Yugoslav Journal of Operations Research
18 (2008), Number 1, 13-22
DOI: 10.2298/YUJOR0801013D

THE INVERSE MAXIMUM FLOW PROBLEM WITH LOWER AND UPPER BOUNDS FOR THE FLOW

Adrian DEACONU
Theoretical Computer Science Department,
Faculty of Mathematics and Informatics,
University "Transilvania" of Brasov, Romania,
am_deaconu@yahoo.com, a.deaconu@unitbv.ro

Received: March 2006 / Accepted: April 2008

Abstract: The general inverse maximum flow problem (denoted GIMF) is considered, where lower and upper bounds for the flow are changed so that a given feasible flow becomes a maximum flow and the distance (considering l_1 norm) between the initial vector of bounds and the modified vector is minimum. Strongly and weakly polynomial algorithms for solving this problem are proposed. In the paper it is also proved that the inverse maximum flow problem where only the upper bound for the flow is changed (IMF) is a particular case of the GIMF problem.

Keywords: Inverse problems, maximum flow, minimum cut, residual network, graph search.

1. INTRODUCTION

In the last years many papers were published in the field of the inverse combinatorial optimization [2, 3, 5-17]. The inverse maximum flow problem (IMF) is one of the problems that have been studied. Strongly polynomial algorithms to solve this problem were presented by *C. Yang, J. Zhang* and *Z. Ma* [14]. The IMF problem is reduced to a minimum cut problem in an auxiliary network with finite and infinite arc capacities. Therefore, weakly and non-polynomial algorithms can not be directly applied.

In the paper of *C. Yang, J. Zhang* and *Z. Ma* [14] only the upper bound for the flow is changed as little as possible in order to make a given feasible flow becomes a maximum flow. That is why in many networks the inverse maximum flow problem does not have solution.

We shall study the more general case where lower and upper bounds for the flow can be modified in order to make the given flow become a maximum flow. This improves the solution because the amount of change can be considerably lower. We shall call this problem as the **general inverse maximum flow problem** (denoted GIMF). As we shall see, if there are not many restrictions in modifying the bounds for the flow, then the GIMF always has solution.

We shall give a simple example to illustrate the difference between the two problems: IMF and GIMF. In the figure 1 on each arc (x, y) the first value is the lower bound $l(x, y)$ for the flow, the second value is the flow $f(x, y)$ of the arc and the third value is the upper bound $c(x, y)$ for the flow.

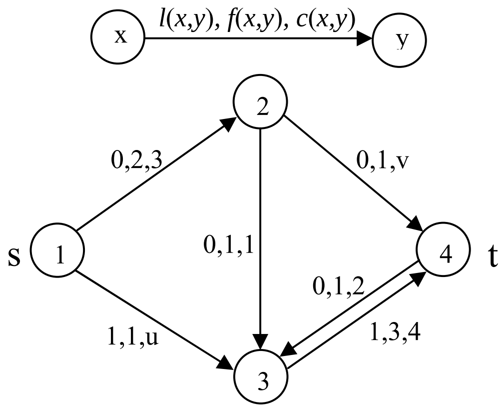

Figure 1: The network *G* and the given flow *f*

We consider the values u and v (which appear in the figure 1) greater than 4, i.e., $u, v > 4$.

In the figures 2 and 3 the solutions for IMF and GIMF are presented, the modified bounds are bolded.

