Yugoslav Journal of Operations Research
12 (2002), Number 1, 61-71

DETERMINISTIC INVENTORY MODEL FOR ITEMS WITH TIME VARYING DEMAND, WEIBULL DISTRIBUTION DETERIORATION AND SHORTAGES

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Abstract: In this paper, an EOQ inventory model is depleted not only by time varying demand but also by Weibull distribution deterioration, in which the inventory is permitted to start with shortages and end without shortages. A theory is developed to obtain the optimal solution of the problem; it is then illustrated with the aid of several numerical examples. Moreover, we also assume that the holding cost is a continuous, non-negative and non-decreasing function of time in order to extend the EOQ model. Finally, sensitivity of the optimal solution to changes in the values of different system parameters is also studied.

Keywords: Inventory, time-varying demand, Weibull distribution, shortage.

1. INTRODUCTION

Deterioration is defined as decay, change or spoilage that prevents the items from being used for its original purpose. Foods, pharmaceuticals, chemicals, blood, drugs are some examples of such products. In many inventory systems, deterioration of goods in the form of a direct spoilage or gradual physical decay in the course of time is a realistic phenomenon and hence it should be considered in inventory modeling.

Deteriorating inventory has been widely studied in recent years. Ghare and Schrader [9] were two of the earliest researchers to consider continuously decaying inventory for a constant demand. Shah and Jaiswal [21] presented an order-level inventory model for deteriorating items with a constant rate of deterioration. Aggarwal [1] developed an order-level inventory model by correcting and modifying the error in Shah and Jaiswal's [21] analysis in calculating the average inventory holding cost. Covert and Philip [4] used a variable deterioration rate of two-parameter Weibull distribution to formulate the model with assumptions of a constant demand rate and no shortages. However, all the above models are limited to the constant demand.

Time-varying demand patterns are commonly used to reflect sales in different phases of a product life cycle in the market. For example, the demand for inventory items increases over time in the growth phase and decreases in the decline phase. Donaldson [8] initially developed an inventory model with a linear trend in demand. After that, many researchers' works (see, for example, Silver [22], Goel and Aggarwal [12], Ritchie [20], Deb and Chaundhuri [7], Dave and Pal [5-6], Chung and Ting [3], Kishan and Mishra [15], Giri et al. [10], Hwang [14], Pal and Mandal [19], Mandal and pal [16], and Wu et al. [25-26]) have been devoted to incorporating a time-varying demand rate into their models for deteriorating items with or without shortages under a variety of circumstances.

Recently, Wu et al. [26] investigated an EOQ model for inventory of an item that deteriorates at a Weibull distribution rate, where the demand rate is a continuous function of time. In their model, the inventory model starts with an instant replenishment and ends with shortages.

In the present paper, the model of Wu et al. [26] is reconsidered. We have revised this model to consider that it starts with zero inventories and ends without shortages. Comparing the optimal solutions for the same numerical examples, we find that both the order quantity and the system cost decrease considerably as a result of its starting with shortages and ending without shortages.

2. ASSUMPTIONS AND NOTATIONS

The proposed inventory model is developed under the following assumptions and notations.

1. Replenishment size is constant and replenishment rate is infinite.
2. Lead time is zero.
3. T is the fixed length of each production cycle.
4. The initial and final inventories are both zero.
5. The inventory model starts with zero inventories and ends without shortages.
6. The demand rate $D(t)$ at any instant t is positive in $(0, T]$ and continuous in $[0, T]$
7. The inventory holding cost c_1 per unit per unit time, the shortage cost c_2 per unit per unit time, and the unit-deteriorated cost c_3 are known and constant during the period T .
8. The deterioration rate function, $\theta(t)$, represents the on-hand inventory deterioration per unit time, and there is no replacement or repair of deteriorated units during the period T. Moreover, in the present model, the function $\theta(t) = \alpha\beta t^{\beta-1}$, $\alpha > 0, \beta > 0, t > 0, 0 \le \theta(t) < 1$ (also see Covert and Philip [4]).

3. MATHEMATICAL MODEL AND SOLUTION

The objective of the inventory problem here is to determine the optimal order quantity in order to keep the total relevant cost as low as possible. The behavior of the inventory system at any time during a given cycle is depicted in Fig. 1. The inventory system starts with zero inventories at $t=0$ and shortages are allowed to accumulate up to t_1. Procurement is done at time t_1. The quantity received at t_1 is used partly to make up for the shortages that accumulated in the pervious cycle from time 0 to t_1. The rest of the procurement accounts for the demand and deterioration in $[t_1, T]$. The inventory level gradually falls to zero at T.

