Yugoslav Journal of Operations Research
27 (2017), Number 3, 367–384
DOI: 10.2298/YJOR160317012E

SOME VARIANTS OF REVERSE SELECTIVE CENTER LOCATION PROBLEM ON TREES UNDER THE CHEBYSHEV AND HAMMING NORMS

Roghayeh ETEMAD
Department of Applied Mathematics, Faculty of Basic Sciences,
Sahand University of Technology, Tabriz, Iran.
r_etemad@sut.ac.ir

Behrooz ALIZADEH*
Department of Applied Mathematics, Faculty of Basic Sciences,
Sahand University of Technology, Tabriz, Iran
alizadeh@sut.ac.ir, brz_alizadeh@yahoo.com

Received: March 2016 / Accepted: June 2016

Abstract: This paper is concerned with two variants of the reverse selective center location problems on tree graphs under the Hamming and Chebyshev cost norms in which the customers are existing on a selective subset of the vertices of the underlying tree. The first model aims to modify the edge lengths within a given modification budget until a prespecified facility location becomes as close as possible to the customer points. However, the other model wishes to change the edge lengths at the minimum total cost so that the distances between the prespecified facility and the customers satisfy a given upper bound. We develop novel combinatorial algorithms with polynomial time complexities for deriving the optimal solutions of the problems under investigation.

Keywords: Center Location Problems, Combinatorial Optimization, Reverse Optimization, Tree Graphs, Time Complexity.

MSC: 90C27; 90B80; 90B85; 90C35.

*Corresponding author

1. INTRODUCTION

Facility location problems are fundamental optimization models in operations research which are concerned with locating facilities on a system in order to serve a given set of customers in an optimal way under certain assessment criteria. In recent years, different variants of these problems have found significant interest due to their applications in theory and practice. Two widely investigated models in location theory are the "center" and "obnoxious center" problems. Whereas the center problem aims to obtain the best locations for establishing one or more desirable facilities such that the maximum of distances from the customers to the closest facility becomes minimum, the obnoxious center problem wishes to determine the best locations for installing some undesirable facilities so that the minimum of distances between the customers and the nearest facility is maximized. Such problems occur when the places of fire stations, hospitals, bank branches and also the locations of undesirable facilities like mega-airports, military bases, chemical plants and nuclear reactors have to be found. For a detailed survey on location problems see e.g. Eiselt [8], Mirchandani and Francis [10] and Zanjirani and Hekmatfar [17].

In contrast to the classical location problems, in practice we may envisage some situations on a facility system where the facilities have already been located at the present and they cannot serve the customers in an optimal way anymore. On the other hand, the replacement of them is not possible for the sake of some available restrictions. In this situation, a decision maker may attempt to improve the underlying system by formulating and solving one of the following improvement problems:

a) Inverse location problem: Modify specific input parameters of the underlying system in the cheapest possible way until the already installed facilities get their optimal positions.

b) Reverse location problem: Modify certain input parameters of the underlying system within a given modification budget so that the locations of the already established facilities are improved as much as possible under the new parameter values. Another variant of the reverse problem wishes to modify the input parameters at the minimum total cost such that the corresponding objective value of the predetermined facility locations obey an upper (a lower) bound.

In 1999, Cai et al. [7] proved that the inverse 1-center location problem with edge length modifications on unweighted directed graphs is $\mathcal{NP}$-hard. Moreover, the $\mathcal{NP}$-hardness of this problem on undirected graphs was proved in [11]. Therefore, the special polynomially solvable cases were considered. In 2008, Yang and Zhang [16] considered the inverse vertex center problem with variable edge lengths on unweighted trees and suggested an $O(n^2 \log n)$ time algorithm for this problem. Later, Alizadeh et al. [4] developed an $O(n \log n)$ time combinatorial algorithm for the inverse 1-center location problem with edge length augmentation on trees. For the inverse absolute (and vertex) 1-center location problems, solution algorithms with time complexities of $O(n^2)$ were designed by Alizadeh and Burkard [1]. The same authors investigated the uniform-cost inverse 1-center location model on trees and showed that the problem can be solved in $O(n \log n)$ time if there exists no topology change [2]. In 2012, Alizadeh and Burkard [3] derived a linear time combinatorial approach for the inverse obnoxious center location problem with edge length variations on general networks. Nguyen and Anh [13] investigated the inverse k-centrum problem on weighted trees with variable vertex weights and showed that this problem is $\mathcal{NP}$-hard. They proposed an $O(n^2)$ time algorithm for the inverse 1-center problem with vertex weight modifications on a tree. The inverse version of the 1-center problem on weighted trees with variable edge lengths under the Chebyshev and bottleneck-type Hamming cost norms was recently studied by Nguyen and Sepasian [12]. The authors presented an $O(n \log n)$ time solution approaches for the case that no topology change is permitted on the underlying tree. Furthermore, they showed that the general model can be solved in $O(n^2)$ time.

Concerning the reverse center (obnoxious center) location models, Berman et al. [6] proved that the reverse 1-center problem on unweighted graphs under the rectilinear norm is $\mathcal{NP}$-hard. In 2000, Zhang et al. [18] developed a solution algorithm with $O(n^2 \log n)$ running time for the reverse 1-center location problem on an unweighted tree. Nguyen [14] considered the uniform cost reverse 1-center location problem with edge length modifications on weighted trees and designed an $O(n^2)$ time method. Recently, Alizadeh and Etemad [5] proposed a linear time combinatorial algorithm for the reverse obnoxious center problem on general networks which is based on a binary search procedure. A variant of the reverse center location problem, called the vertex-to-vertices problem, was investigated in [19]. The authors showed that the problem with uniform modification costs on unweighted networks is solvable in $O(n^3)$ time under the Chebyshev norm, but under the rectilinear and Euclidean norms acheiving an approximation ratio $O(\log n)$ is $\mathcal{NP}$-hard.

In this paper, we investigate two variants of the reverse "selective" center location problem on tree graphs with variable edge lengths under the Chebyshev norm and the bottleneck-type and sum-type Hamming distances in which an arbitrary subset of the vertex set is assumed to be the existing customer points. We develop novel combinatorial algorithms with polynomial time complexities for obtaining the optimal solutions of the problems under the mentioned cost norms.

The organization of the paper is as follows: In the next section, we define and formulate the problems under investigation and discuss some basic properties. The exact solution algorithms are proposed in sections 3 and 4. Finally, the conclusion of the paper is presented in Section 5.

