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APPROXIMATION OF THE STEADY STATE SYSTEM STATE DISTRIBUTION OF THE M/G/1 RETRIAL QUEUE WITH IMPATIENT CUSTOMERS

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Abstract: For M/G/1 retrial queues with impatient customers, we review the results, concerning the steady state distribution of the system state, presented in the literature. Since the existing formulas are cumbersome (so their utilization in practice becomes delicate) or the obtaining of these formulas is impossible, we apply the information theoretic techniques for estimating the above mentioned distribution. More concretely, we use the principle of maximum entropy which provides an adequate methodology for computing a unique estimate for an unknown probability distribution based on information expressed in terms of some given mean value constraints.

Keywords: Retrial queue, steady state distribution, estimation, principle of maximum entropy, impatient customer.

MSC: 60K25, 62G05, 54C70.

1. INTRODUCTION: MODEL DESCRIPTION

The main characteristic of queuing systems with repeated attempts (retrial queues) is that a customer who finds the server busy upon arrival is obliged to leave the service area and join a retrial group (orbit). After some random time, the blocked customer will have a chance to try his luck again. There is an extensive literature on the retrial queues and we refer the reader to [3], [7] and references there. The models in question arise in the analysis of different communication systems: cellular mobile networks, Internet, local area computer networks, see in [2], [4], [6].

In telephone networks, we can observe that a calling subscriber after some unsuccessful retrials gives up further repetitions and leaves the system. In queuing systems with repeated attempts, this phenomenon is represented by the set of probabilities $\{H_k, k \geq 1\}$, called the persistence function, where H_k is the probability that a customer will make the (*k+1*)-th attempt after the *k*-th attempt fails. In general, it is assumed that the probability of a customer reinitiating after failure of a repeated attempt does not depend on the number of previous attempts (i.e. $H_2 = H_3 = H_4 = \ldots$). In the queuing literature, an extensive body research addressing impatience phenomena observed in single or multi server retrial systems can be found, for example in [1], [8]-[9]. An M/G/1 retrial queue with impatient customers (where $H_2 = 1$ and $H_2 < 1$) is analyzed in [7]. In the case of $H_2 = 1$, the authors study the no stationary regime of the system, investigate the embedded Markov chain and obtain the steady state joint distribution of the server state and the number of customers in the retrial group. In the case of $H_2 < 1$, the closed form solution for the steady state distribution of the system state is derived only in the case of exponential service time. For general service time, the authors obtain the partial factorial moments of the size of retrial group in terms of the server utilization, and describe the embedded Markov chain. Recent contributions on this topic include the papers of Senthil Kumar and Arumuganathan (2009) [10], Shin and Choo (2009) [11], Shin and Moon (2008) [12]. In the first paper, the steady state behaviour of an M/G/1 retrial queue with impatient customers ($H_1 < 1$ and $H_2 = 1$) is given, where the first preliminary service is followed by the second additional one; possibility of the server vacation is analyzed, and some performance measures (expected number of customers in the retrial group, expected waiting time of the customers in the retrial group, ...) are obtained. In [11], the authors model the M/M/s retrial queue with balking and reneging as a Markov chain on two-dimensional lattice space $Z^+ \times Z^+$ and present an algorithm to calculate the steady state distribution of the number of customers in retrial group and service facility. The considered model contains the retrial model with finite capacity of service facility by assigning specific values to the probabilities of joining the balking customers and reneging ones the retrial group. In [12], a retrial queuing system limited by a finite number (*m*) of retrials for each customer is analyzed as the model with $H_k = 1$, for $k \leq m$, and $H_k = 0$, for $k > m$.

