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EXTENSIONS OF P-PROPERTY, R_0-PROPERTY AND SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS

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Abstract: In this manuscript, we present some new results for the semidefinite linear complementarity problem in the context of three notions for linear transformations, viz., pseudo w-P property, pseudo Jordan w-P property, and pseudo SSM property. Interconnections with the $P_\#$-property (proposed recently in the literature) are presented. We also study the $R_\#$-property of a linear transformation, extending the rather well known notion of an R_0-matrix. In particular, results are presented for the Lyapunov, Stein, and the multiplicative transformations.

Keywords: Linear Complementarity Problem, P-property, R-property, Semidefinite Linear Complementarity Problem, w-P properties, Jordan w-P property, Moore-Penrose Inverse.

MSC: 90C33, 15A09.

1. INTRODUCTION

Let S^n denote the vector space of all $n \times n$ real symmetric matrices and S^n_+ be the set of all symmetric positive semidefinite matrices in S^n. Given a linear transformation $L : S^n \to S^n$ and a matrix $Q \in S^n$, the *semidefinite linear complementarity problem*, denoted by $SDLCP(L, Q)$, is to find an $X \in S^n$ such that

$$X \in S^n_+, \; Y = L(X) + Q \in S^n_+, \text{ and } \langle X, Y\rangle = tr(XY) = 0,$$

where $tr(A)$ denotes the trace of the square matrix A. If such an X exists, we call X, a *solution* of $SDLCP(L, Q)$. The set of all of solutions $SDLCP(L, Q)$ is denoted by $SOL(L, Q)$. This problem was introduced by Gowda and Song [3] and is a generalization of the linear complementarity problem (LCP). For details on LCP, we refer the reader to the book [1]. In [4], $SDLCP(L, Q)$ has been studied over H^n, the space of all $n \times n$ complex hermitian matrices, in which S^n_+ is replaced by H^n_+, the set of all hermitian positive semidefinite matrices.

Throughout this article, V stands for either S^n or H^n. For a matrix $X \in V$, we write $X \geq 0$ if and only if $X \in S^n_+$ or H^n_+ depending on whether $X \in S^n$ or $X \in H^n$, and $X \leq 0$ when $-X \geq 0$. For $X, Y \in V$, we define $X \circ Y = \frac{1}{2}(XY + YX)$. Many of the results here are proved for the *Lyapunov transformation* L_A, *Stein transformation* S_A, and *multiplicative transformation* M_A which we define below: For a fixed $A \in \mathbb{R}^{n\times n}$ ($A \in \mathbb{C}^{n\times n}$), define L_A, S_A and M_A on S^n (respectively, L_A, S_A and M_A on H^n) by $L_A(X) = AX + XA^T$, $S_A(X) = X - AXA^T$ and $M_A(X) = AXA^T$ (respectively, $L_A(X) = AX + XA^*$, $S_A(X) = X - AXA^*$ and $M_A(X) = AXA^*$) where A^T and A^* denote the transpose and the conjugate transpose of A, respectively. These transformations are extensively studied in the literature and are related to continuous and discrete dynamical systems [3, 4, 5].

Motivated by the concepts of P and strictly semimonotone (SSM) matrices in LCP, Gowda and Song [3] introduced the P and SSM properties to study the existence and uniqueness of solutions of $SDLCP(L, Q)$. A linear transformation $L : V \to V$ is said to have the *P-property* if $XL(X) = L(X)X \leq 0 \implies X = 0$, and *SSM-property* if $X \geq 0$, $XL(X) = L(X)X \leq 0 \implies X = 0$. It is shown that L_A has the P-property if and only if A is positive stable (see Section 2 for the definition). This result is proved for S^n in [3] and H^n in [4]. Gowda and Parthasarathy [4] proved that S_A has the P-property if and only if A is Schur stable (see Section 2 for the definition). Tao ([14], Theorem 3.9) proved that P and SSM-properties are equivalent for L_A and S_A transformations.

We now move to the next class of operators called w-P property which is introduced to study the certain type of uniqueness of solutions of $SDLCP(L, Q)$. Let $L : V \to V$ be linear. We say that L has the *w-P property* [15] if

$$XL(X) = L(X)X \leq 0 \implies L(X) = 0.$$

Note that, if L is invertible, then w-P property is equivalent to P-property. A non-commutative version of the w-P property is also studied in the literature: $L : V \to V$ has the *Jordan w-P property* [15] if

$$X \circ L(X) \leq 0 \implies L(X) = 0.$$

It is easy to see that Jordan w-P property always implies w-P property but the converse is not true, even for the Lyapunov transformation [16]. Tao [15, 16] characterized the w-P property for the Lyapunov and Stein transformations on V. He showed that L_A (S_A) has the w-P property if and only if A is semipostive stable (respectively, generalized Schur table) (see Section 2 for definitions). For S_A, he has shown that w-P and Jordan w-P properties are equivalent.

In order to outline the objectives of this article, we collect certain results from LCP. A real square matrix A is called a *P-matrix* if all its principal minors are positive. The concept of P-matrix is equivalent to the uniqueness of solutions of LCP [1]. A real square matrix A is called a *Z-matrix*, if all its off-diagonal entries are nonpositive. If A is a Z-matrix, then $A = sI - B$, for some $s > 0$ and $B \geq 0$ (all the entries of B are nonnegative). In addition to the above condition if $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B, then A is an *M-matrix*. For a Z-matrix A, it is well known that A is a P-matrix if and only if A is a SSM-matrix (i.e. $x \geq 0$ and $x * Ax \leq 0 \Longrightarrow x = 0$) which is equivalent to A being an invertible M-matrix. Here $*$ denotes the Hadamard entrywise product and $x \geq 0$ means that all the components of x are nonnegative. To study the result for singular M-matrices, a new class of strictly range semimonotone matrices was introduced in a recent work [8].

