*)	\$5.79
Ordered quantity (Q^*)	17.57 quintals
Number of instalments (M^*)	5
Cycle length (T^*)	1.9 months
Average profit (APF^*)	\$586.06

7. RESULT DISCUSSION

There are many investigations in Section 6. Tables 2-5 show the results for four individual cases under crisp environment. Tables 6-9 depict the results for the same four cases under IVTPF environments. There is a higher average profit APF in the IVTPF model than the crisp model for all cases. Cycle time and selling price are lower in the IVTPF model than the crisp model for all cases, but PT investment in the IVTPF model is greater. Hence, the deterioration reduction rate is more in the IVTPF model than the crisp model. Due to low selling price and low deterioration, demand is high which causes more profit in the IVTPF model. Furthermore, an advance payment scheme shows a better result, and it provides the highest average profit under both IVTPF and IVTIF environments. Sometimes, the advance payment option helps the payer in the cancellation of the order if needed. From Tables 2-9, it is observed that the average profit has a maximum value when linear selling price-dependent demand is taken. In real situations, sometimes demand depends on hybrid types (Rahman et al.[50]). Therefore, for deteriorating products' retailers should have hybrid type's selling price dependent demands for smoothly run his (or her) business by considering PT investment and advance payment scheme under IVTPF environment. Furthermore, Tables 2, 6 and 10 refer to a comparison among crisp model, IVTPF model and IVTIF model. The comparison is that the IVTPF model provides the best results than the crisp and IVTIF models. Figure 8 implies that the IVTPF model is better than the others models for all sub-cases. It is noted that the IVTPF model would be beneficial by considering PT investment and advance payment scheme as it increases the average profit.

7.1. Comparison of the results with two existing works

Some special cases of the proposed study are as follows:

1. If this work does not consider advance payment policy, PT investment and uncertain environments then this study is similar to Mishra et al. [1]. The present study can increase 10% profit than the profit of Mishra et al. [1] by reducing deterioration rate.
2. If this work does not consider stock-dependent demand, PT investment and uncertain environments then this study is similar to Khan et al. [4]. The present study can increase 5% profit than the profit of Khan et al. [4] by reducing deterioration rate.

The two sub cases validate the proposed study and show that the proposed study is more robust and flexible.

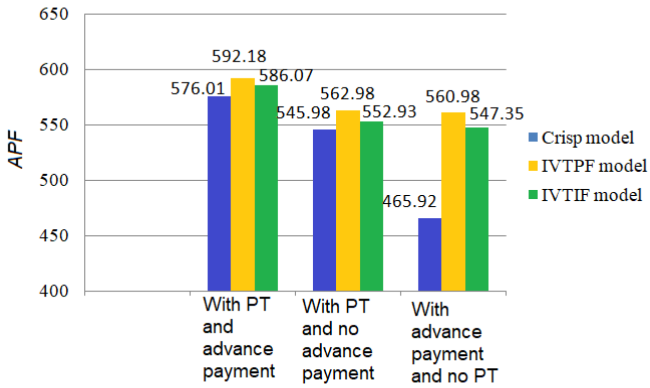

Figure 8: Comparison among the models under crisp, IVTPF and IVTIF environments

7.2. Novelty and Significance of the results

The results show that the IVTPF model outperforms the crisp and IVTIF models in terms of profitability, cycle time, selling price, and degradation reduction. These findings have substantial practical implications, particularly for businesses dealing with degrading products such as food and pharmaceuticals. The significance of the outcomes are as follows:

- Retail and Inventory Management: The findings obtained demonstrate that retailers can optimize their pricing strategies by using a hybrid selling price-dependent demand model. Since the IVTPF model provides higher profits with lower selling prices and reduced degradation, businesses can reduce spoiling losses and boost overall revenue under uncertain information.
- **Supply Chain and Logistics:** Companies that manage perishable or high-turnover inventory can utilize the IVTPF model to improve their pricing and procurement strategy, combining PT investment and advance payment systems while maintaining high demand.
- The benefits of advance payment choices mentioned in the results indicate that firms can use pre-payment tactics to improve their financial stability. This technique not only ensures consistent cash flow, but also allows for order cancellations when necessary.
- Implementing the IVTPF model allows industries that use continuous production, such as food processing or pharmaceuticals, to optimize the balance between production rates and demand changes.

