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AN ALGORITHM FOR SOLVING A CAPACITATED INDEFINITE QUADRATIC TRANSPORTATION PROBLEM WITH ENHANCED FLOW

KAVITA GUPTA
Department of Mathematics, Ramjas College , University of Delhi, Delhi- 110007, India
gupta_kavita31@yahoo.com
S.R. ARORA
Ex-Principal, Hans Raj College, University of Delhi, Delhi-110007, India
srarora @yahoo.com

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Abstract: The present paper discusses enhanced flow in a capacitated indefinite quadratic transportation problem. Sometimes, situations arise where either reserve stocks have to be kept at the supply points say, for emergencies, or there may be extra demand in the markets. In such situations, the total flow needs to be controlled or enhanced. In this paper, a special class of transportation problems is studied, where the total transportation flow is enhanced to a known specified level. A related indefinite quadratic transportation problem is formulated, and it is shown that to each basic feasible solution called corner feasible solution to related transportation problem, there is a corresponding feasible solution to this enhanced flow problem. The optimal solution to enhanced flow problem may be obtained from the optimal solution to the related transportation problem. An algorithm is presented to solve a capacitated indefinite quadratic transportation problem with enhanced flow. Numerical illustrations are also included in support of the theory. Computational software GAMS is also used.

Keywords: Capacitated transportation problem, enhanced flow, quadratic transportation problem, software GAMS.

MSC: 90B06.

1. INTRODUCTION

A class of transportation problems where the objective function to be optimized is a product of two linear functions gives rise to an indefinite quadratic transportation problem, which was first studied by Arora and Khurana [1]. Later, Khurana and Arora [8] studied linear plus linear fractional transportation problem with restricted and enhanced flow.

Another important class of transportation problems consists of capacitated transportation problem. Many researchers, i.e. Bit et.al [6], Arora and Gupta [2-5], Dahiya et.al. [7], have contributed in this field. Sometimes, situations arise due to extra demand of the market that the total flow needs to be enhanced, compelling some factories to increase their production in order to meet the extra demand. The total flow from the factories in the market is now increased by an amount of the extra demand. This motivated us to study enhanced flow in a capacitated indefinite quadratic transportation problem. Khurana and Arora [9] studied enhanced flow in a fixed charge indefinite quadratic transportation problem. In this paper, we shall be discussing the case when the flow gets enhanced due to extra demand in the market for a capacitated indefinite quadratic transportation problem.

2. PROBLEM FORMULATION

Consider the problem of transporting goods from various sources to different destinations. Let c_{ij} be the cost of transportation of one unit from i^{th} source to j^{th} destination. While transporting goods, a part of the goods get damaged. Let d_{ij} be the cost of one unit of the damaged goods. The quantity of damaged goods may be some fraction of the goods transported. We are interested in minimizing both the cost of transportation and the cost of damaged goods simultaneously. Moreover, in practical situations, the two costs, i.e. the cost of transportation and damage cost are always interdependent. Therefore, the objective function of the problem under consideration should be the product of two cost functions so that both of them are minimized simultaneously, and their interdependence is justified. The problem can then, be formulated as a capacitated indefinite quadratic transportation problem given by

$$\text{(P1): } \min\left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)\right\}$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i; \forall i \in I \tag{1}$$

$$\sum_{i\in I} x_{ij} \geq b_j; \forall j \in J \tag{2}$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P \quad \text{where } P > \max\left(\sum_{i\in I} a_{i,} \sum_{j\in J} b_j\right) \tag{3}$$

$$l_{ij} \le x_{ij} \le u_{ij}\,;\ \forall (i, j) \in I \times J \tag{4}$$

$I = \{1, 2, \dots m\}$ is the index set of m origins.

$J = \{1, 2, \dots, n\}$ is the index set of n destinations.

x_{ij} = number of units transported from i^{th} origin to the j^{th} destination .

c_{ij} = variable cost of transporting one unit of commodity from i^{th} origin to the j^{th} destination.

d_{ij} = per unit damage cost or depreciation cost of commodity transported from i^{th} origin to the j^{th} destination.

l_{ij} and u_{ij} are the bounds on number of units to be transported from i^{th} origin to j^{th} destination.

In the problem (P1), we need to minimize the total transportation cost and depreciation cost simultaneously.

In order to solve the problem (P1), we consider the following related problem (P2) with an additional supply point and an additional destination point.