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be strictly range semimonotone if the following implication holds:

$$x \in R(A), x \geq 0 \text{ and } x * Ax \leq 0 \Longrightarrow x = 0.$$

A generalization of the above mentioned result is obtained with the help of the concept of a $P_{\#}$-matrix introduced in [10]. For a symmetric Z-matrix A, it is shown that the $P_{\#}$-matrix notion is equivalent to strictly range semimonotonicity [8]. It is natural to ask whether the result can be generalized in the setting of S^n and H^n. In this work, we answer this question by introducing the concept of pseudo SSM-property. We show (Theorem 3.19) that pseudo *SSM* and $P_{\#}$-properties are equivalent for a self-adjoint Z-transformation, a generalization of the Z-matrix property. We also introduce the notions of pseudo *w-P* property and pseudo Jordan *w-P* property which can be considered as singular analogues of the *w-P* property and Jordan *w-P* property, respectively. Under the assumption on the existence of the group inverses of L_A and S_A, we show that $P_{\#}$ and pseudo *w-P* properties are equivalent for L_A and S_A (Theorem 3.11 and Theorem 3.16). Moreover, if A is normal, then we prove that these properties are equivalent to *w-P* property for L_A (Corollary 3.13).

In the second part of the article, we are concerned with the $R_{\#}$-property of a linear transformation on S^n. This is a generalization of the concept R_0-property introduced in [3]: $L : S^n \to S^n$ is said to have the R_0-property if 0 is the only solution of the $SDLCP(L, 0)$. That is $X \in S^n$ is the matrix satisfying $X \geq 0$, $L(X) \geq 0$ and $XL(X) = 0$, then $X = 0$. It is known that L has the R_0-property if and only if $SOL(L, Q)$ is a (possibly empty) bounded set, for all $Q \in S^n$. In [3], it was shown that if L_A has the R_0-property, then A is non-singular. Motivated by this concept and results, Sivakumar [13] introduced the $R_\#$-property. A linear transformation $L : S^n \to S^n$ is said to have the $R_\#$-property if $X = 0$ is the only matrix $X \in R(L)$ such that $X \geq 0, L(X) \geq 0$ and $XL(X) = 0$. There, it was shown that if M_A or L_A has the $R_\#$-property, then $A^\#$ exists. This result gives a connection between generalized inverse and SDLCP. He has given two characterizations for M_A to be an $R_\#$-operator. In this work, we give one more necessary and sufficient condition for M_A to be an $R_\#$-operator (Theorem 4.4). As a consequence, for a symmetric matrix A, we show that M_A has the $R_\#$-property if and only M_A is strongly monotone on $R(M_A)$ (Theorem 4.7), extending a result of [9].

The contents of the article are organized as follows. In the next section, we provide a brief background for the rest of the material in the article. The subsequent section contains some results for $P_\#$-property which is a generalization of P-property in semidefinite linear complementarity setting. A brief introduction to pseudo w-P property and pseudo Jordan w-P property is given in this section, and the interconnection with $P_\#$-property is also considered here. The main results of this section are Theorem 3.11 and Theorem 3.16. Section 4 is devoted to the $R_\#$-property. Here we obtained a necessary and sufficient condition for the multiplicative transformation to have $R_\#$-property.

2. PRELIMINARIES

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant if $\mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{m\times n}$, the set of all $m \times n$ real matrices, let $R(A)$ denote the range space of A and $N(A)$ stand for the null space of A. The spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A.

Recall that a square matrix A is called positive stable if every eigenvalue has positive real part. We say that A is *semipositive stable* if every eigenvalue has a nonnegative real part. A similar definition will be understood for a linear transformation on a vector space. A is called Schur stable if all the eigenvalues of A lie in the open unit disc of the complex plane. We say that A is *generalized Schur stable* if all the eigenvalues of A lie in the closed unit disc. Motivated by the similarities between the properties of Z-matrices on nonnegative orthant and Lyapunov and Stein transformations on the H^n_+, Gowda and Tao [7], introduced and studied Z-transformations on proper cones. Next, we list certain definitions and results to be used in the sequel.

Let us recall that a linear transformation $L : V \to V$ is said to be *normal* if $LL^* = L^*L$ and *self-adjoint* if $L = L^*$. Let $L : S^n \to S^n$ be linear. We say L is a Z-transformation on S^n_+ if $X \in S^n_+$, $Y \in S^n_+$ and $\langle X, Y\rangle = 0 \implies \langle L(X), Y\rangle \leq 0$. A similar definition applies for linear maps over H^n. For Z-transformations, there is a very useful result, which we recall next.

Theorem 2.1. *(Theorem 6, [12]) If $L : H^n \to H^n$ is a Z-transformation then $\lambda = min\{Re(\mu) : \mu \in \sigma(L)\}$ is an eigenvalue of L with a corresponding eigenvector in H^n_+.*

The group inverse of a matrix $A \in \mathbb{C}^{n\times n}$, if it exists, is the unique matrix X satisfying $A = AXA$, $X = XAX$ and $AX = XA$. The group inverse of A is denoted by $A^{\#}$. A necessary and sufficient condition for $A^{\#}$ to exist is $R(A^2) = R(A)$, which, in turn, is equivalent to the condition $N(A^2) = N(A)$. Another equivalent condition is that the subspaces $R(A)$ and $N(A)$ are complementary. Next, we recall the definition of group inverse of a linear transformation. Let V be a vector space and $L : V \to V$ be linear. Then the group inverse of L, if it exists, is the unique linear transformation $T : V \to V$ such that $TLT = T$, $LTL = L$ and $TL = LT$. The following will be used in a proof: Let $L : V \to V$ be a linear transformation such that $L^{\#}$ exists. If $X \in R(L)$, then $LL^{\#}X = X$.