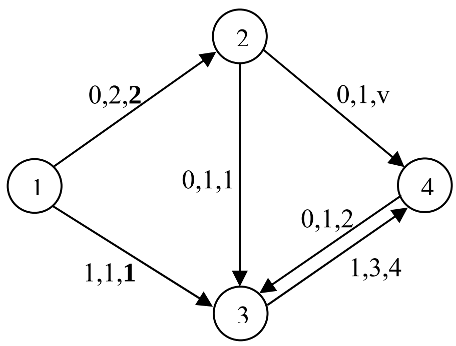

Figure 2: The solution of IMF

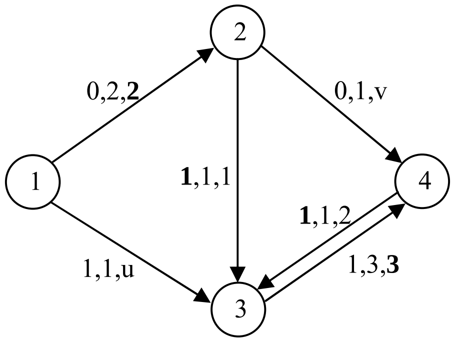

Figure 3: The solution of GIMF

The amount of change brought only to the arc upper bounds for the flow after solving IMF is $3–2 + u–1 > 4$. The amount of change to the upper and lower bounds after solving GIMF is equal to $3–2 + 1–0 + 1–0 + 4–3 = 4$. So, the solution for the GIMF is better than the solution of the IMF problem ($u > 4$ and u has no upper limit). Moreover, the solution of GIMF is at least as good as the solution of IMF in any network and for any given feasible flow.

2. THE GENERAL INVERSE MAXIMUM FLOW PROBLEM

Let $G = (N, A, c, l, s, t)$ be an s-t network, where N is the set of nodes, A is the set of directed arcs, c is the vector of the upper bounds for the flow, l is the vector of lower bounds for the flow, s is the source and t is the sink node. Of course, we have $l(x,y) \leq c(x, y)$, for each arc $(x, y) \in A$.

If a network has more than a source or/and more than a sink node, it can be transformed into an s-t network (introducing a super-source and a super-sink node) [4].

We shall introduce the definitions of the s-t cut, the capacity of an s-t cut and the minimum s-t cut in the network G:

Definition 1. *The set of arcs* $[X, \overline{X}] = (X, \overline{X}) \cup (\overline{X}, X)$ *is called an* ***s-t cut*** *in the network G, where:* $X \subset A$, $\overline{X} = A \setminus X$, $s \in X$, $t \in \overline{X}$, $(X, \overline{X}) = \{(x,y) \in A \mid x \in X \text{ şi } y \in \overline{X}\}$ *and* $(\overline{X}, X) = \{(x,y) \in A \mid x \in \overline{X} \text{ şi } y \in X\}$. $(X, \overline{X})$ *is called the set of the* ***direct arcs*** *of the s-t cut and* $(\overline{X}, X)$ *is the set of the* ***inverse arcs*** *of the s-t cut.*

Definition 2. *The* ***capacity of an s-t cut*** *is*

$$c[X, \overline{X}] = c(X, \overline{X}) - l(\overline{X}, X) = \sum_{(x,y) \in (X, \overline{X})} c(x,y) - \sum_{(x,y) \in (\overline{X}, X)} l(x,y).$$

It is easy to see that, if the network has no lower bounds for the flow, then the capacity of the *s*-*t* cut is $c[X, \overline{X}] = c(X, \overline{X}) = \sum_{(x,y)\in(X,\overline{X})} c(x,y)$.

Definition 3. *An s-t cut* $[X, \overline{X}]$ *is a* ***minimum s-t cut*** *if its capacity is minimum, i.e.,* $c[X, \overline{X}] = \min\left\{c[X', \overline{X}'] \mid [X', \overline{X}'] \text{ is an } s-t \text{ cut}\right\}$.

Let *f* be a given feasible flow in the network *G*. It means that *f* has to satisfy the flow balance condition and the capacity restrictions.

The balance condition for the flow *f* is:

$$\forall x \in N: \sum_{y\in N,(x,y)\in A} f(x,y) - \sum_{y\in N,(y,x)\in A} f(y,x) = \begin{cases} v(f), & x = s \\ -v(f), & x = t \\ 0, & x \in N \setminus \{s,t\} \end{cases}, \tag{1}$$

where *v*(*f*) is the value of the flow *f* from *s* to *t*.