Overall findings highlight the significance of incorporating uncertain environments (IVTPF) into corporate strategy, especially when dealing with deteriorating goods. The findings give a practical framework for decision-makers to increase profitability while minimizing the risks associated with product deterioration.

8. SENSITIVITY ANALYSIS

To inspect the impact of the changes of the known inventory parameters' values on the optimal policy, post optimality analyses are executed with respect to Example 17. The analyses are prepared by changing the known parameters from -20% to $+20\%$. During the analysis, one parameter is changed at a time and the remaining parameters are kept fixed at their original values. The simulated results of these analyses are presented in Table 11.

Table 11: Impact of changed values of known parameters for Example 17

Parameter	% change of parameters	APF	T^*	G^*	S_p^*	Q^*
	-20	605.39	2.62	3.69	96.01	25.81
p_c	-10	598.26	2.61	3.84	96.38	25.58
	10	537.291	2.57	4.03	97.25	24.81
	20	577.17	2.55	4.29	97.46	24.45
	-20	439.004	2.69	3.70	96.78	21.09
φ	-10	515.084	2.62	3.86	96.75	22.96
	10	667.29	2.56	4.1	96.72	27.22
	20	705.34	2.53	4.15	96.71	28.06
	-20	581.15	2.57	4.2	97.25	24.66
α	-10	588.16	2.58	4.05	96.89	24.88
	10	594.21	2.6	3.92	96.58	25.31
	20	606.42	2.62	3.67	95.96	25.84
	-20	624.513	2.3	3.96	96.72	21.96
A	-10	611.18	2.45	3.97	96.73	23.59
	10	574.51	2.72	4	96.75	26.56
	20	557.85	2.85	4.21	96.78	27.91
	-20	297.69	2.63	4.22	78.69	19.31
D_o	-10	435.53	2.61	4.12	87.71	22.08
	10	764.66	2.56	3.84	105.77	27.52
	20	933.83	2.53	3.7	113.79	29.57
	-20	591.04	2.62	3.98	96.76	25.42
b	-10	591.11	2.6	3.98	96.75	25.23
	10	592.21	2.52	3.98	96.73	24.36
	20	592.32	2.5	3.98	96.72	24.14
	-20	594.25	1.96	3.99	96.69	15.64
κ	-10	593.23	2.02	3.99	96.71	19.02
	10	590.65	2.79	3.99	96.75	27.34
	20	590.59	2.95	3.99	76.76	29.14

From Table 11, the following interpretations are observed:

a. The average profit APF is inversely affected by the purchasing cost p_c, ordering cost A and the price-sensitive coefficient κ.

b. Furthermore, when κ rises, PT investment G^* has no change. But, cycle time T^* increases and selling price S_p^* decreases. In contrast, the ordered quantity Q^* increases and the average profit APF reduces due to declining selling price and no change of deterioration rate.

c. Due to expanding original market demand D_o, the selling price as well as the ordered quantity increase. Consequently, the average profit APF grows.

d. The cycle time T^* is highly sensitive with respect to p_c, φ, b and D_o; it is moderately sensitive concerning the remaining parameters α, A and κ.

e. If discount rate α on the purchased cost moves upwards then the selling prices S_p^* decreases. Hence, the demand rate increases and the ordered quantity Q^* is also increased. As a result, the average profit APF increases.

f. Figure 9 implies that the average profit APF decreases for increasing the value of δ. On the other hand, from Figure 10 it is indicated that selling price S_p^* is positively sensitive with the parameter δ.

