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# NEWTON METHOD FOR DETERMINING THE OPTIMAL REPLENISHMENT POLICY FOR EPQ MODEL WITH PRESENT VALUE

Jeff Kuo-Jung WU[1], Hsui-Li LEI [2], Sung-Te JUNG[2], Peter CHU[3]

[1]*Department of Logistics Management,*
*National Defense University, Taiwan, R.O.C.*
*Andy1977661014@hotmail.com*
[2]*Department of Communication Engineering,*
*Oriental Institute of Technology, Taiwan, R.O.C.*
*{FO016, FO015}@mail.oit.edu.tw*
[3]*Department of Traffic Science,*
*Central Police University, Taiwan, R.O.C.*
*una211@mail.cpu.edu.tw*

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**Abstract**: This paper is a response for the paper of Dohi, Kaio and Osaki, that was published in RAIRO: Operations Research, 26, 1-14 (1992) for an EPQ model with present value. The purpose of this paper is threefold. First, the convex and increasing properties for the first derivative of the objective function are proved. Second, we apply the Newton method to find the optimal cycle time. Third, we provide some numerical examples to demonstrate that the Newton method is more efficient than the bisection method.

**Keywords**: Economic production quantity, inventory model, present value, Newton method.

## 1. INTRODUCTION

The Newton method and the bisection method are the two most popular methods to find the zero of an equation. Wan and Chu [15] demonstrated that the Newton method is better than the bisection method for EOQ inventory models with present value. The purpose of this paper is extended discussion for EPQ inventory models. For the application of the bisection method, we need a lower bound and an upper bound for the optimal solution. On the other hand, when utilizing Newton method, we only need one bound. Only an upper bound is required for minimization of monotonically increased convex function. For decreasing and convex function, we need a lower bound. Therefore, intuitively the Newton method should be more easily applied by researchers. However, the explanation of Newton method is more complicated than the bisection method. Hence, many researchers still preferred to use the bisection method. Chung and Tsai [7] studied Goswami and Chaudhuri [12] to reveal that Newton method may diverge with an improper starting point so that an approximated solution of the first derivative could not satisfy constraints. Hence, the analytical structure of the objective function should be examined case by case in order to validate the legitimacy of the Newton method. Moreover, how to decide an appropriate starting point must be checked to insure a fast convergent sequence to the optimal solution. Several papers, for examples, Dohi *et al.* [10], Cormier and Gunn [9], Chu *et al.* [1] Chung *et al.* [3], and others have applied Newton method in order to derive the optimal solution. Dohi *et al.* [10] did not explain how to select a starting point to execute the Newton method. Chung [2] also considered the EPQ inventory model of Dohi *et al.* [10]. He disapproved of using the Newton method first and then adopted a bisection algorithm to compute the optimal cycle time. Chung *et al*. [5] revised Chung [2] to prepare a simple bisection algorithm and obtain the optimal cycle time without unnecessary assumptions. Chung [2] and Chung *et al*. [5] considered that the iterative sequence spanned by Newton method may not converge to an optimal solution when the first derivative has two roots. There are six other papers, Dohi *et al.* [11], Chung *et al.* [6], Chung and Lin [4], Chung and Tsai [8], Huang and Chung [13], and Moon *et al.* [14] that have referred to Dohi *et al.* [10], but none of them provided further discussion of the Newton method for the optimal solution. In this paper, we will show that Newton method is applicable to this kind of EPQ inventory model of Dohi *et al*. [10], with any starting point. Under the consideration of the speed of modern computer, it is unnecessary to consider a good starting point to accelerate the convergence of the iterative procedure. We compare numerical examples in Dohi *et al*. [10], Chung [2] and Chung *et al*. [5], to illustrate that the Newton method is efficient to derive the optimal solution and under a small threshold, the Newton method outperforms both bisection methods suggested in Chung [2] and Chung *et al*. [5] in order to illustrate.