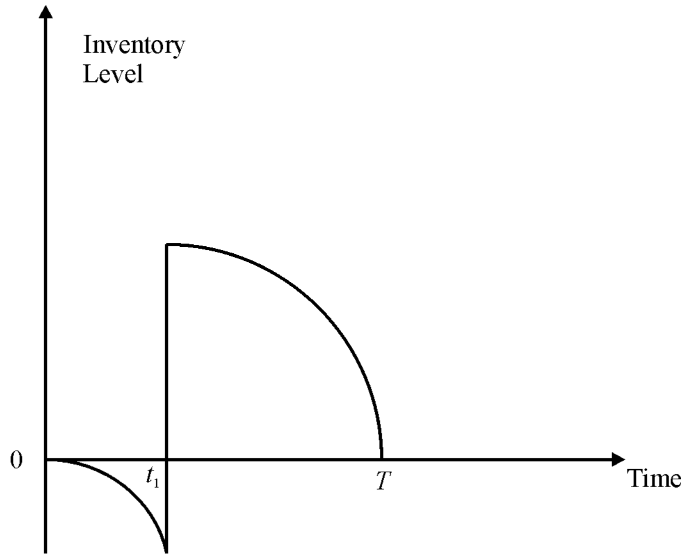

Figure 1: An illustration of inventory cycle

The inventory level of the system at time t over the period $[0, T]$ can be described by the following differential equations:

$$\frac{d}{dt}I(t) = -D(t), \qquad 0 \le t \le t_1 \tag{1}$$

and

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t), \qquad t_1 \le t \le T \tag{2}$$

where

$$\theta(t) = \alpha\beta t^{\beta-1},\ \alpha > 0,\ \beta > 0,\ t > 0 \tag{3}$$

By virtue of equation (3) and (2), we get

$$\frac{dI(t)}{dt}+\alpha\beta t^{\beta-1}I(t)=-D(t), \qquad t_1\le t\le T \tag{4}$$

The solutions of differential equations (1) and (4) with the boundary conditions $I(0)=0$ and $I(T)=0$ are

$$I(t)=\int_0^t -D(u)du, \qquad 0\le t\le t_1 \tag{5}$$

and

$$I(t)=e^{-\alpha t^{\beta}}\int_t^T D(u)e^{\alpha u^{\beta}}du, \qquad t_1\le t\le T \tag{6}$$

Using equation (6), the total number of items that deteriorated during $[t_1,T]$ is

$$D_T=I(t_1)-\int_{t_1}^T D(t)dt=e^{-\alpha t_1^{\beta}}\int_{t_1}^T D(t)e^{\alpha t^{\beta}}dt-\int_{t_1}^T D(t)dt \tag{7}$$

The inventory that accumulates over the period $[t_1,T]$ is

$$I_T=\int_{t_1}^T e^{-\alpha t^{\beta}}\left[\int_t^T D(u)e^{\alpha u^{\beta}}du\right]dt \tag{8}$$

Moreover, from equation (5), the amount of shortage during the interval $[0,t_1)$ is given by

$$B_T=\int_0^{t_1}\int_0^t D(u)dudt=\int_0^{t_1}(t_1-u)D(u)du \tag{9}$$

Using equations (7)-(9), we can get the average total cost per unit time (including holding cost, shortage cost and deterioration cost) as

$$\begin{aligned}C(t_1)&=\frac{1}{T}[c_1I_t+c_2B_T+c_3D_T]\\&=\frac{c_1}{T}\int_{t_1}^T e^{-\alpha t^{\beta}}\left[\int_t^T D(u)e^{\alpha u^{\beta}}du\right]dt+\frac{c_2}{T}\int_0^{t_1}(t_1-u)D(u)du\\&\quad+\frac{c_3}{T}\left\{e^{-\alpha t_1^{\beta}}\int_{t_1}^T D(u)e^{\alpha u^{\beta}}du-\int_{t_1}^T D(u)du\right\}\end{aligned} \tag{10}$$