(P2): minimize the cost function $\left\{\left(\sum_{i\in I'}\sum_{j\in J'} c'_{ij} y_{ij}\right)\left(\sum_{i\in I'}\sum_{j\in J'} d'_{ij} y_{ij}\right)\right\}$

subject to

$$\sum_{j\in J'} y_{ij} = a_i'\,;\forall i \in I'$$

$$\sum_{i\in I'} y_{ij} = b'_j\,;\forall j \in J'$$

$$l_{ij} \le y_{ij} \le u_{ij}\,;\forall (i,j) \in IxJ$$

$$0 \le y_{m+1,\,j} \le \sum_{i\in I} u_{ij} - b_j;\forall j \in J$$

$$0 \le y_{i,\,n+1} \le \sum_{j\in J} u_{ij} - a_i;\forall i \in I$$

$y_{m+1,\,n+1} \ge 0$ and integers.

$$a'_i = \sum_{j\in J} u_{ij};\forall i \in I \quad ,\quad a'_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P = b'_{n+1}\,,\quad b'_j = \sum_{i\in I} u_{ij};\forall j \in J\,,$$

$$c'_{ij} = c_{ij}\ ,\ \forall i\in I, j\in J,\ \ c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i\in I,\ \ \forall j\in J,\ \ c'_{m+1,n+1} = M$$

$$d'_{ij} = d_{ij}\ ,\ \ \forall i\in I, j\in J,\ \ d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i\in I,\ \ \forall j\in J\ \ d'_{m+1,n+1} = M$$

$$I' = I \cup \{m+1\} \qquad J' = J \cup \{n+1\}$$

3. PRELIMINARY RESULT

Result 1: Let $X = \{x_{ij}\}$ be a basic feasible solution of problem (P2) with basis matrix B. Then, it will be an optimal basic feasible solution if

$$R^1_{ij} = \theta_{ij}\left[z_1(d_{ij}-z^2_{ij})+z_2(c_{ij}-z^1_{ij})+\theta_{ij}(c_{ij}-z^1_{ij})(d_{ij}-z^2_{ij})\right] \geq 0 \forall (i,j) \in N_1$$

and

$$R^2_{ij} = \theta_{ij}\left[\theta_{ij}(c_{ij}-z^1_{ij})(d_{ij}-z^2_{ij})-z_1(d_{ij}-z^2_{ij})-z_2(c_{ij}-z^1_{ij})\right] \geq 0 \forall (i,j) \in N_2$$

such that

$$u^1_i + v^1_j = c_{ij} \qquad \forall (i,j) \in B$$

$$u^2_i + v^2_j = d_{ij} \qquad \forall (i,j) \in B$$

$$u^1_i + v^1_j = z^1_{ij} \qquad \forall (i,j) \in N_1 \text{ and } N_2$$

$$u^2_i + v^2_j = z^2_{ij} \qquad \forall (i,j) \in N_1 \text{ and } N_2$$

z_1 = value of $\sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B

z_2 = value of $\sum_{i \in I}\sum_{j \in J} d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B.

θ_{ij} = level at which a non basic cell (i,j) enters the basis replacing some basic cell of B.

N_1 and N_2 denote the set of non basic cells (i,j) which are at their lower bounds and upper bounds, respectively.

Note: $u^1_i, v^1_j, u^2_i, v^2_j$ are dual variables, determined by using the above equations and taking one of the u_i's or v_j's. as zero.

Proof: Let z^0 be the objective function value of the problem (P2).

Let $z^0 = Z_1 Z_2$

Let $\hat{z}$ be the objective function value at the current basic feasible solution $\hat{X} = \{x_{ij}\}$, corresponding to the basis B obtained on entering the non- basic cell $x_{ij} \in N_1$ in to the basis which undergoes change by an amount θ_{ij} and is given by min$\{u_{ij} - l_{ij}$; $x_{ij} - l_{ij}$ for all basic cells (i,j) with a ($-\theta$) entry in the θ- loop; $u_{ij} - x_{ij}$ for all basic cells (i,j) with a ($+\theta$) entry in the θ-loop$\}$.