2. PROBLEM DEFINITION AND PRELIMINARIES

Let an undirected tree network $T = (V(T), E(T))$ with vertex set $V(T)$ and edge set $E(T)$, $|E(T)| + 1 = |V(T)| = n$, be given such that each edge $e \in E(T)$ has a nonnegative length $\ell(e)$. Moreover, let $\mathcal{V}_c \subseteq V(T)$ denote the set of existing customer points and $\mathcal{V}_f \subseteq V(T)$ stand for the set of candidate facility locations. The length of the unique path between two vertices u and v with respect to the edge lengths ℓ is denoted by $d_\ell(u, v)$. In a *classical selective center location problem* on the given tree T, the task is to find a facility location $p^* \in \mathcal{V}_f$ as an optimal solution for

$$\begin{aligned} &\text{minimize} \quad \mathcal{F}_\ell(p) = \max_{v \in \mathcal{V}_c} d_\ell(p, v) \\ &\text{subject to} \quad p \in \mathcal{V}_f. \end{aligned}$$

Note that the above "selective model" is a generalization of the well-known vertex center location problem with $\mathcal{V}_f = V(T)$ and $\mathcal{V}_c = V(T)$ on the underlying network.

In contrast to the classical selective center model, we are going to state two variants of the reverse selective center location problem: Let the underlying tree T with associated edge lengths $\ell = (\ell(e))_{e \in E(T)}$ and the existing customer points $\mathcal{V}_c \subseteq V(T)$ be given. Assume that $s \in \mathcal{V}_f$ is a prespecified facility location on T. We want to change the original lengths ℓ in order to improve the quality of the service center s as much as possible. Let $u^x(e)$ and $u^y(e)$ denote the amounts by which the length $\ell(e)$, $e \in E(T)$, is increased and decreased, respectively. Since the edge lengths of the tree T cannot be modified arbitrarily, the increasing and decreasing amounts $x(e)$ and $y(e)$ have to obey the given upper bounds $u^x(e)$ and $u^y(e)$, respectively. On the other hand, note that any modification imposes us a cost. Hence, suppose that $\mathcal{G}\big((\mathbf{x}, \mathbf{y})\big)$ denotes the cost function for measuring the incurred total cost for modifying the edge lengths by

$$(\mathbf{x}, \mathbf{y}) = \big(x(e), y(e)\big)_{e \in E(T)}.$$

In the first variant of the reverse selective center location problem, so-called budget-constrained reverse selective center problem ($\text{RSCP}_{\text{b-c}}$ for short), on the tree network T, we are given a budget $\mathcal{B}$. The aim is to modify the edge lengths $\ell(e)$ to the new nonnegative lengths

$$\tilde{\ell}(e) = \ell(e) + x(e) - y(e)$$

such that the following three statements hold:

(i) The objective value $\mathcal{F}_{\tilde{\ell}}(s)$ is minimized under the new edge lengths $\tilde{\ell}$.

(ii) The budget constraint $\mathcal{G}\big((\mathbf{x}, \mathbf{y})\big) \leqslant \mathcal{B}$ is satisfied.

(iii) The modifications $x(e)$ and $y(e)$ fulfill the bounds

$$\begin{aligned} &0 \leqslant x(e) \leqslant u^x(e) \quad \forall e \in E(T), \\ &0 \leqslant y(e) \leqslant u^y(e) \quad \forall e \in E(T). \end{aligned}$$

In the second variant of the reverse selective center location problem, so-called objective-bounded reverse selective center problem ($\text{RSCP}_{\text{o-b}}$ for short), on the given tree T, an upper bound λ for the objective value $\mathcal{F}_{\ell}(s)$ is specified. The goal is to modify the edge lengths ℓ to the new lengths $\tilde{\ell}$ so that the following statements are fulfilled:

i) The total modification cost $\mathcal{G}\big((\mathbf{x},\mathbf{y})\big)$ is minimized.

ii) The objective constraint
$$\mathcal{F}_{\tilde{\ell}}(s) \leqslant \lambda$$
is satisfied under the new lengths $\tilde{\ell}$.

iii) The modification amounts $x(e)$ and $y(e)$ obey the bounds
$$\begin{aligned} &0 \leqslant x(e) \leqslant u^x(e) \quad \forall e \in E(T), \\ &0 \leqslant y(e) \leqslant u^y(e) \quad \forall e \in E(T). \end{aligned}$$

In this paper, we concentrate on the $\text{RSCP}_{\text{b-c}}$ and $\text{RSCP}_{\text{o-b}}$ models on the underlying tree T where the cost function $\mathcal{G}(.)$ is defined in the following three cases:

(i) The total modification cost is measured by the weighted Chebyshev norm. In this case, we have
$$\mathcal{G}\big((\mathbf{x}, y)\big) = \max_{e\in E(T)} \big\{c(e)x(e), d(e)y(e)\big\},$$
where $c(e)$ and $d(e)$ are the costs for increasing and decreasing the length of an edge $e \in E(T)$ by one unit, respectively.

(ii) The total modification cost is measured by the weighted sum-type Hamming distance. In this case, we have
$$\mathcal{G}\big((\mathbf{x},\mathbf{y})\big) = \sum_{e\in E(T)} \Big(\hat{c}(e)\mathcal{H}\big(x(e), 0\big) + \hat{d}(e)\mathcal{H}\big(y(e), 0\big)\Big),$$
where $\hat{c}(e)$ and $\hat{d}(e)$ are the costs for increasing and decreasing $\ell(e)$ by any positive amount, respectively. Moreover, $\mathcal{H}(a, b)$ denotes the Hamming distance between a and b, i.e.,
$$\mathcal{H}(a,b) = \begin{cases} 1 & a \neq b, \\ 0 & a = b. \end{cases}$$

(iii) The modification cost is measured by the weighted bottleneck-type Hamming distance. In this case, we have
$$\mathcal{G}\big((\mathbf{x},\mathbf{y})\big) = \max_{e\in E(T)} \Big\{\hat{c}(e)\mathcal{H}\big(x(e), 0\big), \hat{d}(e)\mathcal{H}\big(y(e), 0\big)\Big\}.$$