The first and second order differentials of $C(t_1)$ with respect to t_1 are respectively as follows:

$$\frac{dC(t_1)}{dt_1}=\frac{1}{T}\left\{c_2\int_0^{t_1}D(t)dt-(c_1+c_3\alpha\beta t_1^{\beta-1})e^{-\alpha t_1^{\beta}}\int_{t_1}^{T}D(t)e^{\alpha t^{\beta}}dt\right\} \tag{11}$$

and

$$\frac{d^2C(t_1)}{dt_1^2}=\frac{1}{T}\left\{\left[c_3(1-\beta+\alpha\beta t_1^{\beta})\alpha\beta t_1^{\beta-2}+c_1\alpha\beta t_1^{\beta-1}\right]e^{-\alpha t_1^{\beta}}\int_{t_1}^{T}D(t)e^{\alpha t^{\beta}}dt\right\} + \frac{D(t_1)}{T}(c_1+c_2+c_3\alpha\beta t_1^{\beta-1}) \tag{12}$$

Because $\frac{d^2C(t_1)}{dt_1^2}>0$ for $\beta\le 1$, therefore, the optimal value of t_1 (we denote it by t_1^*) which minimizes the average total cost per unit time can be obtained by solving the equation: $\frac{dC(t_1)}{dt_1}=0$. That is, t_1^* satisfies the following equation:

$$c_2\int_0^{t_1}D(t)dt-(c_1+c_3\alpha\beta t_1^{\beta-1})e^{-\alpha t_1^{\beta}}\int_{t_1}^{T}D(t)e^{\alpha t^{\beta}}dt=0 \tag{13}$$

Now, we let

$$f(t_1)=c_2\int_0^{t_1}D(t)dt-(c_1+c_3\alpha\beta t_1^{\beta-1})e^{-\alpha t_1^{\beta}}\int_{t_1}^{T}D(t)e^{\alpha t^{\beta}}dt$$

because $f(0)<0$ and $f(T)>0$, then by using the Intermediate Value Theorem there exists a unique solution $t_1^*\in(0,T)$ satisfying (13). Equation (13) in general cannot be solved in an explicit form; hence we solve the optimal value t_1^* by using Maple V, a program developed by the Waterloo Maple Software Industry, which can perform the symbolic as well as the numerical analysis.

Substituting $t_1=t_1^*$ in equation (6), we find that the optimal ordering quantity Q (which is denoted by Q^*) is given by

$$Q^*=I(t_1^*)+\int_0^{t_1^*}D(t)dt=e^{-\alpha(t_1^*)^{\beta}}\int_{t_1^*}^{T}D(t)e^{\alpha t^{\beta}}dt+\int_0^{t_1^*}D(t)dt \tag{14}$$

Moreover, from equation (10), we have that the minimum value of the average total cost per unit time is $C^*=C(t_1^*)$.

4. EQQ INVENTORY WITH TIME VARYING OF HOLDING COST

In the Section 3 the holding cost is assumed to be constant. In practice, the holding cost may not always be constant because the price index may increase with time. In order to generalize the EQQ inventory model, various functions describing the holding cost were introduced by several researchers, such as Naddor [18], Van der Veen [23], Muhlemann and Valtis Spanopoulos [17], Weiss [24], Goh [13], Giri et al. [10], Giri and Chaudhuri [11], Beyer and Sethi [2], Wu et al. [26], and among others. Therefore, in this section we assume that the holding cost $h(t)$ per unit per unit time is a continuous, nonnegative and non-decreasing function of time. Then, the average total cost per unit is replaced by

$$\begin{aligned} C(t_1) &= \frac{1}{T}\int_{t_1}^{T} h(t)e^{-\alpha t^\beta}\left[\int_t^T D(u)e^{\alpha u^\beta}du\right]dt + \frac{c_2}{T}\int_0^{t_1}(t_1-u)D(u)du \\ &+ \frac{c_3}{T}\left\{e^{-\alpha t_1^\beta}\int_{t_1}^{T} D(u)e^{\alpha u^\beta}du - \int_{t_1}^{T} D(u)du\right\} \end{aligned} \tag{15}$$

Hence, the necessary condition that the average total cost $C(t_1)$ be minimum is replaced by $\frac{dC(t_1)}{dt_1}=0$, which gives

$$c_2\int_0^{t_1} D(t)dt - (h(t_1) + c_3\alpha\beta t_1^{\beta-1})e^{-\alpha t_1^\beta}\int_{t_1}^{T} D(t)e^{\alpha t^\beta}dt = 0 \tag{16}$$