Then, $\hat{z} = \left[z_1 + \theta_{ij}\left(c_{ij} - z^1_{ij}\right)\right]\left[z_2 + \theta_{ij}\left(d_{ij} - z^2_{ij}\right)\right]$

$$\hat{z} - z^0 = \left[z_1 z_2 + \theta_{ij} z_1 (d_{ij} - z^2_{ij}) + z_2 \theta_{ij} (c_{ij} - z^1_{ij}) + \theta^2_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij}) - z_1 z_2\right]$$

$$= \theta_{ij}\left[z_1 (d_{ij} - z^2_{ij}) + z_2 (c_{ij} - z^1_{ij}) + \theta_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij})\right]$$

This basic feasible solution will give an improved value of z if $\hat{z} < z^0$. It means

$$\text{if } \theta_{ij}\left[z_1 (d_{ij} - z^2_{ij}) + z_2 (c_{ij} - z^1_{ij}) + \theta_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij})\right] < 0 \tag{5}$$

Therefore, one can move from one basic feasible solution to another basic feasible solution on entering the cell (i,j) $\in N_1$ in to the basis for which condition (5) is satisfied.

It will be an optimal basic feasible solution if

$$R^1_{ij} = \theta_{ij}\left[z_1 (d_{ij} - z^2_{ij}) + z_2 (c_{ij} - z^1_{ij}) + \theta_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij})\right] \geq 0; \forall (i, j) \in N_1$$

Similarly, when non basic variable $x_{ij} \in N_2$ undergoes change by an amount θ_{ij} then,

$$\hat{z} - z^0 = \theta_{ij}\left[\theta_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij}) - z_1 (d_{ij} - z^2_{ij}) - z_2 (c_{ij} - z^1_{ij})\right] < 0$$

It will be an optimal basic feasible solution if

$$R^2_{ij} = \theta_{ij}\left[\theta_{ij} (c_{ij} - z^1_{ij})(d_{ij} - z^2_{ij}) - z_1 (d_{ij} - z^2_{ij}) - z_2 (c_{ij} - z^1_{ij})\right] \geq 0; \forall (i, j) \in N_2$$

4. THEORETICAL DEVELOPMENT:

Definition: Corner feasible solution: *A basic feasible solution* $\{y_{ij}\}$ $i \in I'$, $j \in J'$ *to problem (P2) is called a corner feasible solution (cfs) if* $y_{m+1,n+1} = 0$

Theorem 1. *A non corner feasible solution of problem (P2) cannot provide a basic feasible solution to problem (P1).*

Proof: Let $\{y_{ij}\}_{I' \times J'}$ be a non corner feasible solution to problem (P2).Then, $y_{m+1,n+1} = \lambda$ (> 0)

Thus, $\sum_{i \in I'} y_{i,n+1} = \sum_{i \in I} y_{i,n+1} + y_{m+1,n+1}$

$$= \sum_{i\in I} y_{i,\,n+1} + \lambda$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - P$$

Therefore, $\sum_{i\in I} y_{i,n+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - (P+\lambda)$

Now, for $i \in I$,

$$\sum_{j\in J'} y_{ij} = a_i^{'} = \sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J'} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I} y_{i,n+1} = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I}\sum_{j\in J} u_{ij} - (P+\lambda) = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} = P + \lambda$

This implies that total quantity transported from the sources in I to the destinations in J is $P + \lambda > P$, a contradiction to assumption that total flow is P and hence $\{y_{ij}\}_{I' xJ'}$ cannot provide a feasible solution to problem (P1).

Lemma 1: *There is a one –to-one correspondence between the feasible solution to problem (P1) and the corner feasible solution to problem (P2).*

Proof: Let $\{x_{ij}\}_{I\,xJ}$ be a feasible solution of problem (P1).

So by relation (4), we have $x_{ij} \le u_{ij}$ which implies $\sum_{j\in J} x_{ij} \le \sum_{j\in J} u_{ij}$ (6)

By relation (1) and (6), we get

$$a_i \le \sum_{j\in J} x_{ij} \le \sum_{j\in J} u_{ij} = a_i^{'}$$

Similarly, $b_j \le \sum_{i\in I} x_{ij} \le \sum_{i\in I} u_{ij} = b_j^{'}$

Define $\{y_{ij}\}_{I' xJ'}$ by the following transformation

$$y_{ij} = x_{ij} \quad , i \in I,\ j \in J \tag{7}$$

$$y_{i,n+1} = \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij}; \forall i \in I \tag{8}$$

$$y_{m+1,j} = \sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij}; \forall j \in J \tag{9}$$

$$y_{m+1,n+1} = 0 \tag{10}$$

It can be shown that $\{y_{ij}\}$ so defined is a cfs to problem (P2).