In our work, we consider single server queuing systems where primary customers arrive according to a Poisson stream with rate $\lambda > 0$. If the server is busy at the arrival epoch, then the arriving primary customer leaves the system without service with probability $1 - H_1 > 0$ and joins the orbit with probability H_1. In the same situation, any orbiting customer leaves the system forever with probability $1 - H_2 > 0$ and returns to the orbit with probability H_2. If the server is idle at the arrival epoch, the primary/orbiting customer begins his service. The service time follow a general distribution with distribution function $B(t)$ and Laplace-Stieltjes transform

$\widetilde{B}(s)=\int_0^\infty e^{-st}dB(t)$, $\operatorname{Re}(s)>0$. Let $\beta_k=(-1)^k\widetilde{B}^{(k)}(0)$ be the k-th moment of the service time about the origin and $\rho=\lambda H_1\beta_1$ be the traffic intensity. Our system operates under so-called classical retrial policy. In this context, each blocked customer generates a stream of repeated attempts independently of the rest of customers in the orbit. The intervals between successive repeated attempts are exponentially distributed with rate $j\theta+0(\Delta t)$, when the number of customers in the retrial group is j and $\theta>0$. Finally, we accept the hypothesis of mutual independence between all random variables defined above.

For models in question, we review the results concerning the steady state distribution of the system state presented in the literature and compare them with the results we obtained. Since the existing formulas are cumbersome (so their utilization in practice becomes delicate) or the obtaining of these formulas is impossible, we apply the information theoretic techniques for estimating the above mentioned distribution. More concretely, we use the principle of maximum entropy which provides an adequate methodology for computing a unique estimate for an unknown probability distribution based on information expressed in terms of some given mean value constraints.

This paper is organized as follows. The next section contains the existing results on the steady state joint distribution of the server state and the number of customers in the orbit of the M/G/1 retrial queues with impatient customers so as our results (some performance measures, moments). In the third section, we present the maximum entropy estimations of the steady state distribution of the system state. In the last section, we show through numerical results how the considered information of a theoretic method works for the models in question.

2. STEADY STATE DISTRIBUTION OF THE SYSTEM STATE

The state of the system at time t can be described by means of the process $\{C(t),N_o(t),\zeta(t),t\geq 0\}$, where $N_o(t)$ is the number of customers in the retrial group, and $C(t)$ is the state of the server at time t. Depending on the fact that the server is idle or busy, $C(t)$ is 0 or 1. If $C(t)=1$, $\zeta(t)$ represents the elapsed service time of the customer in service at time t.

An important feature of the model under consideration is that the cases $H_2<1$ and $H_2=1$ yield different solutions.

Case $H_2=1$: Under $\rho=\lambda\beta_1H_1<1$, the steady state joint distribution of the server state and the number of the customers in the orbit

$$p_{0n}=\lim_{t\to\infty}P(C(t)=0,N_o(t)=n)$$

and

$$p_{1n} = \int_0^{\infty} \lim_{t\to\infty} \frac{d}{dx} P(C(t)=1, \zeta(t) \le x, N_o(t)=n) \tag{1}$$

has the following partial generating functions [7]

$$P_0(z) = \sum_{n=0}^{\infty} z^n p_{0n} = \frac{1-\rho}{1+\lambda\beta_1-\rho} \exp\left\{ \frac{\lambda}{\theta} \int_1^z \frac{1-K(u)}{K(u)-u} du \right\}, \tag{2}$$

$$P_1(z) = \sum_{n=0}^{\infty} z^n p_{1n} = \frac{1}{H_1} \frac{1-K(z)}{K(z)-z} P_0(z), \tag{3}$$

where $K(z) = \widetilde{B}(\lambda H_1 - \lambda H_1 z)$.

