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ON SOME OPTIMIZATION PROBLEMS IN NOT NECESSARILY LOCALLY CONVEX SPACE

Ljiljana GAJIĆ

Department of Mathematics and Informatics
University of Novi Sad, Serbia
gajic@im.ns.ac.yu

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Abstract: In this note, by using O. Hadžić's generalization of a fixed point theorem of Himmelberg, we prove a non - cooperative equilibrium existence theorem in non - compact settings and a generalization of an existence theorem for non - compact infinite optimization problems, all in not necessarily locally convex spaces.

Keywords: Set valued mapping, quasi - convex map, non-cooperative equilibrium.

1. INTRODUCTION

In paper [5], Kaczynski and Zeidan introduced the concept of the continuous cross - section property and by using the Ky Fan fixed point theorem proved an existence theorem for finite optimization problem in compact convex seting. A few years later S.M. Im and W.K. Kim, by using Himmelberg's [4] non - compact generalization of the K. Fan fixed point theorem, proved a non - cooperative equilibrium existence theorem in non - compact setting. Using O. Hadžić's generalization of Himmelberg's fixed point theorem we shall prove existence theorem for non-cooperative equilibrium and existence theorem to non - compact infinite optimization problems in not necessarily locally convex spaces.

2. PRELIMINARIES

Let I be any (possibly uncountable) index set and for each $i \in I$, let X_i be a Hausdorff topological vector space and $X = \prod_{i \in I} X_i$ be the product space. We shall use the following notations:

$$X^i = \prod_{\substack{k\in I\\ k\neq i}} X_k$$

and $p_i : X \to X_i$, $p^i : X \to X^i$ be the projection of X onto X_i and X^i respectively. For any $x \in X$, we simply denote $p^i(x) \in X^i$ by x^i and $x = (x^i, x_i)$. For any given subset K of X, K_i and K^i denote the image of K under the projection of X onto X_i and X^i, respectively.

For each $i \in I$, let $S_i : X^i \to 2^{X_i}$ be a given set valued map. We are concerned with the existence of a solution $\breve{x} \in K$ to the following system of minimization problems:

$$f_i(\breve{x}) = \min\{f_i(\breve{x}^i, z) \mid z \in S_i(\breve{x}^i)\}, \tag{*}$$

where $f_i : X \to \mathbf{R}$ is a real valued function for each $i \in I$.

Such problems arise from mathematical economics or game theory where the solution $\breve{x} \in X$ is usually called the non - cooperative equilibrium or social equilibrium.

Of course, it is clear that when the functions f_i are continuous and K is compact, the minimum in (*) is obtained for each $i \in I$ but not necessarily at $\breve{x}_i$. Therefore we shall need a consistency assumption between f_i and S_i in order to obtain a solution of a system of minimization problem.

Now let us recall some definitions and results which will be useful later.
Let X and Y be two Hausdorff topological spaces and 2^Y a set of non - empty subset of Y. Under a multivalued mapping of X into Y we mean a mapping $f : X \to 2^Y$. Then f is called:

(1) Lower semicontinuous (l.s.c.) if the set $\{x \in X \mid f(x) \cap V \neq \emptyset\}$ is open in X for every open set V in Y.

(2) Upper semicontinuous (u.s.c.) if the set $\{x \in X \mid f(x) \subset V\}$ is open in X for every open set V in Y.

(3) Continuous if it is both l.s.c. and u.s.c..

Lemma 1. [1] *Suppose that $W : X \times Y \to \mathbf{R}$ is a continuous function and $G : X \to 2^Y$ is continuous with compact values. Then the marginal (set valued) function*

$$V(x) := \{y \in G(x) \mid W(x,y) = \sup_{z\in G(x)} W(x,z)\}$$

is u.s.c. mapping.

Definition 1. *A function $f : K \to \mathbf{R}$, where K is a subset of a vector space, is called quasi - convex on K if the set $\{x \in K \mid f(x) \leq r\}$ is convex set for all $r \in \mathbf{R}$. Of course every convex function is quasi - convex but the converse is not true.*

Definition 2. *Let X be a Hausdorff topological space, $K \subset X$ and $\mathcal{U}$ the fundamental system of neighbourhoods of zero in X. The set K is said to be of $Z-$ type if for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that*

$$conv(U \cap (K-K)) \subset V.$$

($convA =$ convex hull of the set A).