The capacity restrictions are:

$$\forall (x, y) \in A : l(x, y) \le f(x, y) \le c(x, y). \tag{2}$$

The maximum flow problem is:

$$\begin{cases} \max v(f) \\ f \text{ is a feasible flow in } G \end{cases}. \tag{3}$$

The general inverse maximum flow problem (GIMF) consists in changing the lower bound vector *l* and the upper bound vector *c* as little as possible so that the given feasible flow *f* becomes a maximum flow in *G*.

The GIMF problem can be formulated using the following mathematical model:

$$\begin{cases} \min \|\overline{l} - l\| + \|c - \overline{c}\| \\ f \text{ is a maximum flow in } \overline{G} = (N, A, \overline{c}, \overline{l}, s, t) \\ l(x,y) - \gamma(x,y) \le \overline{l}(x,y) \le \min\{\overline{c}(x,y), l(x,y) + \beta(x,y)\} \\ c(x,y) - \delta(x,y) \le \overline{c}(x,y) \le c(x,y) + \alpha(x,y), \ \forall (x,y) \in A \end{cases}, \tag{4}$$

where $\alpha(x, y)$, $\delta(x, y)$, $\beta(x, y)$ and $\gamma(x, y)$ are given non-negative numbers and $\gamma(x, y) \le l(x,y)$, $\delta(x, y) \le c(x, y)$, for each arc $(x, y) \in A$.

We shall consider the l_1 norm in (4).

In order to make the flow *f* become a maximum flow in the network *G*, the lower bounds of some arcs from *A* must be increased or/and upper bounds of some arcs from *A* must be decreased. So, the conditions $l(x,y) - \gamma(x,y) \le \overline{l}(x,y)$ and $\overline{c}(x,y) \le c(x,y) + \alpha(x,y)$, for each arc $(x, y) \in A$ have no effect.

So, the generality of GIMF is not reduced if the following mathematical model is considered, instead of (4):

$$\begin{cases} \min\left\|\bar{l} - l\right\| + \left\|c - \bar{c}\right\| \\ f \text{ is a maximum flow in } \overline{G} = (N, A, \bar{c}, \bar{l}, s, t) \\ \bar{l}(x,y) \le \min\{\bar{c}(x,y), l(x,y) + \beta(x,y)\} \\ c(x,y) - \delta(x,y) \le \bar{c}(x,y), \ \forall (x,y) \in A \end{cases}. \quad (5)$$

When solving GIMF, let's observe that if the lower bound is changed on an arc (x, y), then it will be increased with the amount of $f(x, y) - l(x, y)$. If not so, then the flow f is not stopped from being increased on an augmenting path in G from s to t that contains the inverse directed arc (x, y) and the modification of the lower bound is useless. This means that if $l(x, y) + \beta(x, y) < f(x, y)$ on an arc (x, y), then when solving GIMF the lower bound will not be changed on (x, y).

Similarly, if the upper bound is changed on an arc (x, y), then it will be decreased with the amount of $c(x, y) - f(x, y)$ in order to stop the flow from being increased on a an augmenting path in G from s to t that contains the arc (x, y). So, if $c(x,y) - \delta(x, y) > f(x, y)$ on the arc (x, y), then the value $c(x, y)$ of the upper bound will not be changed on the arc (x, y).

It is easy to see that the GIMF problem has solution if and only if there is no an augmenting path from s to t that contains inverse directed arcs (x, y) with $l(x, y) + \beta(x, y) < f(x, y)$ and/or directed arcs (x, y) with $c(x, y) - \delta(x, y) > f(x, y)$, because on such a path (if exists) the flow f can not be stopped from being increased by modifying the bounds of the arcs.

A graph denoted $\widetilde{G} = (N, \widetilde{A})$ can be constructed to verify if GIMF has solution, where:

$$\widetilde{A} = \{(x,y) \mid l(y,x) + \beta(y,x) < f(y,x) \text{ or } c(x,y) - \delta(x,y) > f(x,y)\}.$$

So, we have the following theorem:

Theorem 1. *In the network G, the GIMF problem has solution for the given flow f, if and only if there is no directed path in the graph $\widetilde{G}$ from the node s to the node t.*

The verification can be done in $O(p)$ time complexity, using a graph search algorithm in the $\widetilde{G}$, where p is the number of arcs in the set $\widetilde{A}$. Of course, we have $p \le m$.

