Yugoslav Journal of Operations Research
27 (2017), Number 1, 125–132
DOI: 10.2298/YJOR160104005B

A MIXED INTEGER LINEAR PROGRAMMING FORMULATION FOR LOW DISCREPANCY CONSECUTIVE K-SUMS PERMUTATION PROBLEM

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Received: January 2016 / Accepted: March 2016

Abstract: In this paper, low discrepancy consecutive k-sums permutation problem is considered. A mixed integer linear programing (MILP) formulation with a moderate number of variables and constraints is proposed. The correctness proof shows that the proposed formulation is equivalent to the basic definition of low discrepancy consecutive k-sums permutation problem. Computational results, obtained on standard CPLEX solver, give 88 new exact values, which clearly show the usefulness of the proposed MILP formulation.

Keywords: Mixed Integer Linear Programming, Permutations with Low Discrepancy Consecutive k-sums.

MSC: 90C10, 68R10.

1. INTRODUCTION

Our description of the problem can start with the question, and its answer, as given in [1]: "Is it possible to arrange the integers 1 through n on a circle so that, for a given k, any sum of k consecutive integers on the circle is close to the expected value of $\frac{k\cdot(n+1)}{2}$? We can do it remarkably well."

1.1. *Problem defintion*

Let n and k be positive integers such that $k \leq n$, and S_n is a set of all permutations $\pi = (\pi_1, \pi_2, ..., \pi_n)$ of the set $\{1, 2, ..., n\}$ viewed as a circle, i.e. indices are always evaluated by modulo n. So, low discrepancy consecutive k-sums permutation problem (LDCkSPP) can be formulated as follows.

$$disc(n, k) = \min_{\pi \in S_n} disc(\pi, k) \tag{1}$$

where

$$disc(\pi, k) = \max_{1 \leq i \leq n} \left| -\frac{k \cdot (n+1)}{2} + \sum_{j=1}^{k} \pi_{i+j} \right| \tag{2}$$

It is easy to see that the only case when $disc(n, k) = 0$ is if $n = k$, while $disc(n, k) > 0$ for all $k < n$. Also, $disc(n, k)$ and $disc(n, n - k)$ are complementary, so it is enough to consider only the case that $k \leq \frac{n}{2}$. Moreover, when n is even and k is odd then $disc(n, k) \in \{0.5 + m \,|\, m \in Z, m \geq 0\}$, while $disc(n, k) \in N$, otherwise.

Example 1. *For $n = 5$ and $k = 2$, permutation $\pi = (1, 4, 3, 2, 5)$ has the corresponding consecutive 2-sums equal to $5, 7, 5, 7, 6$, respectively. Since $\frac{k\cdot(n+1)}{2} = 6$ then, $disc(\pi, 2) = 1$, which is obviously the optimal solution, since for odd n or even k, $disc(n, k) \in N$.*

1.2. *Previous work*

Anstee et al. (2002) in [1] give many theoretical results about permutation discrepancy. They find that, in general, the discrepancy is small, never more than $k + 6$, and independent of n. For $g = gcd(n, k) > 1$, they proved the upper bound of $\frac{7}{2}$, while for $g = 1$, the result is more complicated, which is presented in Table 2. Their constructions show that $disc(n, k) \leq \frac{k}{2} + 9$ for large n, while it is at least $\frac{k}{2}$ for infinitely many n. They also give some theoretical results regarding easier case of linear permutations (non-cyclic).

Stefanović (2010) in [2] determined exact values of $disc(n, k)$ for small n, k using branch-and-bound technique. Exact values are reported, for k up to 10, and for n up to several tenths. Additionally, upper and lower bounds, for k up to 10, and for n up to 100, obtained by theoretical results from the literature are presented.

Stefanović and Živković (2015) in [3] proved that $disc(6t + 3, 3) = 2$, completing previously known results about $disc(n, 3)$. They also found that $disc(2kt, k) = 1.5$, for odd k and $t > 1$.