In the next sections, we try to develop combinatorial algorithms for the $\mathrm{RSCP_{b-c}}$ and $\mathrm{RSCP_{o-b}}$ models under the weighted Chebyshev norm and the weighted sum-type and bottleneck-type Hamming distances. As mentioned, the special models of $\mathrm{RSCP_{b-c}}$ and $\mathrm{RSCP_{o-b}}$ on tree networks with $\mathcal{V}_c = \mathcal{V}_f = V(T)$ under the weighted rectilinear cost norm have been studied in [18] and solution approaches with $O(n^2 \log n)$ time complexities have been presented. From the specific structure of the $\mathrm{RSCP_{b-c}}$ and $\mathrm{RSCP_{o-b}}$ models, it is easy to observe that any augmentation of the edge lengths imposes us an additional cost. Therefore, we immediately conclude that

Lemma 2.1. *In order to solve the* $\mathrm{RSCP_{b-c}}$ *and* $\mathrm{RSCP_{o-b}}$ *models, it is sufficient to decrease the edge lengths of the underlying tree.*

Hence, we set $x(e) = 0$ and try to obtain only the optimal values of $y(e)$ for all $e \in E(T)$ in the following. Let the underlying tree T be rooted at the prespecified vertex s and

$$\mathbf{Lea}(T) = \{z_1, \cdots, z_k\}$$

denote the set of leaves of T. Suppose that q_i is the farthest customer to s on the unique path $P(s, z_i)$ between s and any leaf z_i. If there does not exist any customer on $P(s, z_i)$, then set $q_i = s$. Removing all paths $P(q_i, z_i)$, $i = 1, \cdots, k$, from T, we obtain a subtree T_{cri} which is called the *critical subtree* of T. Observe that

$$\mathcal{F}_\ell(s) = \max\left\{d_\ell(s, z) : \ z \in \mathbf{Lea}(T_{\mathrm{cri}})\right\}.$$

Hence, we get

Lemma 2.2. *In order to solve the* $\mathrm{RSCP_{b-c}}$ *and* $\mathrm{RSCP_{o-b}}$ *models on the tree* T*, it is sufficient to decrease the edge lengths of the critical subtree* T_{cri} *in an optimal way.*

3. OPTIMAL ALGORITHMS FOR $\mathrm{RSCP_{b-c}}$ MODELS

In this section, we first investigate the $\mathrm{RSCP_{b-c}}$ model on the given tree T under the sum-type Hamming distance and prove that this problem is $\mathcal{NP}$-hard. For the uniform-bound case, we develop an exact polynomial time solution algorithm. Then, we show that the $\mathrm{RSCP_{b-c}}$ model under the bottleneck-type Hamming distance and the Chebyshev norm can be solved in linear time.

3.1. The problem under the sum-type Hamming distance

Consider the $\mathrm{RSCP_{b-c}}$ model on the given tree T where the budget constraint under the sum-type Hamming distance is given by

$$\sum_{e \in E(T)} \Big(\hat{c}(e)\mathcal{H}\left(x(e), 0\right) + \hat{d}(e)\mathcal{H}\left(y(e), 0\right) \Big) \leqslant \mathcal{B}.$$

Note that since the Hamming distance is used for measuring the modification cost, any variation of the edge length $\ell(e)$ imposes us a fixed cost $\hat{c}(e)$ or $\hat{d}(e)$ (depending on the augmentation or reduction of $\ell(e)$) regardless its magnitude. We first prove the following important result.

Theorem 3.1. *The* $\mathrm{RSCP_{b-c}}$ *model on a tree under the sum-type Hamming distance is* $\mathcal{NP}$*-hard.*

Proof. Consider an instance of the problem on a path $P = \big(V(P), E(P)\big)$ where one of the end points of P is the prespecified facility location s and the other endpoint stands for the unique existing customer location. This instance of $\mathrm{RSCP_{b-c}}$ model can equivalently be formulated as

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E(P)} u^y(e)p(e) \\ \text{subject to} \quad & \sum_{e \in E(P)} \hat{d}(e)p(e) \leqslant \mathcal{B}, \\ & p(e) \in \{0,1\} \qquad \forall\, e \in E(P). \end{aligned}$$

This optimization model is a binary knapsack problem which is well-known to be $\mathcal{NP}$-hard (see e.g. Korte and Vygen [9]). This result immediately proves the claim of the theorem. □

According to Theorem 3.1, in case that the modification bounds and costs are arbitrary, the problem of selecting the best edges for modifications is $\mathcal{NP}$-hard. However, in the uniform-bound case, the edges will be selected for modifications with respect to their fixed cost coefficients. Based on this fact, we consider the $\mathrm{RSCP_{b-c}}$ model with uniform modification bounds under the sum-type Hamming distance on the tree T and try to derive a solution approach to it. In the uniform-bound model, we suppose that

$$u^x(e) = u^y(e) = \rho \quad \forall e \in E(T).$$

As a subroutine of our solution approach, we have somehow benefited from the solution idea presented in [18] for the reverse center problem under the rectilinear cost norm. But, our algorithm in general carries out different computational operations. In fact, the algorithm is based on a sequence of minimum $s-t$ cuts in an auxiliary network N which is constructed as follows: Add an additional vertex t to the critical subtree T_{cri} rooted at s and connect it to every leaf $z \in \mathbf{Lea}(T_{\mathrm{cri}})$, namely set

$$V(N) = V(T_{\mathrm{cri}}) \cup \{t\}$$

and

$$E(N) = E(T_{\mathrm{cri}}) \cup E_1,$$

where

$$E_1 = \{(z,t) \mid z \in \mathbf{Lea}(T_{\mathrm{cri}})\}.$$

All edges on N are also directed from s to t. Let M be a very big value. The length, bound and cost coefficient of any edge $e \in E(N)$ are defined as

$$\ell_N(e) = \begin{cases} \ell(e) & \text{if } e \in E(T_{\text{cri}}), \\ \mathcal{F}_\ell(s) - d_\ell(s,z) & \text{if } e = (z,t) \in E_1, \end{cases}$$

$$u_N(e) = \begin{cases} \rho & \text{if } e \in E(T_{\text{cri}}), \\ \ell_N(e) & \text{if } e \in E_1, \end{cases}$$

$$c_N(e) = \begin{cases} \hat{d}(e) & \text{if } e \in E(T_{\text{cri}}), \\ 0 & \text{if } e = (z,t) \in E_1,\ d_\ell(s,z) < \mathcal{F}_\ell(s), \\ M & \text{if } e = (z,t) \in E_1,\ d_\ell(s,z) = \mathcal{F}_\ell(s). \end{cases}$$

