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# TWO WAREHOUSE INVENTORY MODEL FOR DETERIORATING ITEM WITH EXPONENTIAL DEMAND RATE AND PERMISSIBLE DELAY IN PAYMENT

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**Abstract:** A two warehouse inventory model for deteriorating items is considered with exponential demand rate and permissible delay in payment. Shortage is not allowed and deterioration rate is constant. In the model, one warehouse is rented and the other is owned. The rented warehouse is provided with better facility for the stock than the owned warehouse, but is charged more. The objective of this model is to find the best replenishment policies for minimizing the total appropriate inventory cost. A numerical illustration and sensitivity analysis are provided.

**Keywords:** Two Warehouse, Deteriorating Item, Exponential Demand Rate, Inventory Model**,** Permissible Delay in Payment

**MSC:** 90B05.

# 1. INTRODUCTION

Inventory modeling is a mathematical approach to decide when to order, and how much to order so as to minimize the total cost. The effect of deterioration is applicable to most of the items, which cannot be neglected in inventory model. Generally, every firm has its own warehouse (OW) with a limited capacity. If the quantity exceeds the capacity of OW then, these quantities should be transferred to another, rented warehouse (RW). The customers are served first from RW, then from OW.

The first two warehouse inventory model was developed by Hartley [8]. Sarma [30] developed the inventory model which included two levels of storage and the optimum release rule. Sarma [23] extended his previous model to the case of infinite refilling rate with shortages. Ghosh and Chakrabarty [6] developed an order level inventory model with two levels of storage for deteriorating items. An EOQ model with two levels of storage was studied by Dave [4], considering distinct stage production schemes. Several researchers developed inventory models for deteriorating goods. The deterioration of goods is defined as damage, spoilage, and dryness of items like groceries, pictographic film, electronic equipment, etc. Pakkala and Acharya [13] developed a two warehouse inventory model for deteriorating items with finite replenishment rate and shortages. Benkherouf [2] developed a two warehouse model with deterioration and continuous release pattern. Lee and Ma [10] studied an optimal inventory policy for deteriorating items with two warehouse and time dependent demand. Zhou [29] developed two warehouse inventory models with time varying demand. Yanlai Liang and Fangming Zhou [28] developed a two warehouse inventory model for deteriorating items under conditionally permissible delay in payment.

In an EOQ model, it is frequently considered that a retailer should pay off as soon as the items are received. In fact, the supplier provides the retailer with a postponement period, known as trade credit period, in paying for purchasing cost as a common business practice. Suppliers frequently propose trade credit as a marketing policy to raise sale and decrease hand stock level. Once a trade credit has been presented, the amount of period for the retailer's capital tied up in stock is reduced, which leads to a decline of the retailer holding cost of funding. In addition, during the trade credit period, the retailer can add revenue by selling goods and by earning interests. Goyal [7] was the first author who established an EOQ model with a constant demand rate under the condition of permissible delay in payments. It was considered that the unit purchase cost is equal to the selling cost per unit. Yang [26] developed a two-warehouse inventory model for deteriorating items with shortages under inflation and constant demand rate, where each cycle begins with shortages and ends without shortages. Wee et al. [25] developed a two warehouse model with constant demand and Weibull distribution deterioration underneath inflation. Yang [27] extended Yang's [26] by including partial backlogging, then compared the two warehouse models, and supported the minimum price approach. In this model, an effort was made to develop a two warehouse inventory model with exponential demand for Weibull deteriorating items, which depends on time and on hand inventory. Since the deterioration depends on protective facility available in warehouse, the different warehouses may have different deterioration rates. Sana [14] developed an economic order quantity inventory model for time varying deterioration and partial backlogging. Sana [15] developed an economic order quantity model for defective items with deterioration. He assumed the percentage of defective items to depend on production time, as well as the elapsed time, if the process does not shift to ‘‘out of control ‘state. Sana [16] developed an economic order quantity model for compatible and unusual quality goods, where a casual proportion of total goods is sold at a reduced cost after a 100% screening process. Sana [17] introduced a price-dependent demand with stochastic selling price into the traditional Newsboy problem, evaluated the anticipated average profit for a universal distribution function of p and found an optimum order size.

