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ON RELATION BETWEEN ONE MULTIPLE AND A CORRESPONDING ONE-DIMENSIONAL INTEGRAL WITH APPLICATIONS

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Abstract: For a given finite positive measure on an interval $I \subseteq \mathbb{R}$, a multiple stochastic integral of a Volterra kernel with respect to a product of a corresponding Gaussian orthogonal stochastic measure is introduced. The Volterra kernel is taken such that the multiple stochastic integral is a multiple iterated stochastic integral related to a parameterized Hermite polynomial, where parameter depends on Gaussian distribution of an underlying one-dimensional stochastic integral. Considering that there exists a connection between stochastic and deterministic integrals, we expose some properties of parameterized Hermite polynomials of Gaussian random variable in order to prove that one multiple integral can be expressed by a corresponding one-dimensional integral. Having in mind the obtained result, we show that a system of multiple integrals, as well as a collection of conditional expectations can be calculated exactly by generalized Gaussian quadrature rule.

Keywords: Multiple Stochastic Integral, Multiple Integral, Gaussian Quadrature Rule.

MSC: 60H05, 60H30, 65R10, 65D32.

1. INTRODUCTION

In many practical problems, for example in dynamic programming and stochastic optimization, we are often faced with computing an expected value of some function of an absolutely continuous random variable (see, e.g. [1], 506-511, 547). If the random variable is scalar, classical methods based on quadrature formulae work quite well, but when the expectation is taken with respect to a random vector and we must integrate over a high-dimensional space, then the quadrature formulae usually become impractical ([1], 209, [9], 146). Therefore, the representation of a multiple integral by an one-dimensional integral could be useful.

On the other hand, it is well known that there exists a connection between stochastic integrals and corresponding deterministic integrals*. Moreover, Ito's generalization of *multiple Wiener integral* claims that a multiple stochastic integral can be expressed by Hermite polynomial of a suitable one-dimensional stochastic integral ([5], 162-166, [10], 51-54). Keeping that in mind, for a given finite positive measure on an interval of real line, we introduce a multiple stochastic integral of a Volterra kernel with respect to a product of a corresponding Gaussian orthogonal stochastic measure. The Volterra kernel is taken such that the multiple stochastic integral is a multiple iterated stochastic integral related to a parameterized Hermite polynomial, where parameter depends on Gaussian probability distribution of the underlying one-dimensional stochastic integral. Considering this fact, we expose some properties of parameterized Hermite polynomials of Gaussian random variable to prove that, for a given finite and positive measure, one multiple integral can be represented by a corresponding one-dimensional integral.

The remained of this paper is organized as follows: an one-dimensional stochastic integral with respect to a Gaussian orthogonal stochastic measure, along with one corresponding generalization of multiple Wiener integral are introduced in *Section 2*. *Section 3* considers Hermite polynomials of the one-dimensional stochastic integral with a suitable parameter, as well as their relation with the defined multiple stochastic integrals and from there, with the corresponding multiple deterministic integrals. *Section 4* is dedicated to a consideration of parameterized Hermite polynomials as random variables, and to the proof of the main theorem that one multiple integral can be expressed by a corresponding one-dimensional integral. The applications of the main theorem, in the case of numerical integration of multiple integrals based on Gaussian quadrature formulae, are presented in *Section 5*. An overview of the research results is given in the concluding section, *Section 6*.

2. ONE GENERALIZATION OF MULTIPLE WIENER INTEGRAL

In what follows, for an arbitrary point $t_0 \in \mathbb{R} = (-\infty, +\infty)$, an interval denoted by $I \subseteq \mathbb{R}$ represents a finite interval $[t_0, T] \subset \mathbb{R}$, $t_0 < T < +\infty$, or a half-open interval $[t_0, T) \subseteq \mathbb{R}$, $t_0 < T \leq +\infty$, if not stated precisely.

Denote by $\mathcal{B}(I)$ the Borel sigma-field of $I \subseteq \mathbb{R}$ and let $\mu = \mu(t)$, $t \in I$, be a left-continuous monotone increasing function such that

$$\mu(\Delta) = \mu(t) - \mu(s),\ \Delta = [s, t) \in \mathcal{B}(I), \tag{1}$$

defines a finite ($\mu(I) < +\infty$) positive measure on $\mathcal{B}(I)$, ($\mu(B) \geq 0, B \in \mathcal{B}(I)$), absolutely continuous with respect to the Lebesgue measure on $\mathcal{B}(I)$, that is $d\mu(u) = \varphi(u)du$, $u \in I$, where $\varphi = \varphi(u)$ is a corresponding non-negative continuous

*In what follows, the term *integral* means exclusively *deterministic integral*, while the notion *stochastic integral* takes its full connotation.

Radon-Nikodym derivative. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian random process $\eta = \{\eta(t), t \in I\}$,

$$\eta(t_0) = 0, \text{ a.c. and } E\eta(t) = 0, \; E\left[\eta(t)\right]^2 = \mu(t) < +\infty, \; t \in I, \; t \neq t_0,$$

corresponding to a left-continuous function in the Hilbert space $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ with orthogonal increments

$$\eta(\Delta) = \eta(t) - \eta(s) \tag{2}$$

on disjoint intervals $\Delta = [s, t) \in \mathcal{B}(I)$, such that

$$E\eta(\Delta) = 0, \quad E\left[\eta(\Delta)\right]^2 = \mu(\Delta) < +\infty, \quad \Delta \in \mathcal{B}(I).$$

