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HEURISTIC ALGORITHM FOR SINGLE RESOURCE CONSTRAINED PROJECT SCHEDULING PROBLEM BASED ON THE DYNAMIC PROGRAMMING

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Abstract: We introduce a heuristic method for the single resource constrained project scheduling problem, based on the dynamic programming solution of the knapsack problem. This method schedules projects with one type of resources, in the non-preemptive case: once started an activity is not interrupted and runs to completion. We compare the implementation of this method with well-known heuristic scheduling method, called Minimum Slack First (known also as Gray-Kidd algorithm), as well as with Microsoft Project.

Keywords: Resource scheduling, dynamic programming, knapsack problem, DELPHI.

1. INTRODUCTION

In practice most organizations work within limited resources, so projects are subject to the same constraint. A new project may seek an additional use of resources, so it is needed to ensure that they really would be available.

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On the other hand, the time constraints are always present, so project manager must work under that boundary.

Resource leveling is a way to resolve having too much work assigned to resources, known as resource over allocation.

The network diagram can be used to find opportunities for shortening the project schedule. This involves looking at where we can cut the amount of time it takes to complete activities on the critical path, for example, by increasing the resources available to these activities. Another solution is to identify where any other routes might have some slack. We may then be able to reallocate resources to reduce the pressure on the team members who are responsible for activities on the critical path.

We consider the problem of resource distribution at the point of minimization of the project makespan, and introduce a heuristic method for solving the single resource constrained project scheduling problem, based on the knapsack problem and dynamic programming. The total number of available resource units is constant and specified in advance. A unit of resource cannot be shared by two or more activities. An activity is ready to be processed only when all its predecessor activities are completed and the number of resource units required by it are free and can be allocated to it. Once started, an activity is not interrupted and runs to completion.

When leveling resources, we do not change resource assignments, nor task information. We only delay tasks.

In a dynamic programming solution to the knapsack problem, we calculate the best combination for all knapsack sizes up to M [12]:

```
for j:=1 to N do
  for i:=1 to M do
    if  i-size[j]>=0  then
      if cost[i]<(cost[i-size[j]]+val[j]) then
        begin
           cost[i]:=cost[i-size[j]]+val[j];
          best[i]:=j
        end;
```

In this program, N is the number of items, *val*[*j*] is the value of *j*th item, *size*[*j*] is its volume, *cost*[*i*] is the highest value that can be achieved with a knapsack of capacity i and *best*[*i*] is the last item that was added to achieve that maximum (this is used to recover the contents of the knapsack). First, we calculate the best that we can do for all knapsack sizes when only items of type A (for $j = 1$) are taken, then we calculate the best that we can do when only A's and B's (for $j = 2$) are taken, etc. The solution reduces to a simple calculation for *cost*[*i*]. Suppose an item j is chosen for the knapsack: then the best value that could be achieved for the total would be *val*[*j*] + *cost*[*i* - *size*[*j*]], where *cost*[*i* - *size*[*j*]] is the optimal filling of the rest of the knapsack. If this value exceeds the best value that can be achieved without an item j, then we update *cost*[*i*] and *best*[*i*]; otherwise we leave them alone. A simple induction proof shows that this strategy solves the problem [12].

In this paper we propose a new strategy to solve the single resource constrained project scheduling problem. In each stage of the scheduling we consider a schedule time t and the corresponding eligible set of activities which could be started at the moment t without violation of given constraints which define the project. As it is proposed in [14] and [9], we activate a subset of activities from the eligible set solving the knapsack problem maximizing the resource utilization. But, instead of Greedy randomized adaptive search procedure (GRASP), used in [9], we apply the dynamic programming and Bellman's principle (see [1]).

The paper is organized as follows. In the second section we state mathematical formulation of the problem and compare the project duration computed by our algorithm with the early finish of the project.

In the third section the algorithm and several implementation details are described.

In the last section we compare the implementation of our algorithm with known software Microsoft Project 2003® and Gray-Kidd algorithm.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

A project consists of a set of activities $J = \{A_i\}_{i\in I}$, partially ordered by precedence constraints, where $I = \{1,\dots,n\}$ is a set of activities indices and n is a number of activities required by the project. It is assumed that the project requires only one type of resources. The entire project is defined as the ordered pair (J, R), where the natural number R denotes the resource maximal units available in the project. Each of activities A_i is defined as the ordered triple

$$A_i = (p_{i,}, r_i, P_i), \ \ i \in I,$$

where $p_i \in N$ represents the processing time (duration) of activity A_i, value $r_i \in \mathbf{N}$ is a number of resources needed for A_i, and $P_i \subset I$ is an array which contains predecessors indices for A_i. An activity A_i is said to be a predecessor of A_j, when A_j cannot start until A_i has finished. This fact is written as $i \in P_j$ or $A_i < A_j$, where '<' defines the precedence relationship. Similarly A_j is said to be a successor activity of A_i. We assume that the number r_i is fixed for the lasting time of activity A_i.