Sana and Chaudhuri [18] established an economic production lot size model in which the production procedure shifts from an ‘in-control’ state to an ‘out-of-control’ state after a definite, exponentially distributed time. Imperfect items are accrued and reworked instantly at certain price for maintaining the quality of the item during the ‘out-of-control’ state. The demand rate of the product is stochastic. Shortages are allowed and backlogged; both partial backlogging and complete backlogging are considered. Sana and Goyal [19] developed an economic order quantity model for varying lead time, purchasing cost that depends on order, partial backlogging depending on lead-time, order size and reorder point. Lead-time, order size and reorder point are the decision variables. Sana and De [20] developed an economic order quantity model for fuzzy variables with promotional effort,t and selling price dependent demand. They observed that demand rate decreases over time during shortage period. Sana et al. [21] analyzed a single-period newsvendor inventory model to define the optimum order quantity where the consumers' flinching occur, and depend on holding cost. Shortages are allowed and partially backlogged. Bhunia and Maity [3] analyzed the deterministic inventory model with different levels of items deterioration in both warehouses. Ishii and Nose [9] discussed two types of consumers in the perishable inventory models under the warehouse capacity constraint. Lee and Hsu [11] developed a two warehouse inventory model for deteriorating items with time dependent demand and constant production rate. Maiti [12] described a model based on two warehouse production system for imperfect quality items with no shortages. He assumed stock dependent demand for perfect items, and time dependent production rate to determine optimal production. Abad [1] was the first author to develop a pricing and ordering policy for a variable rate of deterioration with partially backlogged shortages. Dye et al. [5] modified the Abad [1] model considering the backorder cost and lost sale cost. Shah and Shukla [24] also developed a deterministic inventory model for deteriorating items with partially backlogged shortages.

In this model, a two warehouse inventory model for deteriorating items is developed in which demand rate is exponentially increasing with time and the deterioration rate is constant. It was assumed that the rented warehouse has higher unit holding cost than the owned warehouse. The objective of this model is to find the best replenishment policies for minimizing the total appropriate inventory cost. Numerical example and sensitivity analysis are provided to illustrate the model and the optimal solution.

## 2. ASSUMPTIONS

We need the following assumptions for developing the mathematical model:

1. The inventory model consider single item.
2. The demand rate exponentially increases over time i.e. $D(t) = ae^{bt}$
3. The lead time is negligible.
4. Shortages are not allowed.

5. The OW has the finite capability of S units and the RW has infinite capability.
6. We consider that first the items of RW are used, and the items of OW
7. are the next.
8. The item deteriorates at a fixed rate $\mu$ in OW and at $\eta$ in RW.
9. RW offers improved services, so $\mu > \eta$ and $h_r - h_o > c(\mu - \eta)$
10. Replenishment rate is infinite.
11. The maximum deteriorating quantity of items in OW is $\mu S < D$.

## 3. NOTATIONS

We need the following symbols for developing mathematical model:

$A$: Ordering cost.

$p$: The sale price per unit.

$c$: The purchasing cost.

$h$: The holding cost per unit time, when $h = h_o$ for item in OW and $h = h_r$ for item in RW and $h_r > h_o$.

$Q_r(t)$: Inventory level at time t, $0 \le t \le t_1$ in RW.

$Q_o(t)$: Inventory level at time t, $0 \le t \le T$ in OW.

M: Retailer's trade credit period presented by a supplier per year.

$I_e$: Earned interest per year.

$I_p$: Interest charges per year by the supplier.

$TC_1, TC_2$, and $TC_3$ are the total appropriate costs.

$t_1^*$ is the optimal solution and $TC^*$ is the best minimum total appropriate cost.

## 4. MATHEMATICAL MODEL

At the beginning of the cycle, the inventory level reaches its maximum $S$ units of item at time $t = 0$, which is kept in OW and the remaining in RW. The item of RW is consumed first and next the item of OW. The inventory in RW depletes due to demand and deterioration during $[0, t_1]$, it vanishes at $t = t_1$. The inventory in OW depletes due to deterioration during $[0, t_1]$, but the inventory depletes due to demand and deterioration during $[t_1, T]$. Both the warehouses are empty at time T.