Consequently, the increments (2) of the Gaussian random process η, as the centered Gaussian random variables from the space $\mathcal{L}^2(\Omega)$, satisfy

$$\langle \eta(\Delta_1), \eta(\Delta_2) \rangle_{\mathcal{L}_2(\Omega)} = 0 \text{ when } \Delta_1 \cap \Delta_2 = \emptyset, \; \Delta_1, \Delta_2 \in \mathcal{B}(I) \tag{i}$$

$$\eta(\Delta) = \sum_{k=1}^{+\infty} \eta(\Delta_k), \text{ a.c. when } \Delta = \bigsqcup_{k=1}^{+\infty} \Delta_k \in \mathcal{B}(I), \; \Delta_i \cap \Delta_j = \emptyset \text{ for } i \neq j \tag{ii}$$

$$\left\| \eta(\Delta) \right\|^2_{\mathcal{L}_2(\Omega)} = \mu(\Delta) < +\infty \text{ for all } \Delta \in \mathcal{B}(I), \tag{iii}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{L}^2(\Omega)}$ and $\|\cdot\|_{\mathcal{L}^2(\Omega)}$ denote the inner product and the norm in the space $\mathcal{L}^2(\Omega)$, respectively. Hence, for the given finite positive measure (1), the equality (2) defines a Gaussian orthogonal stochastic measure, $\eta(\Delta)$, $\Delta \in \mathcal{B}(I)$, and with respect to this measure, one can define an one-dimensional stochastic integral

$$\xi^I_{(1)}(f) = \int_{t_0}^{T} f(u) d\eta(u), \tag{3}$$

for any function $f \in \mathcal{L}^2(I, \mu)$ ([6], 36-39, [11], 212-218).

On the other hand, the one-dimensional stochastic integrals (3) generate a system $\mathcal{H}_{(1)} = \left\{ \xi^I_{(1)}(f), \; f \in \mathcal{L}^2(I, \mu) \right\}$ of Gaussian random variables with zero mean, variance

$$E\left[\xi^I_{(1)}(f)\right]^2 = \left\|f\right\|^2_{\mathcal{L}^2(I,\mu)} = \int_{t_0}^{T} f^2(u) d\mu(u) = \mathcal{J}^I_{(1)}(f^2),$$

and covariance

$$E\left[\xi^I_{(1)}(f) \xi^I_{(1)}(g)\right] = \langle f, g \rangle_{\mathcal{L}^2(I,\mu)} = \int_{t_0}^{T} f(u) g(u) d\mu(u) = \mathcal{J}^I_{(1)}(fg),$$

where $f, g \in \mathcal{L}^2(I, \mu)$ and μ is the corresponding measure defined by (1). The symbols $\|\cdot\|_{\mathcal{L}_2(I,\mu)}$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}_2(I,\mu)}$ denote the norm and the inner product in $\mathcal{L}^2(I, \mu)$, respectively. In particular, if $f = 0$ a.e. on $\mathcal{B}(I)$, then $\xi^I_{(1)}(f)$ is a degenerate Gaussian random variable. Hence, if we denote by $\overline{\mathcal{L}}_{hull}(\eta)$ the closed linear hull spanned by $\eta(\Delta)$, $\Delta \in \mathcal{B}(I)$, then the Gaussian system $\mathcal{H}_{(1)}$ is equal to $\overline{\mathcal{L}}_{hull}(\eta)$. The integrals $\mathcal{J}^I_{(1)}(f^2)$ and $\mathcal{J}^I_{(1)}(fg)$, on the right-hand side of above equalities, are defined as the Lebesgue integrals with respect to the corresponding measure μ.

In order to introduce a multiple iterated stochastic integral, as one generalization of multiple Wiener integral, recall that the first generalization of multiple Wiener integral was given by K. Ito ([5]). Instead of defining multiple stochastic integral with respect to a product of stochastic measure given by increments of the Brownian motion process, K. Ito defined it with respect to a product of Gaussian orthogonal stochastic measure ([5], 157-162).

In contrast to K. Ito, who introduced multiple stochastic integral of a symmetric $\mathcal{L}^2$-kernel, we take an m-tuple Volterra kernel $k_{(m)} = k_{(m)}(t_1, ..., t_m)$, $m \geq 1$, from the space

$$\mathcal{L}^2(I^m, \mu^m) = \left\{ k_{(m)};\int_{t_0}^{T}\int_{t_0}^{t_1}...\int_{t_0}^{t_{m-1}} k^2_{(m)}(t_1, ..., t_m)d\mu(t_m)\cdots d\mu(t_2)d\mu(t_1) < +\infty \right\},$$

that is

$$k_{(m)} \in \mathcal{L}^2(I^m, \mu^m) \text{ and } k_{(m)}(t_1, ..., t_m) = 0, \text{ if } t_i < t_j \text{ for some } i < j, \tag{4}$$

where μ is the finite positive measure (1) ([4], 149), and define an m-tuple stochastic integral

$$\xi^I_{(m)}(k_{(m)}) = \int_I\int_I ... \int_I k_{(m)}(t_1, ..., t_m)d\eta(t_m)\cdots d\eta(t_2)d\eta(t_1), \tag{5}$$

as K. Ito did it, where η is the Gaussian orthogonal stochastic measure (2).