Let F denotes the set including all pairs of activities with predecessor and successor relationships. These pairs define a digraph of the project $G = (J, F)$, where $(A_i, A_j) \in F$ if and only if $i \in P_j$, $A_i, A_j \in J$.

A project starts at time $t = 0$. A schedule for the project is an assignment of a start time t_i^s to each activity A_i. An activity is said to be scheduled when it is assigned a start time. The vector defining starts of activities included into the project (J, R) is defined as the ordered n-tuple $t^s = (t_1^s,\dots,t_n^s)$ of natural numbers, and it is called start vector of the project. Similarly denote the vector of activities finishes by $t^f = (t_1^f,\dots,t_n^f)$. The finish of each activity is computed from its start as $t_i^f = t_i^s + p_i$. Vectors t^s and t^f are being calculated in our algorithm.

A feasible schedule is a schedule that satisfies the given precedence and resource constraints. An optimum schedule is a feasible schedule that optimizes the given objective function. Our goal is to find start time t_i^s for each activity A_i due to minimize the project completion time (makespan) of the project, calculated by $Dur := \max\{t_i^f \mid i \in I\}$.

We define the resource units required by A_i in the scheduling time t as follows:

$$r_i^t = \begin{cases} r_i, & t_i^s \le t \le t_i^f \\ 0, & otherwise \end{cases} = x_i^t r_i,$$

where $x_i^t = 1$ when the activity A_i is started and still not finished (in the time interval $t_i^s \le t \le t_i^f$), and $x_i^t = 0$ otherwise.

Formally, the aim of the general single resource constrained project scheduling problem is to find an optimal schedule, and can be formulated by the following mathematical model, in the time t:

$$\min Dur = \max\{t_i^f \mid i \in I\} \tag{2.1}$$

$$s.t. (\forall j \in I)(\forall i \in P_j) \quad t_j^s \ge t_i^f \tag{2.2}$$

$$(\forall t \in \mathbf{N}) \sum_{i \in I} r_i^t \le R \tag{2.3}$$

Our objective is to minimize the makespan of the project. Each activity needs to be started after all its predecessors activities finish (condition (2.2)), and in every moment t, the total number of occupied resources is less than R (condition (2.3)).

Proposition 2.1. *Condition* (2.2) *can be written in the following equivalent form:*

$$(\forall j \in I)(\forall i \in P_j) \quad t_j^f - t_i^f \ge p_j \tag{2.4}$$

Observe that the condition (2.4) is equivalent with the corresponding one from [15].

Therefore, mathematical model (2.1)-(2.3) is equivalent with corresponding one, described in [9], [15], in the case when one type of resources is used.

Schedule t^s of the project (J, R) is feasible if conditions (2.2) and (2.3) are satisfied. A feasible schedule is optimal if (2.1) is fulfilled.

Next lemma can be easily proven by the induction. This lemma gives a stopping criterion for increasing maximal unit's availability of the resource, i.e. the early finish of the project.

Lemma 2.1 *Let t^s be the feasible schedule of the project* (J, R). *Let us consider the early finish τ_j^0 for each activity A_j, $j \in I$, recursively as follows*:

$$\tau_j^0 = \begin{cases} p_j, & P_j = 0; \\ \max\limits_{i \in P_j}\{\tau_i^0 + p_j\}, & otherwise. \end{cases}$$

Then $t_i^f \geq \tau_i^0$ *, for each* $i \in I$*. Also* $Dur \geq \max\limits_{i \in P_j} \tau_i^0 = \tau^0$.