With the help of (2) and (3), we can get the generating function of the number of customers in the orbit

$$P(z) = P_0(z) + P_1(z) = \frac{1-\rho}{1+\lambda\beta_1-\rho} \frac{1-K(z)+H_1(K(z)-z)}{H_1(K(z)-z)} \exp\left\{ \frac{\lambda}{\theta} \int_1^z \frac{1-K(u)}{K(u)-u} du \right\},$$

the steady state distribution of the server state

$$P_0 = \lim_{t\to\infty} P(C(t)=0) = P_0(1) = \frac{1-\rho}{1+\lambda\beta_1-\rho},$$

$$P_1 = \lim_{t\to\infty} P(C(t)=1) = \frac{\lambda\beta_1}{1+\lambda\beta_1-\rho};$$

and the mean number of customers in the orbit

$$\lim_{t\to\infty} E[N_o(t)] = P'(1) = \frac{\lambda^2 H_1}{1-\rho}\left(\frac{\beta_1}{\theta} + \frac{\beta_2}{2(1+\lambda\beta_1-\rho)} \right).$$

When the service time follow an exponential distribution ($B(t) = 1 - e^{-\frac{1}{\beta_1}t}, t \ge 0$), the partial generating functions (2) and (3) become

$$P_0(z) = \frac{1-\rho}{1+\lambda\beta_1-\rho} \left(\frac{1-\rho}{1-\rho z} \right)^{\frac{\lambda}{\theta}}, \tag{4}$$

$$P_1(z) = \frac{\lambda\beta_1}{1+\lambda\beta_1-\rho} \left(\frac{1-\rho}{1-\rho z} \right)^{\frac{\lambda}{\theta}+1} \tag{5}$$

We have also the mean number of customers in the orbit

$$\lim_{t\to\infty} E\left[N_o(t)\right] = \left(P_0(z) + P_1(z)\right)'\Big|_{z=1} = \frac{\lambda\rho}{\frac{1}{\beta_1}(1-\rho)+\lambda}\left(\frac{1}{\theta\beta_1} + \frac{\lambda+\theta}{\theta(1-\rho)}\right)$$

and the mean number of customers in the system

$$\lim_{t\to\infty} E\left[C(t) + N_o(t)\right] = \left(P_0(z) + zP_1(z)\right)'\Big|_{z=1} = \frac{1}{\frac{1}{\beta_1}(1-\rho)+\lambda}\left(\frac{1}{\beta_1} + \lambda + \frac{\lambda\rho(\lambda+\theta)}{\theta(1-\rho)}\right).$$

By differentiation of formulas (4)-(5), after some fastidious algebra, we get out the following expressions for the partial moments

$$M_0^0 = \sum_{n=0}^{\infty} p_{0n} = P_0(1) = \frac{1-\rho}{1+\lambda\beta_1-\rho}\,;\; M_1^0 = \sum_{n=0}^{\infty} p_{1n} = P_1(1) = \frac{\lambda\beta_1}{1+\lambda\beta_1-\rho}\,;$$

$$M_0^1 = \sum_{n=0}^{\infty} np_{0n} = P_0'(1) = \frac{\lambda\rho}{\theta(1+\lambda\beta_1-\rho)}\,;\; M_1^1 = \sum_{n=0}^{\infty} np_{1n} = P_1'(1) = M_0^1 \times \frac{\left(\frac{\lambda}{\theta}+1\right)\beta_1}{1-\rho}\,;$$

$$M_0^2 = \sum_{n=0}^{\infty} n^2 p_{0n} = P_0''(1) + \sum_{n=0}^{\infty} np_{0n} = \frac{\lambda\rho^2}{\theta(1+\lambda\beta_1-\rho)}\frac{\frac{\lambda}{\theta}+1}{1-\rho} + M_0^1\,;$$

$$M_1^2 = \sum_{n=0}^{\infty} n^2 p_{1n} = P_1''(1) + \sum_{n=0}^{\infty} np_{in} = P_1''(1) + M_1^1 = \frac{\lambda\rho^2(\lambda+\theta)}{\theta^2\left(\frac{1}{\beta_1}(1-\rho)+\lambda\right)}\frac{1}{(1-\rho)^2} + M_1^1\,.$$

It is easy to see that

$\lim_{t\to\infty} E\left[N_o(t)\right] = M_0^1 + M_1^1$; $\lim_{t\to\infty} E\left[C(t)+N_o(t)\right] = M_0^1 + M_1^0 + M_1^1$.