Further, notice that for $m \geq 1$ and for any $f \in \mathcal{L}^2(I, \mu)$, one can define a Volterra kernel $k_{(m)} \in \mathcal{L}^2(I^m, \mu^m)$ by

$$k_{(m)}(t_1, ..., t_m) = \begin{cases} f(t_m)\cdots f(t_1), & t_m \leq ... \leq t_2 \leq t_1 \\ 0, & otherwise \end{cases}. \tag{6}$$

Then, taking into account the area of integration, the multiple stochastic integral (5) takes the following form

$$\xi^I_{(m)}(f) = \int_{t_0}^{T}\int_{t_0}^{t_1}...\int_{t_0}^{t_{m-1}} f(t_m)\cdots f(t_2)f(t_1)d\eta(t_m)\cdots d\eta(t_2)d\eta(t_1). \tag{7}$$

Bearing in mind the properties (i)-(iii) of the Gaussian orthogonal stochastic measure (2), the m-times iterated stochastic integral (7) is a centered random variable $\xi^I_{(m)}(f)$ with variance

$$\begin{aligned} E\left[\xi^I_{(m)}(f)\right]^2 &= \int_{t_0}^{T}\int_{t_0}^{t_1}...\int_{t_0}^{t_{m-1}} f^2(t_m)\cdots f^2(t_2)f^2(t_1)d\mu(t_m)\cdots d\mu(t_2)d\mu(t_1) \\ &= \mathcal{J}^I_{(m)}(f^2), \end{aligned}$$

where the corresponding m-times iterated integral $\mathcal{J}^I_{(m)}(f^2)$ is defined as a Lebesgue integral with respect to the product measure

$$\mu^m(\Delta) = \mu(\Delta_1) \times \mu(\Delta_2) \times ... \times \mu(\Delta_m)$$

on Borel sets $\Delta = \Delta_1 \times \Delta_2 \times ... \times \Delta_m$ of I^m such that $\Delta_i \in \mathcal{B}(I)$, $i = 1, ..., m$.

Since the integrals $\mathcal{J}^I_{(1)}(f^2)$ and $\mathcal{J}^I_{(m)}(f^2)$ represent the variance of the random variables $\xi^I_{(1)}(f)$ and $\xi^I_{(m)}(f)$, respectively, a relation between the m-tuple integral $\mathcal{J}^I_{(m)}(f^2)$ and the one-dimensional $\mathcal{J}^I_{(1)}(f^2)$ will follow from a suitable connection between the corresponding stochastic integrals (7) and (3). There Hermite polynomials with a suitable parameter will play an important role.

3. HERMITE POLYNOMIALS WITH PARAMETER AND MULTIPLE STOCHASTIC INTEGRALS

Let P_{σ^2} be a Gaussian measure on Borel sigma-field of $\mathbb{R}$ with zero mean and variance $\sigma^2 \geq 0$. If $\sigma^2 = 0$, P_{σ^2} is a degenerate Gaussian measure on $\mathbb{R}$, concentrated in zero.

Recall that for fixed $\sigma^2 \geq 0$, a system of Hermite polynomials with parameter $\sigma^2 \geq 0$,

$$h_m(\sigma^2, x) = (\sigma^2)^m \frac{(-1)^m}{m!} e^{\frac{x^2}{2\sigma^2}} \frac{d^m e^{-\frac{x^2}{2\sigma^2}}}{dx^m}, \ x \in \mathbb{R}, m = 0, 1, 2, ..., \tag{8}$$

is complete and orthogonal in Hilbert space $\mathcal{L}^2(\mathbb{R}, P_{\sigma^2})$ ([4], 152, Definition A1, Proposition A2(iii)). Moreover, a system

$$\left\{h_{m_1}(\sigma_1^2, x_1), h_{m_2}(\sigma_2^2, x_2), ..., h_{m_d}(\sigma_d^2, x_d); m_1, m_2, ..., m_d = 0, 1, 2, ...\right\}$$

is a complete orthogonal system in Hilbert space $\mathcal{L}^2(\mathbb{R}^d, P_{(\sigma_1^2, \sigma_2^2, ..., \sigma_d^2)})$, where $P_{(\sigma_1^2, \sigma_2^2, ..., \sigma_d^2)}$ is a Gaussian measure on Borel sigma-field of $\mathbb{R}^d$ with zero mean and covariance matrix $\left[\sigma_i^2 \delta_{ij}\right]$ ([4], 152-153, Proposition A2(iii')).

Since the one-dimensional stochastic integral (3), as a Gaussian random variable $\xi^I_{(1)}(f)$, $f \in \mathcal{L}^2(I, \mu)$, has Gaussian distribution $P_{\mathcal{J}^I_{(1)}(f^2)} = \mathcal{N}(0, \mathcal{J}^I_{(1)}(f^2))$ on $\mathbb{R}$, in the next lemma, we consider a Hermite polynomial $h_m(\mathcal{J}^I_{(1)}(f^2), x) \in \mathcal{L}^2(\mathbb{R}, P_{\mathcal{J}^I_{(1)}(f^2)})$ with parameter $\mathcal{J}^I_{(1)}(f^2)$.

Lemma 3.1. *For each fixed $m \geq 0$ and for any $f \in \mathcal{L}^2(I, \mu)$, it holds*

$$E\left[h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f))\right]^2 = \mathcal{J}^I_{(m)}(f^2). \tag{9}$$

Proof. Having in mind that $E\left[\xi^I_{(m)}(f)\right]^2 = \mathcal{J}^I_{(m)}(f^2)$, $f \in \mathcal{L}^2(I, \mu)$, in order to prove the equality (9), one needs to show that $h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f))$ can be expressed as a multiple stochastic integral (7). Therefore, for fixed $m \geq 0$, denote by

$$\overline{\mathcal{L}}_{hull}(h_m, \xi^I_{(1)}, \mathcal{L}^2(I, \mu)) = \overline{\mathcal{L}}_{hull}\left\{h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f)), f \in \mathcal{L}^2(I, \mu)\right\}$$

the closed linear hull spanned by parameterized Hermite polynomials of degree m of the Gaussian variables $\xi^I_{(1)}(f)$, $f \in \mathcal{L}^2(I,\mu)$, and by

$$\overline{\mathcal{L}}_{hull}(h_m,\eta,\mathcal{B}(I)) = \overline{\mathcal{L}}_{hull}\left\{h_m(\mu(\Delta),\eta(\Delta)),\Delta \in \mathcal{B}(I)\right\}$$