Proof. Let us apply the topological sort of G, i.e. find the permutation $i_1,\dots,i_n$ of I such that $P_{i_j} \subseteq \{i_1,\dots,i_{j-1}\}$ for all $j \in I$. Then we sort values τ_i^0 in the non-ascending order $\tau_{i_1}^0 \geq \cdots \geq \tau_{i_n}^0$.We will continue proof by the induction. For $j = 1$, it follows that $P_{i1} = \emptyset$, so $\tau_{i_1}^0 = p_{i_1} \leq p_{i_1} + t_{i_1}^s = t_{i_1}^f$. In the case $j > 1$ suppose that the claim of lemma is satisfied for all $i_1, \dots, i_{j-1}$ and prove it for i_j. In the case $P_{ij} = \emptyset$;, conclusion follows from the same reasons as in the case $j = 1$. Suppose now that $P_{ij} = \emptyset$, and consider $\tau_{i_j}^0 = p_{i_j} + \max\limits_{k \in P_{i_j}} \tau_k^0$. From the inductive hypothesis we have $\tau_k^0 \leq t_k^f$ for every $k \in P_{i_j}$. Also, in view of (2.2) we have

$$t_{i_j}^f = t_{i_j}^s + p_{i_j} \geq t_k^f + p_{i_j} \geq \tau_k^0 + p_{i_j} \tag{2.5}$$

for all $k \in P_{i_j}$. If we take the maximum in (2.5) over all $k \in P_{i_j}$, we obtain $t_{i_j}^f \geq \tau_{i_j}^0$, so we finished the proof by induction. The second statement of the lemma follows immediately. □

The result of this lemma is used as the stopping criterion in the following sense: in each example, the maximal units of the resource is increased until all of three considered algorithms reach the duration equal to early finish of the project.

We restate, in the recursive form, the notion of the late start of activity A_i, known in the literature:

$$\theta_i^1 = \begin{cases} \tau_n^0, & i = n \\ \min\{\theta_j^1 - p_i\}, & i = 1,\dots,n-1. \end{cases}$$

The late finish τ_i^1 of A_i is equal to $\tau_i^1 = \theta_i^1 + p_i$ and its early start θ_i^0 is equal to $\theta_i^0 = \tau_i^0 - p_{i.}$

3. ALGORITHM

Let us first mention some known heuristic resource scheduling methods from [4]. To choose a subset of activities satisfying project constraints into the schedule, several heuristics are known. Metrics for assigning priorities are:

1) Shortest Task First
2) Most Resources First
3) Minimum Slack First (Gray-Kidd algorithm)

4) Most Critical Followers
5) Most Successors

There are many papers comparing alternative heuristic algorithms. Patterson and Davis in [6], [7] compared these heuristics, in serial and parallel modes and achieved the result the most effective algorithm is Minimum Slack First. The similar results are achieved in [10]. The Minimum Slack First method is described in [5] (p. 225–233).

Kochetov and Stolyar [9] devise an evolutionary algorithm which com-bines genetic algorithm, path relinking, and tabu search. In order to select a subset of activities from the eligible set into the schedule, they solve the knapsack problem. The idea of using the knapsack problem with objective function maximizing the re-source utilization ratio is introduced in [13] and [14]. In order to solve the knapsack problem stated for the sake of resource utilization, in [9] and [14] use GRASP (greedy randomized adaptive search procedure) algorithm from [8]. GRASP is an iterative multi-start algorithm. There are two phases in every iteration: a greedy adaptive randomized construction phase and a local search phase. Starting from the feasible solution built during the greedy adaptive randomized construction phase, the local search explores its neighborhood until a local optimum is found. The best solution found overall the different iterations is kept as the result [8]. A solution x is said to be in the basin of attraction of the global optimum if local search starting from x leads to the global optimum. Once the neighborhood and objective function are determined, different starting solutions can be used to start the local search in a multi-start procedure. If the starting solution is in the basin of attraction of the global optimum, local search finds the global optimum. Otherwise, a non-global local optimum is found [11]. Using greedy solutions as starting points for local search in a multi-start procedure will usually lead to good, though, most often, suboptimal solutions. This is because the amount of variability in greedy solutions is small and it is less likely that a greedy starting solution will be in the basin of attraction of a global optimum. If there are no ties in the greedy function values or, if a deterministic rule is used to break ties, there is no variability and a multistart procedure would produce the same solution in each iteration [11].

In this section we will introduce an algorithm for resource scheduling, called *DynamicRes*, which is based on the knapsack problem and the dynamic programming. This heuristic gives better results with respect to Gray-Kidd algorithm in most cases. Algorithm *DynamicRes* is being written in the programming language DELPHI.

The algorithm requires a sequence of activities, and the following parameters for each activity $A_i, i \in I$:

- duration of the activity (integer p_i),
- array of ordinal numbers of its predecessors, denoted by P_i,
- units of resource required (integer r_i).

Also, the input parameter of the algorithm is total number of resource units available in the project, denoted by R.

Output of algorithm are beginnings of activities after the resource scheduling is performed, i.e. the start vector $(t_1^s,...,t_n^s)$ of the project activities as well as the corresponding makespan of the project.