The steady state joint distribution $p_{in} = \lim_{t\to\infty} P(C(t)=i, N_o(t)=n)$, $i=0,1$ and $n \ge 0$, can be calculated using

$$p_{0n} = \frac{\rho^n}{n!\theta^n}\prod_{k=0}^{n-1}(\lambda+k\theta)p_{00} \tag{6}$$

and

$$p_{1n} = \frac{\beta_1\rho^n}{n!\theta^n}\prod_{k=0}^{n}(\lambda+k\theta)p_{00} \tag{7}$$

having $p_{00} = \dfrac{(1-\rho)^{\frac{\lambda}{\theta}+1}}{1+\lambda\beta_1-\rho}$.

Case $H_2 < 1$: For model in question, the closed form solution for

$p_{in} = \lim_{t\to\infty} P(C(t) = i, N_o(t) = n)$, $i = 0,1$

and $n \geq 0$, (8)

and for the corresponding partial generating functions $P_0(z) = \sum_{n=0}^{\infty} z^n p_{0n}$ and $P_1(z) = \sum_{n=0}^{\infty} z^n p_{1n}$ is available only when the service times are exponentially distributed (in the general case a complete closed form solution seems impossible) [7]. That is

$$P_0(z) = \frac{\Phi(a,c,\varsigma z)}{\Phi(a,c,\varsigma) + \lambda\beta_1\Phi(a+1,c,\varsigma)}, \tag{9}$$

$$P_1(z) = \frac{\lambda\beta_1\Phi(a+1,c,\varsigma z)}{\Phi(a,c,\varsigma) + \lambda\beta_1\Phi(a+1,c,\varsigma)}, \tag{10}$$

where $\Phi(a,c,x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \prod_{k=0}^{n-1} \frac{a+k}{c+k}$ with $a = \frac{\lambda}{\theta}$, $c = \frac{\frac{1}{\beta_1} + (1-H_2)(\lambda+\theta)}{\theta(1-H_2)}$ and $\varsigma = \frac{\lambda H_1}{\theta(1-H_2)}$.

We dispose also the joint distributions of the steady state

$$p_{0n} = \frac{\varsigma^n}{n!} \prod_{k=0}^{n-1} \frac{a+k}{c+k} p_{00} \tag{11}$$

and

$$p_{1n} = \frac{\lambda\beta_1\varsigma^n}{n!} \prod_{k=0}^{n-1} \frac{a+1+k}{c+k} p_{00}, \tag{12}$$

where $p_{00} = \frac{1}{\Phi(a,c,\varsigma) + \lambda\beta_1\Phi(a+1,c,\varsigma)}$.

Now, we can get the steady state distribution of the server state $P_0 = \lim_{t\to\infty} P(C(t)=0) = \frac{1}{1+\Lambda}$, $P_1 = \lim_{t\to\infty} P(C(t)=1) = \frac{\Lambda}{1+\Lambda}$, the mean number of customers in the orbit

$$\lim_{t\to\infty} E[N_o(t)] = \left(P_0(z) + P_1(z)\right)'\Big|_{z=1} = \frac{\lambda H_2 + \left(\lambda H_1 - \frac{H_2}{\beta_1}\right)\Lambda}{\theta(1-H_2)(1+\Lambda)}$$

and the mean number of customers in the system

$$\lim_{t\to\infty} E\left[C(t)+N_o(t)\right] = (P_0(z)+zP_1(z))'\Big|_{z=1} = ((\Phi(a,c,\varsigma z)+\lambda\beta_1 z\Phi(a+1,c,\varsigma z))p_{00})'\Big|_{z=1}$$

$$= \frac{1}{1+\Lambda}\left(\frac{\varsigma}{c}(\Psi+c)+\Lambda+\frac{\varsigma}{c}(\lambda\beta_1 R)+\frac{\varsigma}{c}(\Psi+c)\lambda\beta_1\frac{1-c+a}{a}\right).$$