Similarly, there exists a single solution $t_1^* \in [0,T]$ that can be solved from equation (16). Moreover, the sufficient condition for the minimum average total cost is

$$\begin{aligned} \frac{d^2C(t_1)}{dt_1^2} &= \frac{1}{T}\left\{\left[c_3(1-\beta+\alpha\beta t_1^\beta)\alpha\beta t_1^{\beta-2} + h'(t_1) + h(t_1)\alpha\beta t_1^{\beta-1}\right]e^{-\alpha t_1^\beta}\int_{t_1}^{T} D(t)e^{\alpha t^\beta}dt\right\} \\ &+ \frac{D(t_1)}{T}(h(t_1) + c_2 + c_3\alpha\beta t_1^{\beta-1}) > 0 \quad (\text{for } \beta \le 1) \end{aligned} \tag{17}$$

would be satisfied. In addition, from equation (15), we have that the minimum value of the average total cost per unit time is $C^* = C(t_1^*)$. Finally, the optimal order quantity is the same as equation (14).

5. NUMERICAL EXAMPLES

To illustrate the proceeding theory, the following examples are considered.

Example 1. Linear trend in demand

Let the values of the parameters of the inventory model be $c_1 = \$3$ per unit per year, $c_2 = \$15$ per unit per year, $c_3 = \$5$ per unit, $\alpha = 2$, $\beta = 0.5$, $T = 1$ year, and linear demand rate $D(t) = a + bt,\ a = 20,\ b = 2$. Under the given parameter values and according to equation (13), we obtain that the optimal value $t_1^* = 0.49555$ year. Taking

$t_1^* = 0.49555$ into equation (14), we can get that the optimal order quantity Q^* is 25.23619 units. Moreover, from equation (10) we have that the minimum average total cost per unit time is $C^* = \$109.74650$.

Example 2. Constant demand

The parameter's values in the example are identical to example 1 except for the constant demand rate $D(t) = 50$. By using a similar procedure, we obtain that the optimal values $t_1^* = 0.48662$ year, $Q^* = 60.22576$ units and the minimum average total cost per unit time C^* is \$260.87344.

Example 3. Exponential trend in demand

The parameter's values in the example are identical to example 1 except for the exponential demand rate $D(t) = 50e^{-0.98t}$. By using a similar procedure, we obtain that the optimal values $t_1^* = 0.39773$ year, $Q^* = 39.07469$ units and the minimum average total cost per unit time C^* is \$159.60552 .

Example 4. Linear trend in holding cost

The parameter's values in the example are identical to example 3 except for the holding cost rate $h(t) = 3 + 2t$ per unit per year. By using a similar procedure, we obtain that the optimal values $t_1^* = 0.40991$ year, $Q^* = 38.64943$ units and the minimum average total cost per unit time C^* is \$164.60384 .

Example 5.

The parameter's values in the example are identical to example 1 except for $\beta = 0.125$, $\beta = 0.25$ and $\beta = 1$. By using a similar procedure, the computed results are shown in Table 1. Table 1 shows that each of t_1^*, Q^* and C^* increases with an increase in the value of β .

Next, comparison of our results with those of Wu [26] for four examples is shown in Table 2 and 3. They show that Q^* and C^* all decrease in our model. That is, it is established that this model, where the inventory starts with shortages and ends without shortages, is economically better than the model of Wu et al. [26] (where the inventory starts without shortages and ends with shortages).

Table 1: Optimal results of the various values of β

	t_1^*	Q^*	C^*
$\beta = 0.125$	0.33668	23.48148	87.88620
$\beta = 0.25$	0.40965	24.27143	105.25136
$\beta = 0.5$	0.49555	25.23619	109.74650
$\beta = 1$	0.58636	25.99910	115.90109

Table 2: Optimal results of the proposed model

	Q^*	C^*
Example 1	25.23619	109.74650
Example 2	60.22576	260.87344
Example 3	39.07469	159.60552
Example 4	38.64943	164.60384

Table 3: Optimal results of Wu's model

	Q^*	C^*
Example 1	27.66787	116.02575
Example 2	66.36872	278.46337
Example 3	45.57834	189.16897
Example 4	45.31080	189.51684

6. SENSITIVITY ANALYSIS

We are now to study the effects of changes in the system parameters $c_1, c_2, c_3, \ \alpha, a, b$ and T on the optimal value (t_1^*), optimal order quantity (Q^*) and optimal average total cost per unit time (C^*) in the EOQ model of Example 1. The sensitivity analysis is performed by changing each of the parameters by $-50\%, -25\%, +25\%$ and $+50\%$, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 4.