We use the status of each activity, denoted with $Stat_i$:

$$Stat_i = \begin{cases} 0, & A_i \quad \textit{is not started}, \\ 1, & A_i \quad \textit{is finished}. \end{cases}$$

In each stage of scheduling we have a schedule time t, $t \in \mathbf{N}$ and corresponding sets $E(t)$ and $A(t)$, defined in the following definitions, restated from [9]:

Definition 3.1. *The set of activities which could be started at time t without violation of any constraints is equal to*

$$E(t) = \{j \in I \mid Stat_j = 0 \wedge (\forall i \in P_j)\ Stat_i = 1\}. \tag{3.1}$$

Definition 3.2. *The set of activities which are in progress at the time t is*

$$A(t) = \{i \in I \mid t_i^s \le t \le t_i^f\}. \tag{3.2}$$

We now define the notion called *moment of the project*, useful in the algorithm description. This value is represented by variable *Moment* in the algorithm.

Definition 3.3. *The moment of the project is equal to 0 in its start, and in each other case the moment is equal to the minimal time when at least one of started and uncompleted activities is finished:*

$$Moment = \begin{cases} 0, \textit{when project starts } (t = 0), \\ \min\{t_j^f \mid j \in A(t)\}, \textit{otherwise}. \end{cases}$$

We also define *momentary slack*. This notion is actual for activities A_j with indices belonging to $E(t)$ at the time defined by *moment*.

Definition 3.4. *Momentary slack of the activity A_i is equal to*

$$MSlack = t_i^f - Moment.$$

Algorithm 3.1. *DynamicRes*

Step 1. Set the schedule time $t = 0$, *and* $Stat_i = 0$ *for all* $i \in I$.

Step 2. Compute the number of available resource units by

$$R_a(t) = R - \sum_{i \in A(t)} r_i \tag{3.3}$$

For each $j \in E(t)$ *perform the following:*

If

$$r_j \le R_a(t) \wedge t + p_j \ge \tau_j^1, \tag{3.4}$$

where τ_j^1 *denotes the late finish for activity* A_j, *then set*

$$t_j^s = t, \quad t_j^f = t_j^s + p_j, \quad R_a(t) = R_a(t) - r_j; \tag{3.5}$$

otherwise, delay activity A_j.

This means that each activity which exceeds its own late finish, starting from the actual moment of the project, needs to be started if there are enough resource units available as it is required by it.

Step 3. If $E(t) = \varnothing$ *; then go to Step 5, otherwise go to Step 4.*

Step 4. Solve the knapsack problem with capacity $R_a(t)$, *where the values and volumes of articles are equal to* r_i, $i \in E(t)$:

$$\begin{aligned} &\min R_a(t) - \sum_{i\in E(t)\setminus A(t)} x_i r_i \\ &s.t. \sum_{i\in E(t)\setminus A(t)} x_i r_i \le R_a(t), \end{aligned} \tag{3.6}$$

where $x_i \in \{0,1\}$, $i \in E(t)$.
For each i satisfying $x_i = 1$ *set* $t_i^s = t, t_i^f = t_i^s + p_i$.

Step 5. Now set the new value for the variable Moment:

$$Moment = \min\{t_i^f \mid i \in A(t)\} \tag{3.7}$$

and set $Stat_i = 1$ *for all activities satisfying* $t_i^f = Moment$.

Step 6. Set t = Moment. If there are non started activities $((\exists i \in I)\ Stat_i = 0)$, *go to Step 2. Otherwise, the makespan of the project is equal to* $\max\{t_i^f \mid i \in I\}$, *and the schedule of the project is the* $t^s = (t_1^s, \ldots, t_n^s)$.

Remark 3.1 *Gray-Kidd algorithm schedules activities which have a minimal value of the momentary slack MSlack. Opposite of this algorithm, we propose the following: set the beginnings (like in Step 4.) for all activities which coincides with the optimal solution of the knapsack problem* (3.6).