The inventory level in RW and OW at time $t \in [0, t_1]$ is described by the following differential equation:

$$\frac{dQ_r(t)}{dt} = -ae^{bt} - \eta Q_r(t), \ \ 0 \le t \le t_1 \tag{1}$$

With the boundary condition

$$Q_r(t_1) = 0$$

From equation $(1)$, we have

$$Q_r(t) = \frac{a}{(\eta + b)}\left[e^{(b+\eta)t_1}e^{-\eta t} - e^{bt}\right] \tag{2}$$

and

$$\frac{dQ_o(t)}{dt} = -\mu Q_0(t), \quad 0 \le t \le t_1 \tag{3}$$

With boundary condition

$$Q_o(0) = S$$

From equation $(3)$, we have

$$Q_o(t) = Se^{-\mu t},\ 0 \le t \le t_1 \tag{4}$$

The inventory depletes due to demand and deterioration during $[t_1, T]$. At time T, the inventory level becomes zero and both warehouses are empty. The inventory level in OW i.e. $Q_o(t)$ is described by the following differential equation

$$\frac{dQ_o(t)}{dt} = -ae^{bt} - \mu Q_0(t), \quad t_1 \le t \le T \tag{5}$$

With boundary condition

$$Q_o(T) = 0$$

From equation $(5)$, we have

$$Q_o(t) = \frac{a}{(\mu + b)}\left[e^{(\mu+b)T}e^{-\mu t_1} - e^{bt}\right],\ t_1 \le t \le T \tag{6}$$

Consider that the continuity of $Q_o(t)\ at\ t = t_1$, it follows that

$$Q_o(t_1) = Se^{-\mu t_1} = \frac{a}{(\mu + b)}\left[e^{(\mu+b)T}e^{-\mu t_1} - e^{bt_1}\right]$$

Thus,

$$T = \frac{\mu t_1 + \ln\left(e^{bt_1} + \frac{(\mu+b)S}{a}e^{-\mu t_1}\right)}{(\mu+b)} \tag{7}$$

According to the assumption of the model, the total relevant cost per year, TC, includes the following elements:

I. Ordering cost per year

$$= \frac{A}{T} \tag{8}$$

II. Stock holding cost per year:

The increasing inventory in RW during the interval $[0,t_1]$, and in OW during the interval $[0,T]$ is

$$\int_0^{t_1} Q_r(t)dt = \frac{a}{(\eta+b)}\int_0^{t_1}\left[e^{(b+\eta)t_1}e^{-\eta t} - e^{bt}\right]dt = \frac{a}{(\eta+b)}\left[\frac{1}{b} + \frac{e^{(b+\eta)t_1}}{\eta} - e^{bt_1}M_1\right]$$

Where, $M_1 = \frac{1}{b} + \frac{1}{\eta}$, and

$$\int_0^{T} Q_o(t)dt = \int_0^{t_1} Q_o(t)dt + \int_{t_1}^{T} Q_o(t)dt = \int_0^{t_1} Se^{-\mu t}dt + \frac{a}{(\mu+b)}\int_{t_1}^{T}\left[e^{(\mu+b)T}e^{-\mu t_1} - e^{bt}\right]dt$$

$$= \frac{S}{\mu}\left(1 - e^{-\mu t_1}\right) + \frac{a}{(\mu+b)}\left(e^{(\mu+b)T}e^{-\mu t_1}(T-t_1) - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right)$$

The stock holding cost per year in RW is

$$\frac{ah_r}{T(\eta+b)}\left[\frac{1}{b} + \frac{e^{(b+\eta)t_1}}{\eta} - e^{bt_1}M_1\right] \tag{9}$$

The stock holding cost per year in OW is

$$\frac{h_o}{T}\left[\frac{S}{\mu}\left(1 - e^{-\mu t_1}\right) + \frac{a}{(\mu+b)}\left(e^{(\mu+b)T}e^{-\mu t_1}(T-t_1) - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right)\right] \tag{10}$$

III. Deterioration cost per year:

Cost of the deteriorating item per year in RW and OW during the interval $[0,T]$ is

$\eta\int_0^{t_1} Q_r(t)dt$ and $\mu\int_0^{T} Q_o(t)dt$.

Thus, the deterioration cost of item is;

$$\frac{c}{T}\left[\begin{array}{l}\frac{a\eta}{(\eta+b)}\left(\frac{e^{(b+\eta)t_1}}{\eta}-e^{bt_1}M_1+\frac{1}{b}\right)+S\left(1-e^{-\mu t_1}\right)\\+\frac{a\mu}{(\mu+b)}\left(e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\end{array}\right] \tag{11}$$

IV. The opportunity cost with interest:

There are three cases arise:

Case1: $M \le t_1 < T$

The interest payable per year is

$$\frac{cI_p}{T}\left[\int_M^{t_1} Q_r(t)dt+\int_M^{t_1} Q_o(t)dt+\int_{t_1}^{T} Q_o(t)dt\right]$$

$$=\frac{cI_p}{T}\left\{\begin{array}{l}\frac{a}{(\eta+b)}\left(-\frac{e^{(b+\eta)t_1}}{\eta}\left(e^{-\eta t_1}-e^{-\eta M}\right)-\frac{1}{b}\left(e^{bt_1}-e^{bM}\right)\right)+\frac{S}{\mu}\left(e^{-\mu M}-e^{-\mu t_1}\right)\\+\frac{a}{(\mu+b)}\left(e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\end{array}\right\} \tag{12}$$

Case 2: $t_1 < M \le T$

The interest payable per year is

$$\frac{cI_p}{T}\left[\int_M^{T} Q_o(t)dt\right]=\frac{cI_p}{T}\left[\frac{a}{(\mu+b)}\left(e^{(\mu+b)T}e^{-\mu t_1}(T-M)-\frac{1}{b}\left(e^{bT}-e^{bM}\right)\right)\right] \tag{13}$$

Case 3: $M > T$ No interest charges payable per year for the goods.

V.

The interest earned per year:

There are two cases which arise:

Case 1: $M \le T$

The interest earned per year is

$$\frac{pI_e}{T}\left[\int_0^{M} Dtdt\right]=\frac{pI_e}{T}\left[\int_0^{M} ae^{bt}tdt\right]=\frac{apI_e}{T}\left[M\frac{e^{bM}}{b}-\frac{e^{bM}}{b^2}+\frac{1}{b^2}\right]=\frac{apI_eE}{T} \tag{14}$$

Where,

$$E=M\frac{e^{bM}}{b}-\frac{e^{bM}}{b^2}+\frac{1}{b^2}$$

Case 2: $M > T$

The interest earned per year is

$$\frac{pI_e}{T}\left[\int_0^T Dtdt + DT\left(M-T\right)\right] = \frac{pI_e}{T}\left[\int_0^T ae^{bt}tdt + ae^{bt}T\left(M-T\right)\right]$$
$$= \frac{pI_e}{T}\left[T\frac{ae^{bT}}{b} - \frac{ae^{bT}}{b^2} + \frac{a}{b^2} + ae^{bT}T\left(M-T\right)\right] \tag{15}$$

Thus, the total relevant cost per year for the retailer is given by

$TC\left(t_1,T\right) =$ Ordering cost + Stock holding cost in RW + Stock holding cost in OW + Deterioration cost + Opportunity cost with interest + Interest earned

Therefore,

$$TC\left(t_1,T\right) = \begin{cases} TC_1, & M \le t_1 < T \\ TC_2, & t_1 < M \le T \\ TC_3, & M > T \end{cases} \tag{16}$$

Where,

$$TC_1 = \frac{1}{T}\left[\begin{array}{l} A + \dfrac{a\left(h_r + c\eta\right)}{b\left(\eta+b\right)} - apI_eE + \dfrac{Sh_o}{\mu} + Sc + \dfrac{acI_pe^{bM}}{b\left(\eta+b\right)} + \dfrac{cI_pSe^{-\mu M}}{\mu} \\ + \dfrac{a\left(h_r + c\eta\right)}{b\left(\eta+b\right)}\left(\dfrac{e^{(b+\eta)t_1}}{\eta} - e^{bt_1}M_1\right) - e^{-\mu t_1}\left(\dfrac{Sh_o}{\mu} + Sc\right) + \dfrac{acI_p}{\left(\eta+b\right)} \\ \left\{-\dfrac{e^{(b+\eta)t_1}}{\eta}\left(e^{-\eta t_1} - e^{-\eta M}\right) - \dfrac{1}{b}e^{bt_1}\right\} - \dfrac{cI_pSe^{-\mu t_1}}{\mu} \\ + \dfrac{a\left(h_o + \mu c + cI_p\right)}{\left(\mu+b\right)}\left(e^{\left(\mu+b\right)T}e^{-\mu t_1}\left(T-t_1\right) - \dfrac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) \end{array}\right] \tag{17}$$

$$TC_2 = \frac{1}{T}\left[\begin{array}{l} A + \dfrac{a\left(h_r + c\eta\right)}{\left(\eta+b\right)}\left(\dfrac{e^{(b+\eta)t_1}}{\eta} - e^{bt_1}M_1 + \dfrac{1}{b}\right) + \left(h_o + c\mu\right)\dfrac{S}{\mu}\left(1 - e^{-\mu t_1}\right) \\ -apI_eE + \dfrac{a\left(h_o + c\mu\right)}{\left(\mu+b\right)}\left(e^{\left(\mu+b\right)T}e^{-\mu t_1}\left(T-t_1\right) - \dfrac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) \\ + \dfrac{cI_pa}{\left(\mu+b\right)}\left(e^{-\mu t_1}e^{\left(\mu+b\right)T}\left(T-M\right) - \dfrac{1}{b}\left(e^{bT} - e^{bM}\right)\right) \end{array}\right] \tag{18}$$