Relation (4) and (7) imply that $l_{ij} \le y_{ij} \le u_{ij}$ for all $i \in I, j \in J$

Relation (1) and (8) imply that $0 \le y_{i,n+1} \le \sum_{j\in J} u_{ij} - a_i; \forall i \in I$

Relation (2) and (9) imply that $0 \le y_{m+1,j} \le \sum_{i\in I} u_{ij} - b_j; \forall j \in J$

Relation (10) implies that $y_{m+1,n+1} \ge 0$

Also for $i \in I$, relation (7) and (8) imply that

$$\sum_{j\in J'} y_{ij} = \sum_{j\in J} y_{ij} + y_{i,n+1} = \sum_{j\in J} x_{ij} + \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij} = \sum_{j\in J} u_{ij} = a_i$$

For i = m+1

$$\sum_{j\in J'} y_{m+1,j} = \sum_{j\in J} y_{ij} + y_{m+1,n+1} = \sum_{j\in J}\left(\sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij}\right)$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{i\in I}\sum_{j\in J} x_{ij}$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - P = a'_{m+1}$$

Therefore, $\sum_{j\in J'} y_{ij} = a'_i; \forall i \in I'$

Similarly, it can be shown that $\sum_{i\in I'} y_{ij} = b'_j; \forall j \in J'$

Therefore, $\{y_{ij}\}_{I' x J'}$ is a cfs to problem (P2).

Conversely, let $\{y_{ij}\}_{I' x J'}$ be a cfs to problem(P2).Define x_{ij}, $i \in I, j \in J$ by the following transformation.

$$x_{ij} = y_{ij} \quad , i \in I, j \in J \tag{11}$$

It implies that $l_{ij} \le x_{ij} \le u_{ij}$, $i \in I, j \in J$

Now for $i \in I$, the source constraints in problem (P2) imply

$$\sum_{j\in J'} y_{ij} = a_i' = \sum_{j\in J} u_{ij}$$

$$\sum_{j\in J} y_{ij} + y_{i,n+1} = \sum_{j\in J} u_{ij}$$

$$\Rightarrow a_i \le \sum_{j\in J} y_{ij} \le \sum_{j\in J} u_{ij} \qquad (\text{since } 0 \le y_{i,n+1} \le \sum_{j\in J} u_{ij} - a_i; \forall i \in I)$$

Hence, $\sum_{j\in J} y_{ij} \ge a_i$, $i \in I$ and subsequently $\sum_{j\in J} x_{ij} \ge a_i$, $i \in I$

Similarly, for $j \in J$, $\sum_{j\in J} y_{ij} \ge b_j; \forall j \in J$ and subsequently, $\sum_{i\in I} x_{ij} \ge b_j$; $\forall\ j \in J$

For i= m+1

$$\sum_{j\in J'} y_{m+1,j} = a'_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P$$

$$\Rightarrow \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P \quad \text{because } y_{m+1,n+1} = 0 \tag{12}$$

Now, for $j \in J$ the destination constraints in problem (P2) give

$$\sum_{i\in I} y_{ij} + y_{m+1,j} = \sum_{i\in I} u_{ij}$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij}$

By relation (12), we have $\sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{j\in J} y_{m+1,j} = P$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

Therefore, $\{x_{ij}\}_{I\,xJ}$ is a feasible solution to problem (P1).

Remark 1: *If problem (P2) has a cfs ,then since $c'_{m+1,n+1}$=M and $d'_{m+1,n+1}$= M, it follows that non corner feasible solution can not be an optimal solution to problem (P1) .*

Lemma 2: *The value of the objective function of problem (P1) at a feasible solution $\{x_{ij}\}_{I\,x\,J}$ is equal to the value of the objective function of problem (P2) at its corresponding cfs $\{y_{ij}\}_{I'xJ'}$ and conversely.*

Proof: The value of the objective function of problem (P1) at a feasible solution $\{x_{ij}\}_{I \times J}$ is

$$\begin{aligned} z &= \left[\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)\right] \\ &= \left[\left(\sum_{i\in I'}\sum_{j\in J'} c'_{ij}y_{ij}\right)\left(\sum_{i\in I'}\sum_{j\in J'} d'_{ij}y_{ij}\right)\right] \end{aligned} \quad \text{because} \left\{\begin{array}{l} c'_{ij} = c_{ij}, \forall i \in I, j \in J \\ d'_{ij} = d_{ij}, \forall i \in I, j \in J \\ x_{ij} = y_{ij}, \forall i \in I, j \in J \\ c'_{i,n+1} = c'_{m+1,j} = 0; \forall i \in I, j \in J \\ d'_{i,n+1} = d'_{m+1,j} = 0; \forall i \in I, j \in J \\ y_{m+1,n+1} = 0 \end{array}\right\}$$

= the value of the objective function of problem (P2) at the corresponding cfs $\{y_{ij}\}_{I' \times J'}$

The converse can be proved in a similar way.