Remark 3.2. *Problem* (3.6) *can be restated in the following equivalent form:*

$$\begin{aligned} &\max \sum_{i\in E(t)\setminus A(t)} x_i r_i \\ &s.t. \sum_{i\in E(t)\setminus A(t)} x_i r_i \le R_a(t), \end{aligned} \tag{3.8}$$

Problem (3.8) *is a one-dimensional variant of the knapsack problem stated in* [9] *and* [14], *and a variant of the classical* 0-1 *knapsack problem* (*see, for example* [2]). *In order to solve* (3.8)*, instead of the GRASP algorithm used in* [9] *and* [14], *we use the dynamic programming. Therefore, we speak of items representing requirements of the resource. The set of weights* (*volumes*) *as well as the set of values is equal to* $\{r_i \mid i\in I\}$. *To avoid trivial cases we assume* $\sum_{i\in I} r_i \ge R$ *and* $r_i \le R$, *for each* $i\in I$. *Comparing problem* (3.8) *with the dynamic programming solution of the knapsack problem, restated from* [12], *the following analogies are evident:*

- *the objective function cost is equivalent with* $\sum_{i\in E(t)\setminus A(t)} x_i r_i$;
- *the knapsack size M is equivalent with the number of available resource units, denoted by* $R_a(t)$*, which is defined in* (3.3) *and* (3.5);
- *number N of different types of items is replaced by the cardinal number of the set I, denoted by n*;
- $val[j] = size[j] = r_j$;
- *Assignment* $best[i] = j$ *implies* $x_j = 1$ *in* (3.6) *and* (3.8).

It is known that the knapsack problem exhibits optimal substructure, and its optimal solution contains within optimal solution to sub problems [3]. *Typically, the total number of distinct sub problems is a polynomial in the input size. When a recursive algorithm revisits the same problem over and over again, we say that the optimization problem has overlapping sub problems. Dynamic programming algorithms typically take advantage of overlapping sub problems by solving each sub problem once and then storing the solution in a table* [3]. *Also, it is known that greedy algorithms do not always yield optimal solutions* [3].

Remark 3.3 *Using the main idea of the Gray-Kidd algorithm, in the case when we have more solutions for the knapsack problem, we use the solution containing activity with the minimal value for MSlack.*

Here we consider an example to discuss about the difference between the *Gray-Kidd* algorithm and *DynamicRes* algorithm.

Example 3.1 *Durations, units of the resource required and predecessors of all activities in the project are arranged in the following table.*

Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Duration	4	4	2	4	3	3	6	5	2
Units	5	5	3	6	4	5	7	3	4
Predecessors				*A*	*B,C*	*B*	*B*	*E*	*F,G*

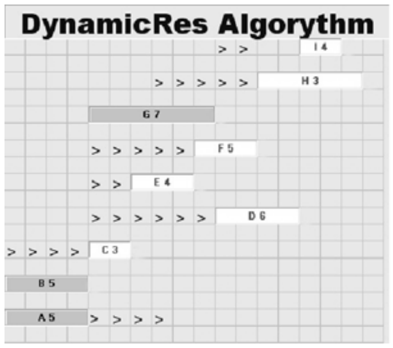
Assume that $R = 12$ *resource units are available in the project.*

Observe the first actual interval: [0, 2]. We should numerate activities A, B and C, in accordance to Gray-Kidd algorithm. Activity B is critical and we mark it by the number 1. Activity C has a slack of 2 days and it is marked by the number 2. Finally, activity A has the biggest slack of 4 days and is assigned by the number 3. Activities A, B and C require all together 13 units of the resource, so we put the start of activity A at the moment 2, and schedule activities B, C.

On the other hand, using algorithm DynamicRes we fill the knapsack of capacity 12 by objects whose values and capacities are $r_1 = 5$, $r_2 = 5$, $r_3 = 3$. Therefore, the maximal volume is filled with volumes $r_1 = 5$, $r_2 = 5$. In accordance with algorithm DynamicRes, it is necessary to move the start of activity C at the moment 2.

In both algorithms, the remainder of the project starts at the moment 2.

Gray-Kidd and DynamicRes give different solutions with the same duration of the project (17 days). Here are given two different solutions of the project scheduling, first using DynamicRes algorithm (Figure 3.1) and the other using Gray-Kidd algorithm (Figure 3.2).



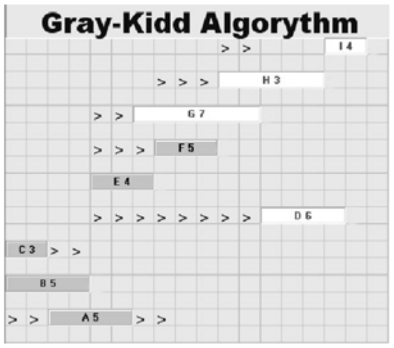


Figure 3.1. Solution given by *DynamicRes* **Figure 3.2.** Solution given by *Gray-Kidd*

Activities are denoted by letters A, B,..., I, and it is shown how many resources each of them requires. Here, cells in StringGrid assigned with the sign '>' represent the total slack for each of activities. Activities that are started after their late finish are colored white, otherwise are blue.