the closed linear hull spanned by parameterized Hermite polynomials of degree m of the Gaussian variables $\eta(\Delta)$, $\Delta \in \mathcal{B}(I)$. Taking into account that the Gaussian system $\mathcal{H}^I_{(1)} = \left\{\xi^I_{(1)}(f),\ f \in \mathcal{L}^2(I,\mu)\right\}$ is equal to the closed linear hull $\overline{\mathcal{L}}_{hull}(\eta^I)$ spanned by $\eta(\Delta)$, $\Delta \in \mathcal{B}(I)$, one can prove that for any fixed $m \geq 0$, $\overline{\mathcal{L}}_{hull}(h_m, \xi^I_{(1)}, \mathcal{L}^2(I,\mu))$ coincides with the $\overline{\mathcal{L}}_{hull}(h_m,\eta,\mathcal{B}(I))$ ([10], 51). Thus, in order to show that

$$h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f)) = \xi^I_{(m)}(f),$$

it is sufficient to prove that for any $m \geq 0$ and for any fixed $\Delta = [s,t) \in \mathcal{B}(I)$ $(s < t)$, it holds

$$h_m(\mu(\Delta),\eta(\Delta)) = \int_s^t \int_s^{t_1} \dots \int_s^{t_{m-1}} d\eta(t_m)\cdots d\eta(t_2)d\eta(t_1). \tag{10}$$

The proof is going to be given using mathematical induction.

Obviously, for $k = 0$ and $k = 1$, we get

$$h_0(\mu(\Delta),\eta(\Delta)) = 1 \quad \text{and} \quad h_1(\mu(\Delta),\eta(\Delta)) = \eta(\Delta) = \int_s^t d\eta(u).$$

Assume that for $k < m$, it holds

$$h_k(\mu(\Delta),\eta(\Delta)) = \int_s^t \int_s^{t_1} \dots \int_s^{t_{k-1}} d\eta(t_k)\cdots d\eta(t_2)d\eta(t_1). \tag{11}$$

For a sufficiently large integer $M > 0$, let Γ be a division of $[s,t] : s = \tau_0 < \tau_1 < \dots < \tau_M = t$ such that $\Delta_{i_j} = \left[\tau_{i_j-1}, \tau_{i_j}\right)$, $j = 1,\dots,m$, are disjoint subintervals of $\Delta = [s,t) \subset [s,t]$. Then,

$$\sum\sum_{M \geq i_1 > i_2 > \dots > i_m \geq 1} \dots \sum \eta(\Delta_{i_1})\eta(\Delta_{i_2})\cdots\eta(\Delta_{i_m}) = \tag{12}$$

$$= \frac{1}{m}\sum_{k=1}^{M}\left\{\eta(\Delta_k)\left(\sum_{\substack{M \geq i_1 > i_2 > \dots > i_{m-1} \geq 1 \\ i_1, i_2, \dots, i_{m-1} \neq k}} \dots \sum \eta(\Delta_{i_1})\cdots\eta(\Delta_{i_{m-1}})\right)\right\}$$

$$= \frac{1}{m} \left\{ \sum_{k=1}^{M} \eta(\Delta_k) \left(\sum_{M \geq i_1 > i_2 > \ldots > i_{m-1} \geq 1} \cdots \sum \eta(\Delta_{i_1}) \cdots \eta(\Delta_{i_{m-1}}) \right) \right.$$

$$- \sum_{k=1}^{M} (\eta(\Delta_k))^2 \left(\sum_{M \geq i_1 > i_2 > \ldots > i_{m-2} \geq 1} \cdots \sum \eta(\Delta_{i_1}) \cdots \eta(\Delta_{i_{m-2}}) \right)$$

$$\left. + \sum_{k=1}^{M} (\eta(\Delta_k))^3 \left(\sum_{\substack{M \geq i_1 > i_2 > \ldots > i_{m-3} \geq 1 \\ i_1, i_2, \ldots, i_{m-3} \neq k}} \cdots \sum \eta(\Delta_{i_1})\eta \cdots \eta(\Delta_{i_{m-3}}) \right) \right\}$$

As the division Γ becomes finer, the left-hand side of (12) converges to

$$\int_s^t \int_s^{t_1} \ldots \int_s^{t_{m-1}} \eta(dt_m) \cdots \eta(dt_1) \tag{13}$$

in $\mathcal{L}_2(\Omega)$. The second and the third term of the right-hand side of (12) are the rectifications of $(\eta(\Delta_k))^\gamma$, $\gamma = 2, 3$, in the first term. Due to the properties (i)-(iii) of the Gaussian orthogonal stochastic measure (2), as a Gaussian random variable $\eta(\Delta) \in \mathcal{L}_2(\Omega)$, we obtain that for any $\Delta \in \mathcal{B}(I)$,

$$E\left[\eta(\Delta)\right]^{2k-1} = 0, \quad E\left[\eta(\Delta)\right]^{2k} = (2k-1)!!\,(\mu(\Delta))^k, \quad k = 1, 2, \ldots$$

and hence,

$$\sum_{k=1}^{M} (\eta(\Delta_k))^2 \to \mu(\Delta) = \mu(t) - \mu(s) \text{ and } \sum_{k=1}^{M} (\eta(\Delta_k))^3 \to 0$$

in $\mathcal{L}_2(\Omega)$, while

$$\sum_{k=1}^{M} \eta(\Delta_k) = \eta(\Delta) = \eta(t) - \eta(s).$$

Therefore, the right-hand side of (12) converges to

$$\begin{aligned} &\frac{\eta(t) - \eta(s)}{m} \int_s^t \int_s^{t_1} \ldots \int_s^{t_{m-2}} d\eta(t_{m-1}) \cdots d\eta(t_2) d\eta(t_1) - \\ &-\frac{\mu(t) - \mu(s)}{m} \int_s^t \int_s^{t_1} \ldots \int_s^{t_{m-3}} d\eta(t_{m-2}) \cdots d\eta(t_2) d\eta(t_1) \end{aligned} \tag{14}$$