Table 1: Theoretical exact values from the literature

Ref.	*Conditions*	$disc(n,k)$
[1]	$n = 2k$, k odd	0.5
	$n = 3k$, k odd	1
	$n \geq 3, k = 2$	1
	$n = 3k$, k odd	1
	k even, $n = \pm 1 (mod\, k)$	$\frac{k}{2}$
[3]	$n = 2kt$, k odd, $t > 1, t \in N$	1.5
	$n = 6t + 3, k = 3, t \geq 2, t \in N$	2

Table 2: Theoretical bounds from the literature

Ref.	*Conditions*	*Bound*
[1]	$n \geq 6$	$disc(n,3) \leq 2$
		$disc(n,pq) \leq p \cdot disc(n,q)$
	k even	$disc(n,k) \leq \frac{k}{2}$
	k odd	$disc(mk,k) \leq 2$
	$g > 1$, g even	$disc(n,k) \leq 2$
	$g > 1$, g odd	$disc(n,k) \leq 3.5$
	$g > 1$, g odd	$disc(n,k) \leq disc(\frac{n}{g}, \frac{k}{g})$
	$g = 1, n > 2k, r \geq 1,$	$disc(n,k) \geq \frac{k}{2s}$
	k odd, $g = 1$	$disc(n,k) \leq k + 6$
	k odd, $g = 1, n > n_0(k)$	$disc(n,k) \leq \frac{k}{2} + 9$

1.3. Theoretical results from literature

The concise survey of theoretical exact values about $disc(n,k)$ in the literature is given in Table 1. The first column contains the reference to the paper in which the theoretical result is introduced, the second column lists the conditions, while the third column gives the exact values of $disc(n,k)$.

Table 2, which is organized in a similar way as Table 1, contains the lower or upper bounds of $disc(n,k)$. In Table 2, the following denotations are used:

g denotes $gcd(n,k)$, i.e. greatest common divisor of n and k;
r denotes the residue of division of n by k, i.e. $r \equiv n\, (mod\, k)$;
s denotes the smallest positive integer for which holds $r \cdot s \equiv \pm 1\, (mod\, k)$.

2. PROPOSED MILP FORMULATION

As it is suggested in literature, it is useful to represent various mathematical problems as integer programming problems in order to use different well-known optimization techniques. Following that idea, we will introduce a mixed linear programming (MILP) formulation of the present problem in order to give theoretical and practical insights and to compare to with the previous branch-and-bound techniques, proposed in the literature.

Let π be the permutation. Decision variables x_{ij} can be defined as:

$$x_{ij} = \begin{cases} 1, & j = \pi_i \\ 0, & j \neq \pi_i \end{cases} \tag{3}$$

and

$$z = disc(\pi, k) = \max_{1 \leq i \leq n} \left| -\frac{k \cdot (n+1)}{2} + \sum_{j=1}^{k} \pi_{i+j} \right| \tag{4}$$

The mixed integer linear programming formulation for solving the low discrepancy consecutive k-sums permutation problem can be stated as:

$$min\ z \tag{5}$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, ..., n \tag{6}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, ..., n \tag{7}$$

$$\frac{k(n+1)}{2} + z \geq \sum_{j=i+1}^{i+k} \sum_{l=1}^{n} l \cdot x_{jl} \qquad i = 1, ..., n \tag{8}$$

$$\frac{k(n+1)}{2} - z \leq \sum_{j=i+1}^{i+k} \sum_{l=1}^{n} l \cdot x_{jl} \qquad i = 1, ..., n \tag{9}$$

$$x_{ij} \in \{0, 1\} \qquad i, j = 1...n \tag{10}$$

$$z \in [0, +\infty) \tag{11}$$

The objective function (5) minimizes the discrepancy, defined by constraints (8) and (9). Constraints (6) and (7) ensure that variables x_{ij} represent a permutation, while (10) and (11) reflect the nature of decision variables x_{ij} and variable z.

The presented MILP model has n^2 binary variables, one continuous, and $4 \cdot n$ constraints. The equivalence of the proposed MILP model (5)-(11) with the basic mathematical formulation (1)-(2) is proved in Theorem 1.