Theorem 2.2. *(Theorem 5, [11]) Let $L : V \to V$ be a linear transformation. Then $L^{\#}$ exists if and only if $V = R(L) \oplus N(L)$.*

Lemma 2.3. *Let $L : V \to V$ be a normal linear transformation, where V is a finite dimensional inner product space. Then $L^{\#}$ exists.*

Proof. If L is normal, then

$$\begin{aligned} N(L) &= N(L^*L) \\ &= N(LL^*) \\ &= N(L^*) \\ &= R(L)^{\perp}, \end{aligned}$$

so that $R(L)$ and $N(L)$ are complementary subspaces. Thus, $L^{\#}$ exists. □

Remark 2.4. *Let $A \in \mathbb{C}^{n\times n}$. It follows that $L_A^* = L_{A^*}$, $M_A^* = M_{A^*}$, and $S_A^* = S_{A^*}$. Consequently, if A is a normal matrix (i.e. $AA^* = A^*A$), then L_A, S_A and M_A are normal operators. Further, $L_A^{\#}$, $M_A^{\#}$ and $S_A^{\#}$ exist, by the result above.*

Next, we collect results concerning the characterization of P-property for the Lyapunov and Stein transformations.

Theorem 2.5. *([3],[7]) Let $A \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*
(a) A is positve stable.
(b) L_A has the P-property.
(c) L_A is positive stable.

Theorem 2.6. *(Theorem 11, [4]) Let $A \in \mathbb{C}^{n\times n}$. The following are equivalent:*
(a) $\rho(S_A) < 1$.
(b) A is Schur stable.
(c) S_A has the P-property.

As an analogue of the results in [6] for P-property, Tao [15] introduced the notion of Jordan w-P property and w-P property for linear transformation, defined on Euclidean Jordan algebras (see the introduction, for the definitions). In particular, he studied these properties for the Lyapunov transformation and the Stein transformation. Note that w-P property implies Jordan w-P property. The following result, proved by Tao [15], presents a characterization of the w-P property for the Lyapunov transformation.

Theorem 2.7. *(Theorem 5.1, [15]) Let $A \in \mathbb{C}^{n\times n}$. Then A is semipositive stable if and only if L_A has the w-P-property.*

Tao [15] has also presented an example of a Lyapunov transformation to show that *w-P*-property does not imply Jordan *w-P*-property.

In [4], it is shown that S_A has P-property if and only if A is Schur stable i.e., all eigenvalues of A lie in the open unit disk. In [16], Tao gave a characterization of the Jordan *w-P* property for S_A.

Theorem 2.8. *(Theorem 3.1, [16]) Let $A \in \mathbb{C}^{n\times n}$. For S_A on V, the following are equivalent:*
(a) A is generalized Schur stable.
(b) S_A has the Jordan w-P property.
(c) S_A has the w-P property.

3. $P_\#$-PROPERTY

In this section, we shall be discussing four classes of operators, viz., operators which satisfy the properties $P_\#$, pseudo *SSM*, pseudo *w-P*, and pseudo Jordan *w-P*. It must be mentioned that in connection with $P_\#$-property, results where obtained only for matrix classes, till now. Here, we extend our study to operators on the spaces of real symmetric matrices and complex hermitian matrices. This property will be the main rallying point for two main results of this section. These are presented in Theorem 3.11 and Theorem 3.16. Corollary 3.13 delineates an important special case of Theorem 3.11. Statements of these results include the notions of pseudo *SSM*, pseudo *w-P*, and pseudo Jordan *w-P* properties, which are introduced in this article. While $P_\#$-property implies pseudo *SSM*-property, we are able to show that for self-adjoint Z-operators, these notions coincide. This is the fourth main result and is presented in Theorem 3.19.

We begin by considering the first two notions and prove some preliminary results. As an analogue of the P-property for the linear transformation, Rajesh and Sivakumar [10] introduced the notion of $P_\#$-property. A linear tranformation $L : V \to V$ is said to have the $P_\#$-property if

$$X \in R(L), XL(X) = L(X)X \leq 0 \implies X = 0.$$

If L has the $P_\#$-property, we may sometimes say that L is a $P_\#$-operator.

Next, we propose the first new class of operators of our study.

Definition 3.1. *Let $L : V \to V$ be linear. We say that L has the pseudo SSM-property if*

$$X \in R(L), X \geq 0, XL(X) = L(X)X \leq 0 \implies X = 0.$$

We may refer to such an operator as a pseudo SSM-operator. Clearly, if L has the $P_\#$-property, then L has the pseudo SSM-property. Let us present two examples.

Example 3.2. *Let* $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Set* $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in S^2$. *Then*

$$L_A(X) = \begin{pmatrix} 2x_{11} & x_{12} \\ x_{12} & 0 \end{pmatrix}.$$

We have

$$XL_A(X) = \begin{pmatrix} 2x_{11}^2 + x_{12}^2 & x_{11}x_{12} \\ 2x_{11}x_{12} + x_{12}x_{22} & x_{12}^2 \end{pmatrix}.$$

So, for any $X \in R(L_A)$ satisfying $XL_A(X) \leq 0$, one has $x_{11} = x_{12} = x_{22} = 0$ so that $X = 0$. This shows that L_A has $P_\#$-property.

Example 3.3. *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in S^2$. Then*

$$M_A(X) = (x_{11} + 2x_{12} + x_{22})\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $U \in R(M_A)$. Then $U = \beta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, for some constant β. If $U \geq 0$, then $\beta \geq 0$. We have

$$UM_A(U) = 8\beta^2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

So, if $UM_A(U) \leq 0$, then $\beta \leq 0$ so that $\beta = 0$. Thus $U = 0$, proving that M_A is a pseudo SSM-operator. It is clear from the calculations that M_A is also a $P_\#$-operator.

Next, we give an example to show that pseudo SSM-property does not imply $P_\#$-property.

Example 3.4. *Consider the transformation $L : S^2 \to S^2$ defined by*

$$L\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{22} \\ x_{22} & 0 \end{pmatrix}.$$

If $0 \leq X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in R(L)$ then $x_{22} = 0$ and $x_{12} = 0$. If, in addition, one has $XL(X) = L(X)X = \begin{pmatrix} x_{11}^2 & 0 \\ 0 & 0 \end{pmatrix} \leq 0$, then $x_{11} = 0$, and hence L has the pseudo SSM-property. Consider $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in R(L)$. Then $L(X) = 0$. Thus, $XL(X) = L(X)X = 0$. But $X \neq 0$. Therefore, L does not have the $P_\#$-property.