Observe that, there exist $|\mathbf{Lea}(T_{\text{cri}})|$ paths from s to t on the network N and all of them have equal lengths $\mathcal{F}_\ell(s)$. For solving the uniform-bound $\text{RSCP}_{\text{b-c}}$ model under the sum-type Hamming distance on the underlying tree T, we propose Algorithm 1 which is based on decreasing the lengths of all edges contained in a finite sequence of minimum $s-t$ cuts on the auxiliary network N. Let $\mathcal{R}$ be a minimum $s-t$ cut on N and $E(\mathcal{R})$ be the set of the edges which are contained in the cut $\mathcal{R}$. The capacity of $\mathcal{R}$ is computed as

$$C(\mathcal{R}) = \sum_{e \in E(\mathcal{R})} c_N(e). \tag{1}$$

If $C(\mathcal{R}) \leqslant \mathcal{B}$ and $C(\mathcal{R}) < M$, then it means that we can decrease the lengths of all edges $e \in E(\mathcal{R})$ by the amount

$$\delta(\mathcal{R}) = \min\left\{u_N(e) :\ e \in E(\mathcal{R})\right\} \tag{2}$$

in order that the objective value $\mathcal{F}_\ell(s)$ of the problem is improved by the amount $\delta(\mathcal{R})$ incurring the minimum cost $C(\mathcal{R})$. Performing the above modification, the remaining budget will be

$$\mathcal{B} = \mathcal{B} - C(\mathcal{R}). \tag{3}$$

If the remaining budget and the modification bounds permit, then we can repeat the above procedure on the auxiliary network N with updated lengths and capacities

$$\ell_N(e) = \begin{cases} \ell_N(e) - \delta(\mathcal{R}) & \text{if } e \in E(\mathcal{R}), \\ \ell_N(e) & \text{else}, \end{cases} \tag{4}$$

$$c_N(e) = \begin{cases} c_N(e) & \text{if } e \notin E(\mathcal{R}), \\ 0 & \text{if } e \in E(\mathcal{R}),\ u_N(e) > \delta(\mathcal{R}), \\ M & \text{if } e \in E(\mathcal{R}),\ u_N(e) = \delta(\mathcal{R}), \end{cases} \tag{5}$$

and the modification bounds

$$u_N(e) = \begin{cases} u_N(e) - \delta(\mathcal{R}) & \text{if } e \in E(\mathcal{R}), \\ u_N(e) & \text{else,} \end{cases} \tag{6}$$

until an optimal modification is achieved. Considering the above discussion, our solution approach is summarized as follows:

Algorithm 1 (solves the uniform-bound $\text{RSCP}_{\text{b-c}}$ model under the sum-type Hamming distance on the tree T)

Begin
Step 1. Construct the critical subtree T_{cri}.

Step 2. Set $\mathcal{F}^* = \mathcal{F}_\ell(s)$.

Step 3. Determine a minimum $s-t$ cut $\mathcal{R}$ in N and compute the corresponding capacity $C(\mathcal{R})$ by (1).

Step 4. If $C(\mathcal{R}) \geqslant M$ or $C(\mathcal{R}) > \mathcal{B}$, then stop; otherwise, compute $\delta(\mathcal{R})$ by (2).

Step 5. Update $\mathcal{B}$, ℓ_N, c_N and u_N according to (3), (4), (5) and (6), respectively.

Step 6. Set $\mathcal{F}^* = \mathcal{F}^* - \delta(\mathcal{R})$ and go to Step 3.
End

By executing Algorithm 1, the optimal objective value $\mathcal{F}^*$ and the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ with

$$x^*(e) = 0, \tag{7}$$

$$y^*(e) = \begin{cases} \ell(e) - \ell_N(e) & \text{if } e \in E(T_{\text{cri}}), \\ 0 & \text{else,} \end{cases} \tag{8}$$

for all $e \in E(T)$ is determined.

We are now going to proceed the correctness arguments of the algorithm: Observe that the objective value $\mathcal{F}_\ell(s)$ of the original problem is decreased by an amount δ if and only if the lengths of all paths $P(s,z) \cup \{(z,t)\}$, $z \in \mathbf{Lea}(T_{\text{cri}})$, is decreased by the amount δ. On the other hand, the modification of the edge lengths must be performed within an associated budget $\mathcal{B}$. Hence, it is necessary to take such edges on every path $P(s,z) \cup \{(z,t)\}$, $z \in \mathbf{Lea}(T_{\text{cri}})$, which have the smallest total capacity. To this end, by finding a minimum $s-t$ cut on the network N, one can decrease the lengths of all paths $P(s,z) \cup \{(z,t)\}$, $z \in \mathbf{Lea}(T_{\text{cri}})$, at the minimum total cost. Let $\mathcal{R}$ be a minimum $s-t$ cut on N. If $C(\mathcal{R}) \geqslant M$, then according to the definition of the capacities $c_N(e)$, we conclude that there exists a path $P(s,z') \cup \{(z',t)\}$, $z' \in \mathbf{Lea}(T_{\text{cri}})$, such that its length cannot be decreased anymore. If $C(\mathcal{R}) > \mathcal{B}$, then it means that there is no enough budget for simultaneous perturbation of the lengths $\ell_N(e)$, $e \in E(\mathcal{R})$, on the paths $P(s,z) \cup \{(z,t)\}$, $z \in \mathbf{Lea}(T_{\mathrm{cri}})$. Therefore, if at least one of the above two cases occurs, then it implies that the objective value $\mathcal{F}_\ell(s)$ cannot be improved any more and its current value is optimal. In case that $C(\mathcal{R}) \leqslant B$ and $C(\mathcal{R}) < M$, the objective value $\mathcal{F}_\ell(s)$ is decreased by any amount δ with $0 < \delta \leqslant \delta(\mathcal{R})$ at the minimum cost $C(\mathcal{R})$, if all lengths $\ell_N(e)$, $e \in E(\mathcal{R})$, are decreased by the amount δ. Let us now suppose that the objective value $\mathcal{F}_\ell(s)$ is decreased by the amount $\delta(\mathcal{R})$ enduring the cost $C(\mathcal{R})$. If the budget $\mathcal{B}$ is not completely spent, i.e.