Lemma 3: *There is a one –to-one correspondence between the optimal solution to problem (P1) and optimal solution among the corner feasible solution to problem (P2).*

Proof: Let $\{\overset{\circ}{x}_{ij}\}_{I \times J}$ be an optimal solution to problem (P1) yielding objective function value z^0 and $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be the corresponding cfs to problem (P2). Then by Lemma 2, the value yielded by $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ is z^0 If possible, let $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be not an optimal solution to problem (P2). Therefore, there exists a cfs $\{y'_{ij}\}$ to problem (P2) with the value $z^1 < z^0$. Let $\{x'_{ij}\}$ be the corresponding feasible solution to problem (P1).Then by lemma 2,

$z^1 = \left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x'_{ij}\right)\right\}$ which is less than z^0, a contradiction to the assumption that $\{\overset{\circ}{x}_{ij}\}_{I \times J}$ is an optimal solution to problem (P1). Hence, $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ must be an optimal solution to problem (P2). Similarly, it can be proved that an optimal corner feasible solution to problem (P2) will give an optimal solution to problem (P1).

Theorem 2: *Optimizing problem (P2) is equivalent to optimizing problem (P1) provided problem (P1) has a feasible solution.*

Proof: As problem (P1) has a feasible solution, by lemma 1, there exists a cfs to problem (P2).Thus by remark 1, an optimal solution to problem (P2) will be a cfs. Hence, by lemma 3, an optimal solution to problem (P1) can be obtained.

5. ALGORITHM

Step 1: Given a capacitated fixed charge indefinite quadratic transportation problem (P1) with enhanced flow form a related transportation problem (P2) by introducing a dummy source and a dummy destination with

$$a_i^{'} = \sum_{j\in J} u_{ij}; \forall i \in I \ , \ a_{m+1}^{'} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P = b_{n+1}^{'} \ , \ b_j^{'} = \sum_{i\in I} u_{ij}; \forall j \in J \ ,$$

$$c'_{ij} = c_{ij} \ , \forall i \in I, j \in J, \ c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i \in I, \ \forall j \in J, \ c'_{m+1,n+1} = M$$

$$d'_{ij} = d_{ij} \ , \ \forall i \in I, j \in J, \ d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i \in I, \ \forall j \in J \ d'_{m+1,n+1} = M$$

Step 2: Find an initial basic feasible solution to (P2) with respect to variable cost only. Let B be its corresponding basis.

Step 3 : Calculate θ_{ij} ,$(c_{ij}-z^1_{ij})$, $(d_{ij} - z^2_{ij})$, z_1, z_2 for all non basic cells such that

$$u_i^1 + v_j^1 = c_{ij} \qquad \forall (i,j) \in B$$

$$u_i^2 + v_j^2 = d_{ij} \qquad \forall (i,j) \in B$$

$$u_i^1 + v_j^1 = z_{ij}^1 \qquad \forall (i,j) \in N_1 \text{ and } N_2$$

$$u_i^2 + v_j^2 = z_{ij}^2 \qquad \forall (i,j) \in N_1 \text{ and } N_2$$

z_1 = value of $\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B

z_2 = value of $\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B.

θ_{ij} = level at which a non basic cell (i,j) enters the basis replacing some basic cell of B.

N_1 and N_2 denote the set of non basic cells (i,j) which are at their lower bounds and upper bounds, respectively.

Note: $u_i^1, v_j^1, u_i^2, v_j^2$ are the dual variables which are determined by using the above equations and taking one of the $u_i{}^{,s}$ or $v_j{}^{,s}$. as zero.