According to the assumption (11), (14) is equal to

$$\frac{\eta(\Delta)}{m} h_{m-1}(\mu(\Delta), \eta(\Delta)) - \frac{\mu(\Delta)}{m} h_{m-2}(\mu(\Delta), \eta(\Delta)). \tag{15}$$

Since the parameterized Hermite polynomials (8) satisfy a recursion formula

$$h_m(\sigma^2, x) = \frac{x}{m} h_{m-1}(\sigma^2, x) - \frac{\sigma^2}{m} h_{m-2}(\sigma^2, x) \tag{16}$$

([4], 152), the term (15) is equal to

$$h_m(\mu(\Delta), \eta(\Delta)) \tag{17}$$

and thus, the equality (10) follows from (12), (13), and (17).

Consequently, for any fixed $m \geq 0$, the equality (10) proves that

$$h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f)) = \xi^I_{(m)}(f).$$

Thus,

$$E\left[h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f))\right]^j = E\left[\xi^I_{(m)}(f)\right]^j, \; j = 0, 1, 2.....$$

and bearing in mind that for $j = 2$, $E\left[\xi^I_{(m)}(f)\right]^2 = \mathcal{J}^I_{(m)}(f^2)$, we obtain the equality (9)■

4. RELATION BETWEEN ONE MULTIPLE AND THE CORRESPONDING ONE-DIMENSIONAL INTEGRAL

For a finite positive measure μ, defined by (1) on the Borel sigma-field of $I \subseteq \mathbb{R}$, we will consider the next multiple integral

$$\mathcal{J}^I_{(m)}(g) = \int_{t_0}^{T}\int_{t_0}^{t_1}...\int_{t_0}^{t_{m-1}} g(t_m)\cdots g(t_2)g(t_1)d\mu(t_m)\cdots d\mu(t_2)d\mu(t_1), \tag{18}$$

where $g : I \to \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, is non-negative continuous function on $I \subseteq \mathbb{R}$, and integrable with respect to the measure μ on I. In order to express the multiple integral (18) by a corresponding one-dimensional

$$\mathcal{J}^I_{(1)}(g) = \int_{t_0}^{T} g(u)d\mu(u), \tag{19}$$

we introduce the next lemma which considers the parameterized Hermite polynomials (8) of a Gaussian variable, as centered random variables with finite second moments. Since for any $\sigma^2 \geq 0$ and $x \in \mathbb{R}$, $h_0(\sigma^2, x) = 1$, we take into account only the parameterized Hermite polynomials of degree $m \geq 1$.

Lemma 4.1. *Let ξ be a real-valued random variable with Gaussian probability distribution $P_{\sigma^2} = \mathcal{N}(0, \sigma^2)$ on the Borel sigma-field of $\mathbb{R}$ and $h_m(\sigma^2, x) \in \mathcal{L}^2(\mathbb{R}, P_{\sigma^2})$ a Hermite polynomial with parameter $\sigma^2 \geq 0$. Then, for any $m \geq 1$, $h_m(\sigma^2, \xi)$ is a centered random variable with finite second moment such that*

$$E\left[h_m(\sigma^2, \xi)\right]^2 = \frac{\left(E\xi^2\right)^m}{m!}. \tag{20}$$

Proof. As $h_1(\sigma^2, \xi) = \xi$, for $m = 1$,

$$E\left[h_1(\sigma^2, \xi)\right] = E\xi = 0 \quad \text{and} \quad E\left[h_1(\sigma^2, \xi)\right]^2 = E\xi^2.$$

Further, since for any $m \geq 2$, the parameterized Hermite polynomials (8) satisfy the recursion formula (16), the next equality

$$E\left[h_m(\sigma^2, \xi)\right]^j = E\left[\frac{h_1(\sigma^2, \xi)}{m} h_{m-1}(\sigma^2, \xi) - \frac{\sigma^2}{m} h_{m-2}(\sigma^2, \xi)\right]^j, \; m \geq 2, \; j = 1, 2,$$

holds. Therefore, considering that parameterized Hermite polynomials of different degrees are mutually orthogonal in $\mathcal{L}^2(\mathbb{R}, P_{\sigma^2})$, the proof that

$$E\left[h_m(\sigma^2, \xi)\right] = 0, \; \text{for any } m \geq 2,$$

is straightfoward.

However, for $m \geq 2$ and $j = 2$, one can notice that

$$\begin{aligned} E\left[h_m(\sigma^2, \xi)\right]^2 &= E\left[\frac{\xi}{m} h_{m-1}(\sigma^2, \xi) - \frac{\sigma^2}{m} h_{m-2}(\sigma^2, \xi)\right]^2 \\ &= \frac{1}{m^2}\Big\{E\left[\xi h_{m-1}(\sigma^2, \xi)\right]^2 - 2\sigma^2 E\left[\xi h_{m-1}(\sigma^2, \xi) h_{m-2}(\sigma^2, \xi)\right] \\ &\quad + \sigma^4 E\left[h_{m-2}(\sigma^2, \xi)\right]^2\Big\}, \end{aligned}$$

that is

$$\begin{aligned} m^2 E\left[h_m(\sigma^2, \xi)\right]^2 + \sigma^4 E\left[h_{m-2}(\sigma^2, \xi)\right]^2 &= E\left[\xi h_{m-1}(\sigma^2, \xi)\right]^2 \\ &\quad - 2\sigma^2 E\left[\xi h_{m-1}(\sigma^2, \xi) h_{m-2}(\sigma^2, \xi)\right] \\ &\quad + 2\sigma^4 E\left[h_{m-2}(\sigma^2, \xi)\right]^2. \end{aligned} \tag{21}$$