It is obviously to see that if the set $\widetilde{A}$ is empty, then GIMF has solution.

Let's consider now the residual network $G_f = (N, A_f, r, s, t)$ attached to the network G for the flow f, where:

$$\forall x, y \in N : r(x, y) = c(x, y) - f(x, y) + f(y, x) - l(y, x). \quad (6)$$

In the relation (6), for a pair of nodes (x, y) which is not a directed arc from A, we consider $l(x, y) = f(x, y) = c(x, y) = 0$. The set A_f contains any arc (x,y) for which the residual capacity is positive, i.e., $r(x, y) > 0$.

If the set $\widetilde{A}$ is empty, then solving GIMF is equivalent to find the set of arcs $B \subseteq A_f$ so that in case that the arcs of B are eliminated from A_f, there is no longer a directed path in $G_f^B = (N, A_f \setminus B)$ from the node s to the node t and $r(B) = \sum_{(x,y)\in B} r(x,y)$ is minimal. This means that B is the set of direct arcs of the minimum s-t cut in the network G_f (see the definitions 1, 2 and 3, where, instead of the network G, we have G_f and this network has no lower bounds).

As we have seen, if the set $\widetilde{A}$ is not empty, then, when solving GIMF, no change will be done to the lower bounds and/or to the upper bounds on the arcs of $\widetilde{A}$. This means that for each arc (x, y) of $\widetilde{A}$ such that $l(x, y) + \beta(x, y) < f(x, y)$ the residual capacity of (y, x) can be set to $+\infty$ and for each arc (x, y) of $\widetilde{A}$ such that $c(x, y) - \delta(x, y) > f(x, y)$ the residual capacity of (x, y) can be also modified to $+\infty$. If the GIMF problem has solution, then by setting the bounds to $+\infty$, we assure that these arcs will not be in the set B, which is the set of direct arcs of the minimum s-t cut in the residual network.

So, if the set $\widetilde{A}$ is not empty, then the minimum s-t cut must be searched in the network $G_f' = (N, A_f, r', s, t)$, where:

$$\forall(x, y) \in A_f : r'(x,y) = \begin{cases} +\infty, \text{ if } l(y,x) + \delta(y,x) < f(y,x) \\ \quad \text{or } c(x,y) - \beta(x,y) > f(x,y) \\ c(x,y) - f(x,y) + f(y,x) - l(y,x), \text{ otherwise} \end{cases} . \tag{7}$$

If the upper bounds of the arcs $(x, y) \in A \cap B$ are increased with the quantity $c(x, y) - f(x, y)$ and the lower bounds of the arcs $(x, y) \in A$ where $(y, x) \in B$ are decreased with quantity $f(x, y) - l(x, y)$, then the flow f is stopped from being increased and it becomes a maximum flow in the network G with the modified bounds for the flow. So, for each arc (x, y) from $B \cap A$ the upper bound must be changed to the value $f(x, y)$ and for each arc (x, y) from A, where $(y, x) \in B$, the lower bound of (x, y) must be changed to the value $f(x, y)$. This means that the solution of GIMF is the pair of vectors (c^*, l^*), where:

$$\forall(x,y) \in A : c^*(x,y) = \begin{cases} f(x,y), \text{ if } (x,y) \in B \\ c(x,y), \text{ otherwise} \end{cases} \tag{8}$$

and

$$\forall(x,y) \in A : l^*(x,y) = \begin{cases} f(x,y), \text{ if } (y,x) \in B \\ l(x,y), \text{ otherwise} \end{cases} \tag{9}$$

So, we have the following result:

Theorem 2. *(c^*, l^*) is the solution for the GIMF problem (4'), where c^* and l^* are defined in (8) and (9) and the set B is the set of direct arcs of a minimum s-t cut in the network G_f'.*

It is easy to see that the amount of change done to the bound vectors c and l is equal to the capacity of the minimum s-t cut in G_f', i.e.:

$$\left\| l^* - l \right\|_1 + \left\| c - c^* \right\|_1 = l^* - l + c - c^* = r'(B) . \tag{10}$$

3. ALGORITHMS FOR THE *GIMF* PROBLEM

As it has been seen so far, after verifying if GIMF has solution (using the network $\widetilde{G}$, see theorem 1), the GIMF problem can be reduced to a minimum *s-t* cut problem in the network G_f' (see theorem 2).