The following result will be useful in the sequel.

Lemma 3.5. *Let $L : V \to V$ be a Z-transformation which also possesses the pseudo SSM-property. Then L is semipositive stable.*

Proof. Suppose that L has the pseudo SSM-property. Since L is a Z-transformation on V, by Theorem 2.1, the number $\lambda = min\{Re(\mu) : \mu \in \sigma(L)\}$ is an eigenvalue of L so that $L(X) = \lambda X$, where $0 \neq X \geq 0$. If $\lambda < 0$, then $X \in R(L)$ and $XL(X) = L(X)X = \lambda X^2 \leq 0$. Since L has the pseudo SSM-property, we have $X = 0$, a contradiction. Therefore $\lambda \geq 0$, and hence L is semipositive stable. □

In the next three results, we collect some properties of $P_\#$-operators.

Theorem 3.6. *Let $L : V \to V$ possess the $P_\#$-property. Then $L^\#$ exists and has the $P_\#$-property.*

Proof. Let $X \in R(L) \cap N(L)$. Then $L(X) = 0$. Now $XL(X) = L(X)X = 0$. Since L has the $P_\#$-property, it follows that $X = 0$. Thus $L^\#$ exists. Let $X \in R(L^\#) = R(L)$ such that X and $L^\#(X)$ commute, and $XL^\#(X) \leq 0$. Then $LL^\#(X) = X$ and one has $0 \geq XL^\#(X) = LL^\#(X)L^\#(X) = L(Y)Y = YL(Y)$, where $Y = L^\#(X) \in R(L^\#) = R(L)$, $YL(Y) \leq 0$. Since L has $P_\#$-property, we then have $Y = L^\#(X) = 0$. Therefore, $X \in N(L^\#) = N(L)$. Thus $X \in N(L) \cap R(L)$ and as $L^\#$ exists, it follows that $X = 0$. We have shown that $L^\#$ has $P_\#$-property. □

The following is an extension of a result that was proved for real matrices in [10]. Since the proof is similar, it is skipped.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$. We have the following: If L_A or M_A has the $P_\#$-property, then $A^\#$ exists. If S_A has the $P_\#$-property, then $(I - A)^\#$ exists.*

Lemma 3.8. *Let $L : H^n \to H^n$ possess the $P_\#$-property. Suppose that $L(X) = \lambda X$, $0 \neq X \in R(L)$ with $\lambda \in \mathbb{R}$. Then $\lambda > 0$.*

Proof. Let L have $P_\#$-property. Suppose that $L(X) = \lambda X$, $0 \neq X \in R(L)$ with $\lambda \leq 0$. Clearly, $L(X)$ and X commute and $XL(X) = \lambda X^2 \leq 0$. By the $P_\#$-property of L, we then have $X = 0$, a contradiction. So, $\lambda > 0$. □

Next, we turn our attention to the third and the fourth properties mentioned earlier viz., the notions of pseudo Jordan *w-P* property and pseudo *w-P* property. These are generalizations of the corresponding properties, studied in [15] and [16].

Definition 3.9. *Given a linear transformation $L : V \to V$, we say that*
(i) *L has pseudo Jordan w-P property if*

$$X \in R(L) \text{ and } X \circ L(X) \leq 0 \implies L(X) = 0.$$

(ii) *L has pseudo w-P property if*

$$X \in R(L) \text{ and } X \text{ and } L(X) \text{ commute and } X \circ L(X) \leq 0 \implies L(X) = 0.$$

Remark 3.10. *It is easy to see that the following implications hold:*

Jordan w-P-property ⇒ pseudo Jordan w-P-property,

w-P-property ⇒ pseudo w-P-property

and

$$\text{pseudo Jordan } w\text{-}P\text{-property} \Rightarrow \text{pseudo } w\text{-}P\text{-property.}$$

Let us note that if the operator under consideration is invertible, then the adjective "pseudo" is redundant. That is, pseudo w-P-property is equivalent to w-P-property and pseudo Jordan w-P-property is equivalent to Jordan w-P-property. In view of this, one might think of these new properties as singular versions of the corresponding properties. The following observation is also noteworthy. Suppose that $L^{\#}$ *exists. If* L *has pseudo Jordan w-P-property or pseudo w-P-property, then* L *has* $P_{\#}$*-property. The next example shows that it is no longer true if the group inverse of* L *does not exist. Let* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*. Then* A *is semipositive stable, and by Theorem 2.7, it follows that* L_A *has w-P-property, and hence it possesses the pseudo w-P-property. Let* $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$*. Then* $L_A(Y) = X$ *and* $L_A(X) = 0$*. Thus* $R(L_A)$ *and* $N(L_A)$ *are not complementary subspaces and so* $L_A^{\#}$ *does not exist. This means that* L_A *does not have* $P_{\#}$*-property, in view of Theorem 3.6.*

In the next result, we present neccesary and sufficient conditions for the Lyapunov transformation to have pseudo *w-P* property. We also discuss the connection with $P_{\#}$-property. This is the first main result of this article and is motivated by Theorem 2.7.

