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Research Article

DUALITY AND OPTIMALITY FOR QUASIDIFFERENTIABLE INTERVAL-VALUED PROBLEMS

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Abstract: In this article, we explore the concept of interval-valued nonsmooth optimization problems using r-invexity in relation to convex compact sets. For the selected nonsmooth interval-valued problem (IP), we derive necessary and sufficient optimality criteria. In addition to that, we establish various duality theorems under r-invex quasidifferentiable with respect to a convex compact set that is equal to the Minkowski sum of their subdifferentials and superdifferentials. We draft a numerical example to support the results obtained in this paper. It is important to note that the Lagrange multipliers are nonconstant for the considered problem.

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1. INTRODUCTION

Optimization deals with the way to find the best possible solution within the feasible region. We commonly use three types of approaches in optimization to tackle the uncertainty occurring in the problem, which give rise to stochastic optimization, deterministic optimization, and interval-valued optimization. An interval-valued optimization problem addresses the uncertainty and imprecision incurred in optimization problems during decision-making. In interval-valued problems, uncertainty arises either in the objective function or in constraints, or in both the objective and constraints. Interval-valued optimization problems can apply in various disciplines such as finance [1], engineering [2], energy systems [3], and stock portfolios [4] where decision-makers need to account for uncertainties and variations in input data. Recently, many researchers have put tremendous effort into the formulation of optimality and duality results for interval-valued optimization problems under different generalized convexity. Wu [5] formulated the Karush-Kuhn-Tucker (KKT) optimality criteria based on interval-valued functions. Jayswal *et al.* [6] focused on the interval-valued Mond-Weir and Wolf problems and derived the sufficient optimality criteria and appropriate duality results in order to find the connection between the efficient solution of the primal and dual problems. Bhurjee and Panda [7] used parametric form to define an interval-valued function and demonstrated the existence of a solution. Ahmad *et al.* [8] derived sufficient optimality criteria along with weak, strict converse, and strong duality theorems for interval-valued Mond-Weir and Wolf problems under (p,r)-ρ-(η,θ)-invexity. Zhang *et al.* [9] proposed the KKT-type optimality criteria for a class of nonconvex problems by magnifying the concept of invexity and preinvexity to interval-valued programming problems. Moreover, he proposed the relation between interval-valued problems and variational-like inequalities problems. Recently, Debnath and Pokharna [10] worked on interval-valued variational problems using B-(p,r)-invexity and established optimality and duality results.

Most of the real-life problems do not satisfy differentiability or smoothness. Nonsmoothness in the problem opens a considerably big platform for scientists. A number of techniques have been developed over a period of time to tackle nonsmooth optimization problems. Most of the nonsmooth optimization problems can be modeled using quasidifferentiable calculus. The nonsmooth vector optimization, where each component is locally Lipschitz, was studied intensively by Clarke [11] which has given incredible results in optimization theory. Another important approach by Demyanov and Rubinov [12] led to the development of quasidifferential calculus, which is extended by many researchers, including Demyanov and Rubinov [13], Gao [14], Kuntz and Scholtes [15] Luderer and R*ö*siger [16], Polyakova [17], Shapiro [18], Uderzo [19], Ward [20]. In a nonsmooth optimization problem, quasidifferential calculus played a constructive role in a variety of problems applied in different fields such as optimal control theory, nonsmooth analysis, engineering, mechanics, economics, and other fields. Antczak [21] derived duality results and optimality criteria of the Slater-type constraints under r-invex function. Antczak [22] has also worked on an alternate approach of modified r-invex functions and derived the duality and optimality results for a constrained programming problem that may or may not contain differentiable functions. Later on, Antczak [23] derived the optimality and duality by considering nonsmooth as well as nonconvex functions under quasidifferentiable r-invexity.

Recently, Singh and Laha [24] examined a class of fractional multiobjective programming problems characterized by quasidifferentiable functions. They extended the concept of (F,ρ)-convexity to the quasidifferentiable domain and utilized it to derive the optimality conditions. Building on this work, Singh and Laha [25] further developed the theory of quasidifferentials by formulating Minty and Stampacchia-type vector variational-like inequalities for optimization problems with invex functions defined over convex compact sets. These formulations were then used to establish optimality conditions in a more generalized framework. Prasad *et al.* [26] focused on interval-valued vector optimization problems. By employing the concept of quasidifferentiable F-convexity in relation to compact convex sets, they established Fritz John and Karush-Kuhn-Tucker (KKT) type necessary optimality conditions and further provided sufficient conditions under similar assumptions, illustrated with a numerical example. Laha *et al.* [27] investigated approximate solutions for interval-valued multiobjective optimization problems with inequality constraints. Utilizing quasidifferential calculus, they derived KKT-type necessary and sufficient optimality conditions based on approximate and generalized approximate convexity defined through quasidifferentials.

Inspired by the aforementioned research, our focus lies in exploring a class of interval-valued optimization problems that are nonsmooth in nature, addressed through the framework of quasidifferentiable r-invex functions in connection with compact convex sets. We derive necessary and sufficient optimality criteria for the formulated problem using quasidifferential calculus, which is suitable for handling nonsmoothness. Furthermore, we formulate the Mond-Weir type dual model and derive the duality theorems under r-invex quasidifferentiable in connection with compact convex sets. The quasidifferentials of the functions are characterized using the Minkowski sum of their subdifferentials and superdifferentials, which plays a fundamental role in both the optimality and duality results. An important theoretical insight emphasized is that the Lagrange multipliers derived in the KKT-type conditions are nonconstant, highlighting the complexity and generality of the proposed model. Finally, to validate the theoretical findings, a numerical example is provided, demonstrating the applicability of the developed results.

The paper is structured as follows: Section 2 provides a review of fundamental definitions and outlines the necessary optimality conditions. In Section 3, we develop the interval-valued problem and derive optimality conditions by employing r-invex quasidifferentiable functions in the context of compact convex sets and η. Finally, Section 4 constructs the Mond-Weir dual problem and derives relevant duality results for the considered Mond-Weir dual problem consisting of r-invex quasidifferentiable function in connection with compact convex sets.

2. PRELIMINARIES

This section begins with a set of interval operations that are fundamental to the developments presented in this paper. Let $\Im$ represent the collection of all bounded and closed

intervals in $\Re$. The symbol $\xi=[\varepsilon^L,\varepsilon^U]$ stands for the bounded and closed interval with ε^L as lower bound and ε^U as upper bound. If $\varepsilon^L=\varepsilon^U=\varepsilon$, then $\xi=[\varepsilon,\varepsilon]=\varepsilon$ reduces to a real number.
If $\xi=[\varepsilon^L,\varepsilon^U],\ \zeta=[\varsigma^L,\varsigma^U]\ \in\Im$, then we define

(i) $\xi+\zeta=\{\varepsilon+\varsigma:\varepsilon\in\xi \text{ and } \varsigma\in\zeta\}\ =\ [\varepsilon^L+\varsigma^L,\varepsilon^U+\varsigma^U]$,

(ii) $-\xi=\{-\varepsilon:\varepsilon\in\xi\}\ =\ [-\varepsilon^U,-\varepsilon^L]$,

(iii) $\xi-\zeta=\{\xi+(-\zeta)\}\ =\ [\varepsilon^L-\varsigma^U,\varepsilon^U-\varsigma^L]$,

(iv) $m+\xi=\{m+\varepsilon:\varepsilon\in\xi\}\ =\ [m+\varepsilon^L,m+\varepsilon^U]$,

(v) $m\xi=\{m\varepsilon:\varepsilon\in\xi\}\ =\begin{cases}[m\varepsilon^L,m\varepsilon^U],\ m\geq 0,\\ [m\varepsilon^U,m\varepsilon^L],\ m<0,\end{cases}$

where m be an arbitrary real number. Let us denote $\Re^k$ by the k-dimensional Euclidean space and let $X\subseteq\Re^k$ be a nonempty set. A function $\aleph:X\to\Im$, where $\Im$ is the set of all closed and bounded intervals in $\Re$, is termed an interval-valued function. For any point $\pi=(\pi_1,\pi_2,\ldots,\pi_k)\in X$, we write $\aleph(\pi)$ in the compact form $\aleph(\pi)=[\aleph^L(\pi),\aleph^U(\pi)]$, where $\aleph^L,\aleph^U:X\to\Re$ and satisfy $\aleph^L(\pi)\leq\aleph^U(\pi)$ for all $\pi\in X$.

We now define a partial ordering $\leq_{LU}$ on the set of closed and bounded intervals $\Im$. Given two intervals $\xi=[\varepsilon^L,\varepsilon^U]$ and $\zeta=[\varsigma^L,\varsigma^U]$ in $\Im$, we say that $\xi\leq_{LU}\zeta$ if both $\varepsilon^L\leq\varsigma^L$ and $\varepsilon^U\leq\varsigma^U$ hold. The strict relation $\xi<_{LU}\zeta$ is defined by $\xi\leq_{LU}\zeta$ and $\xi\neq\zeta$. Therefore, $\xi<_{LU}\zeta$ is satisfied whenever at least one of the following conditions is true:
$\varepsilon^L<\varsigma^L,\ \ \varepsilon^U<\varsigma^U,$
or,
$\varepsilon^L\leq\varsigma^L,\ \ \varepsilon^U<\varsigma^U,$
or,
$\varepsilon^L<\varsigma^L,\ \ \varepsilon^U\leq\varsigma^U.$

Definition 1. [23] *A function $f:\Re^k\to\Re$ is known as directionally differentiable at $\bar{\pi}\in\Re^k$ along a direction $\delta\in\Re^k$ if the limit*

$$f'(\bar{\pi};\delta)=\lim_{\gamma\downarrow 0}\frac{f(\bar{\pi}+\gamma\delta)-f(\bar{\pi})}{\gamma},$$

exists and is finite.
If the directional derivative of the function f exists finitely for each $\delta\in\Re^k$, then the function is said to be directionally differentiable or semi-differentiable at a point $\bar{\pi}$.

Definition 2. [23] *A function $f:\Re^k\to\Re$ is said to be quasidifferentiable at a point $\bar{\pi}\in\Re^k$ if it is directionally differentiable at $\bar{\pi}$ and there exists a pair of convex, compact ordered sets $D_f(\bar{\pi})=[\underline{\partial}f(\bar{\pi}),\overline{\partial}f(\bar{\pi})]$ such that the following condition is fulfilled:*

$$f'(\bar{\pi};\delta)=\max_{\vartheta\in\underline{\partial}f(\bar{\pi})}\vartheta^T\delta+\min_{\lambda\in\overline{\partial}f(\bar{\pi})}\lambda^T\delta,$$

where the term $\underline{\partial}f(\bar{\pi})$ is known as the subdifferential and the term $\overline{\partial}f(\bar{\pi})$ is known as the superdifferential of the function f at a point $\bar{\pi}$. Moreover, the pair of ordered sets $D_f(\bar{\pi})$ $[\underline{\partial}f(\bar{\pi}),\overline{\partial}f(\bar{\pi})]$ is known as the quasidifferential f at a point $\bar{\pi}$.

Note The uniqueness of the quasidifferential to the function f at some particular point $\bar{\pi}$ may not be guaranteed. This reduces the fact that $D_f(\bar{\pi}) = [\underline{\partial} f(\bar{\pi}),\ \overline{\partial} f(\bar{\pi})]$ and $[\underline{\partial} f(\bar{\pi}) + C,\ \overline{\partial} f(\bar{\pi}) - C]$ both are quasidifferential of the function f at $\bar{\pi}$ for each nonempty compact as well as convex set C.