Here $\Lambda = \lambda\beta_1\dfrac{\Phi(a+1,c,\varsigma)}{\Phi(a,c,\varsigma)}$, $R = c\dfrac{\Phi(a+1,c,\varsigma)}{\Phi(a,c,\varsigma)}$, $\Psi = \dfrac{(a-c)\Phi(a,c+1,\varsigma)}{\Phi(a,c,\varsigma)}$.

Note that the system is always in steady state when $\rho = \lambda\beta_1 H_1 < 1$ and $H_2 < 1$.

By differentiation of formulas (9)-(10), we obtain

$$M_0^0 = \sum_{n=0}^{\infty} p_{0n} = P_0(1) = \frac{1}{1+\Lambda};\ M_1^0 = \sum_{n=0}^{\infty} p_{1n} = P_1(1) = \frac{\Lambda}{1+\Lambda};$$

$$M_0^1 = \sum_{n=0}^{\infty} np_{0n} = P_0'(1) = \frac{\varsigma}{c}\frac{\Psi+c}{1+\Lambda};$$

$$M_1^1 = \sum_{n=0}^{\infty} np_{1n} = P_1'(1) = \lambda\beta_1\left(\frac{\varsigma R}{c(1+\Lambda)}+\frac{1-c+a}{a}M_0^1\right);$$

$$M_0^2 = \sum_{n=0}^{\infty} n^2 p_{0n} = P_0''(1)+\sum_{n=0}^{\infty} np_{0n} = P_0''(1)+M_0^1 = \frac{U}{1+\Lambda}+(\varsigma+1)M_0^1;$$

$$M_1^2 = \sum_{n=0}^{\infty} n^2 p_{1n} = P_1''(1)+\sum_{n=0}^{\infty} np_{in} = P_1''(1)+M_1^1$$

$$= \frac{\rho\varsigma(\varsigma-c+a+1)}{a}\left[M_0^1-\frac{\varsigma}{1+\Lambda}\right]-\frac{\rho\varsigma^2(\varsigma+2a-c+1)}{a(1+\Lambda)}+\frac{\Theta}{1+\Lambda}+M_0^1,$$

where $\Theta = \dfrac{\rho\varsigma^2(a+1)(3+3a-c)}{c(c+1)}\dfrac{\Phi(a+2,c+2,\varsigma)}{\Phi(a,c,\varsigma)}$.

Once again

$$\lim_{t\to\infty} E\left[N_o(t)\right] = M_0^1+M_1^1;\ \lim_{t\to\infty} E\left[C(t)+N_o(t)\right] = M_0^1+M_1^0+M_1^1.$$

3. APPROXIMATION OF THE STEADY STATE DISTRIBUTION OF THE SYSTEM STATE

Since the exact formulas of the steady state joint distribution of the server state and the number of customers in the orbit are cumbersome or impossible to get, information theoretic methods (in particular, the principle of maximum entropy) can provide an adequate procedure for approximating the distribution in question [4]-[5].

First we summarize the maximum entropy formalism. Let Q be a system with discrete state space $S = \{s_n\}$, and the available information about Q imposes some number of constraints on the distribution $P = \{p(s_n)\}$. We assume that these constraints take the form of mean values of m functions $\{f_k(s_n)\}_{k=1}^{m}$ ($m < card(S)$). The principle of maximum entropy states that, among all distributions satisfying the mean values constraints, the minimal prejudiced is the one maximizing the Shannon's entropy functional

$$H(P) = -\sum_{s_n \in S} p(s_n) \log_2(p(s_n))$$

subject to the constraints

$$\sum_{s_n \in S} p(s_n) = 1$$

$$\sum_{s_n \in S} f_k(s_n) p(s_n) = f_k, 1 \le k \le m,$$

where $f_k(s_n)$ are known functions and f_k are known values. The maximization of $H(P)$ can be carried out by using the method of Lagrange's multipliers.