Theorem 3.11. *Let* $A \in \mathbb{C}^{n\times n}$ *such that* $L_A^{\#}$ *exists. Then the following statements are equivalent:*
(*i*) *A is semipositive stable.*
(*ii*) *L_A has pseudo w-P property.*
(*iii*) *L_A has $P_{\#}$-property.*
(*iv*) *L_A has pseudo SSM-property.*

Proof. (*i*) ⇒ (*ii*): If A is semipositive stable, then L_A has the *w-P*-property and this in turn implies pseudo *w-P*-property.
(*ii*) ⇒ (*iii*): As mentioned in Remark 3.10, the existence of $L_A^{\#}$ together with pseudo *w-P* property ensure that L_A has $P_{\#}$-property.
(*iii*) ⇒ (*iv*): Follows from the definition.
(*iv*) ⇒ (*i*): By Lemma 3.5, it follows that L_A is semipositive stable. This means that $L_A + \epsilon I$ is positive stable, for any $\epsilon > 0$. Now,

$$(L_A + \epsilon I)X = AX + XA^* + \epsilon X = L_{A+\frac{\epsilon}{2}I}(X).$$

Thus $L_{A+\frac{\epsilon}{2}I}$ is positive stable, and hence from Theorem 2.5, it follows that $A + \frac{\epsilon}{2}I$ is positive stable for every $\epsilon > 0$. Hence, A is semipositive stable. □

Remark 3.12. *From example* 4.8 *in [16], the statements in Theorem 3.11 are equivalent for the (real) Lyapunov transformation* L_A *on* S^n*, where* $A \in \mathbb{R}^{n\times n}$*.*

Next, for a normal matrix A, we show that w-P property, pseudo w-P property, $P_{\#}$-property, and pseudo SSM-property are equivalent. Further, they are equivalent to the monotonicity of L_A.

Corollary 3.13. *Let $A \in \mathbb{R}^{n\times n}$ be normal. Then the following are equivalent.*
(i) *A is semipositive stable.*
(ii) *L_A is monotone.*
(iii) *L_A has w-P property.*
(iv) *L_A has pseudo w-P property.*
(v) *L_A has $P_{\#}$-property.*
(vi) *L_A has pseudo SSM-property.*

Proof. Since A is normal, from example 5.4 in [16], it follows that L_A is monotone if and only if L_A has the w-P property. Now, the proof follows from Theorem 2.7 and Theorem 3.11. □

Next, we give an example to illustrate Theorem 3.11.

Example 3.14. *Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ so that A is semipositive stable. Set $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in S^2$. Then*

$$L_A(X) = \begin{pmatrix} 2(x_{11}+x_{12}) & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix},$$

which is not an invertible operator. Define $T : S^2 \to S^2$ by

$$T(X) := \begin{pmatrix} \frac{x_{11}}{2} - x_{12} - \frac{3}{2}x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}.$$

First, we show that $T = L_A^{\#}$. We have

$$\begin{aligned} TL_A(X) &= T\left(\begin{pmatrix} 2(x_{11}+x_{12}) & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_{11}-x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}. \end{aligned}$$

Thus, one has

$$\begin{aligned} L_ATL_A(X) &= L_A\left(\begin{pmatrix} x_{11}-x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 2(x_{11}+x_{12}) & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix} \\ &= L_A(X). \end{aligned}$$

Also,

$$\begin{aligned} L_AT(X) &= L_A\left(\begin{pmatrix} \frac{x_{11}}{2} - x_{12} - \frac{3}{2}x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_{11}-x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}. \end{aligned}$$

This shows that T commutes with L_A. Finally,

$$\begin{aligned} TL_AT(X) &= T\left(\begin{pmatrix} x_{11}-x_{12} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} \frac{x_{11}}{2}-x_{12}-\frac{3}{2}x_{22} & x_{12}+x_{22} \\ x_{12}+x_{22} & 0 \end{pmatrix} \\ &= T(X), \end{aligned}$$

proving the claim. This shows that $L_A^{\#}$ exists and so Theorem 3.11 is applicable. Let us show that L_A is a $P_{\#}$-operator. If $X \in R(L_A)$, then $x_{22} = 0$. Also,

$$XL_A(X) = \begin{pmatrix} 2x_{11}^2+2x_{11}x_{12}+x_{12}^2 & x_{11}x_{12} \\ 2x_{11}x_{12}+2x_{12}^2 & x_{12}^2 \end{pmatrix}.$$

If, in addition, one has $XL_A(X) \leq 0$, then $x_{11} = x_{12} = 0$, so that $X = 0$. This shows that L_A has the $P_{\#}$-property.

Remark 3.15. *It is natural to ask if the pseudo Jordan w-P property and pseudo w-P property for the Lyapunov transformation are equivalent. We do not have an answer to this question.*

In the next result, we consider the Stein transformation and present an analogue of Theorem 2.8 for the new classes of operators. Furthermore, we are also able to relate these classes to $P_{\#}$-property. Again, the result bears a striking resemblance to Theorem 2.8 and is the second main result.

Theorem 3.16. *Let $A \in \mathbb{C}^{n\times n}$ such that $S_A^{\#}$ exists. Then the following statements are equivalent:*
(*i*) *A is generalized Schur stable.*
(*ii*) *S_A has the pseudo Jordan w-P property.*
(*iii*) *S_A has the pseudo w-P property.*
(*iv*) *S_A has the $P_{\#}$-property.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let $\rho(A) \leq 1$. Then, from Theorem 3.1 in [16], it follows that S_A has Jordan *w-P* property, and hence pseudo Jordan *w-P* property.
(*ii*) $\Rightarrow$ (*iii*): Obvious.
(*iii*) $\Rightarrow$ (*iv*): This follows by Remark 3.10.
(*iv*) $\Rightarrow$ (*i*): Let S_A have $P_{\#}$-property. If $|\lambda| = 1$ for all eigenvalues λ of A, then there is nothing to prove. Suppose $|\lambda| \neq 1$ for some eigenvalue λ. Then $Au = \lambda u$ for some non-zero vector u. Set $X = uu^* \neq 0$. We then have

$$S_A(X) = X - A(uu^*)A^* = X - (Au)(Au)^* = X - |\lambda|^2 X = (1-|\lambda|^2)X.$$

Since $|\lambda| \neq 1$, it follows that $X \in R(S_A)$. Since S_A has $P_{\#}$-property, from Lemma 3.8 we then have $1 - |\lambda|^2 > 0$, and hence $\rho(A) \leq 1$. □

Next, we give an example to show that the condition that $S_A^{\#}$ exists is indespensible in Theorem 3.16.