The convex compact sets $S_f(\bar{\pi})$ are equal to the Minkowski sum of subdifferentials and superdifferentials at a point $\bar{\pi}$.

Definition 3. [23] *Let $S_f(\bar{\pi}) \subseteq \Re^k$ be a nonempty compact convex set and r be scalar. A function $f : \Re^k \to \Re$ is known as the r-invex at a point $\bar{\pi}$ on $\Re^k$ in connection with $S_f(\bar{\pi})$ and η if there exists $\eta : \Re^k \times \Re^k \to \Re^k$ satisfying*

$$\left.\begin{aligned} \tfrac{1}{r}e^{rf(\pi)} &\geq \tfrac{1}{r}e^{rf(\bar{\pi})}[1 + r\omega^T\eta(\pi,\bar{\pi})], \quad \text{if } r \neq 0, \\ f(\pi) &\geq f(\bar{\pi}) + \omega^T\eta(\pi,\bar{\pi}), \quad \text{if } r = 0, \end{aligned}\right\} \tag{1}$$

for each $\pi \in \Re^k$ and $\omega \in S_f(\bar{\pi})$.
Furthermore, if the inequality (1) is strict for each $\pi \in \Re^k$ $(\pi \neq \bar{\pi})$, then the function f is known as strictly r-invex at $\bar{\pi}$ on $\Re^k$ in connection with $S_f(\bar{\pi})$ and η.
If the inequality (1) is satisfied for each $\pi \in X$, where X is a nonempty subset of $\Re^k$, then the function f is r-invex at $\bar{\pi}$ on X in connection with $S_f(\bar{\pi})$ and η.

Remark

(a) If the function f is locally Lipschitz at $\bar{\pi}$, and the compact convex set $S_f(\bar{\pi})$ as well as Clarke subdifferential of f at $\bar{\pi}$ are equivalent, then f is said to be a locally Lipschitz r-invex, which was introduced by Antczak [21].

(b) If we consider $r = 0$, in the definition of a locally Lipschitz r-invex function then it reduces to the definition of a locally Lipschitz invex function, which was given by Reiland [28].

(c) If f is differentiable, then $S_f(\bar{\pi}) = \nabla f(\bar{\pi})$ and the definition of r-invex function in connection with compact convex set reduces to the definition of differentiable r-invex function, given by Antczak [22].

(d) If we consider $r = 0$, in the definition of differentiable r-invex function then it reduces to the definition of invex function given by Hanson [29].

Definition 4. *The interval-valued function $\aleph : X \to \Im$ is known as r-invex at a point $\bar{\pi} \in X$ in connection with convex compact set $S_{\aleph}(\bar{\pi})$ and η, if both the functions $\aleph^L,\ \aleph^U : X \to \Re$ are r-invex at a point $\bar{\pi}$ on X in connection with convex compact set $S_{\aleph^L}(\bar{\pi}), S_{\aleph^U}(\bar{\pi})$, respectively, as well as in connection with η, that is, if there exists a vector-valued function $\eta : X \times X \to \Re^k$ and a scalar r such that the inequalities*

$$\begin{aligned} \frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\bar{\pi})}[1 + r(\omega^L)^T\eta(\pi,\bar{\pi})], \quad \text{if } r \neq 0, \\ \aleph^L(\pi) &\geq \aleph^L(\bar{\pi}) + (\omega^L)^T\eta(\pi,\bar{\pi}), \qquad \text{if } r = 0, \end{aligned}$$

and

$$\frac{1}{r}e^{r\aleph^U(\pi)} \geq \frac{1}{r}e^{r\aleph^U(\bar{\pi})}[1+r(\omega^U)^T\eta(\pi,\bar{\pi})], \quad \text{if } r \neq 0,$$

$$\aleph^U(\pi) \geq \aleph^U(\bar{\pi}) + (\omega^U)^T\eta(\pi,\bar{\pi}), \qquad \text{if } r = 0,$$

hold for all $\pi \in X$ and for all $\omega^L \in S_{\aleph}{}^L(\bar{\pi})$ and $\omega^U \in S_{\aleph}{}^U(\bar{\pi})$.

Now, we recall the definition of weighted r-mean, which will be used to define r-preinvex functions.

Definition 5. [23] *Let $\alpha > 0$ and $\beta \geq 0$ be members of $\Re^m$ and r be any real number. If the component of $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ satisfies $\sum_{i=1}^m \beta_i = 1$, then a weighted r-mean is defined by*

$$M_r(\alpha;\beta) = M_r(\alpha_1,\dots,\alpha_m;\beta) := \begin{cases} \left(\sum_{i=1}^m \beta_i \alpha_i^r\right)^{1/r}, & \text{if } r \neq 0, \\ \Pi_{i=1}^m \alpha_i^{\beta_i}, & \text{if } r = 0. \end{cases}$$

Definition 6. [22] *Let $\Omega\ (\neq \emptyset)$ be an invex subset of $\Re^k$. A real-valued function $f: \Omega \to \Re$ is known as r-preinvex at $\rho \in \Omega$ in connection with η, if there exist real numbers r and $\beta_1 \geq 0,\ \beta_2 \geq 0$ satisfying $\beta_1 + \beta_2 = 1$, and*

$$f(\beta_1\rho + \beta_2(\eta(\pi,\rho)+\rho)) \leq \ln\left(M_r(e^{f(\rho)}, e^{f(\pi)};\beta)\right),\ \forall \pi \in \Omega.$$

Similarly, a function f is known as r-preinvex on Ω in connection with η if the above inequality holds for each point $\rho \in \Omega$.

Definition 7. *An interval-valued function $\aleph : \Omega \to \Im$ is known as r-preinvex at $\rho \in \Omega$ in connection with η if both the functions $\aleph^L,\ \aleph^U : \Omega \to \Re$ are r-preinvex at $\rho \in \Omega$ in connection with η, i.e., if there exist real numbers r and $\beta_1 \geq 0,\ \beta_2 \geq 0$, such that $\beta_1 + \beta_2 = 1$ satisfying*

$$\aleph^L(\beta_1\rho + \beta_2(\eta(\pi,\rho)+\rho)) \leq \ln\left(M_r(e^{\aleph^L(\rho)}, e^{\aleph^L(\pi)};\beta)\right),\ \forall \pi \in \Omega,$$

and

$$\aleph^U(\beta_1\rho + \beta_2(\eta(\pi,\rho)+\rho)) \leq \ln\left(M_r(e^{\aleph^U(\rho)}, e^{\aleph^U(\pi)};\beta)\right),\ \forall \pi \in \Omega.$$

Note Let us consider a particular case of r-preinvex function in connection with η. We substitute $\beta_2 = \rho$ where $\rho \in [0,1]$. The condition $\beta_1 + \beta_2 = 1$ gives $\beta_1 = 1-\rho$, therefore, r-preinvex function in connection with η can be express as

$$\aleph^L(\rho + \rho\eta(\pi,\rho)) \leq \begin{cases} \ln(\rho e^{r\aleph^L(\pi)} + (1-\rho)e^{r\aleph^L(\rho)})^{1/r}, & \text{if } r \neq 0, \\ \rho\,\aleph^L(\pi) + (1-\rho)\,\aleph^L(\rho), & \text{if } r = 0, \end{cases}$$

and

$$\aleph^U(\rho + \rho\eta(\pi,\rho)) \leq \begin{cases} \ln(\rho e^{r\aleph^U(\pi)} + (1-\rho)e^{r\aleph^U(\rho)})^{1/r}, & \text{if } r \neq 0, \\ \rho\,\aleph^U(\pi) + (1-\rho)\,\aleph^U(\rho), & \text{if } r = 0. \end{cases}$$

Proposition 8. *Let Ω $(\neq \emptyset)$ be an invex subset of $\Re^k$ w.r.t. η and $\aleph : \Omega \to \Im$ be an interval-valued function. Suppose that $\aleph^L$, $\aleph^U : \Omega \to \Re$ are r-preinvex functions in connection with η at a point $\rho \in \Omega$ on Ω. Moreover, $\aleph^L$ and $\aleph^U$ are quasidifferentiable functions at $\rho \in \Omega$. Then, both the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at ρ on Ω in connection with η and in connection with the convex compact set $S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho) + \bar{\lambda}^L$ and $S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho) + \bar{\lambda}^U$ where, $\bar{\lambda}^L \in \arg\min_{\lambda^L \in \overline{\partial}\aleph^L(\rho)} (\lambda^L)^T \eta(\pi,\rho)$ and $\bar{\lambda}^U \in \arg\min_{\lambda^U \in \overline{\partial}\aleph^U(\rho)} (\lambda^U)^T \eta(\pi,\rho)$ for any $\pi \in \Omega$.*

Proof. It is given that $\aleph^L$ and $\aleph^U$ are r-preinvex functions at a point $\rho \in \Omega$ on Ω in connection with η. We assume $r > 0$ without loss of generality. Therefore, by the definition of r-preinvex function, the following inequalities are satisfied for all $\pi \in \Omega$ and $\rho \in [0,1]$:

$$\aleph^L\big(\rho + \rho\eta(\pi,\rho)\big) \leq \ln\big(\rho e^{r\aleph^L(\pi)} + (1-\rho)e^{r\aleph^L(\rho)}\big)^{1/r},$$

and

$$\aleph^U\big(\rho + \rho\eta(\pi,\rho)\big) \leq \ln\big(\rho e^{r\aleph^U(\pi)} + (1-\rho)e^{r\aleph^U(\rho)}\big)^{1/r}$$

Using the logarithmic identity $\ln(x)^a = a\ln(x)$, we get

$$\aleph^L\big(\rho + \rho\eta(\pi,\rho)\big) \leq \frac{1}{r}\ln\big(\rho e^{r\aleph^L(\pi)} + (1-\rho)e^{r\aleph^L(\rho)}\big),$$

and

$$\aleph^U\big(\rho + \rho\eta(\pi,\rho)\big) \leq \frac{1}{r}\ln\big(\rho e^{r\aleph^U(\pi)} + (1-\rho)e^{r\aleph^U(\rho)}\big).$$

Multiplying both sides of the inequalities by r, we obtain

$$r\aleph^L\big(\rho + \rho\eta(\pi,\rho)\big) \leq \ln\big(\rho e^{r\aleph^L(\pi)} + (1-\rho)e^{r\aleph^L(\rho)}\big),$$

and

$$r\aleph^U\big(\rho + \rho\eta(\pi,\rho)\big) \leq \ln\big(\rho e^{r\aleph^U(\pi)} + (1-\rho)e^{r\aleph^U(\rho)}\big).$$

Exponentiating both sides yields

$$e^{r\aleph^L\left(\rho+\rho\eta(\pi,\rho)\right)} \leq \rho e^{r\aleph^L(\pi)} + e^{r\aleph^L(\rho)} - \rho e^{r\aleph^L(\rho)},$$

and

$$e^{r\aleph^U\left(\rho+\rho\eta(\pi,\rho)\right)} \leq \rho e^{r\aleph^U(\pi)} + e^{r\aleph^U(\rho)} - \rho e^{r\aleph^U(\rho)}.$$