Theorem 1. *A low discrepancy consecutive k-sums permutation defined by (1)-(2) is optimal if and only if constraints (5)-(11) are satisfied.*

Proof. ($\Rightarrow$) Suppose that π is an optimal low discrepancy consecutive k-sums permutation for fixed n and k. Let variables x_{ij} be defined as in (3), and z is defined as in (4). It means that variables x_{ij} are binary and z is continuous, so constraints (10) and (11) are satisfied by default.

¿From the definition of variables x_{ij}, and the fact that π is well-defined function, it holds that $(\forall i)(\exists_1 j)\ j = \pi_i$,

$$x_{i,\pi_i} = 1 \tag{12}$$

and

$$(\forall j \neq \pi_i)\ x_{ij} = 0, \tag{13}$$

which means $\sum\limits_{j=1}^{n} x_{ij} = 1$ implying that constraints (6) are satisfied.

Similarly, π is permutation, so there exist the inverse permutation π^{-1}. From $(\forall j)(\exists_1 i)\ \pi_j^{-1} = i$, it follows $x_{\pi_j^{-1},j} = 1$, while $(\forall i \neq \pi_j^{-1})\ x_{ij} = 0$, which means $\sum\limits_{i=1}^{n} x_{ij} = 1$, so constraints (7) are satisfied.

Let fix $j \in \{1, 2, ..., n\}$. As it can be seen in (13), $(\forall l \neq \pi_j)\ x_{jl} = 0$, so $(\forall l \neq \pi_j)\ l \cdot x_{jl} = 0$, implying $\sum\limits_{l=1, l\neq\pi_j}^{n} l \cdot x_{jl} = 0$. Since by (12) $x_{j,\pi_j} = 1$, then it holds $\pi_j = \pi_j \cdot x_{j,\pi_j} = \sum\limits_{l=1}^{n} l \cdot x_{jl}$.

¿From $z = \max\limits_{1\le i\le n} \left| -\frac{k\cdot(n+1)}{2} + \sum\limits_{j=1}^{k} \pi_{i+j} \right|$ (formula (4)), implying $(\forall i) z \geq \left| -\frac{k\cdot(n+1)}{2} + \sum\limits_{j=1}^{k} \pi_{i+j} \right| \Rightarrow (\forall i)\, z \geq -\frac{k\cdot(n+1)}{2} + \sum\limits_{j=1}^{k} \pi_{i+j}$ or $(\forall i)\, z \geq -\left(-\frac{k\cdot(n+1)}{2} + \sum\limits_{j=1}^{k} \pi_{i+j} \right)$. First term is equivalent to $(\forall i) \frac{k\cdot(n+1)}{2} + z \geq \sum\limits_{j=1}^{k} \pi_{i+j}$

which means $(\forall i)\frac{k\cdot(n+1)}{2} + z \geq \sum_{j=i+1}^{i+k} \pi_j$. In the previous paragraph, it is proven that $\pi_j = \sum_{l=1}^{n} l \cdot x_{jl}$, so we can replace π_j with $\sum_{l=1}^{n} l \cdot x_{jl}$. The first term is equivalent to $(\forall i)\frac{k\cdot(n+1)}{2} + z \geq \sum_{j=i+1}^{i+k}\sum_{l=1}^{n} l \cdot x_{jl}$, which exactly represents constraints (8). Similarly, the second term is equivalent to $(\forall i)\, z \geq \frac{k\cdot(n+1)}{2} - \sum_{j=1}^{k} \pi_{i+j}$, implying $(\forall i)\ \sum_{j=1}^{k} \pi_{i+j} \geq \frac{k\cdot(n+1)}{2} - z$. Finally, when we replace π_j with $\sum_{l=1}^{n} l \cdot x_{jl}$ it holds $(\forall i)\ \sum_{j=i+1}^{i+k}\sum_{l=1}^{n} l \cdot x_{jl} \geq \frac{k\cdot(n+1)}{2} - z$ which is equivalent to constraints (9). Moreover, from $z = disc(\pi, k)$ and the fact that the objective function (5) is minimum of z, it holds that objective function value of MILP formulation (5)-(11) is less than or equal to the value of minimal low discrepancy consecutive k-sums permutation.