Example 3.17. *Let* $A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. *Then, A is generalized Schur stable. If* $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, *then* $X \in N(S_A)$. *Also, if* $Y = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$, *then one may verify that* $S_A(Y) = X$. *Thus* $N(S_A) \cap R(S_A) \neq \{0\}$, *which shows that* $S_A^{\#}$ *does not exist. Now,* $XS_A(X) = S_A(X)X = 0$ *but* $X \neq 0$. *Thus* S_A *does not have* $P_{\#}$ *property.*

We conclude the last part of this section with a general result for a class of operators for which the notions of $P_{\#}$-property and pseudo SSM-property coincide, obsering that the general problem is open. We need the following lemma to prove our next result. For the sake of completeness and ready reference, we give a proof.

Lemma 3.18. *Let* $L : V \to V$ *be a self-adjoint linear transformation. If L is positive semidefinite and* $\langle X, L(X)\rangle = 0$, *then* $L(X) = 0$.

Proof. *Given* $L : V \to V$ *is self-adjoint. There exists an orthonormal basis* $\{u^1, u^2, ..., u^n\}$ *for V consisting of eigenvectors of L. Let* $L(u^i) = \lambda_i u^i$. *Since L is positive semidefinite,* $\lambda_i \geq 0$ *for all i. Let*

$$X = \sum_{i=1}^{n} \alpha_i u^i.$$

Then

$$L(X) = \sum_{i=1}^{n} \alpha_i L(u^i) = \sum_{i=1}^{n} \alpha_i \lambda_i u^i.$$

Now,

$$\begin{aligned} 0 &= \langle X, L(X)\rangle \\ &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \|u^i\|^2 \\ &= \sum_{i=1}^{n} \alpha^2 \lambda_i. \end{aligned}$$

Then $\alpha_i^2 \lambda_i = 0$ *and hence,* $\alpha_i \lambda_i = 0$ *for each i. Thus,* $L(X) = 0$. □

Theorem 3.19. *Let* $L : V \to V$ *be a self-adjoint Z-transformation. Then L has* $P_{\#}$-*property if and only if L has pseudo SSM-property.*

Proof. Neccessity follows from the definition of $P_{\#}$-property. We prove the sufficiency part. Since L is self-adjoint, all the eigenvalues of L are real. Since L is a Z-transformation and has pseudo SSM-property, from Lemma 3.5, it follows that all eigenvalues of L are nonnegative. Therefore, L is a positive semidefinite operator. That is for all $X \in V$,

$$\langle X, L(X)\rangle \geq 0. \tag{1}$$

Let $X \in R(L)$ such that X and $L(X)$ commute and $XL(X) \leq 0$. We claim that $X = 0$. Now

$$\langle X, L(X)\rangle = tr(XL(X)) \leq 0. \tag{2}$$

From (1) and (2), it follows that $\langle X, L(X)\rangle = 0$. By Lemma 3.18, we then have $L(X) = 0$. Since L is self-adjoint, $L^{\#}$ exists. Therefore, $X = 0$. □

4. $R_{\#}$-PROPERTY

In this section, our main concern will be the recently introduced notion of $R_{\#}$-property [13]. First, we prove a very general result characterizing when a linear operator has $R_{\#}$-property, in Theorem 4.1. This will be followed by results on the multiplicative transformation, notably Theorem 4.7. In the rest of the section, we present improvements to the corresponding results obtained in [13]. These are given in Theorem 4.11 and Theorem 4.13.

Theorem 4.1. *Let $L : S^n \to S^n$ be linear. Then L has the $R_{\#}$-property if and only if $SOL(L, Q) \cap R(L)$ is a (possibly empty) bounded set, for all $Q \in S^n$.*

Proof. Assume that L has the $R_{\#}$-property. Suppose for some $Q \in S^n$, $SOL(L, Q) \cap R(L)$ is unbounded. Then there exists a sequence (X_n) in $SOL(L, Q) \cap R(L)$ such that $X_n \neq 0$ and $\|X_n\| \to \infty$. Let $Y_n = \frac{X_n}{\|X_n\|}$. Then (Y_n) has a convergent subsequence. Without loss of generality, we assume that (Y_n) converges to $Y \in R(L)$. Since $X_n \in SOL(L, Q)$ and S^n_+ is a closed convex cone, the sequence $L(\frac{X_n}{\|X_n\|}) + \frac{Q}{\|X_n\|}$ converges to $L(Y) \in S^n_+$ as $n \to \infty$. Also, we have

$$0 = \lim_{n\to\infty} \langle \frac{X_n}{\|X_n\|}, L(\frac{X_n}{\|X_n\|}) + \frac{Q}{\|X_n\|}\rangle = \langle Y, L(Y)\rangle.$$

Thus $0 \neq Y \in SOL(L, 0)$. This is a contradiction as L has the $R_{\#}$- property. Conversely, if $0 \neq X \in SOL(L, 0) \cap R(L)$ then $\lambda X \in SOL(L, 0) \cap R(L)$ for all $\lambda \geq 0$. Since $0 \in SOL(L, 0)$, hence L has $R_{\#}$-property. □

Definition 4.2. *We say that a linear transformation $L : S^n \to S^n$ is strongly monotone on $R(L)$ (strictly copositive on $S^n_+ \cap R(L)$) if $\langle L(X), X\rangle > 0$ for all nonzero $X \in R(L)$ (respectively, $X \in S^n_+ \cap R(L)$).*

Sivakumar [13] proved that the multiplicative transformation M_A is an $R_{\#}$-operator if and only if A is positive definite on $R(A)$ (i.e., $x^T Ax > 0$ for all $0 \neq x \in R(A)$) or negative definite on $R(A)$ (i.e., $-A$ is positive definite on $R(A)$). We now show that M_A is an $R_{\#}$-operator if and only if M_A is strictly copositive on $S^n_+ \cap R(M_A)$. Further, when A is symmetric, we show that they are equivalent to strong monotonicity on $R(M_A)$. We shall be using a preliminary result in this process, which however is interesting in its own right. This characterizes the satisfaction of the $R_{\#}$-property of a linear operator that leaves the cone of positive semidefinite matrices invariant.