By simplifying the inequalities, we obtain

$$e^{r\aleph^L\left(\rho+\rho\eta(\pi,\rho)\right)} - e^{r\aleph^L(\rho)} \leq \rho\big(e^{r\aleph^L(\pi)} - e^{r\aleph^L(\rho)}\big),$$

and

$$e^{r\aleph^U\left(\rho+\rho\eta(\pi,\rho)\right)} - e^{r\aleph^U(\rho)} \leq \rho\big(e^{r\aleph^U(\pi)} - e^{r\aleph^U(\rho)}\big).$$

Taking $e^{r\aleph^L(\rho)}$ and $e^{r\aleph^U(\rho)}$ as common factors from the left sides of the inequalities, we obtain

$$e^{r\aleph^L(\rho)}\Big[e^{r\aleph^L\left(\rho+\rho\eta(\pi,\rho)\right) - r\aleph^L(\rho)} - 1\Big] \leq \rho\big(e^{r\aleph^L(\pi)} - e^{r\aleph^L(\rho)}\big),$$

and

$$e^{r\aleph^U(\rho)}\Big[e^{r\aleph^U\big(\rho+\rho\eta(\pi,\rho)\big)-r\aleph^U(\rho)}-1\Big]\leq\ \rho\big(e^{r\aleph^U(\pi)}-e^{r\aleph^U(\rho)}\big),$$

which implies

$$e^{r\aleph^L(\pi)}-e^{r\aleph^L(\rho)}\geq\ e^{r\aleph^L(\rho)}\frac{e^{r\aleph^L(\rho+\rho\eta(\pi,\rho))-r\aleph^L(\rho)}-1}{\rho},$$

and

$$e^{r\aleph^U(\pi)}-e^{r\aleph^U(\rho)}\geq\ e^{r\aleph^U(\rho)}\frac{e^{r\aleph^U(\rho+\rho\eta(\pi,\rho))-r\aleph^U(\rho)}-1}{\rho}.$$

As the functions $\aleph^L$ and $\aleph^U$ are quasidifferentiable at a point $\rho\in\Omega$, therefore, it is directionally differentiable at ρ. As $\rho\downarrow 0$, the following inequalities are satisfied for all $\pi\in\Omega$

$$e^{r\aleph^L(\pi)}-e^{r\aleph^L(\rho)}\geq\ re^{r\aleph^L(\rho)}\aleph^{L\prime}(\rho;\eta(\pi,\rho)),$$

and

$$e^{r\aleph^U(\pi)}-e^{r\aleph^U(\rho)}\geq\ re^{r\aleph^U(\rho)}\aleph^{U\prime}(\rho;\eta(\pi,\rho)).$$

Since $r>0$, the above inequalities yield

$$\left.\begin{aligned}&\frac{1}{r}e^{r\aleph^L(\pi)}\geq\ \frac{1}{r}e^{r\aleph^L(\rho)}[1+r\aleph^{L\prime}(\rho;\eta(\pi,\rho))],\\ \text{and}\qquad&\\ &\frac{1}{r}e^{r\aleph^U(\pi)}\geq\ \frac{1}{r}e^{r\aleph^U(\rho)}[1+r\aleph^{U\prime}(\rho;\eta(\pi,\rho))],\end{aligned}\right\}\tag{2}$$

for all $\pi\in\Omega$. Due to the fact that $\aleph^L$ and $\aleph^U$ are quasidifferentiable functions, we get

$$\aleph^{L\prime}(\rho;\eta(\pi,\rho))=\max_{\vartheta\in\underline{\partial}\aleph^L(\rho)}(\vartheta^L)^T\eta(\pi,\rho)+\min_{\lambda\in\overline{\partial}\aleph^L(\rho)}(\lambda^L)^T\eta(\pi,\rho),\ \forall\pi\in\Omega,$$

and

$$\aleph^{U\prime}(\rho;\eta(\pi,\rho))=\max_{\vartheta\in\underline{\partial}\aleph^U(\rho)}(\vartheta^U)^T\eta(\pi,\rho)+\min_{\lambda\in\overline{\partial}\aleph^U(\rho)}(\lambda^U)^T\eta(\pi,\rho),\ \forall\pi\in\Omega,$$

where $\overline{\partial}\aleph^L(\rho)$ and $\overline{\partial}\aleph^U(\rho)$ are nonempty convex compact sets. Therefore, for $\overline{\lambda}^L\in\arg\min_{\lambda^L\in\overline{\partial}\aleph^L(\rho)}(\lambda^L)^T\eta(\pi,\rho)$ and $\overline{\lambda}^U\in\arg\min_{\lambda^U\in\overline{\partial}\aleph^U(\rho)}(\lambda^U)^T\eta(\pi,\rho)$, we can find the value of $\bar{\lambda}$. Using the above relations, the following inequalities are satisfied for all $\pi\in\Omega$

$$\left.\begin{aligned}&\aleph^{L\prime}(\rho;\eta(\pi,\rho))\geq(\vartheta^L)^T\eta(\pi,\rho)+(\bar{\lambda}^L)^T\eta(\pi,\rho),\ \forall\ \vartheta^L\in\underline{\partial}\aleph^L(\rho),\\ \text{and}\qquad&\\ &\aleph^{U\prime}(\rho;\eta(\pi,\rho))\geq(\vartheta^U)^T\eta(\pi,\rho)+(\bar{\lambda}^U)^T\eta(\pi,\rho),\ \forall\ \vartheta^U\in\underline{\partial}\aleph^U(\rho).\end{aligned}\right\}\tag{3}$$

From (2) and (3), we can conclude that

$$\frac{1}{r}e^{r\aleph^L(\pi)} \geq \frac{1}{r}e^{r\aleph^L(\rho)}[1+r(\omega^L)^T\eta(\pi,\rho)],\ \forall\ \omega^L \in \underline{\partial}\aleph^L(\rho)+\bar{\lambda}^L,$$

and

$$\frac{1}{r}e^{r\aleph^U(\pi)} \geq \frac{1}{r}e^{r\aleph^U(\rho)}[1+r(\omega^U)^T\eta(\pi,\rho)],\ \forall\ \omega^U \in \underline{\partial}\aleph^U(\rho)+\bar{\lambda}^U,$$

satisfied for all $\pi \in \Omega$. We say that using the definition of r-invex functions, we arrive at the conclusion that the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at a point ρ on Ω in connection with $S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho)+\bar{\lambda}^L$ and $S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho)+\bar{\lambda}^U$ respectively, as well as in connection with η. Hence, the proof is complete. □

Corollary 9. *Let $\Omega\ (\neq \emptyset)$ be an invex subset of $\Re^k$ w.r.t. η and $\aleph : \Omega \to \Im$ be an interval-valued function. Suppose $\aleph^L,\ \aleph^U : \Omega \to \Re$ are r-preinvex functions in connection with η at a point $\rho \in \Omega$ on Ω. Moreover, $\aleph^L$ and $\aleph^U$ both are quasidifferentiable functions at a point $\rho \in \Omega$. If the convex compact sets $\overline{\partial}\aleph^L(\rho)$ and $\overline{\partial}\aleph^U(\rho)$ both are singleton, then the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at ρ on Ω in connection with η as well as in connection with $S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho)+\overline{\partial}\aleph^L(\rho)$ and $S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho)+\overline{\partial}\aleph^U(\rho)$ respectively.*

Theorem 10. *Let $\aleph : \Omega \to \Im$ be an interval-valued function and ρ be an arbitrary point of Ω. The functions $\aleph^L,\ \aleph^U : \Omega \to \Re$ are r-invex quasidifferentiable at a point $\rho \in \Omega$ in connection with $S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho)+\overline{\partial}\aleph^L(\rho)$ and $S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho)+\overline{\partial}\aleph^U(\rho)$, respectively. Then, the following inequalities are satisfied for all $\pi \in \Omega$:*

$$\left.\begin{aligned}
\frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\rho)}[1+r\aleph^{L'}(\rho;\eta(\pi,\rho))],\ \text{if } r\neq 0,\\
\aleph^L(\pi) &\geq \aleph^L(\rho)+\aleph^{L'}(\rho;\eta(\pi,\rho)),\ \text{if } r=0,\\
\frac{1}{r}e^{r\aleph^U(\pi)} &\geq \frac{1}{r}e^{r\aleph^U(\rho)}[1+r\aleph^{U'}(\rho;\eta(\pi,\rho))],\ \text{if } r\neq 0,\\
\aleph^U(\pi) &\geq \aleph^U(\rho)+\aleph^{U'}(\rho;\eta(\pi,\rho)),\ \text{if } r=0.
\end{aligned}\right\} \tag{4}$$

Proof. Because ρ is an arbitrary point of Ω and the functions $\aleph^L,\ \aleph^U : \Omega \to \Re$ are r-invex quasidifferentiable at a point ρ on Ω in connection with $S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho)+\overline{\partial}\aleph^L(\rho)$ and $S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho)+\overline{\partial}\aleph^U(\rho)$, respectively. By definition of r-invexity the inequalities

$$\left.\begin{aligned}
\frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\rho)}[1+r(\omega^L)^T\eta(\pi,\rho)],\ \text{if } r\neq 0,\\
\aleph^L(\pi) &\geq \aleph^L(\rho)+(\omega^L)^T\eta(\pi,\rho),\ \text{if } r=0,\\
\frac{1}{r}e^{r\aleph^U(\pi)} &\geq \frac{1}{r}e^{r\aleph^U(\rho)}[1+r(\omega^U)^T\eta(\pi,\rho)],\ \text{if } r\neq 0,\\
\aleph^U(\pi) &\geq \aleph^U(\rho)+(\omega^U)^T\eta(\pi,\rho),\ \text{if } r=0,
\end{aligned}\right\} \tag{5}$$

satisfy for each $\pi \in \Omega$ and for each $\omega^L \in S_{\aleph}{}^L(\rho) = \underline{\partial}\aleph^L(\rho) + \overline{\partial}\aleph^L(\rho)$ and $\omega^U \in S_{\aleph}{}^U(\rho) = \underline{\partial}\aleph^U(\rho) + \overline{\partial}\aleph^U(\rho)$. From (5), we deduce that

$$\begin{aligned}
\frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\rho)}[1 + r\big((\vartheta^L)^T\eta(\pi,\rho)\big) + (\lambda^L)^T\eta(\pi,\rho))], \text{ if } r \neq 0,\\
\aleph^L(\pi) &\geq \aleph^L(\rho) + (\vartheta^L)^T\eta(\pi,\rho) + (\lambda^L)^T\eta(\pi,\rho), \text{ if } r = 0,\\
\frac{1}{r}e^{r\aleph^U(\pi)} &\geq \frac{1}{r}e^{r\aleph^U(\rho)}[1 + r\big((\vartheta^U)^T\eta(\pi,\rho)\big) + (\lambda^U)^T\eta(\pi,\rho))], \text{ if } r \neq 0,\\
\aleph^U(\pi) &\geq \aleph^U(\rho) + (\vartheta^U)^T\eta(\pi,\rho) + (\lambda^U)^T\eta(\pi,\rho), \text{ if } r = 0,
\end{aligned}$$

for each $\pi \in \Omega$ and for each $\vartheta^L \in \underline{\partial}\aleph^L(\rho), \vartheta^U \in \underline{\partial}\aleph^U(\rho)$ and $\lambda^L \in \overline{\partial}\aleph^L(\rho), \lambda^U \in \overline{\partial}\aleph^U(\rho)$. Therefore, for some $\lambda^L(\pi,\rho) \in \overline{\partial}\aleph^L(\rho)$ and $\lambda^U(\pi,\rho) \in \overline{\partial}\aleph^U(\rho)$, we have