## 2. REVIEW OF PREVIOUS RESULTS

To be compatible with Dohi *et al.* [10], we apply the same notation and assumptions for a deterministic production inventory model without shortage under the condition of present value. The notations used in this paper are listed below. $D$ is the demand rate; $H$ is the holding cost; $K$ is the order cost; $r$ is the interest rate; $S$ is the production rate, with $S > D$; and $TC_r(t)$ is the present value of total cost. With the following assumptions, the EPQ model with infinite planning horizon is developed. A single item is considered and the replenishment lead time is assumed to be zero. There is no shortage. The beginning and ending inventory level are both zero. Since the interest rate, $r$ is assumed stable over the time, so the replenishment cycle, say $t_0$, is constant over time. $t_0 = t_d + t_s$ where $t_d$ is the production period and $t_s$ is the consumption period.

Let us recall the Eq. (4) of Dohi *et al*. [10], the present value of total costs over the entire time horizon is expressed as

$$TC_r(t) = \frac{K + \frac{H}{r^2}\left\{S\left(1 - \exp\left(\frac{-rDt}{S}\right)\right) - D\left(1 - \exp(-rt)\right)\right\}}{1 - \exp(-rt)}. \quad (1)$$

Hence, in Eqs (5) and (6) of Dohi *et al*. [10], they obtained that

$$\frac{dTC_r(t)}{d\,t} = \frac{\exp(-rt)}{\left(1 - \exp(-rt)\right)^2}\xi(t) \quad (2)$$

where

$$\xi(t) = \frac{HD}{r}\left(\exp\left(\frac{r(S-D)t}{S}\right) - \exp\left(\frac{-rDt}{S}\right)\right) - rK - \frac{HS}{r}\left(1 - \exp\left(\frac{-rDt}{S}\right)\right). \quad (3)$$

In Theorem 1 of their paper, they derived that, with the condition $S > D$, $\xi(t)$ is a strictly increasing and continuous function. According to $\xi(0) = -rK < 0$ and $\xi(\infty) = \infty$, they obtained that there is a unique solution, say $t^*$, for $\xi(t) = 0$.

## 3. OUR ANALYTICAL APPROACH

In the following, we begin to develop some properties for $\xi(t)$ to derive that

$$\xi'(t) = HD\left(\frac{S-D}{S}\right)\left[\exp\left(\frac{r(S-D)t}{S}\right) - 1\right] + HD\left(\frac{S-D}{S}\right)\left[1 - \exp\left(\frac{-rDt}{S}\right)\right] \quad (4)$$

and

$$\xi''(t) = rHD\left(\frac{S-D}{S}\right)^2 \exp\left(\frac{r(S-D)t}{S}\right) + rH(S-D)\left(\frac{D}{S}\right)^2 \exp\left(\frac{-rDt}{S}\right). \quad (5)$$

Recall the properties of exponential function, we know that $\xi'(t) > 0$ so $\xi(t)$ is an increasing function, for $t > 0$. Moreover, $\xi''(t) > 0$ then $\xi(t)$ is a convex function, for $t > 0$. We summarize our findings in the next Theorem.

**Theorem 1.** *Under the condition* $S > D$, *for* $t > 0$, $\xi(t)$ *is an increasing and a convex function.*

To find the optimal solution for the objection function, $TC_r(t)$, that is to solve

$\frac{dTC_r(t)}{d\ t} = 0$. According to Eq (2), since exponential function is always positive, we have to consider the solution of the following equation

$$\xi(t) = 0 . \tag{6}$$

The traditional approach is to discuss how to select a good starting point as close to the optimal cycle time, $t^*$, as possible. On the other hand, if we really understand the properties of $\xi(t)$ being continuous and convex, then we will demonstrate that selecting any starting point will converge to the desired optimal cycle time, $t^*$.