Lemma 4.3. *Let $L : S^n S^n$ be linear with $L(S^n_+) \subseteq S^n_+$. Then L is an $R_\#$-operator if and only if L is strictly copositive on $S^n_+ \cap R(L)$.*

Proof. Let L be an $R_\#$-operator. Let $0 \neq X \in S^n_+ \cap R(L)$. Suppose $\langle X, L(X)\rangle \leq 0$. Since $L(S^n_+) \subseteq S^n_+$, $L(X) \in S^n_+$. As S^n_+ is self-dual, we have $\langle X, L(X)\rangle \geq 0$. This implies that $\langle X, L(X)\rangle = 0$, which is a contradiction to L being an $R_\#$-operator. Thus $\langle X, L(X)\rangle > 0$.

For the converse, if $0 \neq X \in S^n_+ \cap R(L)$ then $\langle X, L(X)\rangle > 0$. Note that we have not used the hypothesis $L(S^n_+) \subseteq S^n_+$. $\square$

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$ be given. Then the following statements are equivalent:*
(i) M_A is an $R_\#$-operator.
(ii) M_A is strictly copositive on $S^n_+ \cap R(M_A)$.
(iii) A is either positive definite or negative definite on $R(A)$.

Proof. Since $M_A(S^n_+) \subseteq S^n_+$, the equivalence of (*i*) and (*ii*) follows from Lemma 4.3. The equivalence of (*i*) and (*iii*) is proved in [13]. We now present another proof for the implication (*ii*) $\Rightarrow$ (*iii*).

(*ii*) $\Rightarrow$ (*iii*): Suppose that A is neither positive definite nor negative definite on $R(A)$. Then there exists a nonzero $z \in R(A)$ such that $z^TAz = 0$ (see Theorem 4.16 in [13]). Since $z \in R(A)$, we have $z = Aw$. Let $B = zz^T$ and $C = ww^T$. Since $z \neq 0$, we have $0 \neq B \geq 0$. Now $M_A(C) = Aww^TA^T = zz^T = B$. Thus $B \in S^n_+ \cap R(M_A)$. Since M_A is strictly copositive on $S^n_+ \cap R(M_A)$, $\langle B, M_A(B)\rangle = tr(BM_A(B)) > 0$. On the other hand, $BM_A(B) = BABA^T = zz^TAzz^TA^T = 0$. This implies that $\langle B, M_A(B)\rangle = 0$, which is a contradiction. Hence, A is either positive definite or negative definite on $R(A)$. $\square$

Let us illustrate Theorem 4.4 by means of an example.

Example 4.5. *Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in S^2$ be arbitrary. Then*

$$M_A(X) = \begin{pmatrix} x_{11} + 2x_{12} + x_{22} & 0 \\ 0 & 0 \end{pmatrix}$$

and M_A is an $R_\#$-operator [13, Example 4.12]. If $Y \in R(M_A)$, then Y has the form

$$Y = \begin{pmatrix} y_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $Y \in S^2_+$ is nonzero, we then have $y_{11} > 0$ and so $\langle Y, M_A(Y)\rangle = y^2_{11} > 0$. This shows that M_A is strictly copositive on $S^2_+ \cap R(M_A)$. Let us also note that if $0 \neq y \in R(A)$, then $Ay = (y_1, 0)^T$ with $y_1 \neq 0$. Thus one has $\langle y, Ay\rangle = y_1^2 > 0$, so that A is positive definite on $R(A)$.

When A is symmetric, we prove a stronger result for M_A. We recall a result from [13].

Theorem 4.6. *[13, Theorem 4.16] Let $A \in \mathbb{R}^{n\times n}$ be given. Then the following statements are equivalent:*
(i) X is symmetric, $XAX = 0$, $R(X) \subseteq R(A) \Rightarrow X = 0$.
ii) A is either positive definite or negative definite on $R(A)$.

Theorem 4.7. *Let A be a symmetric matrix. Then the following are equivalent for the multiplicative transformation M_A:*
(i) M_A is strongly monotone on $R(M_A)$.
(ii) M_A is strictly copositive on $S^n_+ \cap R(M_A)$.
(iii) M_A is an $R_\#$-operator.
(iv) A is positive definite or negative definite on $R(A)$.

Proof. It is enough to prove the implication (iv) $\Rightarrow$ (i). Since $M_A = M_{-A}$, without loss of generality we assume that A is positive definite on $R(A)$. Let $0 \neq X \in R(M_A)$. Then $X = AYA$ for some $Y \in S^n$. It follows that $R(X) \subseteq R(A)$. Suppose $\langle X, M_A(X)\rangle = \langle X, AXA\rangle = tr(AXAX) \leq 0$. Let $x \in \mathbb{R}^n$. Then $x = y + z$, where $y \in R(A)$ and $z \in R(A)^\perp$. Since A is symmetric and $N(A^T) = R(A)^\perp$, we have $z \in N(A)$. Now $x^T Ax = x^T Ay = y^T Ay \geq 0$. Thus A is positive semidefinite and so is the matrix XAX. Since S^n_+ is self-dual, $\langle A, XAX\rangle = tr(AXAX) \geq 0$. Hence, $tr(AXAX) = 0$. This implies that $AXAX = 0$. Let $u \in \mathbb{R}^n$. Consider $v = XAu \in R(X) \subseteq R(A)$. Then $v^T Av = u^T AXAXAu = 0$. Since A is positive definite on $R(A)$, $XAu = 0$ for all $u \in \mathbb{R}^n$. Thus $XA = 0$ and hence, $XAX = 0$. Then by Theorem 4.6, we have $X = 0$, which is a contradiction. Hence, $\langle X, M_A(X)\rangle > 0$ and therefore, M_A is strongly monotone on $R(M_A)$. □

Remark 4.8. *For a symmetric matrix A, Parthasarathy et al. (Theorem 6, [9]) proved that A is positive definite or negative definite if and only if M_A is strongly monotone. Therefore, the above theorem generalizes their result.*

Next, we present an example.