Remark. *Every subset $K \subset X$, where X is a locally convex topological vector space, is of $Z-$ type. In [3] examples of subset $K \subset X$ of $Z-$ type, where X is not locally convex topological vector spaces, are given.*

The next fixed point theorem will be an essential tool for proving the existence of solution in our optimization problems.

Theorem A. [3]. *Let K be a convex subset of a Hausdorff topological vector space X and D is a nonempty compact subset of K. Let $S: K \to 2^D$ be an u.s.c. mapping such that for each $x \in K$, $S(x)$ is nonempty closed convex subset of D and $S(K)$ is of $Z-$ type. Then there exists a point $\breve{x} \in D$ such that $\breve{x} \in S(\breve{x})$.*

3. RESULTS

We begin with the following:

Proposition 1. *Let $\{K_i\}_{i\in I}$ be a family of nonempty compact convex subsets of Hausdorff topological vector spaces $\{X_i\}_{i\in I}$ ($K_i \subset X_i$, for every $i \in I$), $X = \prod_{i\in I} X_i$ and $K = \prod_{i\in I} K_i$. If for every $i \in I$ the set K_i is of $Z-$ type in X_i then K is of $Z-$ type in X.*

Proof: For each $i \in I$ let $\mathcal{U}_i$ be a fundamental system of zero neighbourhoods in space X_i and let us denote by $\mathcal{U}$ the fundamental system of zero neighbourhoods in the product (Tihonov) topology on $X = \prod_{i\in I} X_i$. For any $V \in \mathcal{U}$ we have to prove that there exists $U \in \mathcal{U}$ such that $conv(U \cap (K-K)) \subset V$. Suppose that $V \in \mathcal{U}$. Then there exists a finite set $\{i_1, i_2, ..., i_n\} \subset I$ such that $V = \prod_{i\in I} X_i'$, where

$$X_i' = \begin{cases} X_i, & i \in I \setminus \{i_1, i_2, \dots, i_n\}, \\ V_i, & i \in \{i_1, i_2, \dots, i_n\}, \end{cases}$$

and $V_i \in \mathcal{U}_i$, for each $i \in \{i_1, i_2, ..., i_n\}$. Since $K_i \subset X_i$, $i \in I$, and K_i is of $Z-$ type, there exists $U_i \in \mathcal{U}_i$ where $i \in \{i_1, ..., i_n\}$ such that

$$conv(U_i \cap (K_i - K_i)) \subset V_i.$$

Let $U = \prod_{i\in I} X_i''$. for

$$X_i'' = \begin{cases} X_i, & i \in I \setminus \{i_1, i_2, ..., i_n\}, \\ U_i, & i \in \{i_1, i_2, ..., i_n\}, \end{cases}$$

Now, suppose that $z \in conv(U \cap (K - K))$. This implies that there exist $r_k, k = 1,2,...,m$, and $u^k \in U \cap (K - K), i = 1,2,...,m,$ so that $r_k \geq 0, k = 1,2,...,m, \sum_{k=1}^{m} r_k = 1$ and $z = \sum_{k=1}^{m} r_k u^k$. Let us prove that $z \in V$. It is enough to prove that $p_i(z) \in V_i$ for every $i \in \{i_1,...,i_n\}$.

For $z = \sum_{k=1}^{m} r_k u^k$ it follows that $p_i(z) = \sum_{k=1}^{m} r_k p_i(u^k)$ for all $i \in I$. Suppose now that $i \in \{i_1,...,i_n\}$.

Since $p_i(u^k) \in X_i'' \cap (K_i - K_i) = U_i \cap (K_i - K_i)$ it follows that

$$p_i(z) \in conv(U_i \cap (K_i - K_i)) \subset V_i.$$

Now, we shall prove our main result.

Theorem 1. *Let K be a non-empty convex subset of Hausdorff topological vector space X and D be a nonempty compact subset o K. Suppose that $\phi : X \times X \to \mathbf{R}$ is continuous function and $S : K \to 2^D$ a continuous set valued map such that*

(1) far each, $x \in K$, $S(x)$ is a nonempty closed convex subset of D;

(2) $S(K)$ is of $Z-$ type subset;

(3) for each $x \in K$, $y \to \phi(x,y)$ is quasi - convex on $S(x)$.