$$\begin{aligned}
\frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\rho)}\Big[1 + r\Big(\max_{\vartheta^L \in \underline{\partial}\aleph^L(\rho)}(\vartheta^L)^T\eta(\pi,\rho)\Big) + (\lambda^L(\pi,\rho))^T\eta(\pi,\rho))\Big],\\
&\qquad\qquad \text{if } r \neq 0,\\
\aleph^L(\pi) &\geq \aleph^L(\rho) + \max_{\vartheta^L \in \underline{\partial}\aleph^L(\rho)}(\vartheta^L)^T\eta(\pi,\rho) + (\lambda^L(\pi,\rho))^T\eta(\pi,\rho), \text{ if } r = 0,\\
\frac{1}{r}e^{r\aleph^U(\pi)} &\geq \frac{1}{r}e^{r\aleph^U(\rho)}\Big[1 + r\Big(\max_{\vartheta^U \in \underline{\partial}\aleph^U(\rho)}(\vartheta^U)^T\eta(\pi,\rho)\Big) + (\lambda^U(\pi,\rho))^T\eta(\pi,\rho))\Big],\\
&\qquad\qquad \text{if } r \neq 0,\\
\aleph^U(\pi) &\geq \aleph^U(\rho) + \max_{\vartheta^U \in \underline{\partial}\aleph^U(\rho)}(\vartheta^U)^T\eta(\pi,\rho) + (\lambda^U(\pi,\rho))^T\eta(\pi,\rho), \text{ if } r = 0.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{1}{r}e^{r\aleph^L(\pi)} &\geq \frac{1}{r}e^{r\aleph^L(\rho)}\Big[1 + r\Big(\max_{\vartheta^L \in \underline{\partial}\aleph^L(\rho)}(\vartheta^L)^T\eta(\pi,\rho)) + \min_{\lambda^L \in \overline{\partial}\aleph^L(\rho)}(\lambda^L)^T\eta(\pi,\rho)\Big)\Big],\\
&\qquad\qquad \text{if } r \neq 0,\\
\aleph^L(\pi) &\geq \aleph^L(\rho) + \max_{\vartheta^L \in \underline{\partial}\aleph^L(\rho)}(\vartheta^L)^T\eta(\pi,\rho) + \min_{\lambda^L \in \overline{\partial}\aleph^L(\rho)}(\lambda^L)^T\eta(\pi,\rho) \text{ if } r = 0,\\
\frac{1}{r}e^{r\aleph^U(\pi)} &\geq \frac{1}{r}e^{r\aleph^U(\rho)}\Big[1 + r\Big(\max_{\vartheta^U \in \underline{\partial}\aleph^U(\rho)}(\vartheta^U)^T\eta(\pi,\rho)\Big) + (\lambda^U(\pi,\rho))^T\eta(\pi,\rho))\Big],\\
&\qquad\qquad \text{if } r \neq 0,\\
\aleph^U(\pi) &\geq \aleph^U(\rho) + \max_{\vartheta^U \in \underline{\partial}\aleph^U(\rho)}(\vartheta^U)^T\eta(\pi,\rho) + (\lambda^U(\pi,\rho))^T\eta(\pi,\rho), \text{ if } r = 0.
\end{aligned}$$

Therefore, using the definition of directionally differentiable, we see that (4) is satisfied for each $\pi \in \Omega$. Hence the proof is complete. $\square$

3. OPTIMALITY CRITERIA

Let us construct the nonsmooth interval-valued programming problem:

$$\text{(IP)} \qquad \text{minimize} \qquad \aleph(\pi) = [\aleph^L(\pi), \aleph^U(\pi)]$$

$$\text{subject to} \qquad \psi_j(\pi) \leq 0;\ j \in \mathfrak{I}_m = \{1, \ldots, m\};\ \pi \in X,$$

where X is a nonempty subset of $\mathfrak{R}^k$. The function $\aleph : X \to \mathfrak{I}$ is an interval-valued whereas $\aleph^L(\pi)$, $\aleph^U(\pi)$ and $\psi_j(\pi) : X \to \mathfrak{R}$, $\forall j \in \mathfrak{I}_m$ are quasidifferentiable functions on X.
The set $\Omega = \{\pi \in X : \psi_j(\pi) \leq 0,\ j \in \mathfrak{I}_m\}$ represents the set of all feasible solutions to the problem (IP). Moreover, the set of active constraints at a feasible point $\bar{\pi} \in \Omega$ is denoted by $\mathfrak{I}_m(\bar{\pi})$, that is, $\mathfrak{I}_m(\bar{\pi}) = \{j \in \mathfrak{I}_m : \psi_j(\bar{\pi}) = 0\}$.

Definition 11. [8] *A feasible point $\bar{\pi} \in \Omega$ is known as an LU-optimal solution to (IP) if there does not exist any point $\pi \in \Omega$ satisfying*

$$\aleph(\pi) <_{LU} \aleph(\bar{\pi}).$$

Theorem 12. (KKT-type necessary optimality criteria) *Let the feasible point $\bar{\pi} \in \Omega$ be the an LU-optimal solution to the problem* (IP). *Furthermore, suppose that the functions $\aleph^L$, $\aleph^U$ and ψ_j, $\forall \in \mathfrak{I}_m$ are quasidifferentiable at a point $\bar{\pi}$ entangled with the quasidifferentials $D_{\aleph}{}^L(\bar{\pi}) = [\underline{\partial}\aleph^L(\bar{\pi}), \overline{\partial}\aleph^L(\bar{\pi})]$, $D_{\aleph}{}^U(\bar{\pi}) = [\underline{\partial}\aleph^U(\bar{\pi}), \overline{\partial}\aleph^U(\bar{\pi})]$ and $D_{\psi_j}(\bar{\pi}) = [\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})]$, respectively. If Kuntz-Scholtes constraint qualification (given by Kuntz and Scholtes [15]) is satisfied at a point $\bar{\pi}$, then for $\lambda_0{}^L \in \overline{\partial}\aleph^L(\bar{\pi})$, $\lambda_0{}^U \in \overline{\partial}\aleph^U(\bar{\pi})$ and $\lambda_j \in \overline{\partial}\psi_j(\bar{\pi})$, for all j in $\mathfrak{I}_m$, there exist scalars $(\mu^L(\lambda), \mu^U(\lambda)) \in \mathfrak{R}^2$ and $\bar{\rho}_j(\lambda) \in \mathfrak{R}^m$ in such a way that*

$$0 \in \mu^L(\lambda)(\underline{\partial}\aleph^L(\bar{\pi}) + \lambda_0^L) + \mu^U(\lambda)(\underline{\partial}\aleph^U(\bar{\pi}) + \lambda_0^U) + \sum_{j=1}^{m} \bar{\rho}_j(\lambda)(\underline{\partial}\psi_j(\bar{\pi}) + \lambda_j), \quad (6)$$

$$\bar{\rho}_j(\lambda)\psi_j(\bar{\pi}) = 0,\ \forall j \in \mathfrak{I}_m, \quad (7)$$

$$(\mu^L(\lambda), \mu^U(\lambda)) > 0,\ \bar{\rho}_j(\lambda) \geq 0,\ \forall j \in \mathfrak{I}_m, \quad (8)$$

where $\mu^L(\lambda), \mu^U(\lambda), \bar{\rho}_1(\lambda), \ldots, \bar{\rho}_m(\lambda)$ rely on the particular selected $\lambda = (\lambda_0^L, \lambda_0^U, \lambda_1, \ldots, \lambda_m)$.

Theorem 13. (Sufficiency) *A feasible solution $\bar{\pi}$ becomes an LU-optimal solution to* (IP) *if it fulfills the following two conditions:*

(i) *The point $\bar{\pi}$ must satisfy the necessary optimality criteria as given by the condition* (6)-(8).

(ii) *The functions $\aleph^L$, $\aleph^U$ and ψ_j, for all j in $\mathfrak{I}_m(\bar{\pi})$ are r-invex quasidifferentiable at $\bar{\pi}$ on Ω in connection with $S_{\aleph}{}^L(\bar{\pi}) = \underline{\partial}\aleph^L(\bar{\pi}) + \overline{\partial}\aleph^L(\bar{\pi})$, $S_{\aleph}{}^U(\bar{\pi}) = \underline{\partial}\aleph^U(\bar{\pi}) + \overline{\partial}\aleph^U(\bar{\pi})$ and $S_{\psi_j}(\bar{\pi}) = \underline{\partial}\psi_j(\bar{\pi}) + \overline{\partial}\psi_j(\bar{\pi})$, respectively, as well as in connection with η.*

Proof. Since the feasible solution $\bar{\pi} \in \Omega$ satisfies the conditions (6)-(8), thus, for $\lambda_0^L \in \overline{\partial} \aleph^L(\bar{\pi})$, $\lambda_0^U \in \overline{\partial} \aleph^U(\bar{\pi})$ and $\lambda_j \in \overline{\partial} \psi_j(\bar{\pi})$, $\forall j \in \mathfrak{J}_m$, there exists $(\mu^L(\lambda), \mu^U(\lambda)) \in \mathfrak{R}^2$, and $\bar{\rho}(\lambda) \in \mathfrak{R}^m$ which satisfies the conditions (6)-(8). Consequently, by KKT-type necessary criteria (6), there exist $\vartheta_0^L \in \underline{\partial} \aleph^L(\bar{\pi})$, $\vartheta_0^U \in \underline{\partial} \aleph^U(\bar{\pi})$ and $\vartheta_j \in \underline{\partial} \psi_j(\bar{\pi})$, $\forall\ j \in \mathfrak{J}_m$, such that

$$0 = \mu^L(\lambda)(\vartheta_0^L + \lambda_0^L) + \mu^U(\lambda)(\vartheta_0^U + \lambda_0^U) + \sum_{j=1}^{m} \bar{\rho}_j(\lambda)(\vartheta_j + \lambda_j). \tag{9}$$