$$\mathcal{B} - C(\mathcal{R}) > 0,$$

and the associated bounds permit for further improvement, then we update the parameters of the network N according to (4), (5) and (6). Iterating the above process, we obtain a finite sequence of minimum $s-t$ cuts, let say $\mathcal{R}_1, \cdots, \mathcal{R}_t$, which lead successively to the reduction of $\mathcal{F}_\ell(s)$ by the amounts $\delta(\mathcal{R}_1), \cdots, \delta(\mathcal{R}_t)$ until an optimal objective value

$$\mathcal{F}^* = \mathcal{F}_\ell(s) - \sum_{j=1}^{t} \delta(\mathcal{R}_j)$$

is derived for the problem under investigation. As an important remark, recall that the cost function $\mathcal{G}\big((\mathbf{x}, \mathbf{y})\big)$ is defined under the Hamming distance in this subsection. Hence, we should set $c_N(e) = 0$ for every $e \in E(\mathcal{R})$ with $u_N(e) > \delta(\mathcal{R})$, in order to appearing these edges in the next minimum $s-t$ cut. Otherwise, we may incur an additional cost not leading to an optimal solution.

Let us now study the time complexity of the algorithm. The critical subtree T_{cri} is constructed in $O(n)$ time. In each iteration of the algorithm, at least one edge e of the minimum $s-t$ cut $\mathcal{R}$ in N reaches its lower bound and its capacity is updated to $c_N(e) = M$, then it will not be contained in the next minimum $s-t$ cuts, except in the last iteration. Then, the total number of iterations of the algorithm is bounded by n. On the other hand, every minimum $s-t$ cut in N can be found in $O(n)$ time, since $N - \{t\}$ is an arborescence (see e.g. Vygen [15]). Moreover, the updating of the network N takes $O(n)$ time. Therefore, we conclude

Theorem 3.2. *The uniform-bound* $\mathrm{RSCP}_{\mathrm{b-c}}$ *model under the sum-type Hamming distance is solvable in* $O(n^2)$ *time on a tree with n vertices.*

3.2. The problem under the bottleneck-type Hamming distance

Suppose that the budget constraint of the $\mathrm{RSCP}_{\mathrm{b-c}}$ model is defined as

$$\max_{e \in E(T)} \left\{ \hat{c}(e)\mathcal{H}\left(x(e), 0\right), \hat{d}(e)\mathcal{H}\left(y(e), 0\right) \right\} \leqslant \mathcal{B}.$$

Due to Lemma 2.2, the above inequality is equivalently reduced to

$$\max_{e \in E(T_{\mathrm{cri}})} \hat{d}(e)\mathcal{H}\left(y(e), 0\right) \leqslant \mathcal{B}.$$

We can clearly observe that an optimal modification for the problem under investigation is to decrease the length of any edge $e \in E(T_{\text{cri}})$ with $\hat{d}(e) \leqslant \mathcal{B}$ to its modification bound $u^y(e)$. Therefore, an optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the problem is obtained by

$$x^*(e) = 0,$$
$$y^*(e) = \begin{cases} u^y(e) & \text{if } e \in E(T_{\text{cri}}),\ \hat{d}(e) \leqslant \mathcal{B}, \\ 0 & \text{else}, \end{cases}$$

for all $e \in E(T)$ in linear time. Since the critical subtree T_{cri} is also constructed in linear time, we have

Theorem 3.3. *The* $\text{RSCP}_{\text{b-c}}$ *model on trees under the bottleneck-type Hamming distance can be solved in* $O(n)$ *time.*

3.3. The problem under the Chebyshev norm

Consider the $\text{RSCP}_{\text{b-c}}$ model on the given tree T in which the budget constraint is given as

$$\max_{e \in E(T)} \left\{ c(e)x(e), d(e)y(e) \right\} \leqslant \mathcal{B}. \tag{9}$$

According to Lemma 2.2, our modification is limited to the reduction of the edge lengths of the critical subtree T_{cri}. Hence, the constraint (9) is equivalently reduced to

$$\max_{e \in E(T_{\text{cri}})} d(e)y(e) \leqslant \mathcal{B}.$$

Considering the structure of the above constraint, we can observe that for any edge $e \in E(T_{\text{cri}})$, the modification $y(e)$ must satisfy the bound

$$y(e) \leqslant \min \left\{ u^y(e), \frac{\mathcal{B}}{d(e)} \right\}.$$

Recall that we want to decrease the edge lengths ℓ as big as possible to minimize the objective value $\mathcal{F}_{\tilde{\ell}}(s)$ under the new edge lengths $\tilde{\ell}$. Therefore, an optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the $\text{RSCP}_{\text{b-c}}$ model under the Chebyshev norm on the given tree T can be found by

$$x^*(e) = 0$$
$$y^*(e) = \begin{cases} u^y(e) & \text{if } d(e)u^y(e) \leqslant \mathcal{B}, \\ \frac{\mathcal{B}}{d(e)} & \text{else}, \end{cases}$$

for all $e \in E(T)$ in $O(n)$ time. Recall that the critical subtree T_{cri} can also be constructed in $O(n)$ time. Therefore, we get the following result.

Theorem 3.4. *The* $\text{RSCP}_{\text{b-c}}$ *model on trees under the Chebyshev norm is solvable in* $O(n)$ *time.*

4. OPTIMAL ALGORITHMS FOR RSCP$_{\mathrm{o-b}}$ MODELS

This section is dedicated to the RSCP$_{\mathrm{o-b}}$ model on the underlying tree T under the sum-type Hamming and bottleneck-type Hamming distances and the Chebyshev norm with edge lengths variations. Recall that the aim is to increase or decrease the edge lengths at the minimum total cost subject to the given modification bounds $u^x(e)$ and $u^y(e)$ for $e \in E(T)$ such that the maximum of distances from the prespecified facility location s to the existing customer points $v \in \mathcal{V}_c$ does not exceed the given objective bound λ.

According to Lemma 2.2, it is sufficient to decrease the edge lengths ℓ on the critical subtree T_{cri} to $\tilde{\ell}$ at the minimum total cost so that the inequality

$$d_{\tilde{\ell}}(s,z) \leqslant \lambda$$

holds for any leaf $z \in \mathbf{Lea}(T_{\mathrm{cri}})$. Note that the problem is feasible if and only if

$$d_\ell(s,z) - \sum_{e \in E(P(s,z))} u^y(e) \leqslant \lambda$$

is satisfied for all $z \in \mathbf{Lea}(T_{\mathrm{cri}})$. Furthermore, assume that there exists a leaf $z' \in \mathbf{Lea}(T_{\mathrm{cri}})$ such that $d_\ell(s,z') > \lambda$. Otherwise, the problem is trivial.