At present, we can get the first and second order estimations for the steady state joint distributions (1) and (8).

First order estimation

According to the principle of maximum entropy, the first order estimation of the steady state distributions p_{in}, $i \in \{0,1\}$ and $n \ge 0$, (defined by (1) and (8)) can be obtained by maximizing Shannon's entropy

$$H(P_i) = -\sum_{n=0}^{\infty} p_{in} \log_2 p_{in} , \; i \in \{0,1\},$$

subject to the constraints

$$\sum_{i=0}^{1} \sum_{n=0}^{\infty} p_{in} = 1, \; M_i^k = \sum_{n=0}^{\infty} n^k p_{in} , \; i \in \{0,1\} \text{ and } k \in \{0,1\}.$$

Theorem 1. *If the available information is given by* M_i^k, $i \in \{0,1\}$ *and* $k \in \{0,1\}$, *then according to the principle of maximum entropy, the first order estimation of the steady state distribution of the system state is*

$$\hat{p}_{0n}^{(1)} = \frac{(M_0^0)^2}{M_0^0 + M_0^1}\left(\frac{M_0^1}{M_0^0 + M_0^1}\right)^n, \tag{13}$$

$$\hat{p}_{1n}^{(1)} = \frac{(M_1^0)^2}{M_1^0 + M_1^1}\left(\frac{M_1^1}{M_1^0 + M_1^1}\right)^n, n \geq 0 \tag{14}$$

Proof: *First, we construct the Lagrange function*

$$L_0(\{p_{0n}\}, \alpha_0, \alpha_0^0, \alpha_0^1) = -\sum_{n=0}^{\infty} p_{0n} \log_2 p_{0n} - \alpha_0\left(\sum_{n=0}^{\infty} p_{0n} + \sum_{n=0}^{\infty} p_{1n} - 1\right)$$
$$-\alpha_0^0\left(M_0^0 - \sum_{n=0}^{\infty} p_{0n}\right) - \alpha_0^1\left(M_0^1 - \sum_{n=0}^{\infty} np_{0n}\right)$$

Then we follow the method of Lagrange's multipliers and find the first order estimation $\hat{p}_{0n}^{(1)}$ of the steady state distribution p_{0n}

$$\hat{p}_{0n}^{(1)} = \exp(-1 + \alpha_0 \ln 2 - \alpha_0^0 \ln 2 - n\alpha_0^1 \ln 2) = \exp(-1 + \alpha_0 \ln 2 - \alpha_0^0 \ln 2)(\exp(-\alpha_0^1 \ln 2))^n$$

One can see that $\hat{p}_{0n}^{(1)} = uv^n$. Since $\{\hat{p}_{0n}^{(1)}\}$ verifies the constraints for M_0^0 and M_0^1,

$$M_0^0 = \sum_{n=0}^{\infty} \hat{p}_{0n}^{(1)} = \sum_{n=0}^{\infty} uv^n = u\frac{1}{1-v} \text{ and}$$

$$M_0^1 = \sum_{n=0}^{\infty} n\hat{p}_{0n}^{(1)} = \sum_{n=0}^{\infty} nuv^n = uv\sum_{n=1}^{\infty} nv^{n-1} = uv\frac{1}{(1-v)^2} = u\frac{1}{1-v}v\frac{1}{1-v} = M_0^0 v\frac{1}{1-v}$$

.