From the assumptions, the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at $\bar{\pi}$ on Ω in connection with $S_{\aleph}{}^L(\bar{\pi}) = \underline{\partial} \aleph^L(\bar{\pi}) + \overline{\partial} \aleph^L(\bar{\pi})$, $S_{\aleph}{}^U(\bar{\pi}) = \underline{\partial} \aleph^U(\bar{\pi}) + \overline{\partial} \aleph^U(\bar{\pi})$, respectively, as well as in connection with η and the functions ψ_j, $j \in \mathfrak{J}_m(\bar{\pi})$, are r-invex quasidifferentiable at $\bar{\pi}$ on Ω in connection with $S_{\psi_j}(\bar{\pi}) = \underline{\partial} \psi_j(\bar{\pi}) + \overline{\partial} \psi_j(\bar{\pi})$ and in connection with η. Then the inequalities

$$\frac{1}{r} e^{r \aleph^L(\pi)} \geq \frac{1}{r} e^{r \aleph^L(\bar{\pi})} \Big[1 + r(\omega_0^L)^T \eta(\pi, \bar{\pi}) \Big]\ \forall\ \omega_0^L \in S_{\aleph}{}^L(\bar{\pi}), \tag{10}$$

$$\frac{1}{r} e^{r \aleph^U(\pi)} \geq \frac{1}{r} e^{r \aleph^U(\bar{\pi})} \Big[1 + r(\omega_0^U)^T \eta(\pi, \bar{\pi}) \Big]\ \forall\ \omega_0^U \in S_{\aleph}{}^U(\bar{\pi}), \tag{11}$$

$$\frac{1}{r} e^{r \psi_j(\pi)} \geq \frac{1}{r} e^{r \psi_j(\bar{\pi})} \Big[1 + r\omega_j^T \eta(\pi, \bar{\pi}) \Big],\ \forall\ \omega_j \in S_{\psi_j}(\bar{\pi}), \tag{12}$$

can be easily established using the definition of r-invex function, for all $\pi \in \Omega$. Since the above inequalities are fulfilled for any sets $\omega_0^L \in S_{\aleph}{}^L(\bar{\pi})$, $\omega_0^U \in S_{\aleph}{}^U(\bar{\pi})$ and $\omega_j \in S_{\psi_j}(\bar{\pi})$, for all j in $\mathfrak{J}_m(\bar{\pi})$, respectively. Therefore, by definition of $S_{\aleph}{}^L(\bar{\pi})$, $S_{\aleph}{}^U(\bar{\pi})$ and $S_{\psi_j}(\bar{\pi})$, it is also satisfied for $\omega_0^L = \vartheta_0^L + \lambda_0^L \in S_{\aleph}{}^L(\bar{\pi})$, $\omega_0^U = \vartheta_0^U + \lambda_0^U \in S_{\aleph}{}^U(\bar{\pi})$ and $\omega_j = \vartheta_j + \lambda_j \in S_{\psi_j}(\bar{\pi})$, and it gives

$$\frac{1}{r} \Big[e^{r(\aleph^L(\pi) - \aleph^L(\bar{\pi}))} - 1 \Big] \geq (\vartheta_0^L + \lambda_0^L)^T \eta(\pi, \bar{\pi}) \tag{13}$$

$$\frac{1}{r} \Big[e^{r(\aleph^U(\pi) - \aleph^U(\bar{\pi}))} - 1 \Big] \geq (\vartheta_0^U + \lambda_0^U)^T \eta(\pi, \bar{\pi}) \tag{14}$$

$$\frac{1}{r} \Big[e^{r(\psi_j(\pi) - \psi_j(\bar{\pi}))} - 1 \Big] \geq (\vartheta_j^T + \lambda_j^T) \eta(\pi, \bar{\pi}),\ \forall j \in \mathfrak{J}_m(\bar{\pi}). \tag{15}$$

Using the definition of $\mathfrak{J}_m(\bar{\pi})$, and $\psi_j(\pi) \leq \psi_j(\bar{\pi})$, $\forall\ j \in \mathfrak{J}_m(\bar{\pi})$; $\pi,\ \bar{\pi} \in \Omega$, we have

$$\frac{1}{r} \Big[e^{r(\psi_j(\pi) - \psi_j(\bar{\pi}))} - 1 \Big] \leq 0, \tag{16}$$

which holds for all point π of Ω. From inequalities (15) and (16) we get

$$(\vartheta_j^T + \lambda_j^T) \eta(\pi, \bar{\pi}) \leq 0, \quad j \in \mathfrak{J}_m(\bar{\pi}). \tag{17}$$

Applying conditions $\bar{\rho}_j(\lambda) > 0$, $\forall\ j \in \mathfrak{J}_m(\bar{\pi})$ and $\bar{\rho}_j(\lambda) = 0$, $\forall\ j \notin \mathfrak{J}_m(\bar{\pi})$ to the above inequalities, we have

$$\sum_{j=1}^{m} \bar{\rho}_j(\lambda)({\vartheta_j}^T + {\lambda_j}^T) \eta(\pi, \bar{\pi}) \leq 0. \tag{18}$$

Using inequalities (9) and (18), we get

$$\left[\mu^L(\lambda)(\vartheta_0^L+\lambda_0^L)^T+\mu^U(\lambda)(\vartheta_0^U+\lambda_0^U)^T\right]\eta(\pi,\bar{\pi})\geq 0. \tag{19}$$

Multiplying the inequality (13) by $\mu^L(\lambda)$ and (14) by $\mu^U(\lambda)$ and summing up we get

$$\frac{1}{r}\mu^L(\lambda)\left[e^{r(\aleph^L(\pi)-\aleph^L(\bar{\pi}))}-1\right]+\frac{1}{r}\mu^U(\lambda)\left[e^{r(\aleph^U(\pi)-\aleph^U(\bar{\pi}))}-1\right]$$
$$\geq\left[\mu^L(\lambda)(\vartheta_0^L+\lambda_0^L)^T+\mu^U(\lambda)(\vartheta_0^U+\lambda_0^U)^T\right]\eta(\pi,\bar{\pi}). \tag{20}$$

Combining (19) with (20) gives

$$\frac{1}{r}\mu^L(\lambda)\left[e^{r(\aleph^L(\pi)-\aleph^L(\bar{\pi}))}-1\right]+\frac{1}{r}\mu^U(\lambda)\left[e^{r(\aleph^L(\pi)-\aleph^L(\bar{\pi}))}-1\right]\geq 0, \tag{21}$$

which holds for all point $\pi\in\Omega$. Thus, we conclude that $\aleph(\pi)\geq_{LU}\aleph(\bar{\pi})$ for all $\pi\in\Omega$. Therefore, the feasible solution $\bar{\pi}$ becomes an *LU*-optimal solution for the problem (IP). Hence, the proof is complete. □

Now, let us formulate an example of the nonsmooth interval-valued problem considered in this paper and interpret it using *r*-invex quasidifferentiable functions in connection with compact convex sets, which are equivalent to the Minkowski sum of its subdifferentials and superdifferentials as well as in connection with η.

Example 1 Let us construct the following nondifferentiable interval-valued programming problem:

$$\begin{aligned}(\mathrm{IP}_1)\quad \text{minimize}\quad & \aleph(\pi)=[\aleph^L(\pi),\aleph^U(\pi)]\\ &=\Big[\ln\Big(\pi_1^4+\pi_2^2+|\pi_1|+|\pi_2|-\pi_1-\pi_2+1\Big),\\ &\quad\ln\Big(\big||\pi_1|+\pi_2\big|+\pi_1^4+1\Big)\Big],\\ \text{subject to,}\quad & \psi_1(\pi)=\ln\Big(\pi_1^2+\pi_2^2+2|\pi_1|+\pi_2+1\Big)\leq 0;\quad \pi\in X,\end{aligned}$$

where $X\subset\Re^2$ defined by $X:=\{(\pi_1,\pi_2):0\leq\pi_1\leq 1,\ 0\leq\pi_2\leq 1\}$. The set $\Omega=\{\pi=(\pi_1,\pi_2)\in X:\ln(\pi_1^2+\pi_2^2+2|\pi_1|+\pi_2+1)\leq 0\}$ and $\bar{\pi}=(0,0)$ represents the set of all feasible solutions and *LU*-optimal solution respectively to the problem (IP$_1$). Furthermore, we will show that the functions $\aleph^L(\pi)$, $\aleph^U(\pi)$ and $\psi_1(\pi)$ are quasidifferentiable at a point $\bar{\pi}$. Using definition of directional derivative, we get $\aleph^{L'}((0,0);\delta)=|\delta_1|+|\delta_2|-\delta_1-\delta_2$ and $\aleph^{U'}((0,0);\delta)=\big||\delta_1|+\delta_2\big|$ where, $\delta=(\delta_1,\delta_2)\in\Re^2$. Hence,

$$\aleph^{L'}((0,0);\delta)=\max_{\vartheta_0^L\in\mathrm{conv}\{(1,1),(-1,-1)\}}(\vartheta_0^L)^T\delta+\min_{\lambda_0^L\in\mathrm{conv}\{(-1,-1)\}}(\lambda_0^L)^T\delta,$$

where $\underline{\partial}\aleph^L(0,0)=\mathrm{conv}\{(1,1),(-1,-1)\}$, $\overline{\partial}\aleph^L(0,0)=\mathrm{conv}\{(-1,-1)\}$ and

$$\aleph^{U'}((0,0);\delta)=\max_{\vartheta_0^U\in\mathrm{conv}\{(0,0),(-2,2),(2,2)\}}(\vartheta_0^U)^T\delta+\min_{\lambda_0^U\in\{(-1,-1),(1,-1)\}}(\lambda_0^U)^T\delta,$$

where $\underline{\partial}\aleph^U(0,0) = \text{conv}\{(0,0),(-2,2),(2,2)\}$, $\overline{\partial}\aleph^U(0,0) = \{(-1,-1),(1,-1)\}$. Therefore, by Definition 2, we can conclude that the functions $\aleph^L$ and $\aleph^U$ are quasidifferentiable at a point $\bar{\pi} = (0,0)$. Moreover, in a similar way, we have $\psi_1'((0,0),\delta) = 2|\delta_1| + \delta_2$ and hence

$$\psi_1'((0,0);\delta) = \max_{\vartheta_1 \in \text{conv}\{(2,0),(-2,0)\}} (\vartheta_1)^T \delta + \min_{\lambda_1 \in \{(0,1)\}} (\lambda_1)^T \delta,$$

where $\underline{\partial}\psi_1(0,0) = \text{conv}\{(2,0),(-2,0)\}$, $\overline{\partial}\psi_1(0,0) = \{(0,1)\}$.
Now, we will verify that the necessary criteria of KKT-type are satisfied at the feasible point $\bar{\pi}$ with nonconstant Lagrange multipliers. It can be demonstrated that for any set of $\lambda_0^L \in \overline{\partial}\aleph^L(\bar{\pi})$, $\lambda_0^U \in \overline{\partial}\aleph^U(\bar{\pi})$ and $\lambda_1 \in \overline{\partial}\psi_1(\bar{\pi})$ there exists a Lagrange multipliers $(\mu^L(\lambda),\ \mu^U(\lambda)) > 0$, $\bar{\rho}_1(\lambda) > 0$ which satisfies the conditions (6)-(8), such as if $\lambda_0^L = (-1,-1)$, $\lambda_0^U = (-1,-1)$ and $\lambda_1 = (0,1)$, then by substituting $\mu^L(\lambda) = 1$, $\mu^U(\lambda) = 1$, and $\bar{\rho}_1(\lambda) = 1$ we can observe that it satisfies the KKT necessary criteria. On the other hand, if $\lambda_0^L = (-1,-1)$, $\lambda_0^U = (1,-1)$ and $\lambda_1 = (0,1)$, then by substituting $\mu^L(\lambda) = 1$, $\mu^U(\lambda) = 1$, and $\bar{\rho}_1(\lambda) = 2$ it satisfies the KKT necessary criteria.
Since the necessary criteria of KKT-type have been satisfied at a point $\bar{\pi}$. Now, to verify the sufficiency criteria of KKT-type, that is, to show that the feasible point $\bar{\pi}$ is an *LU*-optimal solution to the problem (IP$_1$) it is enough to show that the functions $\aleph^L$, $\aleph^U$ and ψ_1 are r-invex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η and a compact convex set, which is equivalent to the Minkowski sum of its subdifferentials as well as superdifferentials.