and

$$TC_3 = \frac{1}{T}\left[\begin{array}{l} A + \dfrac{S\left(h_0 + c\mu\right)\left(1 - e^{-\mu t_1}\right)}{\mu} + \dfrac{a\left(h_0 + c\mu\right)}{\left(\mu + b\right)} e^{\left(\mu+b\right)T} e^{-\mu t_1}\left(T - t_1\right) \\ -\dfrac{a\left(h_0 + c\mu\right)}{b\left(\mu + b\right)}\left(e^{bT} - e^{bt_1}\right) + \dfrac{a\left(h_r + c\eta\right)}{\left(\eta + b\right)}\left(\dfrac{e^{\left(b+\eta\right)t_1}}{\eta} - e^{bt_1} M_1 + \dfrac{1}{b}\right) \\ + pI_e\left(T\dfrac{ae^{bT}}{b} - \dfrac{ae^{bT}}{b^2} + \dfrac{a}{b^2} + ae^{bT} T\left(M - T\right)\right) \end{array}\right] \quad (19)$$

## 5. SOLUTION PROCEDURE

Our aim is to find out the best possible values of $t_1^*$ such that $TC\left(t_1\right)$ is minimal. For the optimum solution of $TC_1$, we have

$$\frac{\partial TC_1}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial^2 TC_1}{\partial {t_1}^2} > 0$$

$$TC_1 = \frac{1}{T}\left[\begin{array}{l} N_1 + N_2\left(\dfrac{e^{\left(b+\eta\right)t_1}}{\eta} - e^{bt_1} M_1\right) - N_4\left\{\dfrac{1}{b} e^{bt_1} + \dfrac{e^{\left(b+\eta\right)t_1}}{\eta}\left(e^{-\eta t_1} - e^{-\eta M}\right)\right\} \\ -N_5 e^{-\mu t_1} + N_6\left(e^{\left(\mu+b\right)T} e^{-\mu t_1}\left(T - t_1\right) - \dfrac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) - N_3 e^{-\alpha t_1} \end{array}\right] \quad (20)$$

Where,

$$N_1 = A + \frac{a\left(h_r + c\eta\right)}{b\left(\eta + b\right)} - apI_e E + \frac{Sh_o}{\mu} + Sc + \frac{acI_p}{b\left(\eta + b\right)} e^{bM} + cI_p \frac{S}{\mu} e^{-\mu M}, N_2 = \frac{a\left(h_r + c\eta\right)}{b\left(\eta + b\right)},$$

$$N_3 = \left(\frac{Sh_o}{\mu} + Sc\right), N_4 = \frac{acI_p}{\left(\eta + b\right)}, \; N_5 = \frac{cI_p S}{\mu}, N_6 = \frac{a}{\left(\mu + b\right)}\left(h_o + \mu c + cI_p\right)$$

$$\frac{\partial TC_1}{\partial t_1} = \frac{1}{T}\left[\begin{array}{l} N_2\left(\dfrac{\left(b + \eta\right)e^{\left(b+\eta\right)t_1}}{\eta} - be^{bt_1} M_1\right) + N_4\left\{-\dfrac{\left(b + \eta\right)e^{\left(b+\eta\right)t_1}}{\eta}\left(e^{-\eta t_1} - e^{-\eta M}\right)\right\} \\ + \mu N_5 e^{-\mu t_1} + N_6\left(-\mu e^{\left(\mu+b\right)T} e^{-\mu t_1}\left(T - t_1\right) - e^{\left(\alpha+b\right)T} e^{-\mu t_1} + e^{bt_1}\right) + \mu N_3 e^{-\mu t_1} \end{array}\right] \quad (21)$$

We consider $\frac{\partial TC_1}{\partial t_1} = 0$, which gives us the value of $t_1$.