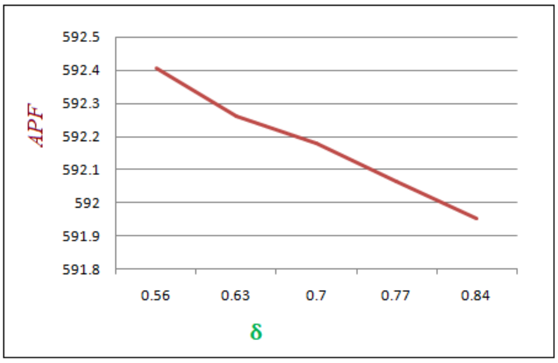

Figure 9: Change of the average profit APF w. r. t. δ for Example 17

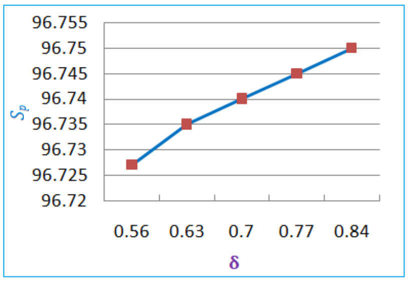

Figure 10: Change of selling price S_p w.r.t. δ for Example 17

g. If the value of φ increases then the average profit APF and PT investment G^* are increased (shown in Figure 11 and Figure 12). But, selling price S_p^* is declined due to increasing the value of φ (see Figure 13). The average profit APF is increased due to lowering deterioration rate and increasing demand rate.

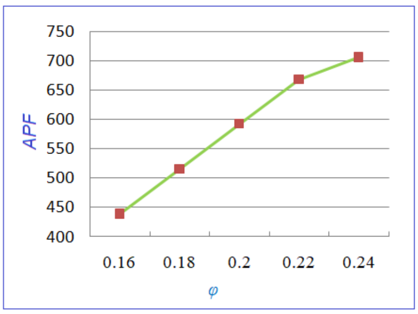

Figure 11: Change of the average profit APF w. r. t. φ for Example 17

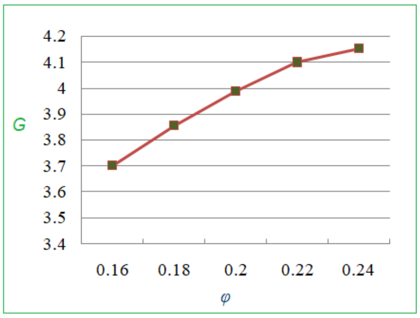

Figure 12: Change of PT investment G w. r. t. φ for Example 17

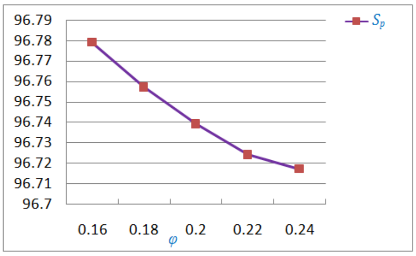

Figure 13: Change of selling price S_p w. r. t. φ for Example 17

8.1. Managerial insights

From the discussed numerical examples and sensitivity analyses, the following opinions can be recommended to a DM of a shop:

(A) In the actual world, a DM should be aware of the rate of product demand before starting a shop. The quantity of actual consumers who will buy specific products on a given day is unknown to a DM in advance. Therefore, a DM is sent to an expert asking for their opinion on the statement "the demand rate for a product is 90 units per month". Suppose, declaration of the expert is that the possibility in favour of the statement is $\frac{\sqrt{5}}{3}$ and the possibility in against of the statement is $\frac{2}{3}$ between $((0,1)$. As a result, the nature of the demand rate and other numerous parameters become ambiguous. One could classify this type of uncertainty as non-standard IF system. For the time being, the membership and non-membership grades change to be high or low, and as a result, their sum changes to be more than or less than one, respectively. IVPF takes these kinds of circumstances rigorously in order to handle difficult decision-making scenarios. The profit function is getting maximum for IVPF among all crisp and IF environments, so from a DM's viewpoint choosing IVPF will be a wiser decision to get more profit.