We will apply the Newton method to find the solution of $\xi(t) = 0$. If we arbitrarily select a positive number, say $t_1$, then there cases may happen: Case (a) $\xi(t_1) = 0$, Case (b) $\xi(t_1) > 0$, and Case (c) $\xi(t_1) < 0$.

For Case (a), $t_1$ is the optimal solution so we already find it as $t_1 = t^*$.

For Case (b), we will show that there exists a decreasing sequence with

$$t_{j+1} = t_j - \frac{\xi(t_j)}{\xi'(t_j)} \tag{7}$$

for $j = 1, 2, ...$ and $t^* < t_{j+1} < t_j$.

If we can show that $t^* < t_2 < t_1$ then it yields that $\xi(t_2) > 0$ and $t^* < t_3 < t_2$. By induction, it will imply that $t^* < t_{j+1} < t_j$ for $j = 1, 2, ...$. We assume that $y(t)$ is the tangent line of $\xi(t)$ with the tangent point $(t_1, \xi(t_1))$ so that

$$y(t) = \xi(t_1) + \xi'(t_1)(t - t_1) . \tag{8}$$

We will show that $y(t) \le \xi(t)$ and if $t \ne t_1$, then $y(t) < \xi(t)$. We compute

$$\begin{aligned} &\xi(t) - y(t) \\ &= \xi(t) - \xi(t_1) - \xi'(t_1)(t - t_1) \\ &= \xi'(\alpha)(t - t_1) - \xi'(t_1)(t - t_1) \\ &= \xi''(\beta)(\alpha - t_1)(t - t_1) , \end{aligned} \tag{9}$$

where $\alpha$ is derived from Mean Value Theorem with $\alpha$ is between $t$ and $t_1$, and $\beta$ is again derived from Mean Value Theorem with $\beta$ is between $\alpha$ and $t_1$. We consider the relations among $t$, $t_1$ and $\alpha$ to divide the problem into following three cases: Case (1) $t > t_1$, Case (2) $t = t_1$, and Case (3) $t < t_1$.

For Case (1), $t > t_1$, then $t_1 < \alpha < t$ to imply that $\alpha - t_1 > 0$ and $t - t_1 > 0$, hence according to Eq (9), it yields that $\xi(t)-y(t)>0$.

For Case (2), $t=t_1$, since $y(t_1)=\xi(t_1)$, it shows that $\xi(t_1)-y(t_1)=0$.

For Case (3), $t<t_1$, then $t<\alpha<t_1$ to imply that $\alpha-t_1<0$ and $t-t_1<0$, therefore based on Eq (9) again, it derives that $\xi(t)-y(t)>0$.

We combine our results in the next lemma.

**Lemma 1.** *If f(t) is an increasing and a convex function and y(t) is the tangent line of f(t) with tangent point* $(t_1, f(t_1))$, *then* $f(t)>y(t)$ *for* $t\neq t_1$.

According to Eq. (7), $t_2$ is the intersection of the tangent line, $y(t)$, with tangent point, $(t_1,\xi(t_1))$, with the x-axis so $y(t_2)=0$. From Lemma 1, it shows that $f(t_2)>y(t_2)=0$ so $t^*<t_2$.

On the other hand, using Eq (7) again, and $\xi'(t_1)>0$, under the condition of Case (b) $\xi(t_1)>0$, such that $t_2<t_1$. Consequently, it derives that $t^*<t_2<t_1$. We summarize our findings in the next lemma.

**Lemma 2.** *If* $f(t)$ *is an increasing and a convex function, and we select a point, say* $t_1$, *with* $f(t_1)>0$ *then the Newton method will generate a decreasing sequence,* $(t_n)$ *with* $t^*<t_{n+1}<t_n$ *for* $n=1,2,\ldots$.

Now we consider Case (c), under the constraint $\xi(t_1)<0$, and we apply the Newton method of Eq. (7), then we will prove that $\xi(t_2)>0$.