Theorem 3.1 *Schedule* t^S *produced by DynamicRes algorithm is feasible* (*satisfies conditions* (2.2) *and* (2.3)).

Proof. Start time t^s_j for each activity A_j can be determined in two different ways, applying Step 2 and Applying Step 4.

Firstly we verify Condition (2.2). Using Step 2 in a fixed schedule time t we schedule all activities whose indices satisfy $j \in E(t)$ and (3.4) and set $t^s_j = t$ for these activities. On the other hand, from Step 5, it is satisfied $Moment \geq \max\{t_i^f \mid Stat_i = 1\}$. After setting t = *Moment* in Step 6 it is satisfied $t \geq \max\{t_i^f \mid Stat_i = 1\}$. From (3.5) we conclude that $t_j^s = t \geq \max\{t_i^f \mid Stat_i = 1\}$, for each $j \in E(t)$ satisfying (3.4). Therefore, we conclude $(\forall i \in P_j)\ t_j^s \geq t_i^f$.

In the second case, according to Step 4 of *DynamicRes* algorithm, we conclude that in the knapsack problem are included only these activities A_j satisfying $j \in E(t) \backslash A(t)$. Therefore, in accordance with (3.1), all predecessors included in the knapsack are finished $((\forall i \in P_j) Stat_i = 1)$. Moreover, in Step 4 we schedule activities corresponding to the optimal solution of the knapsack problem. Analogous to the previous case, according to Step 5, condition (2.2) is satisfied for all activities included.

We now verify condition (2.3). It is clear that condition (2.3) can be written as

$$(\forall t \in N) \quad \sum_{i \in A(t)} r_i \leq R \tag{3.9}$$

In view of (3.3), this condition is later equivalent with

$$(\forall t \in N) \quad R_a(t) \geq 0. \tag{3.10}$$

Now, Step 2 satisfies condition (3.10) because of (3.5) and condition (3.4).

In the sequel we prove that Step 4 also satisfies (3.10). Denote by $A^{'}(t)$ the set of indices of just started activities:

$$A'(t) = \{i \in I \mid t_i^s = t\}.$$

Since $t^s_i = t$ implies $i \in A^{'}(t)$, we conclude

$$A(t) = A(t) \cup A'(t)$$

after Step 4. At this moment, the number of available resources is

$$R_a(t) = R_a(t) - \sum_{i \in A'(t)} r_i \tag{3.11}$$

According to condition in the knapsack problem (3.6), we conclude $R_a(t) \geq 0$. Taking into account (3.3), we get

$$R_a(t) = R - \sum_{i \in A(t)} r_i - \sum_{i \in A'(t)} r_i \geq 0 \tag{3.12}$$

Denote by $A^{''}(t)$ the set of just finished activities in Step 6:

$$A''(t) = \{i \in I \mid t_i^f = Moment\}.$$

Now, in Step 2 it is satisfied $A(t) = A(t) \bigcup A'(t) \setminus A''(t)$ and applying (3.12) we obtain

$$R_a(t) = R - \sum_{i \in A(t)} r_i - \sum_{i \in A'(t)} r_i + \sum_{i \in A''(t)} r_i \geq R - \sum_{i \in A(t)} r_i - \sum_{i \in A'(t)} r_i \geq 0.$$

The proof is complete. □

4. NUMERICAL EXPERIMENTS

In this section we compare three algorithms for resources scheduling: algorithm included in MS Project, *Gray-Kidd* algorithm and *DynamicRes* algorithm. According to our assumptions of the algorithm, in MS Project it is assumed that resources leveling cannot split task and check box *Level only within available slack* in *Resource Leveling* options is cleared.

Example 4.1. Consider the project defined as follows:

Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Duration	5	4	4	9	5	5	4	4
Units	4	5	2	6	2	3	7	8
Predecessors				*A*	*B*	*C*	*A*	*G*

Increasing the number of available resource units we get the following table. The early finish for this example is 14 days, so we stop searching for new solutions when all three algorithms reach this limit, using the result of lemma 2.1.