($\Leftarrow$) For fixed $i \in \{1, 2, ..., n\}$, let π_i be defined as $\pi_i = j$ if $x_{ij} = 1$. From constraints (10) it follows that variable x_{ij} has binary nature, which, with constraints (6), implies that $(\forall i)\ \sum_{j=1}^{n} x_{ij} = 1 \Rightarrow (\forall i)(\exists_1 j)\ x_{ij} = 1$, so π is a well-defined function. Similarly, from constraints (10) and (7), it follows $(\forall j) \sum_{i=1}^{n} x_{ij} = 1 \Rightarrow (\forall j)(\exists_1 i)\ x_{ij} = 1$, so π is bijection. Since $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$, it means that π is permutation.

From the definition of permutation π, it means that $\sum_{l=1}^{n} l \cdot x_{jl} = \pi_j \cdot x_{j,\pi_j} = \pi_j$, since $x_{j,\pi_j} = 1$. Therefore, from constraints (8) and (9), it holds $(\forall i)\ \frac{k(n+1)}{2} - z \leq \sum_{j=i+1}^{i+k} \pi_j \leq \frac{k(n+1)}{2} + z$. The last term is equivalent to $z \geq \frac{k(n+1)}{2} - \sum_{j=i+1}^{i+k} \pi_j$ and $z \geq -\frac{k(n+1)}{2} + \sum_{j=i+1}^{i+k} \pi_j$, which means $z \geq \left| -\frac{k(n+1)}{2} + \sum_{j=i+1}^{i+k} \pi_j \right| = \left| -\frac{k(n+1)}{2} + \sum_{j=1}^{k} \pi_{i+j} \right|$. Therefore, $(\forall i)\ z \geq \left| -\frac{k(n+1)}{2} + \sum_{j=1}^{k} \pi_{i+j} \right| \Rightarrow$

$z \geq \max_{1\leq i\leq n} \left| -\frac{k\cdot(n+1)}{2} + \sum_{j=1}^{k} \pi_{i+j} \right|$, implying $z \geq disc(\pi, k)$. Moreover, from $disc(\pi, k) \leq z \Rightarrow disc(n, k) = \min_{\pi\in S_n} disc(\pi, k) \leq z$, it holds that the value of minimal low discrepancy consecutive k-sums permutation is smaller than or equal to the optimal value of MILP formulation (5)-(11).

Therefore, the minimal value of low discrepancy consecutive k-sums permutation is equal to the optimal value of MILP formulation (5)-(11). □

3. COMPUTATIONAL RESULTS

In this section, experimental results obtained by the CPLEX 12.5.1 solver, using proposed MILP formulation will be presented. All computations were executed