Let $S_{\aleph}{}^L(\bar{\pi}) = \underline{\partial}\aleph^L(\bar{\pi}) + \overline{\partial}\aleph^L(\bar{\pi})$, $S_{\aleph}{}^U(\bar{\pi}) = \underline{\partial}\aleph^U(\bar{\pi}) + \overline{\partial}\aleph^U(\bar{\pi})$, $S_{\psi_1}(\bar{\pi}) = \underline{\partial}\psi_1(\bar{\pi}) + \overline{\partial}\psi_1(\bar{\pi})$ and the vector-valued function $\eta : \Omega \times \Omega \to \Re^2$ be defined by $\eta(\pi,\bar{\pi}) =$

$$\begin{bmatrix} \frac{\pi_1+\pi_2}{4} \\ \frac{\pi_1+\pi_2}{4} \end{bmatrix}.$$

Using Definition 3, it can be stated that the functions $\aleph^L$, $\aleph^U$ and ψ_1 are 1-invex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η as well as in connection with compact convex sets $S_{\aleph}{}^L(\bar{\pi})$, $S_{\aleph}{}^U(\bar{\pi})$ and $S_{\psi_1}(\bar{\pi})$ respectively. As all the assumptions of the sufficiency criteria of KKT-type are satisfied at a point $\bar{\pi}$, one can conclude that the feasible point $\bar{\pi}$ is an *LU*-optimal solution to the problem (IP$_1$).

Now, for the constructed nonsmooth interval-valued programming problem (IP$_1$), we point out the fact that the Lagrange multipliers depend on the choice of λ. Evidently, for the given choice of λ, sometime the necessary optimality criteria of KKT-type may not be satisfied at a feasible point $\bar{\pi}$ for some Lagrange multipliers.
Let us demonstrate the KKT necessary conditions by taking an arbitrary value of Lagrange multipliers. For particularly specified $\lambda = (\lambda_0^L,\ \lambda_0^U,\ \lambda_1)$ the KKT condition (6) is denoted by

$$W_\lambda = \mu^L(\lambda)(\underline{\partial}\aleph^L(\bar{\pi}) + \lambda_0^L) + \mu^U(\lambda)(\underline{\partial}\aleph^U(\bar{\pi}) + \lambda_0^U) + \bar{\rho}_1(\lambda)\big(\underline{\partial}\psi_1(\bar{\pi}) + \lambda_1\big). \quad (22)$$

(1) For $\lambda' = (\lambda_0^L,\ \lambda_0^U,\ \lambda_1) = \big((-1,-1),(-1,-1),(0,1)\big)$ and $\mu^L(\lambda) = 1$, $\mu^U(\lambda) = 1$ $\bar{\rho}_1(\lambda) = 1$, we get,
$W_{\lambda'} = \text{conv}\{(-5,2),(-1,2),(3,2),(1,0),(-1,-2),(-5,-2),(-7,0),(-3,0)\}$

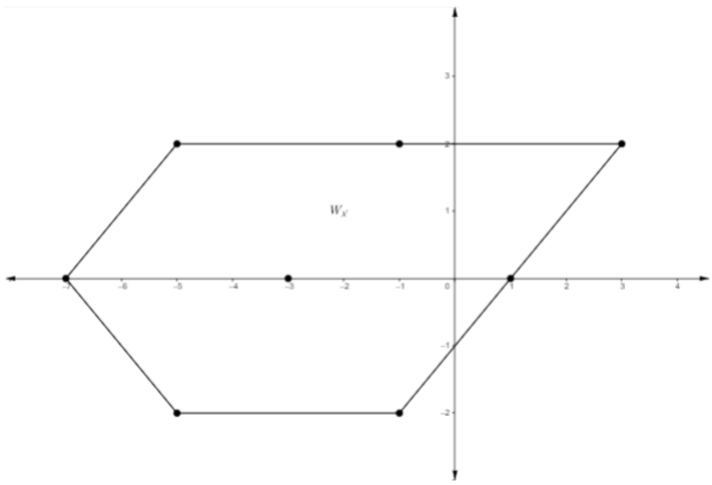

Figure 1: Here $0 \in W_{\lambda'}$, thus we can state that it satisfies the KKT conditions for $\bar{\rho}_1(\lambda) = 1$.

(2) For $\lambda'' = (\lambda_0^L,\ \lambda_0^U,\ \lambda_1) = \big((-1,-1),(1,-1),(0,1)\big)$ and $\mu^L(\lambda) = 1,\ \mu^U(\lambda) = 1,$ $\bar{\rho}_1(\lambda) = 2$, we get,
$W_{\lambda''} = \text{conv}\{(-5,3),(-3,1),(-1,3),(-7,1),(-5,-1),(3,3),(5,1),(7,3),$ $(1,1),(3,-1)\}$

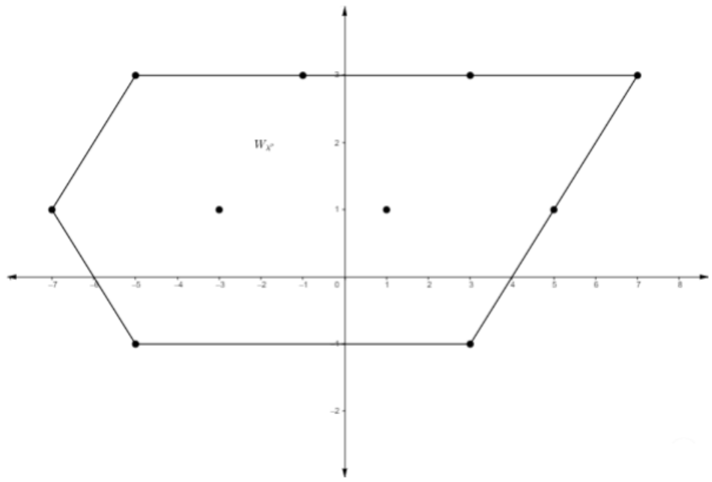

Figure 2: Here also $0 \in W_{\lambda''}$, therefore, we can state that it also satisfies the KKT conditions for $\bar{\rho}_1(\lambda) = 2$.

(3) For $\lambda'' = (\lambda_0^L,\ \lambda_0^U,\ \lambda_1) = \big((-1,-1),(-1,-1),(0,1)\big)$ and $\mu^L(\lambda) = 1$, $\mu^U(\lambda) = 1$, $\bar{\rho}_1(\lambda) = 4$, we get,

$W_{\lambda''} = \text{conv}\{(-11,5),(-7,5),(-9,3),(-13,3),(-11,1),(5,5),(9,5),(7,3),$
$(3,3),(5,1)\}$

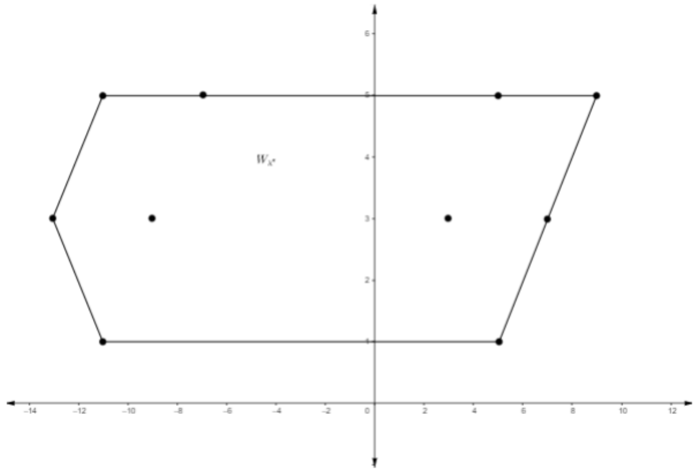

Figure 3: Here $0 \notin W_{\lambda''}$, hence it does not satisfies the KKT conditions for $\bar{\rho}_1(\lambda) = 4$.

Therefore, we can conclude that the Lagrange multipliers depend on the choice of λ.

4. DUAL PROBLEM

Let us discuss the duality of the Mond-Weir type nonsmooth interval-valued programming problem (IP).

(IDP) maximize $\aleph(\sigma) = [\aleph^L(\sigma), \aleph^U(\sigma)]$

subject to $(\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho) \in \Gamma$.

The set of all pairs $(\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho)$ is symbolized by Γ with $\sigma \in X$ and $\bar{\mu}^L, \bar{\mu}^U : \Re^{m+2} \to \Re$, $\rho : \Re^{m+2} \to \Re^m$, $\rho(\lambda) = (\rho_1(\lambda), \ldots, \rho_m(\lambda))$ satisfying for $\lambda_0^L \in \overline{\partial}\aleph^L(\sigma)$, $\lambda_0^U \in \overline{\partial}\aleph^U(\sigma)$ and $\lambda_j \in \overline{\partial}\psi_j(\sigma)$, $\forall\, j \in \Im_m$ the conditions are given below

$$0 \in \bar{\mu}^L(\lambda)(\underline{\partial}\aleph^L(\sigma) + \lambda_0^L) + \bar{\mu}^U(\lambda)(\underline{\partial}\aleph^U(\sigma) + \lambda_0^U) + \sum_{j=1}^{m} \bar{\rho}_j(\lambda)(\underline{\partial}\psi_j(\sigma) + \lambda_j), \quad (23)$$

$$\bar{\rho}_j(\lambda)\psi_j(\sigma) \geq 0,\ \forall j \in \Im_m, \quad (24)$$

$$(\bar{\mu}^L(\lambda),\ \bar{\mu}^U(\lambda)) > 0,\ \bar{\rho}_j(\lambda) \geq 0,\ \forall j \in \Im_m,\ \sigma \in X, \quad (25)$$

where, $\lambda = (\lambda_0^L, \lambda_0^U, \lambda_1, \ldots, \lambda_m)$. Γ denotes the set of all feasible solutions of dual problem (IDP). Furthermore, $Y = pr_{\Re^k}\Gamma$ stands for the projection of the set Γ on X.

Theorem 14. (Weak duality for r-invex function quasidifferentiable) *Let π be the feasible point to the problem (IP) and $(\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho)$ be the feasible point to its Mond-Weir dual (IDP). Furthermore, suppose that*

(i) *the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at a point σ defined on $\Omega \cup Y$ in connection with $S_{\aleph}{}^L(\sigma) = \underline{\partial}\aleph^L(\sigma) + \overline{\partial}\aleph^L(\sigma)$, and $S_{\aleph}{}^U(\sigma) = \underline{\partial}\aleph^U(\sigma) + \overline{\partial}\aleph^U(\sigma)$, respectively, as well as in connection with η.*

(ii) *The functions ψ_j, for all j in $\mathfrak{J}_m(\sigma)$ are r-invex quasidifferentiable at a point σ defined on $\Omega \cup Y$ in connection with $S_{\psi_j}(\sigma) = \underline{\partial}\psi_j(\sigma) + \overline{\partial}\psi_j(\sigma)$ and η.*

Then, $\aleph(\pi) \geq_{LU} \aleph(\sigma)$.