An algorithm for the GIMF problem has the following steps:

Step 1: Construct the network $\widetilde{G} = (N, \widetilde{A})$ (see (5));

If there is a directed path in $\widetilde{G}$ from the node *s* to the node *t*
then the GIMF problem does not have solution; **STOP**.
else **goto** step 2;

Step 2: Construct the network $G_f' = (N, A_f, r', s, t)$ (see (7));
Find the minimum cut $[X, \overline{X}]$ in the network G_f';
B := $(X, \overline{X})$;
Construct the vector c^* using (8);
Construct the vector l^* using (9);
(c^*, l^*) is the solution of the GIMF problem;
f is a maximum flow in the network $G^* = (N, A, c^*, l^*, s, t)$.

Let's apply the algorithm for the network in the figure 1. We shall consider $\widetilde{A} = \varnothing$. So, the GIMF problem has solution and $G_f' = G_f$.

The network $G_f' = G_f$ is presented in the figure 4. The set of direct arcs of the minimum cut in G_f' is $B = (\{1, 3\}, \{2, 4\}) = \{(1, 2), (3, 2), (3, 4)\}$ and $r'(B) = 4$. It follows that the upper bounds for the arcs (*1, 2*) and (*3, 4*) from *G* and the lower bounds of the arcs (*2, 3*) and (*4, 3*) will be changed. The network G^* (the solution of GIMF) is in the figure 3.

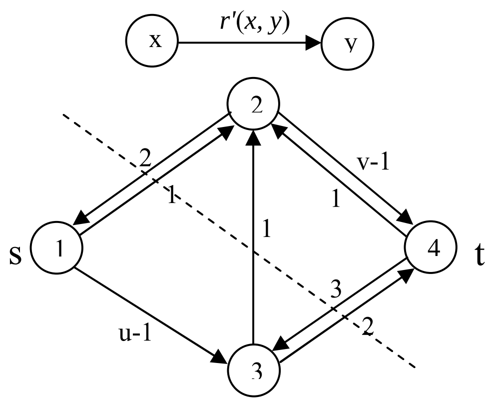

Figure 4: The network $G_f' = G_f$

The complexity of an algorithm having the two steps presented above for the GIMF problem is given by the complexity of the method used to find the minimum *s-t* cut in the network G_f'.

A strongly polynomial algorithm for minimum cut can be applied. For instance, the algorithm for maximum flow (and for minimum cut) due to Goldberg and Tarjan (1988) can be considered. It has the time complexity of $O(n \cdot m \cdot log(n^2/m))$, where $n = |N|$ and $m = |A_f|$.

Weakly polynomial (and non-polynomial) algorithms for minimum cut can not be applied directly, because there can be arcs with infinite capacities in the set A_f (if $\widetilde{A} \neq \emptyset$). It is not necessarily to set the capacities of these arcs to $+\infty$. They can be set to a value big enough. It is easy to see that it is sufficient to set the capacity of these arcs to the value of the maximum flow in the network G_f'. If GIMF has solution, then the value of the maximum flow in G_f' is not greater than $m \cdot R$, where:

$$R = max\{ c(x, y)-f(x, y) + f(y, x)-l(y, x) \mid x, y \in N \}. \quad (11)$$

If the weakly polynomial algorithm for the maximum flow and for the minimum cut due to Goldberg and Rao (1997) is applied in the network G_f', then the time complexity of the algorithm for the GIMF problem becomes $O(min\{n^{2/3},m^{1/2}\} \cdot m \cdot log(n^2/m) \cdot log(R'))$, where $R' = max\{r(x,y) | (x,y) \in A_f\} = m \cdot R$. So, the complexity of the weakly polynomial algorithm for GIMF is:

$$O(min\{n^{2/3},m^{1/2}\} \cdot m \cdot log(n^2/m) \cdot log(m \cdot R)) =$$

$$O(min\{n^{2/3},m^{1/2}\} \cdot m \cdot log(n^2/m) \cdot log(max\{n, R\})). \quad (12)$$

Of course, if the set $\widetilde{A}$ is empty, then the time complexity of the algorithm is even less, it is $O(min\{n^{2/3},m^{1/2}\} \cdot m \cdot log(n^2/m) \cdot log(R))$.