4.1. The problem under the sum-type Hamming distance

Consider the RSCP$_{\mathrm{o-b}}$ model under the sum-type Hamming distance on the given tree T. This problem is equivalently formulated as the following optimization model:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E(T_{\mathrm{cri}})} \hat{d}(e)p(e) & & \\ \text{subject to} \quad & \sum_{e \in E(P(s,z))} u^y(e)p(e) \geqslant d_\ell(s,z) - \lambda \qquad \forall\, z \in \mathbf{Lea}(T_{\mathrm{cri}}), & & (10) \\ & p(e) \in \{0,1\} \qquad\qquad \forall\, e \in E(T_{\mathrm{cri}}). & & (11) \end{aligned}$$

The above problem is a multi-dimensional binary knapsack problem which is strongly $\mathcal{NP}$-hard (see e.g. Korte and Vygen [9]). Then, we immediately get

Theorem 4.1. *The* RSCP$_{\mathrm{o-b}}$ *model on trees under the sum-type Hamming distance is* $\mathcal{NP}$*-hard.*

Considering the above theorem and the discussion given in Subsection 3.1, we try to develop an exact $O(n^2)$-time algorithm for the problem with uniform modification bounds

$$u^x(e) = u^y(e) = \rho, \quad \forall\, e \in E(T).$$

Now, define the gap

$$\mathcal{D} = \mathcal{F}_\ell(s) - \lambda.$$

If the problem is feasible, then we should decrease the edge lengths ℓ on the subtree $T_{\rm cri}$ until the gap $\mathcal{D} \leqslant 0$ is gotten under the new lengths $\tilde{\ell}$. In fact, our solution approach relies on a sequence of minimum $s-t$ cuts $\mathcal{R}_1, \cdots, \mathcal{R}_t$ with corresponding edge sets $E(\mathcal{R}_1), \cdots, E(\mathcal{R}_t)$ on the auxiliary network N as introduced in Section 3. The optimal modification is done by decreasing the lengths of the edges contained in any obtained minimum cut $\mathcal{R}$ by the amount

$$\delta(\mathcal{R}) = \min\left\{\min\{u_N(e):\ e \in E(\mathcal{R})\}, \mathcal{D}\right\} \tag{12}$$

provided that the modification bounds permit, i.e., $C(\mathcal{R}) < M$ and the objective value gap $\mathcal{D}$ is positive. Our proposed solution algorithm is outlined in the following.

Algorithm 2 (solves the uniform-bound RSCP$_{\rm o-b}$ model under the sum-type Hamming distance on the tree T)

Begin

Step 1. Set $C^* = 0$.

Step 2. Construct the critical subtree $T_{\rm cri}$.

Step 3. Find a minimum $s-t$ cut $\mathcal{R}$ of N and obtain its capacity $C(\mathcal{R})$ by (1).

Step 4. If $C(\mathcal{R}) \geqslant M$, then stop; otherwise, compute $\delta(\mathcal{R})$ by (12) and set

$$C^* = C^* + C(\mathcal{R}).$$

Step 5. If $\delta(\mathcal{R}) = \mathcal{D}$, then update the lengths ℓ_N by (4) and stop.

Step 6. If $\delta(\mathcal{R}) < \mathcal{D}$, then update ℓ_N, c_N and u_N according to (4), (5) and (6), respectively. Set

$$\mathcal{D} = \mathcal{D} - \delta(\mathcal{R})$$

and go to Step 3.

End

The correctness arguments for Algorithm 2 is analogous to Algorithm 1. However, note that in the RSCP$_{\rm o-b}$ model, we do not wish to decrease the edge lengths and consequently the objective value $\mathcal{F}_\ell(s)$ as much as possible even if the modification bounds permit. We only want to decrease the gap $\mathcal{D}$ to zero value at the minimum total cost with respect to the given bounds. Hence, for any minimum cut $\mathcal{R}$, the edges $e \in E(\mathcal{R})$ should exactly be decreased by the amount $\delta(\mathcal{R})$, if the problem is feasible. If the algorithm is terminated at Step 4 when $C(\mathcal{R}) \geqslant M$, then it means that we have not succeeded to decrease the gap $\mathcal{D}$ to zero and then the problem is infeasible. But, when the algorithm is terminated at Step 5, it implies that we have decreased the gap $\mathcal{D}$ to zero and consequently the optimal objective value C^* of the RSCP$_{\text{o-b}}$ model is determined and an optimal solution is found by (7) and (8).

Since Algorithm 2 also requires at most n minimum $s-t$ cuts on the tree-like network N and every minimum cut is determined in $O(n)$ time, then the time complexity of the algorithm is bounded by $O(n^2)$.

Therefore, we get

Theorem 4.2. *The uniform-bound* RSCP$_{\text{o-b}}$ *model on trees under the sum-type Hamming distance can be solved in $O(n^2)$ time.*

4.2. The problem under the bottleneck-type Hamming distance

Based on Lemma 2.2, the RSCP$_{\text{o-b}}$ model under the bottleneck-type Hamming distance on the underlying tree T is equivalently transformed to the problem

$$\begin{aligned} &\text{minimize} \quad \max_{e \in E(T_{\text{cri}})} \hat{d}(e)p(e) \\ &\text{subject to} \quad (10) - (11) \end{aligned}$$

on the critical subtree T_{cri}. It can be observed that the optimal objective value of the problem is equal to $\hat{d}(e)$ for some $e \in E(T_{\text{cri}})$. The specific structure of the problem helps us to develop a solution algorithm based on a binary search approach. Let $E(T_{\text{cri}}) = \left\{e_1, \cdots, e_k\right\}$ and define

$$E_i = \left\{e \in E(T_{\text{cri}}) : \ \hat{d}(e) \leqslant \hat{d}(e_i)\right\}$$

for $i = 1, \cdots, k$. Let $D^i(s,z)$ denote the modified distance between the prespecified facility location s and any leaf $z \in \mathbf{Lea}(T_{\text{cri}})$ after decreasing the lengths $\ell(e)$, $e \in E_i \cap E\,(P(s,z))$ by the amounts $u^y(e)$ on the path $P(s,z)$, namely, define

$$D^i(s,z) = d_\ell(s,z) - \sum_{e \in E_i \cap E(P(s,z))} u^y(e).$$

Now, let us consider the following definition and let

$$D^i_{\max} = \max_{z \in \mathbf{Lea}(T_{\text{cri}})} D^i(s,z). \tag{13}$$

Definition 4.3. *After renumbering the edges of the subtree T_{cri} such that*

$$\hat{d}(e_1) \leqslant \hat{d}(e_2) \leqslant \cdots \leqslant \hat{d}(e_k),$$

an edge $e_b \in \{e_i : \ i = 1, \cdots, k\}$ is called a break edge for the RSCP$_{\text{o-b}}$ *model under the bottleneck-type Hamming distance if and only if*

$$D^{b-1}_{\max} > \lambda \quad \textit{and} \quad D^b_{\max} \leqslant \lambda.$$