Step 4: Find $R_{ij}^1 \forall (i,j) \in N_1$ and $R_{ij}^2 \forall (i,j) \in N_2$ where

$$R_{ij}^1 = \theta_{ij}\left[z_1(d_{ij}-z_{ij}^2)+z_2(c_{ij}-z_{ij}^1)+\theta_{ij}(c_{ij}-z_{ij}^1)(d_{ij}-z_{ij}^2)\right]; (i,j) \in N_1 \quad \text{and}$$

$$R_{ij}^2 = \theta_{ij}\left[\theta_{ij}(c_{ij}-z_{ij}^1)(d_{ij}-z_{ij}^2)-z_1(d_{ij}-z_{ij}^2)-z_2(c_{ij}-z_{ij}^1)\right]; \forall (i,j) \in N_2$$

N_1 and N_2 denote the set of non basic cells (i,j), which are at their lower bounds and upper bounds, respectively.

Step 4: If $R^1_{ij} \geq 0 \forall (i,j) \in N_1$ and $R^2_{ij} \geq 0 \forall (i,j) \in N_2$ then, the current solution is optimal to (P2) and subsequently to (P1). Go to step 5. Otherwise, some (i,j) $\in N_1$ for which $R^1_{ij} < 0$ or some (i,j) $\in N_2$ for which $R^2_{ij} < 0$ will enter the basis. Go to step 2.

Step 5: Find the optimal cost $Z = z_1 z_2$

6. NUMERICAL ILLUSTRATION

Illustration 1. Consider a 2 x 3 capacitated indefinite quadratic transportation problem with enhanced flow. Table 1 gives the values of c_{ij}, d_{ij}, a_i ,b_j for i=1,2 and j=1,2,3

Table 1: Cost matrix of problem (P1)

	D_1	D_2	D_3	a_i
O_1	2 3	3 4	1 5	40
O_2	1 4	2 4	2 6	30
b_j	20	10	30	

Note: values in the upper left corners are c_{ij}’s and values in the lower left corners are d_{ij}’s for i=1,2,3.and j=1,2,3.

$1 \leq x_{11} \leq 20$, $2 \leq x_{12} \leq 10$, $0 \leq x_{13} \leq 20$,$0 \leq x_{21} \leq 10$, $2 \leq x_{22} \leq 20$, $1 \leq x_{23} \leq 30$

Let the enhanced flow be P = 80, where $P = 80 > \max\left(\sum_{i=1}^{2} a_i = 70, \sum_{j=1}^{3} b_j = 60\right)$.

Introduce a dummy origin and a dummy destination in Table 1 with $c_{i4} = 0 = d_{i4}$ for all i = 1,2 and $c_{3j} = 0 = d_{3j}$ for all j = 1,2,3 . $c_{34} = d_{34} = M$ where M is a large positive number. Also, we have $0 \leq x_{14} \leq 10$, $0 \leq x_{24} \leq 30$, $0 \leq x_{31} \leq 10$, $0 \leq x_{32} \leq 20$, $0 \leq x_{33} \leq 20$ In this way , we form the problem (P2). Now, we find an initial basic feasible solution of problem (P2), which is given in table 2 below.

Table 2: Initial basic feasible solution of problem (P2)

	D_1	D_2	D_3	D_4	$a_i^{'}$	u_i^1	u_i^2
O_1	2 $\overline{20}$ 3	3 $\underline{2}$ 4	1 $\overline{20}$ 5	0 **8** 0	50	0	0
O_2	1 **10** 4	2 **8** 4	2 **20** 6	0 **22** 0	60	0	0
O_3	0 0	0 $\overline{20}$ 0	0 **10** 0	M M	30	-2	-6
$b_j^{'}$	30	30	50	30			
v_j^1	1	2	2	0			
v_j^2	4	4	6	0			

Note: Entries of the form $\underline{a}$ and $\overline{b}$ represent non basic cells which are at their lower and upper bounds, respectively. Entries in bold are basic cells.

$z_1 = 132$, $z_2 = 360$

Table 3: Computation of R_{ij}^1, R_{ij}^2

NB	O_1D_1	O_1D_2	O_1D_3	O_3D_1	O_3D_2
θ_{ij}	0	6	2	10	10
$c_{ij} - z_{ij}^1$	1	1	-1	1	0
$d_{ij} - z_{ij}^2$	-1	0	-1	2	2
R_{ij}^1		2160		6440	
R_{ij}^2	0		988		-2640

Since $R_{ij}^2 < 0$ for O_3D_2 therefore, O_3D_2 will enter in to the basis. Continuing like this, we get the optimal solution of problem (P2), which is shown below in table 4

Table 4: Optimal solution of problem (P2)