Owing to Eq (7), we recall that

$$t_2=t_1-\frac{\xi(t_1)}{\xi'(t_1)}=t_1+\frac{-\xi(t_1)}{\xi'(t_1)}. \tag{10}$$

From $\xi(t)$ increases then $\xi'(t)>0$ and then $\xi'(t_1)>0$. Under the condition of $\xi(t_1)<0$, hence, Eq (10) implies that $t_2>t_1$. From Lemma 1, $\xi(t_2)>y(t_2)=0$. Therefore, $(t_n)$ for $t\geq 2$, becomes a decreasing sequence, $t^*<t_{n+1}<t_n$ for $n=2,3,\ldots$.

Now, we consider the decreasing sequence, $(t_n)$ for $n=2,3,\ldots$, to prove that is indeed convergent to $t^*$. From the completeness axiom of real number, the bounded below decreasing sequence with $t^*<t_{n+1}<t_n$ for $n=2,3,\ldots$, then it must converges to its most lower bound, say $t^\Phi$, as $\lim_{n\to\infty} t_n=t^\Phi$. According to Eq (7), it yields that

$$\lim_{n\to\infty} t_{n+1} = \lim_{t\to\infty}\left( t_n - \frac{\xi(t_n)}{\xi'(t_n)} \right) \tag{11}$$

We know that $\xi(t)$ and $\xi'(t)$ are both continuous functions so that they can interchange with the limit to imply that

$$t^{\Phi} = t^{\Phi} - \frac{\xi\left(t^{\Phi}\right)}{\xi'\left(t^{\Phi}\right)}. \tag{12}$$

Based on Eq (12), it obtains that $\xi\left(t^{\Phi}\right) = 0$. On the other hand, from the strictly increasing property of $\xi(t)$, we know that $\xi(t)$ has a unique root at $t^*$. Hence, $t^{\Phi} = t^*$ and $\lim_{n\to\infty} t_n = t^*$. We summarize our findings in the next theorem.

**Theorem 2.** *If we select a point, say* $t_1$, *as the starting point for the Newton method of an increasing and a convex function,* $f(t)$ *,with unique zero at* $t^*$ *then the decreasing sequence,* $(t_n)_{n\geq 2}$ *will converge to* $t^*$.

Based on the above discussion, since $\xi(t)$ is an increasing and a convex function with $\xi(0) < 0$ and $\xi(\infty) = \infty$, so $\xi(t)$ has a unique root, say $t^*$. In Theorem 2, we prove that $(t_n)_{n\geq 2}$ will converge to $t^*$ that is independent of the starting point $t_1$. However, if we select the starting point that satisfies $\xi(t_1) > 0$ that will accelerate the convergence. Therefore, we develop an algorithm to find the optimal cycle time. We will use a threshold, say $\varepsilon$, to control the iterated computation for the Newton method.

**Step 1:** Given $\varepsilon > 0$ as the threshold, and pick a point, say $t_1$ with $\xi(t_1) > 0$ as the starting point for the Newton method.

**Step 2:** To assume that $t_j = t_{j-1} - \dfrac{\xi(t_{j-1})}{\xi'(t_{j-1})}$ for $j = 2, 3, \ldots$

**Step 3:** To define that $n = \min\left\{ j : \left| TC_r(t_{j-1}) - TC_r(t_j) \right| < \varepsilon \right\}$

**Step 4:** To accept that the optimal cycle time is $t_n$ and optimal value is $TC_r(t_n)$.

Since the computer computation ability is improved so that the computation step is no longer an important issue for researchers. We will demonstrate this fact by the CPU time for our numerical examples.

## 4. NUMERICAL EXAMPLES

We consider the following example, with the data of parameters, $r$ = 0.3, $S$ = 9, $K$ = 36.5, $H$ = 60.5 and $D$ = 3 that is related to examples in Dohi *et al*. [10], Chung [2] and Chung *et al*. [5]. For example 1, we take $\varepsilon$ = 0.001, with the starting point, $t_1 = 1$, then after two iteration, it shows that $t_4 = t^* = 0.766429$ and $TC_r(t_4) = 334.0771$.