We immediately get

Lemma 4.4. *If the break edge e_b for the* $\mathrm{RSCP}_{\mathrm{o-b}}$ *model under the bottleneck-type Hamming distance is known, then an optimal solution* $(\mathbf{x}^*, \mathbf{y}^*)$ *of the problem is given by*

$$x^*(e) = 0, \tag{14}$$

$$y^*(e) = \begin{cases} u^y(e) & \text{if } e \in E_b, \\ 0 & \text{if } e \in E(T) \setminus E_b, \end{cases} \tag{15}$$

for all $e \in E(T)$ *with the corresponding optimal value*

$$C^* = \hat{d}(e_b).$$

The break edge e_b can be determined by a combination of the linear time algorithm for finding the median of a finite set with a binary search approach (Procedure BrE).

Procedure BrE (finds the break edge e_b)

Step 1. Let $\mathcal{I} = E(T_{\mathrm{cri}})$.

Step 2. Find the median med $= \hat{d}(e_m)$ of the set $\left\{\hat{d}(e_i) : e_i \in \mathcal{I}\right\}$.

Step 3. Let

$$\mathcal{I}^> = \left\{e_i \in \mathcal{I} : \hat{d}(e_i) > \mathrm{med}\right\},$$
$$\mathcal{I}^< = \left\{e_i \in \mathcal{I} : \hat{d}(e_i) < \mathrm{med}\right\}.$$

Step 4. For any $z \in \mathbf{Lea}(T_{\mathrm{cri}})$, compute the distances $D^m(s,z)$ and $D^{m'}(s,z)$, where

$$e_{m'} = \operatorname{argmax}\left\{\hat{d}(e_i) : e_i \in \mathcal{I}^<\right\}.$$

Step 5. If $D^m_{\max} \leqslant \lambda$ and $D^{m'}_{\max} > \lambda$, then $e_b = e_m$ is a break edge and stop.

Step 6. If $D^m_{\max} \leqslant \lambda$ and $D^{m'}_{\max} \leqslant \lambda$, then set $\mathcal{I} = \mathcal{I}^<$ and go to Step 2. If $D^m_{\max} > \lambda$, then set $\mathcal{I} = \mathcal{I}^>$ and return to Step 2.

Let us now discuss the running time of Procedure BrE. In each iteration, the median med and the parameters $D^m(s,z)$ and $D^{m'}(s,z)$ are computed in $O(n)$ time. On the other hand, the procedure terminates at most in $O(\log(|E(T_{\mathrm{cri}})|)$ iterations with $|E(T_{\mathrm{cri}})| \leqslant n$. Then, the time complexity of Procedure BrE is bounded by $O(n \log n)$.

When the break edge e_m is determined by Procedure BrE, an optimal solution of the $\mathrm{RSCP}_{\mathrm{o-b}}$ model is attained by (14) and (15) in $O(n)$ time. Therefore, considering the fact that the time needed to construct the critical subtree T_{cri} is $O(n)$, we conclude

Theorem 4.5. *The* $\mathrm{RSCP}_{\mathrm{o-b}}$ *model on trees under the bottleneck-type Hamming distance is solvable in* $O(n \log n)$ *time.*

4.3. The problem under the Chebyshev norm

Now, we deal with the $\text{RSCP}_{\text{o-b}}$ model on the given tree T under the Chebyshev norm where according to Lemma 2.2, the aim is to minimize

$$\max_{e \in E(T_{\text{cri}})} d(e)y(e).$$

Let $E(T_{\text{cri}}) = \left\{e_1, \cdots, e_k\right\}$ and define the edge sets E_i, $i = 1, \cdots, k$, as

$$E_i = \{e \in E(T_{\text{cri}}) : \ f(e) \leqslant f(e_i)\},$$

where

$$f(e) = d(e)u^y(e) \quad \forall e \in E(T_{\text{cri}}).$$

Let $D^i(s, z)$, $i = 1, \cdots, k$, denote the modified distance between the prespecified location s and any leaf $z \in \mathbf{Lea}(T_{\text{cri}})$ after decreasing the edge lengths by

$$y(e) = \begin{cases} u^y(e) & \text{if } e \in E_i, \\ \frac{f(e_i)}{d(e)} & \text{if } e \in E(T_{\text{cri}}) \setminus E_i, \\ 0 & \text{else.} \end{cases}$$

Moreover, let $D^i_{\max}$ be defined as (13) and consider the following definition.

Definition 4.6. *After renumbering the edges of the subtree T_{cri} such that*

$$f(e_1) \leqslant f(e_2) \leqslant \cdots \leqslant f(e_k),$$

an edge $e_b \in \{e_i : \ i = 1, \cdots, k\}$ is called a break edge for the $\text{RSCP}_{\text{o-b}}$ *model under the Chebyshev norm if and only if*

$$D^b_{\max} > \lambda \quad \textit{and} \quad D^{b+1}_{\max} \leqslant \lambda.$$

The connection between the break edge e_b and the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ is given by the following lemma:

Lemma 4.7. *If the break edge e_b for the* $\text{RSCP}_{\text{o-b}}$ *model under the Chebyshev norm is known, then the optimal solution can be found by*

$$x^*(e) = 0, \tag{16}$$

$$y^*(e) = \begin{cases} u^y(e) & \textit{if } e \in E_b, \\ \frac{C^*}{d(e)} & \textit{if } e \in E(T_{\text{cri}}) \setminus E_b, \\ 0 & \textit{else,} \end{cases} \tag{17}$$

for all $e \in E(T)$ with the corresponding optimal value

$$C^* = \max_{z \in \boldsymbol{Lea}(T_{\text{cri}})} \frac{\Delta(z)}{\Delta'(z)}, \tag{18}$$

where

$$\Delta(z) = d_\ell(s,z) - \sum_{e\in E_b\cap E(P(s,z))} u^y(e) - \lambda,$$

$$\Delta'(z) = \sum_{e\in (E(T_{\mathrm{cri}})\setminus E_b)\cap E(P(s,z))} \frac{1}{d(e)}$$

for all $z \in \textbf{Lea}(T_{\mathrm{cri}})$.