	D_1	D_2	D_3	D_4	u_i^1	u_i^2
O_1	2 $\overline{20}$ 3	3 $\underline{2}$ 4	1 $\overline{20}$ 5	0 **8** 0	0	0
O_2	1 **10** 4	2 **18** 4	2 **10** 6	0 **22** 0	0	0
O_3	0 0	0 **10** 0	0 $\overline{20}$ 0	M M	-2	-4
v_j^1	1	2	2	0		
v_j^2	4	4	6	0		

$z_1 = 132$, $z_2 = 340$

Table 5: Computation of R_{ij}^1, R_{ij}^2

NB	O_1D_1	O_1D_2	O_1D_3	O_3D_1	O_3D_3
θ_{ij}	0	8	2	2	10
$c_{ij}-z_{ij}^1$	1	1	-1	1	0
$d_{ij}-z_{ij}^2$	-1	0	-1	0	-2
R_{ij}^1		2720		680	
R_{ij}^2	0		948		2640

Since $R_{ij}^1 \geq 0 \ \forall (i, j) \in N_1$ and $R_{ij}^2 \geq 0 \forall (i, j) \in N_2$, the solution in table 4 is an optimal solution of (P2) and hence yields an optimal solution of (P1).Therefore minimum cost = (132 x 340) = 44880.

Computing Software

We used the software Gams to solve the above numerical and obtained the same solution in 0.07 seconds. The solution shows the minimum variable cost = 44880. The summary of the results obtained on GAMS is as follows.

```
MODEL STATISTICS


BLOCKS OF EQUATIONS        5    SINGLE EQUATIONS        32
BLOCKS OF VARIABLES        2    SINGLE VARIABLES        13
NON ZERO ELEMENTS         55    NON LINEAR N-Z           6
DERIVATIVE POOL           10    CONSTANT POOL           17
CODE LENGTH               26    DISCRETE VARIABLES      12
GENERATION TIME    =     0.047 SECONDS     4 Mb WEX240-240 Dec 18, 2012
EXECUTION TIME     =     0.047 SECONDS     4 Mb WEX240-240 Dec 18, 2012
GAMS Rev 240 WEX-WEI 24.0.1 x86_64/MS Windows       03/07/13 00:35:23 Page 5
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE transportation Using RMIQCP From line 53
S O L V E     S U M M A R Y
   MODEL   transportation     OBJECTIVE z
   TYPE    RMIQCP             DIRECTION MINIMIZE
   SOLVER  CONOPT             FROM LINE 53

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      2 Locally Optimal
**** OBJECTIVE VALUE          44880.0000
 RESOURCE USAGE, LIMIT         0.000      5000.000
 ITERATION COUNT, LIMIT        7    2000000000
 EVALUATION ERRORS             0           0
The model has 13 variables and 32 constraints
 with 55 Jacobian elements, 6 of which are nonlinear.
The Hessian of the Lagrangian has 6 elements on the diagonal,
15 elements below the diagonal, and 6 nonlinear variables.
```

```
 ** Optimal solution. There are no superbasic variables.
  CONOPT time Total                      0.002 seconds
   of which: Function evaluations          0.001 = 50.0%
         1st Derivative evaluations      0.000 = 0.0%

**** REPORT SUMMARY:        0    NONOPT
                          0 INFEASIBLE
                          0 UNBOUNDED
                          0    ERRORS
GAMS Rev 240 WEX-WEI 24.0.1 x86_64/MS Windows         02/25/13 08:28:08 Page 6
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
E x e c u t i o n
----    54 VARIABLE x.L


           1          2          3          4


1      20.000      2.000      20.000      8.000
2      10.000     18.000      10.000     22.000
3                    10.000     20.000
```

Illustration 2: Consider a 5× 6 capacitated indefinite quadratic problem with the following data.

Table 6: Cost matrix of problem (P1)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2 10 3 1	3 20 4 2	1 10 5 3	3 10 2 1	7 5 5 2	6 5 4 3	50
O_2	1 5 4 0	2 10 4 1	2 5 6 2	3 10 2 1	6 20 3 2	7 30 4 1	75
O_3	6 10 3 0	7 20 4 1	4 10 5 2	5 40 6 0	3 50 2 1	8 30 7 2	140
O_4	7 20 6 1	8 10 5 1	9 10 8 1	10 20 6 1	6 40 7 1	5 40 3 1	90
O_5	8 10 7 1	6 20 4 2	4 30 5 1	5 30 3 2	3 20 1 1	1 25 1 2	110
b_j	25	60	55	90	125	100	

Note: The entries in the upper left corner of each cell shows c_{ij} and entries in the lower left corner of each cell show d_{ij} . Lower bounds and upper bounds in each cell are shown in the lower and upper right corners of each cell.