Therefore, $v = \frac{M_0^1}{M_0^0 + M_0^1}$, $u = \frac{(M_0^0)^2}{M_0^0 + M_0^1}$ and the equation (13) follows. In the same way we find the equation (14). **End of proof**

Second order estimation

It is necessary to maximize the Shannon's entropy

$$H(P_i) = -\sum_{n=0}^{\infty} p_{in} \log_2 p_{in}, \ i \in \{0,1\}, \tag{15}$$

subject to the constraints

$$\sum_{i=0}^{1}\sum_{n=0}^{\infty} p_{in} = 1,\ M_i^k = \sum_{n=0}^{\infty} n^k p_{in}\ ,\ i \in \{0,1\} \text{ and } k \in \{0,1,2\}.$$

Theorem 2. *If the available information is given by* M_i^k, $i \in \{0,1\}$ *and* $k \in \{0,1,2\}$, *then according to the principle of maximum entropy, the second order estimation of the steady state distribution of the system state (defined by (1) and (8)) is*

$$\hat{p}_{in}^{(2)} = \frac{1}{Z_i}\exp(-n\beta_i^1 - n^2\beta_i^2) \text{ with } Z_i = \frac{1}{M_i^0}\left(\sum_{n=0}^{\infty}\exp(-n\beta_i^1 - n^2\beta_i^2)\right), i \in \{0,1\} \quad (16)$$

Here, β_i^1 and β_i^2 are the Lagrangian coefficients corresponding to the constraints for M_i^1 and M_i^2, $i \in \{0,1\}$.

Proof: Again, the method of Lagrange's multiplier is used, and to this end we consider the following Lagrange function

$$L_i(\{p_{in}\},\alpha_0,\alpha_0^0,\alpha_0^1,\alpha_i^2) = -\sum_{n=0}^{\infty} p_{in}\log_2 p_{in} - \alpha_i\left(\sum_{n=0}^{\infty} p_{0n} + \sum_{n=0}^{\infty} p_{1n} - 1\right)$$
$$-\alpha_i^0\left(m_i^0 - \sum_{n=0}^{\infty} p_{in}\right) - \alpha_i^1\left(m_i^1 - \sum_{n=0}^{\infty} np_{in}\right) - \alpha_i^2\left(m_i^2 - \sum_{n=0}^{\infty} n^2 p_{in}\right), i \in \{0,1\}$$

By applying the above mentioned method, it is easy to obtain the second order estimations $\hat{p}_{in}^{(2)}$ of the steady state distributions p_{in} :

$$\hat{p}_{in}^{(2)} = \exp(-1 + \alpha_i \ln 2 - \alpha_i^0 \ln 2 - n\alpha_i^1 \ln 2 - n^2\alpha_i^2 \ln 2) = \frac{1}{Z_i}\exp(-n\beta_i^1 + n^2\beta_i^2)$$

where

$$Z_i = \exp(1 - \alpha_i \ln 2 + \alpha_i^0 \ln 2),\ \beta_i^1 = \alpha_i^1 \ln 2,\ \beta_i^2 = \alpha_i^2 \ln 2 .$$

Since $M_i^0 = \sum_{n=0}^{\infty}\hat{p}_{in}^{(2)} = \frac{1}{Z_i}\sum_{n=0}^{\infty}\exp(-n\beta_i^1 - n^2\beta_i^2)$, $Z_i = \frac{1}{M_i^0}\sum_{n=0}^{\infty}\exp(-n\beta_i^1 - n^2\beta_i^2)$.

End of proof

4. APPLICATION

In this section, we illustrate numerically the use of the principle of maximum entropy to get the estimations for the steady state distributions (1) and (8). To this end we consider M/M/1 retrial queues with $H_1 < 1$ and $H_2 = 1$ (model M1) so as with $H_1 < 1$ and $H_2 < 1$ (model M2). To examine the accuracy of the maximum entropy estimations, we compare the numerical outcomes from (13)-(14) and (16) against the classical solutions given by (6)-(7) (for model M1) and by (11)-(12) (for model M2). The obtained numerical results are presented in Tables 1 and 2. The last row of each table gives the value of the Shannon entropy (SE). We can observe that the entropy decreases when the number of known moments increases ($k \in \{0,1,2\}$).