	Duration of the project		
Max. units	*Ms Project*	*Gray-Kidd*	*DynamicRes*
8	26	26	26
9	25	25	22
10	22	22	22
11	22	22	22
12	22	22	22
13	22	18	18
14	18	17	18
15	18	17	18
16	18	17	14
17	18	17	14
18	14	14	14

Data presented in the table are illustrated by the following chart:

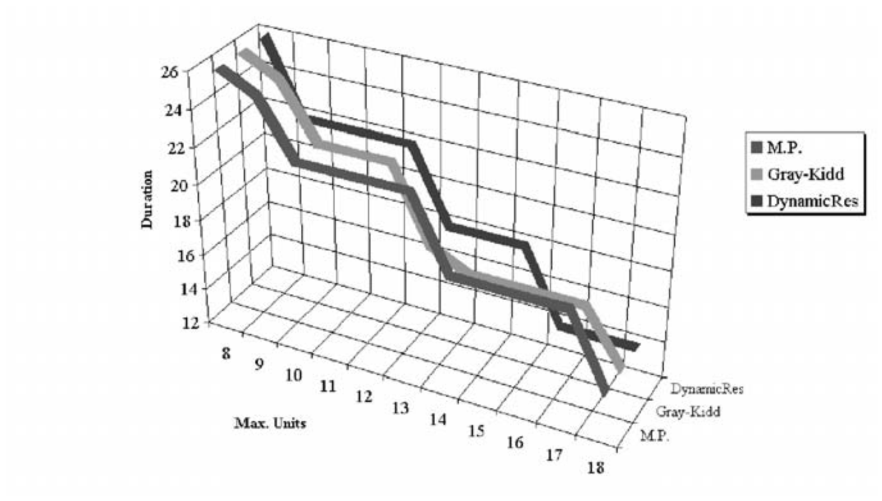


Figure 4.1. Comparison of three methods

Average values, geometrical means and standard deviations for project durations are presented in the table.

Statistical function	*Ms Project*	*Gray-Kidd*	*DynamicRes*
AVERAGE	20.45454545	19.72727273	19.09090909
Geometrical mean	20.1617992	19.39008818	18.69455204
Standard deviation	3.559877424	3.849439156	4.036199833

The average values are minimal for *DynamicRes* algorithm. This means that *DynamicRes* produces, generally, minimal values for makespan of the project. Also, method *DynamicRes* produces maximal value for the standard deviation, because the project makespan corresponding to this algorithm decreases most rapidly.

Example 4.2. *Activities, their durations, maximal units of the resource and predecessors of the project are defined in the table:*

Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Duration	5	4	4	9	5	5	4	4
Units	4	5	2	6	2	3	7	8
Predecessors				*A*	*B*	*C*	*D,F*	*G*

Increasing the number of available resource units we get the following table. It can easily be calculated that the early finish here is 22 days.

	Duration of the project		
Max. units	*Ms Project*	*Gray-Kidd*	*DynamicRes*
8	26	26	31
9	27	27	22
10	26	26	22
11	22	22	22

We illustrate the data in the table by the following chart:

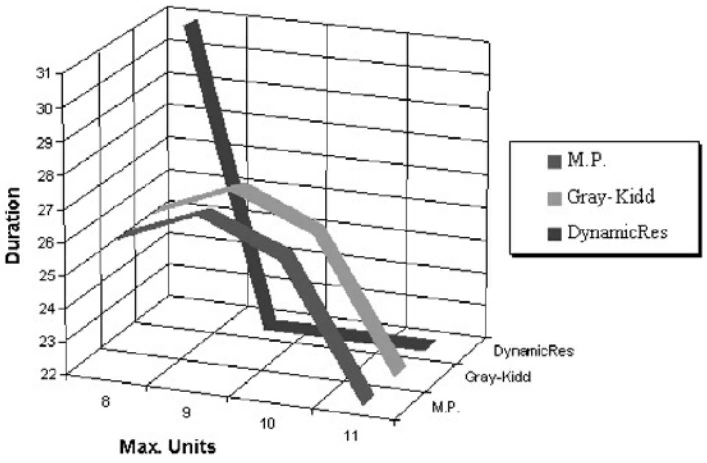


Figure 4.2. Comparison of three methods

Average values, geometrical means and standard deviations for project durations are presented in the table. The conclusions are similar as in the previous example.

Statistical function	*Ms Project*	*Gray-Kidd*	*DynamicRes*
AVERAGE	25.25	25.25	24.25
Geometrical mean	25.17290201	25.17290201	23.96941496
Standard deviation	2.217355783	2.217355783	4.5

Example 4.3. *Project is defined by the following data:*

Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Duration	4	4	2	4	3	3	6	5	2
Units	3	5	3	6	4	5	7	3	4
Predecessors			*A*	*A,C*	*B*	*D*	*B*	*E*	*F,G*

Increasing the number of available units of the resource, we get the following table. The early finish for this example is 15 days.

	Duration of the project		
Max. units	*Ms Project*	*Gray-Kidd*	*DynamicRes*
7	31	29	29
8	24	22	24
9	22	22	22
10	21	21	19
11	21	21	19
12	20	18	19
13	18	18	18
14	16	16	16
15	16	16	16
16	16	16	16
17	15	15	15

The data in the table are illustrated by the following chart.