Example 4.9. *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. If $y \in R(A)$, then $y_1 = y_2$ and so if $y \neq 0$, then one has*

$$\langle y, Ay\rangle = 4y_1^2 > 0,$$

showing that A is positive definite on $R(A)$. By Theorem 4.4, it follows that M_A is an $R_\#$-operator.
We have, for $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in S^2$,

$$M_A(X) = \alpha \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where α is the sum of all the entries of X. Let $0 \neq Y \in R(M_A)$. Then Y has the form

$$Y = \beta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

for some $0 \neq \beta \in \mathbb{R}$. *We then have*

$$M_A(Y) = 4\beta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that

$$\langle Y, M_A(Y) \rangle = tr(YM_A(Y)) = 16\beta^2 > 0,$$

showing that M_A *is strongly monotone on* $R(M_A)$.

The following example shows that the implication $(iv) \Rightarrow (i)$ in Theorem 4.7 does not hold if A is not symmetric.

Example 4.10. *Let* $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* A *is positive definite on* $R(A) = \{(x, y, 0)^T : x, y \in \mathbb{R}\}$. *Let* $X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AYA^T \in R(M_A)$ *where* $Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $M_A(X) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $\langle X, M_A(X) \rangle = tr(XM_A(X)) = 0$. *Thus,* M_A *is not strongly monotone on* $R(M_A)$.

In this last part, we present sufficient conditions for certain classes of Lyapunov and Stein operators to be $R_\#$-operators. These extend the results of [13]. Let $B \in \mathbb{R}^{n\times n}$ and consider $A = \begin{pmatrix} B & 0 \\ 0^T & d \end{pmatrix}$, where 0 is the column vector in $\mathbb{R}^n$ and $d \in \mathbb{R}$. Sivakumar [13] proved that if B is positive definite and $d = 0$, then L_A is an $R_\#$-operator. Generalizing this result, we show that if B is positive stable and $d \geq 0$, then L_A is an $R_\#$-operator. We recall that if B is positive stable, then $X = 0$ is the only matrix satisfying the conditions $X \geq 0$, $L_B(X) \geq 0$ and $XL_B(X) = 0$ (Theorem 5, [3]).

Theorem 4.11. *Let* $B \in \mathbb{R}^{n\times n}$ *be positive stable. Let* $A = \begin{pmatrix} B & 0 \\ 0^T & d \end{pmatrix}$ *where* $d \geq 0$. *Then,* L_A *is an* $R_\#$*-operator.*

Proof. Case (i): $d > 0$. It is easy to see that $\sigma(A) \subseteq \sigma(B) \cup \{d\}$, where $\sigma(A)$ is the set of all eigenvalues of A. Since B is positive stable and $d > 0$, A is positive stable. As a result mentioned before the theorem, we have L_A is an $R_\#$-operator.
Case (ii): $d = 0$. Let $Y \in R(L_A)$, $Y \geq 0$, $L_A(Y) \geq 0$ and $YL_A(Y) = 0$. We claim that $Y = 0$. Let $Y = \begin{pmatrix} X & x \\ x^T & z \end{pmatrix} \in S^{n+1}$ where $X = (x_{ij}) \in S^n$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Then $L_A(Y) = \begin{pmatrix} BX + XB^T & Bx \\ x^TB^T & 0 \end{pmatrix}$. Since $Y \in R(L_A)$, we have $z = 0$. Note that the positive semidefiniteness of Y implies that determinant of all

principal submatrices of Y are non-negative. Consider the principal submatrix $\begin{pmatrix} x_{ii} & x_i \\ x_i & 0 \end{pmatrix}$. This implies that $x_i = 0$. Thus, $Y = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ and $0 = YL_A(Y) = \begin{pmatrix} X(BX + XB^T) & 0 \\ 0 & 0 \end{pmatrix}$. Since $Y \geq 0$ ($L_A(Y) \geq 0$), we have $X \geq 0$ (respectively, $BX + XB^T \geq 0$). Therefore, from the result mentioned before the theorem, we have $X = 0$. Hence, L_A is an $R_\#$-operator. □

It was shown in [13] that the Stein transformation $S_A : S^n S^n$ is an $R_\#$-operator if $A = \begin{pmatrix} B & 0 \\ 0^T & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$, where $B \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\rho(B) < 1$. Next, we extend this result. We collect a result from Gowda and Parthasarathy [4], which will be useful.

Theorem 4.12. *[4, Theorem 11] Let $B \in \mathbb{R}^{k\times k}$. If $\rho(B) < 1$, then $X = 0$ is the only matrix that satisfies the conditions $X \geq 0$, $S_B(X) \geq 0$ and $XS_B(X) = 0$.*

Theorem 4.13. *Let $B \in \mathbb{R}^{n\times n}$ be such that $\rho(B) < 1$. Let $A = \begin{pmatrix} B & 0 \\ 0^T & d \end{pmatrix}$ where $d \in \mathbb{R}$ and $-1 \leq d \leq 1$. Then S_A is an $R_\#$-operator.*

Proof. Case(i): $-1 < d < 1$. Then $\sigma(A) \subseteq \sigma(B) \cup \{d\}$. Since $\rho(B) < 1$ and $|d| < 1$, $\rho(A) < 1$. From Theorem 4.12, we have S_A is an $R_\#$-operator.

Case(ii): $d = 1$ or $d = -1$. Let $Y = \begin{pmatrix} X & x \\ x^T & z \end{pmatrix} \in S^{n+1}$ where $X \in S^n$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Since $d = \pm 1$, we have $S_A(Y) = \begin{pmatrix} X - BXB^T & x - dBx \\ x^T - dx^TB^T & 0 \end{pmatrix}$. Similar to the arguments of case (ii) in Theorem 4.11 and from Theorem 4.12, it follows that S_A is an $R_\#$-operator. □

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