Again, we have the value of

$$\frac{\partial^2 TC_1}{\partial t_1^2}=\frac{1}{T}\left[\begin{array}{l} N_2\left(\frac{(b+\eta)^2 e^{(b+\eta)t_1}}{\eta}-b^2 e^{bt_1}M_1\right)+ \\ N_4\left\{(b+\eta)e^{bt_1}-\frac{(b+\eta)^2 e^{(b+\eta)t_1}}{\eta}\left(e^{-\eta t_1}-e^{-\eta M}\right)\right\}-\mu^2 N_5 e^{-\mu t_1} \\ +N_6\left(\mu^2 e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)+2\mu e^{(\mu+b)T}e^{-\mu t_1}+be^{bt_1}\right)-\mu^2 N_3 e^{-\mu t_1} \end{array}\right] \tag{22}$$

Thus, $\frac{\partial^2 TC_1}{\partial t_1^2}>0$ grasps at $t_1$, and then $TC_1$ is minimal.

For the optimum solution of $TC_2$, we have

$\frac{\partial TC_2}{\partial t_1}=0$ and $\frac{\partial^2 TC_2}{\partial t_1^2}>0$

$$TC_2=\frac{1}{T}\left[\begin{array}{l} A+K_1\left(\frac{e^{(b+\eta)t_1}}{\eta}-e^{bt_1}M_1+\frac{1}{b}\right)+K_2\left(1-e^{-\mu t_1}\right)-K_3 \\ +K_4\left(e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)+K_5\left(e^{-\mu t_1}K_6-K_7\right) \end{array}\right] \tag{23}$$

Where,

$K_1=\frac{a(h_r+c\eta)}{(\eta+b)}, K_2=(h_o+c\mu)\frac{S}{\mu},\ K_3=apI_eE, K_4=\frac{a(h_o+c\mu)}{(\mu+b)}, K_5=\frac{cI_pa}{(\mu+b)},$

$K_6=e^{(\mu+b)T}(T-M), K_7=\frac{1}{b}\left(e^{bT}-e^{bM}\right)$

$$\frac{\partial TC_2}{\partial t_1}=\frac{1}{T}\left[\begin{array}{l} K_1\left(\frac{(b+\eta)e^{(b+\eta)t_1}}{\eta}-be^{bt_1}M_1\right)+K_2\mu e^{-\mu t_1} \\ +K_4\left(-\mu e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)-e^{(\mu+b)T}e^{-\alpha\mu_1}+e^{bt_1}\right)-\mu K_5 e^{-\mu t_1}K_6 \end{array}\right] \tag{24}$$

We consider $\frac{\partial TC_2}{\partial t_1}=0$, which gives us the value of $t_1$.

Again, we have the value of

$$\frac{\partial^2 TC_2}{\partial t_1^2}=\frac{1}{T}\left[\begin{array}{l} K_1\left(\frac{(b+\eta)^2 e^{(b+\eta)t_1}}{\eta}-b^2 e^{bt_1}M_1\right)-K_2\mu^2 e^{-\mu t_1}+\mu^2 K_5 e^{-\mu t_1}K_6 \\ +K_4\left(\mu^2 e^{(\mu+b)T}e^{-\mu t_1}(T-t_1)+2\mu e^{(\mu+b)T}e^{-\mu t_1}+be^{bt_1}\right) \end{array}\right] \tag{25}$$

Thus, $\frac{\partial^2 TC_2}{\partial t_1{}^2} > 0$ grasps at $t_1$, and then $TC_2$ is minimal.

For the optimum solution of $TC_3$, we have

$\frac{\partial TC_3}{\partial t_1} = 0$ and $\frac{\partial^2 TC_3}{\partial t_1{}^2} > 0$

$$TC_3 = \frac{1}{T}\begin{bmatrix} A + K_1\left(\frac{e^{(b+\eta)t_1}}{\eta} - e^{bt_1}M_1 + \frac{1}{b}\right) + K_2\left(1 - e^{-\mu t_1}\right) \\ +K_4\left(e^{(\mu+b)T}e^{-\mu t_1}\left(T - t_1\right) - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) + K_8 \end{bmatrix} \quad (26)$$

Where,

$$K_1 = \frac{a\left(h_r + c\eta\right)}{\left(\eta + b\right)}, K_2 = \frac{S\left(h_o + \mu\right)}{\mu}, \; K_4 = \frac{a\left(h_o + c\mu\right)}{\left(\mu + b\right)},$$

$$K_8 = pI_e\left(\frac{Tae^{bT}}{b} - \frac{ae^{bT}}{b^2} + \frac{a}{b^2} + ae^{bT}T\left(M - T\right)\right)$$

$$\frac{\partial TC_3}{\partial t_1} = \frac{1}{T}\begin{bmatrix} K_1\left(\frac{(b+\eta)e^{(b+\eta)t_1}}{\eta} - be^{bt_1}M_1\right) - \mu K_2 e^{-\mu t_1} \\ +K_4\left(-\mu e^{(\mu+b)T}e^{-\mu t_1}\left(T - t_1\right) - e^{(\mu+b)T}e^{-\mu t_1} + e^{bt_1}\right) \end{bmatrix} \quad (27)$$

We consider $\frac{\partial TC_3}{\partial t_1} = 0$, which gives us the value of $t_1$.