On the other hand, taking into account the orthogonality of parameterized Hermite polynomials of different degrees in $\mathcal{L}^2(\mathbb{R}, P_{\sigma^2})$ and the recursion formula (16), we get

$$\begin{aligned} m^2 E\left[h_m(\sigma^2, \xi)\right]^2 + \sigma^4 E\left[h_{m-2}(\sigma^2, \xi)\right]^2 &= E\left[m h_m(\sigma^2, \xi) - \sigma^2 h_{m-2}(\sigma^2, \xi)\right]^2 \\ &= E\left[\xi h_{m-1}(\sigma^2, \xi) - 2\sigma^2 h_{m-2}(\sigma^2, \xi)\right]^2 \\ = E\left[\xi h_{m-1}(\sigma^2, \xi)\right]^2 & \\ - 4\sigma^2 E\left[\xi h_{m-1}(\sigma^2, \xi) h_{m-2}(\sigma^2, \xi)\right] &+ 4\sigma^4 E\left[h_{m-2}(\sigma^2, \xi)\right]^2. \end{aligned} \tag{22}$$

Thus,

$$E\left[\xi h_{m-1}(\sigma^2,\xi)h_{m-2}(\sigma^2,\xi)\right] = \sigma^2 E\left[h_{m-2}(\sigma^2,\xi)\right]^2, \ m \geq 2. \tag{23}$$

follows from (21) and (22). On the basis of the equality (23), we obtain a recursion formula

$$\begin{aligned}
E\left[h_m(\sigma^2,\xi)\right]^2 &= \frac{1}{m^2}E\left[\xi^2 h_{m-1}^2(\sigma^2,\xi) - 2\sigma^2\xi h_{m-1}(\sigma^2,\xi)h_{m-2}(\sigma^2,\xi)\right.\\
&\quad \left. +\sigma^4 h_{m-2}^2(\sigma^2,\xi)\right]\\
&= \frac{1}{m^2}E\left\{\xi h_{m-1}(\sigma^2,\xi)\left[\xi h_{m-1}(\sigma^2,\xi) - \sigma^2 h_{m-2}(\sigma^2,\xi)\right]\right.\\
&\quad \left. -\sigma^2\xi h_{m-1}(\sigma^2,\xi)h_{m-2}(\sigma^2,\xi) + \sigma^4 h_{m-2}^2(\sigma^2,\xi)\right\}\\
&= \frac{1}{m^2}\left\{mE\left[\xi h_m(\sigma^2,\xi)h_{m-1}(\sigma^2,\xi)\right]\right.\\
&\quad -\sigma^2 E\left[\xi h_{m-1}(\sigma^2,\xi)h_{m-2}(\sigma^2,\xi)\right]\\
&\quad \left. +\sigma^4 E\left[h_{m-2}(\sigma^2,\xi)\right]^2\right\}\\
&= \frac{1}{m^2}\left\{m\sigma^2 E\left[h_{m-1}(\sigma^2,\xi)\right]^2 - \sigma^4 E\left[h_{m-2}(\sigma^2,\xi)\right]^2\right.\\
&\quad \left. +\sigma^4 E\left[h_{m-2}(\sigma^2,\xi)\right]^2\right\}\\
&= \frac{E\xi^2}{m}E\left[h_{m-1}(\sigma^2,\xi)\right]^2, \ m \geq 2,
\end{aligned}$$

and hence, the equality (20) has been proven.■

The proof of the main theorem is based on Lemma 3.1 and Lemma 4.1.

Theorem 4.2. *For an interval $I \subseteq \mathbb{R}$, let μ be a finite positive measure defined by (1) on the Borel sigma-field of I. If $g : I \to \mathbb{R}_+$, is a non-negative continuous function on I and integrable on I with respect to the measure μ, then for any $m \geq 1$,*

$$\mathcal{J}_{(m)}^I(g) = \frac{\left(\mathcal{J}_{(1)}^I(g)\right)^m}{m!}, \tag{24}$$

where $\mathcal{J}_{(m)}^I(g)$ is the multiple integral (18) and $\mathcal{J}_{(1)}^I(g)$ is the corresponding one-dimensional integral (19).

Proof. Since $g : I \to \mathbb{R}_+$ is a non-negative continuous function on I, and integrable with respect to the measure μ on $\mathcal{B}(I)$, there exists a corresponding function $f \in \mathcal{L}^2(I,\mu)$, continuous on I, such that $g = f^2$ a.e on $\mathcal{B}(I)$. Therefore

$$\mathcal{J}_{(1)}^I(g) = \mathcal{J}_{(1)}^I(f^2) \text{ and } \mathcal{J}_{(m)}^I(g) = \mathcal{J}_{(m)}^I(f^2). \tag{25}$$

On the other hand, as for any $f \in \mathcal{L}^2(I,\mu)$, where μ is the finite positive measure (1), there exist the stochastic integrals (3) and (7), which are the random variables

$\xi^I_{(1)}(f)$ and $\xi^I_{(m)}(f)$ such that $E\left[\xi^I_{(1)}(f)\right]^2 = \mathcal{J}^I_{(1)}(f^2)$ and $E\left[\xi^I_{(m)}(f)\right]^2 = \mathcal{J}^I_{(m)}(f^2)$, respectively.