(B) The investment in PT decreases the deterioration rate of perishable product. Thus, the loss due to more deterioration declines and hence, profit of the retailer increases. Perishable products' deterioration rates depend on environmental temperature. Hence, a perishable product's retailer needs to utilize a better PT to preserve products for long time. Furthermore, high preservation cost has reverse effect on the profit. So, a DM should set up a well-equipped preservation facility by deciding an optimal PT investment to preserve the products for a more time or the retailer should sell the products as early as possible to achieve more profit.

(C) The average profit of the retailer with advance payment scheme is greater than the average profit of the retailer without advance payment scheme. In an advance payment scheme, a retailer can cancel the order if there occurs any problematic situation. This fact reduces unnecessary loss. An advance payment policy in inventory management increases profit by allowing retailers to receive price reductions from suppliers, reducing overall procurement costs. Additionally, it improves cash flow management, allowing companies to strategically reinvest money in marketing to increase sales, preservation technologies, or inventory restocking. Additionally, retailers can maximize order amounts and lower financial risks by establishing more favorable payment conditions with suppliers, which will increase profitability.

(D) In a real-situation, market demand becomes hybrid price dependent. For seasonal products, demand decreases non-linearly with the increasing of selling price in the main season whereas, in off-season, demand decreases linearly with the increasing of selling price. Therefore, for deteriorating products' retailers should have a hybrid type's selling price-dependent demand for smoothly running his (or her) business. The dependency of the demand rate on selling price is considered in both linear and non-linear forms to capture diverse market behaviors and pricing sensitivities. Linear form provides a straightforward approach where demand falls proportionally with price, while in non-linear form demand exhibits exponential decay, power-law behavior, or saturation effects (e.g., luxury goods or necessity products). Different industries exhibit varying price-demand relationships, making both forms essential for accurate demand forecasting and revenue optimization. Incorporating that type of demand function enhances decision-making flexibility, ensuring adaptability to real-world economic conditions and competitive market dynamics. This dual consideration enables businesses to develop optimal pricing strategies that balance profitability and customer demand effectively.

(E) If the scaling parameter D_o, i.e., the potential demand rate is increased then the ultimate demand rate will rise. The retailer should increase the order quantity to satisfy the rising demand and for avoiding shortages.

9. CONCLUSION

It is not always the case that the demand function depends on selling price with a linear, non-linear, or constant rate. Sometimes, a product's demand suddenly decreases at a high rate and then decreases slowly due to an increase in selling price, especially in the main season of the product. Furthermore, the same product's demand decreases at a constant rate in another season. So, any item's demand rate depends on selling price both linearly and non-linearly in the market. This study observes that the price and stock of products have an impact on demand. Thus, a hybrid price-stock dependent demand model with an advanced payment policy is developed here. To optimize profit, the ideal cycle time and selling price are determined. A DM hesitates to calculate the profit of an EOQ model due to some uncontrollable factors such as the product's deterioration rate, holding cost, delivery lead time, etc. In most cases, these parameters have been used as imprecise data, but practically, these are imprecise and ambiguous. To handle these imprecise parameters, an EOQ model has been developed under an IVTPF environment. In this present study, IVTPFN has been defined, and then a linear ranking function has been proposed to defuzzify the IVTPFNs. Firstly, the EOQ model has been formulated with hybrid selling price- and stock-dependent demand under advance payment along with a discount facility and PT investment for reducing the deterioration rate in a crisp environment. Then, the model has been reformulated by assuming deterioration rate, holding cost, and delivery lead-time in an IVTPF environment. Several application examples have been investigated to illustrate proposed models in both environments. Always the sum of membership and non-membership degrees of a number does not lie in [0,1], where the sum of their squares lies in [0,1]. In this type of situation, IVTIF set cannot handle the uncertainty. But IVTPF can solve this type of situation. IVTPF set can be defined as an augmented and amplified version of IVTIF set for measuring the impreciseness of a real-life complication. Furthermore, three application examples reveal that

- PFS is more capable of handling the vagueness of the real world than IFS, especially when the sum of membership and non-membership is greater than one, and it gives more profit than crisp as well as IVTIF environments.
- PFS gathers more information than the general fuzzy set and IFS.
- The non-membership grade value is not necessarily smaller than the membership grade value all the time.