On the basis of the results of Table 4, the following observations can be made.

1. t_1^* and C^* increase while Q^* decreases with an increase in the value of the model parameter c_1. However, t_1^*, Q^* and C^* are lowly sensitive to changes in c_1.
2. Q^* and C^* increase while t_1^* decreases with an increase in the value of the model parameter c_2. The obtained results show that t_1^* and C^* are moderately sensitive whereas Q^* is lowly sensitive to changes in the value of c_2.
3. t_1^* and C^* increase while Q^* decreases with an increase in the value of the model parameter c_3. Moreover, t_1^* and C^* are moderately sensitive whereas Q^* is lowly sensitive to changes in the value of c_3.
4. Each of t_1^*, Q^* and C^* increases with an increase in the value of the parameter α. The obtained results show that t_1^* and C^* are moderately sensitive whereas Q^* is lowly sensitive to changes in the value of α.

5. Each of t_1^*, Q^* and C^* increases with an increase in the value of the parameter T. Moreover, t_1^*, Q^* and C^* are very highly sensitive to changes in T.
6. Q^* and C^* increase while t_1^* decreases with an increase in the value of the parameter a. The obtained results show that Q^* and C^* are highly sensitive whereas Q^* is most insensitive to changes in the value of a.
7. Each of t_1^*, Q^* and C^* increases with an increase in the value of the parameter b. Moreover, t_1^*, Q^* and C^* are lowly sensitive to changes in b.

Table 4: Effect of changes in the parameters of the inventory model

parameters	%change	%change in		
		t_1^*	Q^*	C^*
c_1	+50	+5.11	–2.05	+2.94
	+25	+2.62	–1.07	+1.53
	–25	–2.75	+1.19	–1.65
	–50	–5.63	+2.50	–3.43
c_2	+50	–14.26	+6.89	+18.13
	+25	–7.95	3.61	+9.91
	–25	+10.50	–4.00	–12.43
	–50	+25.41	–8.45	–28.97
c_3	+50	+10.85	–4.12	+22.34
	+25	+5.84	–2.23	+11.75
	–25	–6.98	+3.14	–13.21
	–50	–15.67	+7.67	–28.30
α	+50	+15.92	+1.31	+11.24
	+25	+8.64	+0.87	+6.68
	–25	–10.44	–1.55	–9.95
	–50	–23.39	–4.18	–24.82
T	+50	+48.08	+62.62	+102.48
	+25	+23.91	+30.36	+47.03
	–25	–23.74	–28.38	–38.66
	–50	–47.51	–54.65	–68.85
a	+50	–0.58	+47.73	+47.55
	+25	–0.35	+23.87	+23.78
	–25	+0.56	–23.87	–23.78
	–50	+1.64	–47.73	–47.57
b	+50	+0.84	+2.27	+2.44
	+25	+0.42	+1.13	+1.22
	–25	–0.43	–1.13	–1.22
	–50	–0.44	–2.27	–2.45

7. CONCLUSIONS

In this paper we consider that the inventory model is depleted not only by time-varying demand but also by Weibull distribution deterioration, in which the inventory model starts with shortages and ends without shortages. Therefore, the proposed model can be used in inventory control of certain deteriorating items such as food items, electronic components, and fashionable commodities, and others. Moreover, the advantage of the proposed inventory model is illustrated with four examples. On the other hand, as is shown by Table 4, the optimal order quantity (Q^*) and the minimum average total cost per unit time (C^*) are highly sensitive to changes in the value of T.

Acknowledgments: The author would like to thank anonymous referees for helpful comments and suggestions.