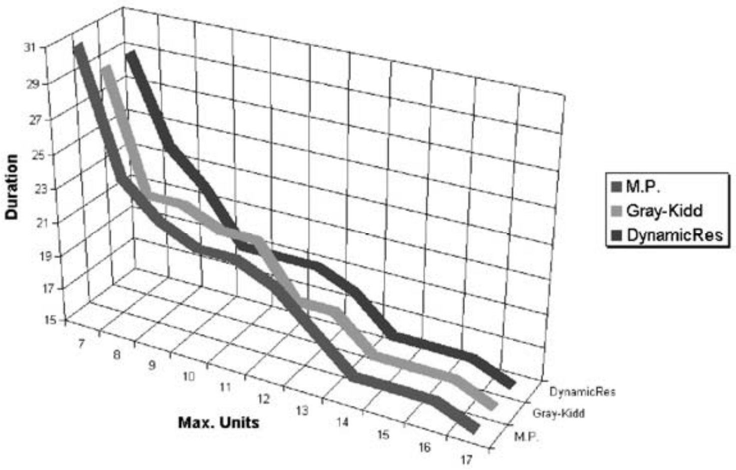


Figure 4.3. Comparison of three methods

Average values, geometrical means and standard deviations for project durations are presented in the table.

Statistical function	*Ms Project*	*Gray-Kidd*	*DynamicRes*
AVERAGE	20	19.45454545	19.36363636
Geometrical mean	19.55234723	19.097246	18.99493701
Standard deviation	4.69041576	4.107642544	4.201731245

In this case, the maximal standard deviation produces the Ms Project. But, this fact is caused by the maximal makespan (equal to 31) for the minimal number of *Max. units* (equal to 7).

Example 4.4. *Consider the following project:*

Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Duration	6	3	6	2	4	1	4	2	4	2
Units	6	5	7	3	4	5	7	3	2	3
Predecessors			*A*	*B*	*A,B*	*C,E*	*D,E*	*D,F*	*G,H*	*G,H*

Increasing the number of available resource units, we get the following table.

	Duration of the project		
Max. units	*Ms Project*	*Gray-Kidd*	*DynamicRes*
7	30	30	30
8	30	30	30
9	30	30	30
10	28	30	28
11	21	23	21
12	20	20	20
13	20	20	20
14	19	19	19

The following chart illustrates the data in the table.

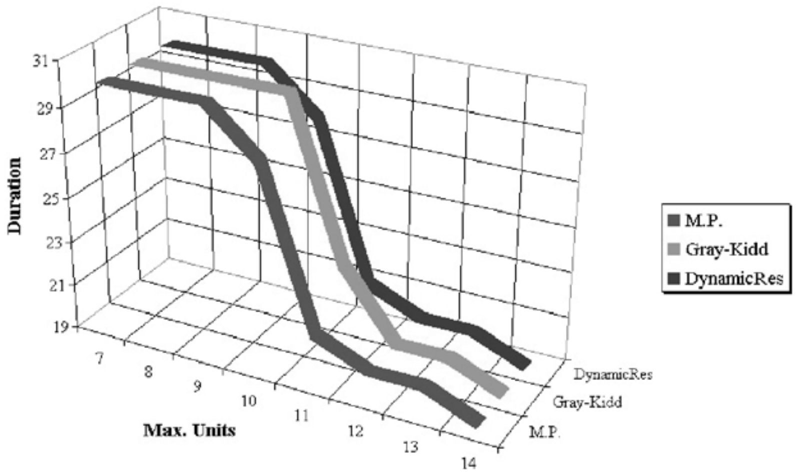


Figure 4.4. Comparison of three methods

Here are given some of statistical functions for project durations represented in the table.

Statistical function	*Ms Project*	*Gray-Kidd*	*DynamicRes*
AVERAGE	24.75	25.25	24.75
Geometrical mean	24.27696182	24.76727954	24.27696182
Standard deviation	5.14781507	5.2030211	5.14781507

5. CONCLUSION

A new algorithm for solving the single resource constrained project scheduling problem is introduced. At each schedule time t we consider the eligible set of activities which could be started without violation of given constraints (defined by predecessor relationships). Using activities from this set we consider the adequate knapsack problem and use the Bellman's principle. The possibility for the farther research can be the generalization of *DynamicRes* algorithm for solving the multiple resource constrained project scheduling problems, and the comparison with GRASP algorithm from [11].

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