Again, we have the value of

$$\frac{\partial^2 TC_3}{\partial t_1{}^2} = \frac{1}{T}\begin{bmatrix} K_1\left(\frac{(b+\eta)^2 e^{(b+\eta)t_1}}{\eta} - b^2 e^{bt_1}M_1\right) + \mu^2 K_2 e^{-\mu t_1} \\ +K_4\left(\mu^2 e^{(\mu+b)T}e^{-\mu t_1}\left(T - t_1\right) + 2\mu e^{(\mu+b)T}e^{-\mu t_1} + be^{bt_1}\right) \end{bmatrix} \quad (28)$$

Thus, $\frac{\partial^2 TC_3}{\partial t_1{}^2} > 0$ grasps at $t_1$, and then $TC_3$ is minimal.

Thus, we obtained the optimal solution of $t_1$ i.e. $t_1^*$ by solving equations $(21)$, $(24)$, and $(27)$, respectively. We present an optimization algorithm to find the best possible solution of this model.

## 6. ALGORITHM

Step 1: Input the initial parameters.

Step 2: Solving equation $(21)$ by using Newton Raphson method to find the optimal solution of $t^{*}{}_{1}^{1}$. Let $t_1^* = t_1^{1*}, TC^* = TC_1^*\left(t_1^{1*}\right)$, if $M \le t_1^* < T_1$, otherwise go to step 3.

Step 3: Solving equation $(24)$ by using Newton Raphson method to find the optimal solution of $t_1^{*2}$. Let $t_1^* = t_1^{*2}, TC^* = TC_2^*\left(t_1^{*2}\right)$, if $t_1^{*2} \le M \le T^*{}_2$, otherwise go to step 4.

Step 4: Solving equation $(27)$ by using Newton Raphson method to find the optimal solution of $t_1^{*3}$. Let $t_1^* = t_1^{*3}, TC^* = TC_3^*\left(t_1^{*3}\right)$, if $t_1^{*3} < M \le T_3^*$, otherwise go to step 5.

Step 5: Let $t_1^* = \arg\min\left\{TC_1\left(t^1{}_1^*\right), TC_2\left(t^2{}_1^*\right), TC_3\left(t^3{}_1^*\right)\right\}$, the optimal value of $t_1^*$ and $TC^*$

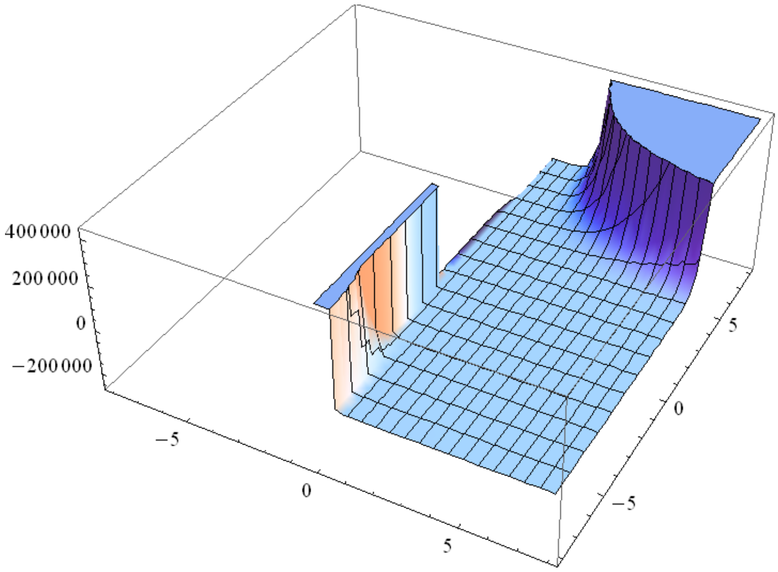


Figure1: The convexity of the total relevant cost $TC_1$

## 7. NUMERICAL EXAMPLE

We illustrate the inventory model with the following parameters:
$A = \$50/$ Order, $a = 10$, $b = 1, h_r = \$3/unit/$ year, $h_o = \$1/unit/$ year, $p = \$12, c = \$2/unit, I_p = \$0.15/$ year, $I_e = \$0.12/$ year, $T = 1$ year, $M = 0.25$ year, $S = 10$ units, $\mu = 0.1$, $\eta = 0.06$.