Table 1: M/M/1 retrial queue with impatient customers $(\lambda = 0.9, \theta = 5, \gamma = 1, H_1 = 0.9, H_2 = 1)$

j	P_{0j}	P_{1j}	P_{0j}^1	P_{1j}^1	P_{0j}^2	P_{1j}^2	$P_{0j}^{1,2}$	$P_{1j}^{1,2}$
0	0.12927	0.11634	0.09862	0.13691	0.10362	0.12827	0.12013	0.11741
1	0.01884	0.11120	0.04282	0.11421	0.03709	0.10990	0.01931	0.11102
2	0.00900	0.09818	0.01859	0.09527	0.01511	0.09889	0.01341	0.09823
3	0.00530	0.08429	0.00807	0.07947	0.00520	0.07998	0.00639	0.08400
4	0.00341	0.07135	0.00360	0.06629	0.00300	0.06794	0.00330	0.07098
5	0.00231	0.05987	0.00345	0.05530	0.00217	0.05755	0.00235	0.05914
6	0.00161	0.04995	0.00096	0.04613	0.00096	0.04861	0.00142	0.04901
7	0.00115	0.04150	0.00078	0.03848	0.00088	0.04094	0.00101	0.04182
8	0.00084	0.03437	0.00069	0.03210	0.00067	0.03438	0.00079	0.03436
SE	3.10522		3.16868		3.13633		3.11455	

Table 2: M/M/1 retrial queue with impatient customers
($\lambda = 0.5, \theta = 5, \gamma = 1, H_1 = 0.9, H2 = 0.8$)

j	P_{0j}	P_{1j}	P_{0j}^1	P_{1j}^1	P_{0j}^2	P_{1j}^2	$P_{0j}^{1,2}$	$P_{1j}^{1,2}$
0	0.44189	0.35351	0.43979	0.37219	0.44021	0.35892	0.44100	0.35426
1	0.15612	0.17231	0.18001	0.19341	0.17314	0.17645	0.15934	0.17462
2	0.07181	0.09452	0.09610	0.10895	0.07542	0.09913	0.07291	0.09734
3	0.02356	0.04315	0.02739	0.05623	0.02300	0.04521	0.02354	0.04324
4	0.00811	0.03363	0.00523	0.03143	0.00724	0.03305	0.00823	0.03388
5	0.00388	0.00613	0.00170	0.00962	0.00283	0.00782	0.00379	0.00629
6	0.00092	0.00088	0.00110	0.00108	0.00105	0.00090	0.00103	0.00085
7	0.00004	0.00010	0.00010	0.00042	0.00005	0.00026	0.00004	0.00011
8	0.00001	0.00001	0.00006	0.00003	0.00004	0.00002	0.00002	0.00003
SE	3.14154		3.30995		3.18720		3.15960	

For the first and the second order estimations, the moments M_i^k were calculated by taking derivates of the partial generating functions (4)-(5) and (9)-(10) at the point $z = 1$. To improve the estimation, for the problem (15) we add another constraint providing information related to another point $z = z_0$, that is $P_i(z_0) = \sum_{n=0}^{\infty} p_{in} z_0^n$, $i = 0,1$.

In the same way, we obtain a new estimation

$$\hat{p}_{in}^{(2,z_0)} = \frac{1}{Z_i} \exp(-\alpha_i^0 - n\alpha_i^1 - n^2\alpha_i^2 - \alpha_i^{2,z_0} z_0^n),$$

where $Z_i = \sum_{n=0}^{\infty} \exp(-\alpha_i^0 - n\alpha_i^1 - n^2\alpha_i^2 - \alpha_i^{2,z_0} z_0^n)$.

From tables 1 and 2, it is easy to see that the estimation improves when we use $\hat{p}_{in}^{(2,z_0)}$ (with $z_0 = 0.55$) instead of $\hat{p}_{in}^2$.

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