Proof. Given π and $(\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho)$ are feasible solutions of the problem (IP) and its dual (IDP), respectively. Therefore, it satisfies the conditions (23)-(25) for $\lambda_0^L \in \overline{\partial}\aleph^L(\sigma)$, $\lambda_0^U \in \overline{\partial}\aleph^U(\sigma)$ and $\lambda_j \in \overline{\partial}\psi_j(\sigma)$, $\forall\, j \in \mathfrak{J}_m$, and $\rho(\lambda) = (\rho_1(\lambda), \rho_2(\lambda), \ldots, \rho_m(\lambda)) \in \mathfrak{R}^m$. On the contrary, let us suppose that

$$\aleph(\pi) <_{LU} \aleph(\sigma), \tag{26}$$

that is, $\begin{array}{l}\aleph^L(\pi) < \aleph^L(\sigma)\\ \aleph^U(\pi) < \aleph^U(\sigma)\end{array}$ or, $\begin{array}{l}\aleph^L(\pi) \leq \aleph^L(\sigma)\\ \aleph^U(\pi) < \aleph^L(\sigma)\end{array}$ or, $\begin{array}{l}\aleph^L(\pi) < \aleph^L(\sigma)\\ \aleph^L(\pi) \leq \aleph^L(\sigma).\end{array}$

From the assumptions that the functions $\aleph^L$ and $\aleph^U$ are r-invex quasidifferentiable at a point σ on $\Omega \cup Y$ in connection with $S_{\aleph}{}^L(\sigma) = \underline{\partial}\aleph^L(\sigma) + \overline{\partial}\aleph^L(\sigma)$ and $S_{\aleph}{}^U(\sigma) = \underline{\partial}\aleph^U(\sigma) + \overline{\partial}\aleph^U(\sigma)$, respectively, and in connection with η, we get

$$\frac{1}{r}e^{r\aleph^L(\pi)} \geq \frac{1}{r}e^{r\aleph^L(\sigma)}\Big[1 + r(\omega_0^L)^T\eta(\pi,\sigma)\Big],\ \forall\ \omega_0^L \in S_{\aleph}{}^L(\sigma), \tag{27}$$

$$\frac{1}{r}e^{r\aleph^U(\pi)} \geq \frac{1}{r}e^{r\aleph^U(\sigma)}\Big[1 + r(\omega_0^U)^T\eta(\pi,\sigma)\Big],\ \forall\ \omega_0^U \in S_{\aleph}{}^U(\sigma). \tag{28}$$

On combining inequality (26) with (27) and (28), respectively, we get

$$(\omega_0^L)^T\eta(\pi,\sigma) < 0,\ \forall\ \omega_0^L \in S_{\aleph}{}^L(\sigma),$$
$$(\omega_0^U)^T\eta(\pi,\sigma) < 0,\ \forall\ \omega_0^U \in S_{\aleph}{}^U(\sigma).$$

or

$$(\omega_0^L)^T\eta(\pi,\sigma) \leq 0,\ \forall\ \omega_0^L \in S_{\aleph}{}^L(\sigma),$$
$$(\omega_0^U)^T\eta(\pi,\sigma) < 0,\ \forall\ \omega_0^U \in S_{\aleph}{}^U(\sigma).$$

or

$$(\omega_0^L)^T\eta(\pi,\sigma) < 0,\ \forall\ \omega_0^L \in S_{\aleph}{}^L(\sigma),$$
$$(\omega_0^U)^T\eta(\pi,\sigma) \leq 0,\ \forall\ \omega_0^U \in S_{\aleph}{}^U(\sigma).$$

The above inequalities with the condition (25) give

$$\Big[\bar{\mu}^L(\lambda)(\omega_0^L)^T + \bar{\mu}^U(\lambda)(\omega_0^U)^T\Big]\eta(\pi,\sigma) < 0 \tag{29}$$

The functions ψ_j, for all j in $\mathfrak{J}_m(\sigma)$ are r-invex quasidifferentiable at a point σ on $\Omega \cup Y$ in connection with $S_{\psi_j}(\sigma) = \underline{\partial}\psi_j(\sigma) + \overline{\partial}\psi_j(\sigma)$ and η. Therefore, using the definition of

r-invex function, we get

$$\frac{1}{r}e^{r\psi_j(\pi)} \geq \frac{1}{r}e^{r\psi_j(\sigma)}\left[1+r\omega_j^T\eta(\pi,\sigma)\right], \,\forall\, \omega_j \in S_{\psi_j}(\sigma). \tag{30}$$

Using condition (24), and the fact that $\pi \in \Omega$ and $\sigma \in Y$, we obtain

$$\rho_j(\lambda)\psi_j(\pi) \leq \rho_j(\lambda)\psi_j(\sigma), \,\forall j \in \mathfrak{I}_m. \tag{31}$$

Since $\rho_j(\lambda) > 0$, for all j in $\mathfrak{I}_m(\sigma)$, therefore the inequality (30) can be rewritten as

$$\frac{1}{r}\left(e^{\frac{r}{\rho_j(\lambda)}(\rho_j(\lambda)\psi_j(\pi)-\rho_j(\lambda)\psi_j(\sigma))}-1\right) \geq \omega_j^T\eta(\pi,\sigma), \,\forall \omega_j \in S_{\psi_j}(\sigma). \tag{32}$$

Combining inequality (31) and (32) yields

$$\omega_j^T\eta(\pi,\sigma) \leq 0, \,\forall \omega_j \in S_{\psi_j}(\sigma), \,\forall j \in \mathfrak{I}_m(\sigma). \tag{33}$$

Using inequality (25), we obtain

$$\sum_{j=1}^{m}\rho_j(\lambda)\omega_j^T\eta(\pi,\sigma) \leq 0, \,\forall \omega_j \in S_{\psi_j}(\sigma), \,\forall j \in \mathfrak{I}_m. \tag{34}$$

On summing the inequalities (29) and (34), we obtain

$$\left[\bar{\mu}^L(\lambda)(\omega_0^L)^T + \bar{\mu}^U(\lambda)(\omega_0^U)^T + \sum_{j=1}^{m}\rho_j(\lambda)\omega_j^T\right]\eta(\pi,\sigma) < 0,$$

for each $\omega_0^L \in S_{\aleph}{}^L(\sigma)$, $\omega_0^U \in S_{\aleph}{}^U(\sigma)$, $\omega_j \in S_{\psi_j}(\sigma)$, and $\omega_0^L \in \underline{\partial}\aleph^L(\sigma)+\lambda_0^L$, $\omega_0^U \in \underline{\partial}\aleph^U(\sigma)+\lambda_0^U$, $\omega_j \in \underline{\partial}\psi_j(\sigma)+\lambda_j$. In the light of the the definitions of $S_{\aleph}{}^L(\sigma)$, $S_{\aleph}{}^U(\sigma)$, and $S_{\psi_j}(\sigma)$, for all j in $\mathfrak{I}_m$, we find that $\lambda_0^L \in \overline{\partial}\aleph^L(\sigma)$, $\lambda_0^U \in \overline{\partial}\aleph^U(\sigma)$ and $\lambda_j \in \overline{\partial}\psi_j(\sigma)$ satisfying the inequality

$$\left[\bar{\mu}^L(\vartheta_0^L+\lambda_0^L)^T + \bar{\mu}^U(\vartheta_0^U+\lambda_0^U)^T + \sum_{j=1}^{m}\rho_j(\lambda)(\vartheta_j+\lambda_j)^T\right]\eta(\pi,\sigma) < 0, \tag{35}$$

for all $\vartheta_0^L \in \underline{\partial}\aleph^L(\sigma)$, $\vartheta_0^U \in \underline{\partial}\aleph^U(\sigma)$ and $\vartheta_j \in \underline{\partial}\psi_j(\sigma)$, for all j in $\mathfrak{I}_m$. The condition (23) assurs that for the sets $\lambda_0^L \in \overline{\partial}\aleph^L(\sigma)$, $\lambda_0^U \in \overline{\partial}\aleph^U(\sigma)$, and $\lambda_j \in \overline{\partial}\psi_j(\sigma)$, there exist $\vartheta_0^L \in \underline{\partial}\aleph^L(\sigma)$, $\vartheta_0^U \in \underline{\partial}\aleph^U(\sigma)$, and $\vartheta_j \in \underline{\partial}\psi_j(\sigma)$, for all j in $\mathfrak{I}_m$ such that

$$\left[\bar{\mu}^L(\vartheta_0^L+\lambda_0^L)^T + \bar{\mu}^U(\vartheta_0^U+\lambda_0^U)^T + \sum_{j=1}^{m}\rho_j(\lambda)(\vartheta_j+\lambda_j)^T\right]\eta(\pi,\sigma) = 0,$$

satisfies which contradicts (35). Hence, the proof is complete. □

Theorem 15. (Weak duality for strict r-invex quasidifferentiable function) *Let π be the feasible point to the problem (IP) and $(\sigma,\bar{\mu}^L,\bar{\mu}^U,\rho)$ be the feasible point to its Mond-Weir dual (IDP). Furthermore, suppose that*

(i) *The functions* $\aleph^L$ *and* $\aleph^U$ *are strict r-invex quasidifferentiable at a point* σ *defined on* $\Omega \cup Y$ *in connection with* $S_{\aleph}{}^L(\sigma) = \underline{\partial}\aleph^L(\sigma) + \overline{\partial}\aleph^L(\sigma)$, *and* $S_{\aleph}{}^U(\sigma) = \underline{\partial}\aleph^U(\sigma) + \overline{\partial}\aleph^U(\sigma)$, *respectively, as well as in connection with* η.

(ii) *The functions* ψ_j, *for all* j *in* $\Im_m(\sigma)$ *are r-invex quasidifferentiable at a point* σ *defined on* $\Omega \cup Y$ *in connection with* $S_{\psi_j}(\sigma) = \underline{\partial}\psi_j(\sigma) + \overline{\partial}\psi_j(\sigma)$ *and* η.

Then, $\aleph(\pi) >_{LU} \aleph(\sigma)$.

Theorem 16. (Direct duality) *Let the feasible point* $\bar{\pi}$ *be an LU-optimal solution to the problem* (IP) *and* $(\bar{\pi}, \mu^L, \mu^U, \bar{\rho})$ *be the feasible point of its dual problem (IDP), where* $\mu^L, \mu^U : \Re^{m+2} \to \Re$, *and* $\bar{\rho} : \Re^{m+2} \to \Re^m$. *Furthermore, suppose that*

(i) *The functions* $\aleph^L$ *and* $\aleph^U$ *are r-invex quasidifferentiable at a point* σ *defined on* $\Omega \cup Y$ *in connection with* $S_{\aleph}{}^L(\sigma) = \underline{\partial}\aleph^L(\sigma) + \overline{\partial}\aleph^L(\sigma)$, *and* $S_{\aleph}{}^U(\sigma) = \underline{\partial}\aleph^U(\sigma) + \overline{\partial}\aleph^U(\sigma)$, *respectively, as well as in connection with* η.

(ii) *The functions* ψ_j, *for all* j *in* $\Im_m(\sigma)$ *are r-invex quasidifferentiable at a point* σ *defined on* $\Omega \cup Y$ *in connection with* $S_{\psi_j}(\sigma) = \underline{\partial}\psi_j(\sigma) + \overline{\partial}\psi_j(\sigma)$ *as well as in connection with* η.