We get

$t_1^* \to 0.03\, and\, TC_1 = 66.16, t_1^* \to 0.24\, and\, TC_2 = 69.97, t_1^* \to 0.5\, and\, TC_3 = 80.13$

According to the algorithm 6, it is found that the optimal solution

$t_1^* = \arg\min\left\{TC_1\left(t^{1}{}_1^{*}\right), TC_2\left(t^{2}{}_1^{*}\right), TC_3\left(t^{3}{}_1^{*}\right)\right\}$, which described

$t_1^* \to 0.03$ and the optimal cost is $TC_1 = 66.16$

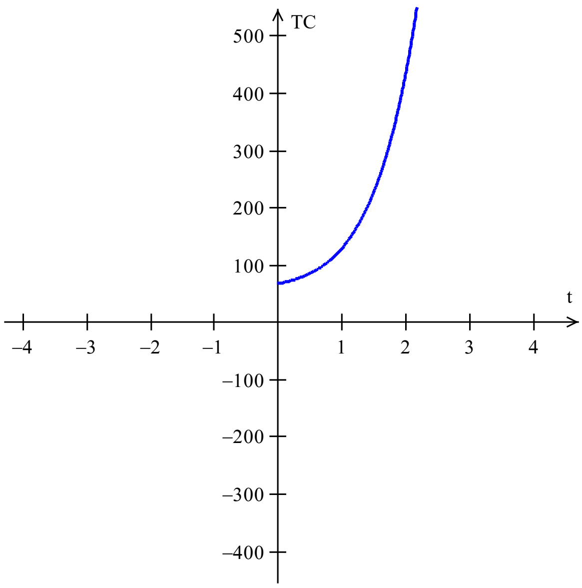


Figure 2: Graphical representation of $t_1^*$ and $TC_1$

## 8. SENSITIVITY ANALYSIS

To know how the optimal solution is affected by the values of parameters, we derive the sensitivity analysis for some of the parameters. The particular values of some parameters are increased or decreased by $+5\%$, $-5\%$ and $+10\%$, $-10\%$. After that, we derive the value of $t_1$ and $TC_1$ with the help of increased or decreased values of some parameters. The result of the minimum relevant cost is in the following table:

| Parameters | Values | Values | Values | Values | Values |
|---|---|---|---|---|---|
| | | +5% increased | +10% increased | –5% decreased | –10% decreased |
| S | 10 | 10.5 | 11 | 9.5 | 9 |
| A | 50 | 52.5 | 55 | 47.5 | 45 |
| a | 10 | 10.5 | 11 | 9.5 | 9 |
| $b$ | 1 | 1.05 | 1.1 | 0.95 | 0.9 |
| $h_r$ | 3 | 3.15 | 3.3 | 2.85 | 2.7 |
| $h_o$ | 1 | 1.05 | 1.1 | 0.95 | 0.9 |
| M | 0.25 | 0.26 | 0.27 | 0.24 | 0.22 |
| $\mu$ | 0.1 | 0.10 | 0.11 | 0.09 | 0.09 |
| $\eta$ | 0.06 | 0.063 | 0.066 | 0.057 | 0.054 |
| $t_1^*$ | 0.03 | 0.25 | 0.31 | 0.25 | 0.4 |
| $TC_1$ | 66.16 | 69 | 70.89 | 60.14 | 57.95 |

## 9. CONCLUSION

We developed a two warehouse inventory model for deteriorating item with exponential demand rate under conditionally permissible delay in payment. Shortage is not allowed and deterioration rate is constant. It was measured that charge of holding cost per unit for a rented warehouse is higher than for the owned warehouse, but we provided the best protective facility in a rented warehouse to reduce the rate of deterioration. We stored the goods, first in the owned warehouse than in the rented warehouse, and first consume the goods from the rented warehouse and then from the owned warehouse. This model gives us the most favorable replenishment policies for minimizing the total appropriate inventory cost. Numerical example is provided to evaluate the proposed model. Sensitivity analysis of the optimal solution with respect to key parameters is carried out. The model which is designed and analyzed can be extended in several ways such as quantity discount, time dependent holding cost, time varying deterioration rate, time-proportional backlogging rate, time dependent demand etc.

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