REFERENCES

[1] Aggarwal, S.P., "A note on an order-level model for a system with constant rate of deterioration", *Opsearch*, 15 (1978) 184-187.
[2] Beyer, D., and Sethi, S., "A proof of the EOQ formula using quasi-variation inequalities", *International Journal of Systems Science*, 29 (1998) 1295-1299.
[3] Chung, K.J., and Ting, P.S., "A heuristic for replenishment of deteriorating items with a linear trend in demand", *Journal of the Operational Research Society*, 44 (1993) 1235-1241.
[4] Covert, R.P., and Philip, G.C., "An EOQ model for items with Weibull distribution deterioration", *AIIE Transaction*, 5 (1973) 323-326.
[5] Dave, U., and Patel, L.K., "Order level inventory system with power demand pattern for items with variable rate of deterioration", *Indian Journal of Pure and Applied Mathematics,* 19 (1988) 1043-1053.
[6] Dave, U., and Patel, L.K., "A note on a replenishment policy for an inventory model with linear trend in demand and shortage", *Journal of the Operational Research Society*, 43 (1992) 993-1001.
[7] Deb, M., and Chaudhuri, K.S., "An EOQ model for items with finite rate of production and variable rate of deterioration", *Journal of the Operational Research Society*, 43 (1992) 175-181.
[8] Donaldson, W.A., "Inventory replenishment policy for a linear trend in demand: an analytic solution", *Operational Research Quarterly*, 28 (1977) 663-670.
[9] Ghare, P.M., and Schrader, G.P., "A model for exponentially decaying inventory", *Journal of Industrial Engineering*, 14 (1963) 238-243.
[10] Giri, B.C., Goswami, A., and Chaudhuri, K.S., "An EOQ model for deteriorating items with time varying demand and costs", *Journal of the Operational Research Society*, 47 (1996) 1398-1405.
[11] Giri, B.C., and Chaudhuri, K.S., "Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost", *European Journal of Operational Research*, 105 (1998) 467-474.
[12] Goel, V.P., and Aggarwal, S.P., "Order level inventory system with power demand pattern for deteriorating items", *Proceedings all India Seminar on Operational Research and Decision Making*, University of Delhi, Delhi, 1981.

[13] Goh, M., "EOQ models with general demand and holding cost function", *European Journal of Operational Research*, 73 (1994) 50-54.
[14] Hwang, H.S., "A study on an inventory model for items with Weibull ameliorating", *Computers and Industrial Engineering*, 33 (1997) 701-704.
[15] Kishan, H., and Mishra, P.N., "An inventory model with exponential demand and constant detrioration with shortages", *Indian Journal of Mathematics*, 37 (1995) 275-279.
[16] Mandal, B., and Pal, A.K., "Order level inventory system with ramp type demand rate for deteriorating items", *Journal of Interdisciplinary Mathematics*, 1 (1998) 49-66.
[17] Muhlemann, A.P., and Valtis-Spanopoulos, N.P., "A variable holding cost rate EOQ model", *European Journal of Operational Research*, 4 (1980) 132-135.
[18] Naddor, E., *Inventory Systems*, Wiley, New York, 1966.
[19] Pal, A.K., and Mandal, B., "An EOQ model for deteriorating inventory with alternating demand rates", *The Korean Journal of Computational and Applied Mathematics*, 4 (1997) 397-407.
[20] Ritchie, E., "The EOQ for linear increasing demand, a simple solution", *Journal of the Operational Research Society*, 35 (1984) 949-952.
[21] Shah, Y.K., and Jaiswal, M.C., "An order-level inventory model for a system with constant rate of deterioration", *Opsearch*, 14 (1977) 174-184.
[22] Silver, E.A., "A simple inventory replenishment decision rule for a linear trend in demand", *Journal of the Operational Research Society*, 30 (1979) 71-75.
[23] Van Der Veen, B., *Introduction to the Theory of Operational Research*, Philip Technical Library, Springer, New York, 1967.
[24] Weiss, H.J., "Economic order quantity models with nonlinear holding cost", *European Journal of Operational Research*, 9 (1982) 56-60.
[25] Wu Jong-Wuu, Lin Chinho, Tan Bertram, and Lee Wee-Chuan, "An EOQ inventory model with ramp type demand rate for items with Weibull deterioration", *International Journal of Information and Management Science*, 10 (1999) 41-51.
[26] Wu Jong-Wuu, Lin Chinho, Tan Bertram, and Lee Wee-Chuan, "An EOQ inventory model with type-varying demand and Weibull deterioration with shortages", *International Journal of Systems Science*, 31 (2000) 677-684.