Table 3: New exact values of $disc(n,k)$

n	*k*	*Liter.*		*MILP*		*n*	*k*	*Liter.*		*MILP*		*n*	*k*	*Liter.*		*MILP*	
		LB	*UB*	*Opt*	*t[sec]*			*LB*	*UB*	*Opt*	*t[sec]*			*LB*	*UB*	*Opt*	*t[sec]*
10	4	1	2	2	0.078	38	10	1	2	2	55.84	32	14	1	2	2	835.8
14	4	1	2	2	0.218	23	11	6	17	6	0.812	33	14	3	7	3	18.84
18	4	1	2	2	0.562	24	11	1.5	16.5	1.5	2.500	34	14	1	2	2	193.5
14	6	1	2	2	0.203	25	11	2	17	2	20.83	35	14	1	1	1	397.3
16	6	1	2	2	0.250	27	11	3	17	3	5.296	38	14	1	2	2	994.5
40	6	1	2	2	10131	29	11	2	17	2	127.6	39	14	2	7	2	372.3
35	7	1	2	1	512.9	31	11	2	17	2	3.218	40	14	1	2	2	8.390
18	8	1	2	2	0.312	34	11	5.5	16.5	5.5	4.171	31	15	8	10	8	5.140
34	8	1	2	2	23.03	35	11	2	17	2	3.687	32	15	1.5	7.5	1.5	8.687
35	8	2	4	2	655.9	39	11	3	17	3	388.6	33	15	1	3	2	5.531
37	8	2	4	2	171.5	26	12	1	2	2	2.390	35	15	1	2	1	112.9
38	8	1	2	2	366.7	27	12	1	2	2	2.734	36	15	0.5	3.5	1.5	28.36
24	9	0.5	1.5	1.5	1.890	28	12	1	2	1	110.5	37	15	4	10	4	2734
25	9	2	6	2	1.625	29	12	2	6	2	3.578	39	15	1	3	1	5.625
29	9	2	6	2	107.3	30	12	1	2	1	7.265	34	16	1	2	2	4.562
31	9	3	6	3	7.078	31	12	2	6	2	61.84	35	16	2	8	2	673.9
33	9	1	2	1	32.66	32	12	1	2	2	1467	36	16	1	2	1	183.2
34	9	1.5	4.5	1.5	128.0	33	12	1	2	2	7.656	38	16	1	2	2	1104
35	9	5	6	5	65.17	34	12	1	2	2	7.906	39	16	2	8	2	6.406
39	9	1	2	2	346.5	38	12	1	2	2	47.56	40	16	1	2	1	940.3
22	10	1	2	2	1.906	40	12	1	2	2	9461	35	17	9	23	9	5.187
23	10	2	5	2	2.000	27	13	7	19	7	1.609	36	17	1.5	22.5	1.5	37.50
24	10	1	2	2	14.31	28	13	1.5	18.5	1.5	1.328	37	17	2	23	2	797.0
26	10	1	2	2	5.718	29	13	2	19	2	47.70	39	17	2	23	2	927.3
27	10	2	5	2	33.75	31	13	2	19	2	97.23	38	18	1	2	2	7.468
28	10	1	2	2	1.859	33	13	4	19	4	61.38	40	18	1	2	2	7231
32	10	1	2	2	3.000	35	13	3	19	3	1449	39	19	10	25	10	6.578
33	10	2	5	2	219.7	37	13	2	19	2	423.5	40	19	1.5	24.5	1.5	246.9
34	10	1	2	2	377.2	30	14	1	2	2	4.265						
36	10	1	2	2	340.6	31	14	2	7	2	42.09						

on HP i5-3470, 3.2GHz PC with 8GB RAM, using single core. The MILP model used by CPLEX 12.5.1 solver was coded in C programming language.

In order to clearly present the effectiveness of the proposed MILP formulation, all previously known exact solutions are omitted, so Table 3 contains only data for new exact values of $disc(n,k)$. It should be mentioned that branch-and-bound approach [2] obtained optimal value $disc(56,4) = 1$, while CPLEX based on the presented model could not obtain an exact result in 7200 seconds. In the first two columns the n and k are given. The third and fourth columns are labeled with *LB* and *UB* and present the values of lower and upper bound, taken from the literature. The fifth and the sixth column are labeled with *Opt* and t, containing the corresponding optimal solution values and total running time (in seconds), obtained by CPLEX 12.5.1 solver. The following six columns have the same meaning as the first six columns.

Experimental results given in Table 3 show that 88 new exact values of $disc(n,k)$, with various n and k, are obtained. Although running time can be large (for example, in the case $n = 40$, $k = 6$ CPLEX needs 10131 seconds which is almost 3 hours), many results are obtained in less than 10 seconds. Advantages of the presented model over the previous exact approaches are more visible for large k values.

4. CONCLUSIONS

This paper is devoted to the low discrepancy consecutive k-sums permutation problem. The mixed integer linear programing formulation with a moderate number of variables and constraints is introduced. We also give the formal proof that the proposed model is equivalent to the basic problem definition. From computational results, it is evident that the proposed model has theoretical and practical significance.

One direction for future work can be to design an exact method by using the proposed MILP formulation. The second direction may be solving some similar problems.

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