Let the enhanced flow be P = 600 where

$P = 600 > \max\left(\sum_{i=1}^{5} a_i = 465, \sum_{j=1}^{6} b_j = 455 \right)$ .In order to solve this problem, form the related transportation problem which is as follows:

Table7: Related problem (P2)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	$a_i^{'}$
O_1	2 10 3 1	3 20 4 2	1 10 5 3	3 10 2 1	7 5 5 2	6 5 4 3	0 10 0 0	60
O_2	1 5 4 0	2 10 4 1	2 5 6 2	3 10 2 1	6 20 3	7 30 4 1	0 5 0 0	80
O_3	6 10 3 0	7 20 4 1	4 10 5 2	5 40 6 0	3 50 2	8 30 7 2	0 20 0 0	160
O_4	7 20 6 1	8 10 5 1	9 10 8 1	10 20 6 1	6 40 7	5 40 3 1	0 50 0 0	140
O_5	8 10 7	6 20 4 2	4 30 5 1	5 30 3 2	3 20 1 1	1 25 1 2	0 25 0 0	135
O_6	0 30 0 0	0 20 0 0	0 10 0 0	0 20 0 0	0 10 0 0	0 20 0 0	M M	95
$b_j^{'}$	55	80	65	110	135	130	95	

Solving this problem on GAMS we obtain the following report summary.

MODEL STATISTICS

BLOCKS OF EQUATIONS	5	SINGLE EQUATIONS	98
BLOCKS OF VARIABLES	2	SINGLE VARIABLES	43

```
NON ZERO ELEMENTS          199     NON LINEAR N-Z           30
DERIVATIVE POOL             10     CONSTANT POOL            20
CODE LENGTH                119     DISCRETE VARIABLES       42
GENERATION TIME     =       0.297 SECONDS     4 Mb WEX240-240 Dec 18, 2012
EXECUTION TIME      =       0.297 SECONDS     4 Mb WEX240-240 Dec 18, 2012
GAMS Rev 240 WEX-WEI 24.0.1 x86_64/MS Windows         03/06/13 23:35:37 Page 5
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE transportation Using RMIQCP From line 62
S O L V E      S U M M A R Y
    MODEL   transportation      OBJECTIVE z
    TYPE    RMIQCP              DIRECTION MINIMIZE
    SOLVER CONOPT                FROM LINE  62

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      2 Locally Optimal
**** OBJECTIVE VALUE          3979596.0000

 RESOURCE USAGE, LIMIT          0.047     50000.000
 ITERATION COUNT, LIMIT        11    2000000000
 EVALUATION ERRORS              0           0
CONOPT 3         Dec 18, 2012 24.0.1 WEX 37366.37409 WEI x86_64/MS Windows
       The model has 43 variables and 98 constraints
   With 199 Jacobian elements, 30 of which are nonlinear.
   The Hessian of the Lagrangian has 30 elements on the diagonal,
   435 elements below the diagonal, and 30 nonlinear variables.

 ** Optimal solution. There are no superbasic variables.
  CONOPT time Total                         0.031 seconds
   of which: Function evaluations             0.000 =  0.0%
           1st Derivative evaluations       0.000 = 0.0%
 **** REPORT SUMMARY:        0     NONOPT
```

```
                 0 INFEASIBLE
                 0 UNBOUNDED
                 0    ERRORS
GAMS Rev 240 WEX-WEI 24.0.1 x86_64/MS Windows       03/06/13 23:35:37 Page 6
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
E x e c u t i o n
----  63 VARIABLE x.L

         1        2          3              4         5        6

1    10.000    20.000    10.000    10.000     2.000     5.000
2     5.000    10.000     5.000    10.000    18.000    30.000
3    10.000    20.000    10.000    40.000    50.000    10.000
4     3.000    10.000     1.000     1.000    35.000    40.000
5     1.000     9.000    30.000    30.000    20.000    25.000
6    26.000    11.000     9.000    19.000    10.000    20.000

+         7

1     3.000
2     2.000
3    20.000
4    50.000
5    20.000
```

CONCLUSION

In order to solve a capacitated indefinite quadratic transportation problem, a related transportation problem is formed and it is shown that the optimal solution to enhanced flow problem may be obtained from the optimal solution to the related transportation problem.

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