Having in mind the equality (9) from Lemma 3.1 and applying Lemma 4.1 on the one-dimensional stochastic integral (3), as a Gaussian random variable $\xi^I_{(1)}(f)$ with probability distribution $P_{\mathcal{J}^I_{(1)}(f^2)} = \mathcal{N}(0, \mathcal{J}^I_{(1)}(f^2))$ on $\mathbb{R}$, we obtain

$$\mathcal{J}^I_{(m)}(f^2) = E\left[h_m(\mathcal{J}^I_{(1)}(f^2), \xi^I_{(1)}(f))\right]^2 = \frac{\left(E\left[\xi^I_{(1)}(f)\right]^2\right)^m}{m!} = \frac{\left(\mathcal{J}^I_{(1)}(f^2)\right)^m}{m!}. \tag{26}$$

Thus, taking into account the equalities (25), (24) follows from (26).■

5. NUMERICAL INTEGRATION OF MULTIPLE INTEGRALS BY GAUSSIAN QUADRATURE

5.1. Systems of multiple integrals and Gaussian quadrature rule

The one-dimensional integral (19), which corresponds to the multiple integral (18) in Theorem 4.2, is defined with respect to the finite positive measure (1) on an interval $I \subseteq \mathbb{R}$. This measure is absolutely continuous with respect to the Lebesgue measure on $\mathcal{B}(I)$ with a non-negative continuous Radon-Nikodym derivative $\varphi = \varphi(u)$, $u \in I$. Thus, depending on the set conditions relative to the non-negative continuous integrand $g : I \to \mathbb{R}_+$, integration of the multiple integral (18) can be considered from the viewpoint of numerical integration based on quadrature formulae.

For a *finite interval* $I = [t_0, T] \subset \mathbb{R}$ let

$$g_k : I \to \mathbb{R}_+,\ k = 1, ..., 2n, \tag{27}$$

be a set of $2n$ non-negative continuous functions on I. If the functions (27) satisfy

$$\det\begin{pmatrix} g_1(t_1) & g_1(t_2) & \dots & g_1(t_{2n}) \\ g_2(t_1) & g_2(t_2) & \dots & g_2(t_{2n}) \\ \vdots & \vdots & \dots & \vdots \\ g_{2n}(t_1) & g_{2n}(t_2) & \dots & g_{2n}(t_{2n}) \end{pmatrix} \neq 0,$$

for any set of $2n$ points $t_1, ..., t_{2n} \in I$ with $t_i \neq t_j$ whenever $i \neq j$, then the collection of functions (27) constitutes a Chebyshev system on $I \subset \mathbb{R}$ ([8], 972). In that case, according to Karlin-Studden theorem ([8], Theorem 2.1, 973), there exists a unique generalized n - point Gaussian quadrature rule with respect to the Chebyshev system of functions (27), where the nodes $\widetilde{r}_1, \widetilde{r}_2, ..., \widetilde{r}_n$ are in I and all the weights $\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_n$ are positive. It means that all functions from the collection (27) are integrable with respect to a given finite positive measure $\widetilde{\mu}$ on I, and the corresponding generalized Gaussian quadrature rule integrates them exactly, such that

$$\int_I g_k(u) d\widetilde{\mu}(u) = g_k(\widetilde{r}_1)\widetilde{\lambda}_1 + .. + g_k(\widetilde{r}_n)\widetilde{\lambda}_n, \quad k = 1, ..., 2n,$$

where $\widetilde{\lambda}_1+\widetilde{\lambda}_2+...+\widetilde{\lambda}_n = \widetilde{\mu}(I)$ ([8], Definition 2.4, 973). Thus, on the basis of Theorem 4.2 and Karlin-Studden theorem, the following statement can be proved.

Corollary 5.1. *For a finite interval $I = [t_0, T] \subset \mathbb{R}$, let μ be a finite positive measure defined by (1) on the Borel sigma-field of I. If the collection of functions (27) constitutes a Chebyshev system on I, then for any fixed $m \geq 1$, there exist $n \geq 1$ points $r_1, ..., r_n$ in I, and corresponding positive numbers $\lambda_1, ..., \lambda_n$ with $\lambda_1 + ... + \lambda_n = \mu(I)$, such that*

$$\mathcal{J}^I_{(m)}(g_k) = \frac{1}{m!}\left[g_k(r_1)\lambda_1 + .. + g_k(r_n)\lambda_n\right]^m, \quad k = 1, ..., 2n,$$

where $\mathcal{J}^I_{(m)}(g_k)$, $k = 1, ..., 2n$, is a system of multiple integrals (18) with respect to the collection of functions (27).

In particular, the collection of functions

$$g_k(x) = x^{k-1},\ k = 1, 2, ...2n, \tag{28}$$

is both Chebyshev and Hermite on any finite interval of $\mathbb{R}$ ([8], Definition 2.2, 972). Then, a generalized Gaussian quadrature rule with respect to the collection of functions (28) on any finite interval of $\mathbb{R}$ is actually a classical n-point Gaussian quadrature rule. Moreover, if the functions $1, x, x^2, ...$ are integrable with respect to a given finite positive measure $\widetilde{\mu}$ on an *arbitrary interval* $J \subseteq \mathbb{R}$, then there exists a sequence of orthogonal polynomials in the space $\mathcal{L}^2(J, \widetilde{\mu})$** and a suitable classical n-point Gaussian quadrature rule, $n \geq 1$, which uses those polynomials to calculate the nodes $\widetilde{r}_1, ..., \widetilde{r}_n$ and the weights $\widetilde{\lambda}_1, ..., \widetilde{\lambda}_n$ in a corresponding quadrature formula, where $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 + ... + \widetilde{\lambda}_n = \widetilde{\mu}(I)$. Hence, since the functions $1, x, x^2, ...$ are non-negative and continuous on any interval $I \subseteq \mathbb{R}_+$, the next conclusion follows from Theorem 4.2 and classical Gaussian quadrature rule.