**Table 1:** The Newton method for $\varepsilon$ = 0.001

| | $k$=1 | $k$=2 | $k$=3 | $k$=4 |
|---|---|---|---|---|
| $t_k$ | 1 | 0.795012 | 0.766965 | 0.766429 |
| $\xi(t_k)$ | 7.852 | 0.844 | $1.55\times10^{-2}$ | $5.66\times10^{-6}$ |
| $TC_r(t_k)$ | 345.5118 | 334.2920 | 334.0772 | 334.0771 |

For example 2, we take $\varepsilon = 0.000001$, with the starting point, $t_1 = 1$, then after three iteration, it shows that $t_5 = t^* = 0.766429$ and $TC_r(t_5) = 334.0771335$.

**Table 2:** The Newton method for $\varepsilon = 0.000001$

| | $k$=1 | $k$=2 | $k$=3 | $k$=4 | $k$=5 |
|---|---|---|---|---|---|
| $t_k$ | 1 | 0.795012 | 0.766965 | 0.766429 | 0.766429 |
| $\xi(t_k)$ | 7.852 | 0.844 | $1.55\times10^{-2}$ | $5.66\times10^{-6}$ | $8.52\times10^{-13}$ |
| $TC_r(t_k)$ | 345.5118109 | 334.2920631 | 334.0772119 | 334.0771335 | 334.0771335 |

From the above three examples, it reveals that the Newton method is an efficient method to find the optimal solution. Moreover, the value of $\varepsilon$ will slightly influence the iteration number and computer CPU seconds.

Here, we consider the impact of different starting point to list the results in the next table.

**Table 3:** The effect of arbitrary starting point

| Item | Initial value | Step | Time |
|---|---|---|---|
| 1 | $t_1$ =0.1 | 6 | 0.047 |
| 2 | $t_1$ =1 | 3 | 0.027 |
| 3 | $t_1$ =2 | 4 | 0.031 |
| 4 | $t_1$ =20 | 10 | 0.051 |
| 5 | $t_1$ =200 | 46 | 0.094 |

Based on Table 3, we may say that the Newton method is an effective method for finding the optimal solution that is independent of the starting point.

Next, we compared the Newton method with the bisection method. First, in Chung [2], under the restriction $S \ge 2D$, he found an upper bound, $t_U^* = \sqrt{\frac{2KS}{HD(S-D)}}$, and took zero as the lower bound for $t^*$. On the other hand, in Chung *et al*. [5], they tried to improve the bisection method to derive an upper bound, $T_U = \frac{S}{S-D}\sqrt{\frac{2K}{HD}}$ and a lower bound, $T_L = \frac{2}{r+\sqrt{r^2+\frac{2HD}{K}\left(1+\frac{S}{D}-\frac{D}{S}\right)}}$.

To compare it with bisection method, we take the upper bound of Chung [2] as our starting point for the Newton method to list the iteration number in the next table.

**Table 4:** Comparison of iteration numbers

| Method | $\varepsilon = 10^{-3}$ | $\varepsilon = 10^{-6}$ |
|---|---|---|
| Newton method | 2 | 3 |
| Bisection [2] | 8 | 20 |
| Bisection [5] | 7 | 18 |

From Table 4, it demonstrates that when the accuracy is emphasized then the Newton method is more efficient than the bisection method.

## 5. CONCLUSIONS

In this paper, when the objective function is increasing and convex, we prove that the Newton method is efficient to find the root of the objective function. Based on numerical examples that are related to Dohi *et al*. [10], Chung [2] and Chung *et al*. [5], we show that Newton method with arbitrary starting point will generate a very fast converge sequence to the optimal solution.

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