4. THE *IMF* PROBLEM AS A PARTICULAR CASE OF *GIMF*

The algorithm for GIMF can be adapted to solve the IMF problem by setting more capacities of arcs from A_f to $+\infty$. The flow f must not be stopped from being increased in G on a path from s to t that contains inverse arc(s) (x, y) with $f(x, y) > l(x, y)$ and these arcs must not be in the set of direct arcs of the minimum *s-t* cut of G_f'. So, for each arc (x, y) with $f(x, y) > l(x, y)$ the capacity of the arc (y, x) in the network G_f' can be set to $+\infty$. The network $G_f'' = (N, A_f, r'', s, t)$ is constructed, where the capacity for an arc $(x,y) \in A_f$ is defined as follows:

$$r''(x,y) = \begin{cases} +\infty, \text{ if } l(y,x) < f(y,x) \text{ or } c(x,y) - \beta(x,y) > f(x,y) \\ c(x,y) - f(x,y) + f(y,x) - l(y,x), \text{ otherwise} \end{cases}. \quad (13)$$

The minimum *s-t* cut in the network G_f'' gives the solution of IMF. It is easy to see that in (13) the particular case of GIMF with $\delta(x, y) = 0, \ \forall (x, y) \in A$ has been obtained.

Finally, let's observe that *C. Yang, J. Zhang* and *Z. Ma* [14] proposed only strongly polynomial algorithms for the IMF problem. We showed that weakly and non-polynomial algorithms can also be adapted for GIMF and, particularly, for the IMF problem. Moreover, our network G_f'' (where the minimum cut is searched) has fewer arcs and the capacities of the arcs are less.

In the article of *C. Yang, J. Zhang* and *Z. Ma* the minimum cut is searched in the network $G''=(N, A'', c'', s, t)$, where:

$$c''(x,y) = \begin{cases} c(x,y), \text{ if } c(x,y) - \delta(x,y) \leq f(x,y) \\ +\infty, \text{ otherwise} \end{cases}. \tag{14}$$

If weakly polynomial algorithms are applied for minimum cut, then they have better complexity in G_f'' than in an adapted network from G'' (where infinite capacities of arcs are reduced to $m \cdot C$), because the complexity depends on $log(max\{n, R\})$ instead of $log(max\{n, C\})$, where $C = max\{c(x, y)|(x, y) \in A\}$ and $R = max\{c(x, y) - f(x, y)|(x, y) \in A\}$.

Now, let's apply our algorithm for the IMF problem in the network from the figure 1. The network G_f'' is presented in the figure 5. As we have seen, the value $m \cdot R = 8 \cdot max\{u\text{-}1, v\text{-}1\}$ can be considered instead of $+\infty$ in the network from the figure 5.

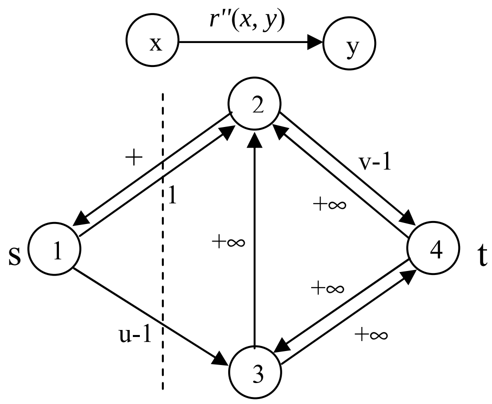

Figure 5: The network G_f''

The set of direct arcs of the minimum cut in G_f'' is $B = (\{1\}, \{2, 3, 4\}) = \{(1, 2), (1, 3)\}$ with $r''(B) = u$ and on the arcs from B the capacities of G will be changed. The solution of the IMF problem is presented in the figure 2.