Then there exists a point $\breve{x} \in D$ such tha $\breve{x} \in S(\breve{x})$ and $\phi(\breve{x}, y) \geq \phi(\breve{x}, \breve{x})$ for all $y \in S(\breve{x})$.

Proof: Define a set valued mapping $V : K \to 2^D$ by

$$V(x) := \{z \in S(x) \mid \phi(x,z) = \inf_{y \in S(x)} \phi(x,y)\}$$

for all $x \in K$. Since ϕ is continuous and $S(x)$ is non-empty compact, $V(x)$ is nonempty compact subset of D for all $x \in K$. For each $z_1, z_2 \in V(x)$ and $t \in [0,1]$, $tz_1 + (1-t)z_2 \in S(x)$.

Since $\phi(x, z_1) = \phi(x, z_2) = \inf_{y \in S(x)} \phi(x,y) = r$ and $\{z \in S(x) \mid \phi(x,z) \leq r\}$ is convex, one can see that $tz_1 + (1-t)z_2 \in V(x)$ so $V(x)$ is convex for every $x \in K$. By Lemma 1., V is u.s.c. mapping. Now, by Theorem A, there exists a fixed point $\breve{x} \in D$ of V, i.e. $\breve{x} \in V(\breve{x})$. But this point is just what we need to find.

In [6] S.M. Im and W.K. Kim give an example which show that, even when X is a locally convex Hausdorff topological spaces, the lower semicontinuity of S is essential in Theorem 1.

Next we shall prove a generalization of Kaczynski - Zeidan's result to non - compact infinite optimizations problems in not necessarily locally convex space.

Theorem 2. *Let* I *any (possibly uncountable) index set and for each* $i \in I$, *let* K_i *be a convex subset of Hausdorff topological vector space* X_i *and the* D_i *be a non - empty compact subset of* K_i. *For each* $i \in I$, *let* $f_i : K = \prod_{i \in I} K_i \to \mathbf{R}$ *be a continuous function and* $S_i : K^i \to 2^{D_i}$ *be a continuous set valued mapping such that for each* $i \in I$

(1) $S_i(x^i)$ *is non - empty closed convex subset of* D_i;

(2) $S_i(X^i)$ *is of* $Z-$ *type;*

(3) $x_i \to f_i(x^i, x_i)$ *is quasi - convex on* $S_i(x^i)$.

Then there exists a point $\breve{x} \in D = \prod_{i \in I} D_i$ *such that for each* $i \in I$, $\breve{x}_i \in S_i(\breve{x}^i)$ *and*

$$f_i(\breve{x}^i, \breve{x}_i) = \inf_{z \in S_i(\breve{x}^i)} f_i(\breve{x}^i, z).$$

Proof: For each $i \in I$, let us define a set valued function $V_i : K^i \to 2^{K_i}$ by

$$V_i(x^i) := \{y \in S_i(x^i) \mid f_i(x^i, y) = \inf_{z \in S_i(x^i)} f_i(x^i, z)\}.$$

As in the proof of Theorem 1. $V_i(x^i)$ is non- empty compact convex set and V_i is u.s.c. mapping. Now, we define $V : K \to 2^D$ by

$$V(x) := \prod_{i \in I} V_i(x^i),$$

for each $x \in K$.

Then $V(x)$ is non - empty compact convex subset of D and V is u.s.c. mapping. By Proposition 1 subset $V(K)$ is of $Z-$ type. Using Theorem A again one can see that there exists a point $\breve{x} \in D$ such that $\breve{x} \in V(\breve{x})$ i.e. $\breve{x}^i \in V_i(\breve{x}^i)$ and

$$f_i(\breve{x}^i, \breve{x}_i) = \inf_{z \in S_i(\breve{x}^i)} f_i(\breve{x}^i, z)$$

for al $i \in I$.

In special case of Theorem 2, when K_i is a compact convex and S_i is the cross section of $K = \prod_{i \in I} K_i$ (i.e. $S_i : K^i \to 2^{K_i}$ is defined by $S_i(x_i) = \{z \in X_i \mid (x^i, z) \in K\}$), then the continuous cross - section property from [5] clearly implies the assumption of Theorem 2 by letting $K_i = D_i$ for each $i \in I$. Therefore, Theorem 2 is an infinite generalization of Theorem in [6] to non - compact setting in not necessarily locally convex space.

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