Proof. According to definition of the break edge, the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ can obviously be obtained by (16) and (17). Since the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ is feasible, the inequality

$$d_\ell(s,z) - \sum_{e\in E_b\cap E(P(s,z))} u^y(e) - \sum_{e\in (E(T_{\mathrm{cri}})\setminus E_b)\cap E(P(s,z))} \frac{C^*}{d(e)} - \lambda \leqslant 0$$

holds for every leaf $z \in \textbf{Lea}(T_{\mathrm{cri}})$. Hence, we conclude

$$C^* \geqslant \frac{\Delta(z)}{\Delta'(z)} \qquad \forall\, z \in \textbf{Lea}(T_{\mathrm{cri}}).$$

These inequalities immediately imply the equation (18). □

Obviously, we can find the break edge e_b in $O(n \log n)$ time by applying a combination of the linear time algorithm for finding the median of a finite set with a binary search approach similar to Procedure BrE. The values $\Delta(z)$ and $\Delta'(z)$ for all $z \in \textbf{Lea}(T_{\mathrm{cri}})$ can be computed in linear time using a breadth-first search algorithm. Then, the optimal value C^* is obtained in linear time if the break edge is identified. Moreover, the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ is obtained according to (16) and (17) in $O(n)$ time. Recall that the critical subtree T_{cri} is constructed in linear time.

Altogether, we get

Theorem 4.8. *The* $\mathrm{RSCP_{o-b}}$ *model on trees under the Chebyshev norm can be solved in* $O(n \log n)$ *time.*

5. CONCLUSION

In this paper, we investigated two variants of the reverse selective center location problem, the so-called $\mathrm{RSCP_{b-c}}$ and $\mathrm{RSCP_{o-b}}$, on tree networks. We showed that the $\mathrm{RSCP_{b-c}}$ and $\mathrm{RSCP_{o-b}}$ models under the sum-type Hamming distance are $\mathcal{NP}$-hard on graphs even on trees. So, we considered the special case of uniform modification bounds and outlined $O(n^2)$ time solution algorithms. Moreover, we showed that the $\mathrm{RSCP_{b-c}}$ model under the bottleneck-type Hamming distance and the Chebyshev norm can be solved in linear time. Finally, we developed two solution methods with $O(n \log n)$ time complexities for the $RSCP_{o-b}$ model under the bottleneck-type Hamming distance and the Chebyshev norm.

For future research, it is interesting to consider the reverse selective center problem on other special networks like cacti, cycles, wheels, unicyclic graphs and etc. Another direction of future research is the investigation of the problem under other cost norms.

REFERENCES

[1] Alizadeh, B., and Burkard, R.E., "Combinatorial algorithms for inverse absolute and vertex 1-center location problems on trees", *Networks*, 58 (2011) 190-200.

[2] Alizadeh, B., and Burkard, R.E., "Uniform-cost inverse absolute and vertex center location problems with edge length variations on trees", *Discrete Applied Mathematics*, 159 (2011) 706-716.

[3] Alizadeh, B., and Burkard, R.E., "A linear time algorithm for inverse obnoxious center location problems on networks", *Central European Journal of Operations Research*, 21 (2012) 585-594.

[4] Alizadeh, B., Burkard, R.E., and Pferschy, U., "Inverse 1-center location problems with edge length augmentation on trees", *Computing*, 86 (2009) 331-343.

[5] Alizadeh, B., and Etemad, R., "The linear time optimal approaches for reverse obnoxious center location problems on networks", *Optimization* 65 (11) (2016) 2025–2036.

[6] Berman, O., Ingco, D.I., and Odoni, A.R., "Improving the location of minmax facility through network modification", *Networks*, 24 (1994) 31-41.

[7] Cai, M.C., Yang, X.G., and Zhang, JZ., "The complexity analysis of the inverese center location problem", *Journal of Global Optimization*, 15 (1999) 213-218.

[8] Eiselt, H.A., *Foundation of Location Analysis*, Springer Verlag, New York, 2011.

[9] Korte, B., and Vygen, J., *Combinatorial Optimization, Theory and Algorithms*, Springer Verlag, New York, 2011.

[10] Mirchandani, P.B., and Francis, R.L., *Discrete Location Theory*, John Wiley, New York, 1990.

[11] Nguyen, K.T., and Chassein, A., "Inverse eccentric vertex problem on networks", *Central European Journal of Operations Research*, 23 (2015) 687-698.

[12] Nguyen, K.T., and Sepasian, A.R., "The inverse 1-center problem on trees with variable edge lengths under Chebyshev norm and Hamming distance", *Journal of Combinatorial Optimization*, 32 (3) (2016) 872–884.

[13] Nguyen, K.T., and Anh, L.Q., "Inverse k-centrum problem on trees with variable vertex weights", *Mathematical Methods of Operations Research*, 82 (2015) 19-30.

[14] Nguyen, K.T., "Reverse 1-center problem on weighted trees", *Optimization*, 65 (2015) 253-264.

[15] Vygen, J., "On dual minimum cost flow algorithms", *Mathematical Methods of Operations Research*, 56 (2002) 101-126.

[16] Yang, X., and Zhang, J., "Inverse center location problem on a tree", *Journal of System Science and Complexity*, 21 (2008) 651-664.

[17] Zanjirani, R., and Hekmatfar, M., *Facility location: concepts, models, algorithms and case studies*, Physica-Verlag, Berlin, 2009.

[18] Zhang, J.Z., Liu, Z.H., and Ma, Z.F., "Some reverse location problems", *European Journal of Operational Research*, 124 (2000) 77-88.

[19] Zhang, J.Z., Yang, X.G., and Cai, M.C., "A network improvement problem under different norms", *Computational Optimization and Applications*, 27 (2004) 305-319.