5. CONCLUSION

We have proposed strongly and weakly polynomial algorithms for solving the GIMF problem. GIMF is reduced to the problem for finding efficiently the minimum *s-t* cut in the modified residual network G_f'. It is possible from the beginning to decide fast (in linear time and space complexity) if the GIMF problem has solution. If the problem does not have solution it is no need to apply the algorithm for minimum cut to see this (with much greater effort). We have showed that IMF is a particular case of the GIMF problem and using the algorithms for GIMF, the IMF problem can be solved more efficiently than using the algorithms proposed by *C. Yang, J. Zhang* and *Z. Ma* in their article [14].

REFERENCES

[1] Ahuja, R.K., and Orlin, J.B., "A faster algorithm for the inverse spanning tree problem", *Journal of Algorithms*, 34 (2000) 177-193.

[2] Ahuja, R.K., and Orlin, J.B., "Inverse optimization", Working Paper, Sloan School of Management, MIT, Cambridge, MA, 1998.

[3] Ahuja, R.K., and Orlin, J.B., *Combinatorial Algorithms for Inverse Network Flow Problems*, Sloan School of Management, MIT, Cambridge, MA, submitted for publication, 1998.

[4] Ahuja, R.K., Magnanti, T.L., and Orlin, J.B., *Network flows: Theory, Algorithms and Applications*, Prentice Hall, Englewood Cliffs, NJ, 1993.

[5] Burton, D., and Toint, Ph.L., "On an instance of the inverse shortest path problem", *Math. Programming*, 53 (1992) 45-61.

[6] Burton, D., Paulleyblank, B., and Toint, Ph.L., "The inverse shortest paths problem with upper bounds on shortest paths costs", in: P. Pardalos, D.W. Hearn, and W.H. Harger (eds.), *Network Optimization, Lecture Notes in Econom., Math. Systems 450*, Springer-Verlang, Berlin/New York, 1997, 156-171.

[7] Burton, D., Toint, Ph.L., "On the use of an inverse shortest path algorithm for recovering linearly correlated costs", *Math. Programming*, 63 (1994) 1-22.

[8] Cai, M., and Yang, X., "Inverse shortest path problems", Technical Report, Institute of Systems Sciences, Academia Sinica, Beijing, China, 1994.

[9] Ciurea, E., and Deaconu, A., "Inverse minimum flow problem", *Journal of Applied Mathematics and Computing*, 23 (2007) 193-203.

[10] Heuberger, C., "Inverse combinatorial optimization: A survey on problems, methods, and results", *Journal of Combinatorial Optimization*, 8 (2004) 329-361.

[11] Huang, S., and Liu, Z., "On the inverse version of the minimum cost flow problem", *Mathematical Programming*, 33, 187-203.

[12] Sokkalingam, P.T., Ahuja, R.K., and Orlin, J.B., "Solving inverse spanning tree problems through network flow techniques", *Oper. Res.*, 47 (1999) 291-298.

[13] Yang, C., and Zhang, J., "Inverse maximum capacity path with upper bound constraints", *OR Spektrum*.

[14] Yang, C., Zhang, J., and Ma, Z., "Inverse maximum flow and minimum cut problems", *Optimization*, 40 (1997) 147-170.

[15] Zhang J., and Cai, C., "Inverse problems of minimum cuts", *ZOR-Math. Methods Oper. Res.*, 48 (1998) 51-58.

[16] Zhang, J., Liu, Z., Ma, Z., "On the inverse problem of minimum spanning tree with partition constrains", *Math. Methods Oper. Res.*, 44 (1996) 171-188.

[17] Zhang, J., Ma, Z., and Yang, C., "A column generation method for inverse shortest path problems", *ZOR-Math. Methods Oper. Res*., 41 (1995) 347-358.