Moreover, an advance payment scheme shows a better result, and it provides the highest average profit. The average profit has a maximum value when linear selling price-dependent demand is taken. But, in real situations, sometimes demand depends on hybrid types. Therefore, for deteriorating products, retailers should have hybrid-type selling price-dependent demands to smoothly run his (or her) business. Case study and sensitivity analysis are considered for validation of the model for application in the fruit shop. In advance payment, a retailer pays some portion of the purchasing cost before delivery of products by taking permission from the supplier. This practice not only helps in securing the desired quantity of goods but also establishes a stronger relationship with suppliers, which can lead to better pricing and more favorable terms in future transactions. By implementing these strategies, retailers can optimize their inventory management and enhance customer satisfaction, ultimately driving higher sales and profitability. from a manufacturer. Such a tactic of allowing for payment advances will motivate customers to make more orders through select suppliers and retailers and through customers who profit independently. As more equally spaced payments are made throughout the lead time, the overall profit grows. The management must therefore choose the supplier or manufacturer who permits a greater number of equally spaced advance payment arrangements with a lower percentage of the entire purchasing cost.

So, incorporation of an advanced payment scheme is profitable and effective.

The investment in PT decreases the deterioration rate of perishable products. Thus, the loss due to more deterioration declines, and hence, the profit of the retailer increases.

Perishable products' deterioration rates depend on environmental temperature. Hence, a perishable product's retailer needs to utilize a better PT to preserve products for a long time. Furthermore, high preservation cost has a reverse effect on the profit. So, a DM should set up a well-equipped preservation facility by deciding on an optimal PT investment to preserve the products for a longer time, or the retailer should sell the products as early as possible to achieve more profit.

This study highlights the significance of a hybrid price-stock dependent demand model in unpredictable circumstances, offering retailers useful insights for managing perishable items. The results imply that demand fluctuates according to seasonal trends rather than always following a set pattern, necessitating a flexible pricing strategy. By managing uncertainty in variables like lead time, holding cost, and deterioration rate, IVTPF sets improve decision-making. Furthermore, it has been demonstrated that expenditures in PT and advance payment plan have a major influence on profitability. By carefully choosing suppliers who have better choices for upfront payments and by striking a balance between preservation expenses and selling time to reduce losses from product deterioration, retailers can maximize profit. The results of this research are highly applicable to inventory management, particularly for retailers of perishable goods who face demand fluctuations due to seasonal variations and pricing strategies. By incorporating a hybrid price-stock dependent demand model under an IVTPF environment, businesses can optimize profit through strategic pricing, preservation investment, and advance payment schemes. These findings offer valuable insights for decision-makers in retail and supply chain management to enhance profitability while mitigating uncertainties in deterioration, holding costs, and lead time.

Despite its contributions, this research has certain limitations: the proposed model assumes that price and stock are the primary factors that influence demand, while other external factors, such as competitor pricing, consumer preferences, and macroeconomic conditions, are not explicitly incorporated; the study focuses on perishable goods, and it is unclear whether the model can be applied to durable products with different demand patterns; and the assumption of a fixed preservation cost, which may vary in practice depending on operational constraints and technological advancements.

One can extend the proposed work by taking into account carbon tax and cap policy, green technology as discussed by Paul et al. [51] and Barman et al. [52] for reducing carbon emission during transportation and production time, and after that, the obtained multi-objective model can be solved using a meta-heuristic algorithm, which was studied by Goli et al. [53] and Goli and Tirkolaee [54]. This study can be analyzed in an interval type 2 Pythagorean fuzzy environment, which was defined by Mondal and Roy [48], etc. Furthermore, one can extend the models in a neutrosophic environment (as discussed by Barman et al. [55] and Kamran et al. [56]) instead of a Pythagorean fuzzy environment.

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