Then, the feasible point $(\bar{\pi}, \mu^L, \mu^U, \bar{\rho})$ *is an LU-optimal solution to its dual problem (IDP).*

Proof. Let us suppose the feasible point $\bar{\pi}$ is an *LU*-optimal solution of the problem (IP), and there exist functions $\mu^L, \mu^U : \Re^{m+2} \to \Re$ and $\bar{\rho} : \Re^{m+2} \to \Re^m$ in such a manner that $(\bar{\pi}, \mu^L, \mu^U, \bar{\rho})$ is the feasible point of its dual problem (IDP). Moreover, as it satisfies all the assumptions of the weak duality theorem for *r*-invex quasidifferentiable functions and$\bar{\pi} \in \Omega$, then

$$\aleph(\bar{\pi}) \geq_{LU} \sup\{\aleph(\sigma) : (\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho) \in \Gamma\}.$$

Therefore, one can say that the feasible point $(\bar{\pi}, \mu^L, \mu^U, \bar{\rho})$ is an *LU*-optimal solution to the dual problem (IDP). □

Theorem 17. (Converse duality) *Suppose the feasible points* $(\bar{\sigma}, \mu^L, \mu^U \bar{\rho})$ *is an LU-optimal solution to the dual problem (IDP) and* $\bar{\sigma} \in \Omega$. *Furthermore, assume that*

(i) *The functions* $\aleph^L$ *and* $\aleph^U$ *are r-invex quasidifferentiable at* $\bar{\sigma}$ *defined on* $\Omega \cup Y$ *in connection with* $S_{\aleph}{}^L(\bar{\sigma}) = \underline{\partial}\aleph^L(\bar{\sigma}) + \overline{\partial}\aleph^L(\bar{\sigma})$, *and* $S_{\aleph}{}^U(\bar{\sigma}) = \underline{\partial}\aleph^U(\bar{\sigma}) + \overline{\partial}\aleph^U(\bar{\sigma})$, *respectively, as well as in connection with* η.

(ii) *The functions* ψ_j, *for all* j *in* $\Im_m(\bar{\sigma})$ *are r-invex quasidifferentiable at a point* $\bar{\sigma}$ *defined on* $\Omega \cup Y$ *in connection with* $S_{\psi_j}(\bar{\sigma}) = \underline{\partial}\psi_j(\bar{\sigma}) + \overline{\partial}\psi_j(\bar{\sigma})$ *as well as in connection with* η.

Then, the feasible point $\bar{\sigma}$ *is an LU-optimal solution to* (IP).

Proof. Its proof is similar to that of the weak duality theorem for the *r*-invex quasidifferentiable function. □

Example 2: Portfolio Optimization with Interval-Valued Objective
In this example, we consider a simplified financial portfolio optimization problem under uncertainty. An investor allocates capital into two assets with uncertain returns. The goal is to minimize an interval-valued objective function representing a trade-off between uncertain return and risk, where both components include nonsmooth features.

We formulate the following nonsmooth interval-valued programming problem:

$$
\begin{aligned}
(\mathrm{IP}_2)\quad \text{Minimize}\qquad & \aleph(\pi) = [\aleph^L(\pi), \aleph^U(\pi)] \\
& \quad = \left[\ln\left(\pi_1^2 + |\pi_2| + 1\right), \ln\left(|\pi_1| + 2\pi_2^2 + 1\right)\right] \\
\text{Subject to}\quad & \psi_1(\pi) = \pi_1 + \pi_2 - 1 \leq 0 \\
& \pi \in X := \{(\pi_1, \pi_2) \in \mathbb{R}^2 : 0 \leq \pi_1 \leq 1, 0 \leq \pi_2 \leq 1\},
\end{aligned}
$$

where $\pi = (\pi_1, \pi_2)$ represents the proportion of capital invested in Asset 1 and Asset 2, respectively. The set of feasible solutions to the problem (IP_1) is given by $\Omega = \{\pi = (\pi_1, \pi_2) \in X : \pi_1 + \pi_2 - 1 \leq 0\}$.

The corresponding Mond-Weir-type dual problem transforms the minimization problem into a maximization type and the constraint into a set-valued inclusion using quasidifferential calculus to account for marginal contributions of return, risk, and the budget constraint. The Mond-Weir dual model for the problem (IP_2) is formulated as

$$
\begin{aligned}
(\mathrm{IDP}_2)\ \text{Maximize}\quad & \aleph(\sigma) = [\aleph^L(\sigma), \aleph^U(\sigma)] \\
& \quad = \left[\ln\left(\sigma_1^2 + |\sigma_2| + 1\right), \ln\left(|\sigma_1| + 2\sigma_2^2 + 1\right)\right]
\end{aligned}
$$

$$
\text{Subject to}\ \ 0 \in \bar{\mu}^L(\partial \aleph^L(\sigma) + \lambda_0^L) + \bar{\mu}^U(\partial \aleph^U(\sigma) + \lambda_0^U) + \rho_1(\partial \psi_1(\sigma) + \lambda_1), \tag{36}
$$

$$
\rho_1 \psi_1(\sigma) \geq 0, \tag{37}
$$

$$
\bar{\mu}^L + \bar{\mu}^U = 1, \quad \bar{\mu}^L > 0,\ \bar{\mu}^U > 0,\ \rho_1 \geq 0, \quad \sigma \in X. \tag{38}
$$

Assume a feasible solution set to (IDP_2) is given by

$$
(\sigma, \bar{\mu}^L, \bar{\mu}^U, \rho_1) = \left((0,0), \frac{1}{2}, \frac{1}{2}, 0\right).
$$

We now verify the quasidifferentiability of $\aleph^L(\sigma)$, $\aleph^U(\sigma)$, and $\psi_1(\sigma)$ at $\sigma = (0,0)$. Using directional derivatives, we have $\aleph^{L'}((0,0);\delta) = |\delta_2|$, $\aleph^{U'}((0,0);\delta) = |\delta_1|$ and $\psi_1'((0,0),\delta) = \delta_1 + \delta_2$ where, $\delta = (\delta_1, \delta_2) \in \Re^2$. Hence

$$
\aleph^{L'}((0,0);\delta) = \max_{\vartheta_0^L \in \mathrm{conv}\{(0,1),(0,-1)\}} (\vartheta_0^L)^T \delta + \min_{\lambda_0^L \in \{(0,0)\}} (\lambda_0^L)^T \delta,
$$

where $\underline{\partial} \aleph^L(0,0) = \mathrm{conv}\{(0,1),(0,-1)\}$, $\overline{\partial} \aleph^L(0,0) = \mathrm{conv}\{(0,0)\}$,

$$
\aleph^{U'}((0,0);\delta) = \max_{\vartheta_0^U \in \mathrm{conv}\{(1,0),(-1,0)\}} (\vartheta_0^U)^T \delta + \min_{\lambda_0^U \in \{(0,0)\}} (\lambda_0^U)^T \delta,
$$

where $\underline{\partial}\aleph^U(0,0) = \text{conv}\{(1,0),(-1,0)\}$, $\overline{\partial}\aleph^U(0,0) = \{(0,0)\}$, and

$$\psi_1'((0,0);\delta) = \max_{\vartheta_1 \in \{(1,1)\}} (\vartheta_1)^T\delta + \min_{\lambda_1 \in \{(0,0)\}} (\lambda_1)^T\delta,$$

where $\underline{\partial}\psi_1(0,0) = \{(1,1)\}$, $\overline{\partial}\psi_1(0,0) = \{(0,0)\}$. Therefore, by Definition 2, we conclude that $\aleph^L$, $\aleph^U$ and ψ_1 are quasidifferentiable functions at a point $\bar{\pi} = (0,0)$. For $\lambda_0^L = \lambda_0^U = \lambda_1 = (0,0)$, together with $\bar{\mu}^L(\lambda) = \bar{\mu}^U(\lambda) = 1/2$, and $\rho_1(\lambda) = 0$ the dual conditions (36)-(38) hold. The condition (36)

$$0 \in \bar{\mu}^L(\partial\aleph^L(\sigma) + \lambda_0^L) + \bar{\mu}^U(\partial\aleph^U(\sigma) + \lambda_0^U) + \rho_1(\partial\psi_1(\sigma) + \lambda_1)$$

by substituting $\lambda_0^L = \lambda_0^U = \lambda_1 = (0,0), \bar{\mu}^L(\lambda) = \bar{\mu}^U(\lambda) = \frac{1}{2}$, and $\rho_1(\lambda) = 0$ gives

$$0 \in \text{conv}\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(-\frac{1}{2},\frac{1}{2}\right),\left(-\frac{1}{2},\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{1}{2}\right)\right\},$$

which can be viewed graphically as follows:

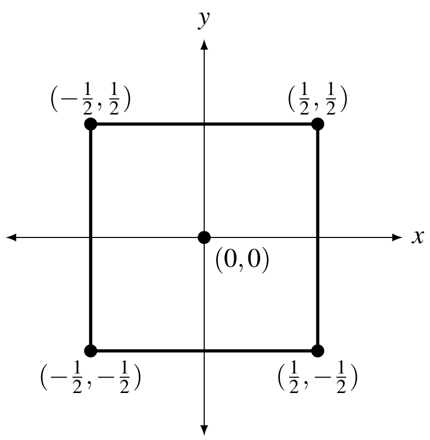

Figure 4: $0 \in \bar{\mu}^L(\partial\aleph^L(\sigma) + \lambda_0^L) + \bar{\mu}^U(\partial\aleph^U(\sigma) + \lambda_0^U) + \rho_1(\partial\psi_1(\sigma) + \lambda_1)$

Further, we will show that the functions $\aleph^L$, $\aleph^U$ and ψ_1 are r-invex quasidifferentiable at a point σ in connection with η and a compact convex set, which is equivalent to the Minkowski sum of its subdifferentials as well as superdifferentials. Let $S_{\aleph}{}^L(\sigma) = \underline{\partial}\aleph^L(\sigma) + \overline{\partial}\aleph^L(\sigma)$, $S_{\aleph}{}^U(\sigma) = \underline{\partial}\aleph^U(\sigma) + \overline{\partial}\aleph^U(\sigma)$, $S_{\psi_1}(\sigma) = \underline{\partial}\psi_1(\sigma) + \overline{\partial}\psi_1(\sigma)$ and the vector-valued function $\eta : \Omega \times \Omega \to \Re^2$ be defined by $\eta(\pi,\sigma) = |\sigma_1| + |\sigma_2| - 1$. Using Definition 3, it can be stated that the functions $\aleph^L$, $\aleph^U$ and ψ_1 are 1-invex quasidifferentiable at a point σ in connection with η as well as in connection with compact convex sets $S_{\aleph}{}^L(\sigma)$, $S_{\aleph}{}^U(\sigma)$ and $S_{\psi_1}(\sigma)$ respectively. As all the assumptions of the Theorem 14 satisfied at σ, therefore $\aleph(\pi) \geq_{LU} \aleph(\sigma)$. This validates that the dual approach via the Mond-Weir framework, under the assumption of quasidifferentiability r-invexity, successfully identifies an optimal strategy for investment allocation in the presence of interval uncertainty and nonsmooth behavior.

5. CONCLUSION

In the present article, we have worked on a class of nonsmooth interval-valued optimization problems with inequality constraints by employing the framework of r-invex quasidifferentiable functions in connection with compact convex sets and η. The main outcomes of our work include the derivation of necessary and sufficient optimality conditions using quasidifferential calculus, which effectively handles the nonsmooth nature of the problem. Furthermore, we constructed a Mond-Weir-type dual model and established duality theorems under the assumption of r-invexity. A key aspect of our analysis is the characterization of quasidifferentials through the Minkowski sum of subdifferentials and superdifferentials, which underpins both the optimality and duality results. Notably, our findings reveal that the Lagrange multipliers in the KKT-type conditions are nonconstant, thereby reflecting the complexity and generality of the proposed model. Finally, the theoretical developments were supported by a numerical example.

The present study opens several avenues for further research in the field of nonsmooth interval-valued optimization. One potential direction is the extension of the present framework to multiobjective or vector-valued interval optimization problems, incorporating the concept of quasidifferentiable r-invexity in higher-dimensional settings. Another potential area involves the relaxation of the convexity assumptions, such as exploring generalized forms like approximate or preinvex quasidifferentiable functions, which would broaden the applicability of the results. We shall investigate these questions in subsequent papers.

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