Corollary 5.2. *For an arbitrary interval $I \subseteq \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, let μ be a finite positive measure defined by (1) on the Borel sigma-field of I. If all functions $1, x, x^2, ...$ are integrable on I with respect to the measure μ, then for any fixed $m \geq 1$, there exist $n \geq 1$ points $r_1, ..., r_n$ in I, and positive numbers $\lambda_1, ..., \lambda_n$ with $\lambda_1 + ... + \lambda_n = \mu(I)$, such that*

$$\mathcal{J}^I_{(m)}(x^{k-1}) = \frac{1}{m!}\left[(r_1)^{k-1}\lambda_1 + .. + (r_n)^{k-1}\lambda_n\right]^m, \quad k = 1, ..., 2n, \tag{29}$$

where $\mathcal{J}^I_{(m)}(x^{k-1})$, $k = 1, ..., 2n$, is a system of multiple integrals (18) with respect to the collection of functions (28).

**In principle, if the functions $1, x, x^2, ...$ are integrable with respect to an arbitrary finite positive measure $\widetilde{\mu}$ on an interval $J \subseteq \mathbb{R}$, then there exists a sequence of orthogonal polynomials in the space $\mathcal{L}^2(J, \widetilde{\mu})$. However, in practice, we are faced mostly with specific sequences of orthogonal polynomials which could be connected to certain probability measures.

Furthermore, if the conditions from Corollary 5.2 are satisfied and $g : I \to \mathbb{R}_+$ is an arbitrary non-negative continuous function on an interval $I \subseteq \mathbb{R}_+$, integrable on I with respect to the measure μ, then one can use a suitable classical n-point Gaussian quadrature formula to approximate the value of the corresponding multiple integral (18), such that

$$\mathcal{J}^I_{(m)}(g) \approx \frac{1}{m!}\left[g(r_1)\lambda_1 + .. + g(r_n)\lambda_n\right]^m . \tag{30}$$

5.2. A probabilistic interpretation of Theorem 4.2

Suppose that measure μ, defined by (1), is a probability measure on the Borel sigma-field of $I \subseteq \mathbb{R}$ with $\mu(I) = 1$. If $\xi_1, ..., \xi_m$ are i.i.d. random variables, concentrated on $I \subseteq \mathbb{R}$, with probability distribution μ on $\mathcal{B}(I)$, then for any non-negative continuous function $g : I \to \mathbb{R}_+$ on I, integrable with respect to the measure μ on I,

$$\begin{aligned} E\left[g(\xi_i)\right] &= \mathcal{J}^I_{(1)}(g), \quad i = 1, ..., m, \\ E\left[g(\xi_1)\cdots g(\xi_m) \mid \xi_i \le \xi_{i-1},\ i = 2, ..., m\right] &= \mathcal{J}^I_{(m)}(g), \end{aligned}$$

where $E\left[\cdot \mid \cdot\right]$ denotes conditional expectation of the product $g(\xi_1)\cdots g(\xi_m)$ under condition that $\xi_i \le \xi_{i-1}$, $i = 2, ..., m$. Hence, a probabilistic statement that directly corresponds to Theorem 4.2 is the next one.

Corollary 5.3. *For an interval $I \subseteq \mathbb{R}$, let $g : I \to \mathbb{R}_+$, be a non-negative continuous function on I and integrable on I with respect to the probability measure μ concentrated on I. If $\xi_1, ..., \xi_m$ is a sequence of $m \ge 1$ stochastically independent random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for each $i = 1, ..., m$, μ is the probability distribution of ξ_i on I, then it holds*

$$E\left[g(\xi_1)\cdots g(\xi_m) \mid \xi_i \le \xi_{i-1},\ i = 2, ..., m\right] = \frac{\left(E\left[g(\xi_1)\right]\right)^m}{m!}.$$

Taking into account the Corollary 5.1, if the probability distribution μ is concentrated on a finite interval $I = [t_0, T] \subset \mathbb{R}$, then a collection of conditional expectations $E\left[g_k(\xi_1)\cdots g_k(\xi_m) \mid \xi_i \le \xi_{i-1},\ i = 2, ..., m\right]$, $k = 1, ..., 2n$, with respect to the Chebyshev system of functions (27) can be calculated exactly by a generalized Gaussian quadrature rule. Moreover, if the probability distribution μ, concentrated on an interval $I \subseteq \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, has finite moments of all orders, then there exists a sequence of orthogonal polynomials with respect to the probability measure μ on $I \subseteq \mathbb{R}_+$. In that case, having in mind the Corollary 5.3, if $g : I \to \mathbb{R}_+$ is an arbitrary non-negative continuous function on an interval $I \subseteq \mathbb{R}_+$, integrable on I with respect to the measure μ, then one can use a suitable classical n-point Gaussian quadrature formula to approximate the conditional expectation $E\left[g(\xi_1)\cdots g(\xi_m) \mid \xi_i \le \xi_{i-1},\ i = 2, ..., m\right]$ by the right-hand side of the expression (30).

6. CONCLUDING REMARKS

Using the properties of the multiple stochastic integral (7), defined with respect to the Volterra kernel (6) and the product of Gaussian orthogonal stochastic measure (2), we proved that the multiple integral (18) can be expressed by the corresponding one-dimensional integral (19). This result allows us to integrate exactly a system of multiple integrals (18), defined with respect to the Chebyshev system of functions (27) on any finite interval of $\mathbb{R}$, by a suitable generalized Gaussian quadrature rule, as well as to approximate a multiple integral (18), defined on an arbitrary interval $I \subseteq \mathbb{R}_+$, by a classical Gaussian quadrature rule. Moreover, the given probabilistic interpretation of Theorem 4.2 could be useful to calculate exactly a collection of conditional expectations or to approximate a certain conditional expectation by Gaussian quadrature.

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