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STRONG METRIC DIMENSION: A SURVEY[1]

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Abstract: The strong metric dimension has been a subject of considerable amount of research in recent years. This survey describes the related development by bringing together theoretical results and computational approaches, and places the recent results within their historical and scientific framework.

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1. INTRODUCTION

In this paper we give a survey of the results related to the strong metric dimension of graphs. The strong metric dimension (SMD) is a recently introduced graph invariant [11], connected to well known metric dimension [1,12] that has been widely investigated.

Metric dimension of a graph can be defined as follows. Let G be a simple connected undirected graph $G = (V, E)$, where V is a set of vertices, and E is a set of edges. The distance between vertices u and v, i.e. the length of a shortest u-v path is denoted by $d(u,v)$. A vertex x of the graph G **resolves** two vertices u and v of G if $d(x,u) \neq d(x,v)$. An ordered vertex set $S = \{x_1, ..., x_k\}$ is a **resolving set** of G if for every two distinct vertices of G there exists a vertex of S which resolves them. For a given vertex t, the k-touple $r(t,S) = (d(t, x_1), ..., d(t, x_k))$ is called **vector of metric coordinates** of t with respect to S. By the definition of the resolving set, S is a resolving set if and only if all vectors $r(t,S)$, $t \in V$, are mutually different. The **metric basis** of G is a resolving set of minimal cardinality. The cardinality of the metric basis, denoted by $\beta(G)$, is called **metric dimension** of G.

Strong metric dimension of G is a more restricted invariant than $\beta(G)$. A vertex w **strongly resolves** two vertices u and v if u belongs to a shortest v-w path, or if v belongs to a shortest u-w path. A vertex set S of G is a **strong resolving set** of G if every two distinct vertices of G are strongly resolved by some vertex of S. The **strong metric basis** of G is a strong resolving set with minimal cardinality. Now, **strong metric dimension** of G, denoted by $sdim(G)$, is defined as the cardinality of its strong metric basis. Notice that if a vertex w strongly resolves vertices u and v then, w also resolves these vertices. Indeed, if for example u belongs to a shortest v-w path, then $d(w,u) < d(w,v)$ and, therefore $d(w,u) \neq d(w,v)$. Hence, every strong resolving set is a resolving set and $\beta(G) \leq sdim(G)$. The problem of finding $sdim(G)$ is called the **strong metric dimension problem** (SMDP).

Example 1. For graph G_1 on Fig. 1, set $S_1 = \{A,B\}$ is a resolving set since vectors of metric coordinates for the vertices of G_1 with respect to S_1 are different: $r(A,S_1)=(0,1)$; $r(B,S_1)=(1,0)$; $r(C,S_1)=(1,2)$; $r(D,S_1)=(2,1)$. On the other hand, singleton sets are not resolving sets. For example, $\{A\}$ is not a resolving set since $d(A,B) = d(A,C) = 1$. Therefore, S_1 is a metric basis of G_1 and $\beta(G_1) = 2$. Set S_1 is also a strong resolving set of G_1. Indeed, each pair of vertices which contain vertex A or vertex B is strongly resolved by A, i.e. B. Vertices C and D are strongly resolved by both A and B since C belongs to a shortest A-D path, and D belongs to a shortest B-C path. Hence, $2 = \beta(G_1) \leq sdim(G_1) \leq |S_1| = 2$, which implies $sdim(G_1) = 2$.

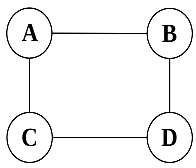

Figure 1: Graph G_1 from *Example 1*

Example 2. For graph G_2 on Fig. 2, it is easy to check that $\{A,B\}$ is a resolving set of minimal cardinality, while $\{A,B,E\}$ is a strong resolving set of minimal cardinality. Therefore, $\beta(G_2)=2$, $sdim(G_2) = 3$.

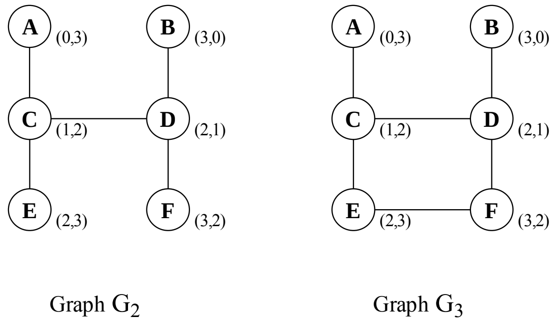

Figure 2: Graphs G_2 and G_3

In the sequel, we introduce several definitions which will be used in the next sections.

Direct product of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = V_1 \times V_2$, and two vertices (a,b), (c,d) are adjacent in $G_1 \times G_2$ if and only if $\{a,c\} \in E_1$ and $\{b,d\} \in E_2$.

Cartesian product of two graphs G_1 and G_2 is a graph $G_1 \square G_2$ such that $V(G_1 \square G_2) = V_1 \times V_2$, and two vertices (a,b), (c,d) are adjacent in $G_1 \square G_2$ if and only if either $a=c$ and $\{b,d\} \in E_2$, or $b=d$ and $\{a,c\} \in E_1$.

Corona product of graphs G_1 and G_2 is a graph $G_1 \odot G_2$ obtained from G_1 and G_2 by taking one copy of G_1 and $|V_1|$ copies of G_2 and joining by an edge each vertex from the i-th copy of G_2 with the i-th vertex of G_1, $i = 1,..., |V_1|$.

Diameter $diam(G)$ of graph G is defined as $max_{u,v\in V}\, d(u,v)$. Graph $G = (V, E)$ is **2-antipodal** if for each vertex $v\in V$ there exists exactly one vertex $w \in V$ such that $d(v,w)=diam(G)$. A vertex of graph is a **simplicial vertex** if the subgraph induced by its neighbors is a complete graph. Set $\sigma(G)$ is the set of all simplicial vertices of G. Graph G is **vertex-transitive** graph if for any two vertices u and v, there is an automorphism $f{:}V\to V$ such that $f(u) = v$. A connected graph G is **distance-regular** if, for any vertices u,v of G and any integers $i,j=0,1, ..., diam(G)$, the number of vertices at distance i from u, and distance j from v depends only on i, j and the distance between u and v, independently of the choice of u and v.

Vertex cover of graph G is a set C of vertices of G such that every edge of G is incident with at least one vertex of C. **Vertex covering number** of G, denoted by $\alpha(G)$, is the minimal cardinality of the vertex cover. **Independence number** $in(G)$ of graph G is the largest cardinality of a set of vertices of G, where neither two of them are adjacent. **Clique** of graph G is a complete subgraph of G. **Clique number** $\omega(G)$ of graph G is the maximal cardinality of a clique in G. A set X of vertices of G is the **twin-free clique** in G if the subgraph induced by X is a clique, and for each $u,v \in X$ there exists a vertex from V adjacent to u but not adjacent to v, or vice versa. **Twin-free clique number** $\varpi(G)$ of G is the maximal cardinality of a twin-free clique in G. **Matching** in graph G is a subset of edges from E without common vertices. The size of a largest matching in G is the **maching number** of graph G, denotead by $\mu(G)$.

The paper is organized as follows. In Section 2, we give an overview of theoretical properties of the strong metric dimension, and summarize the existing explicit expressions for the strong metric dimension of some general classes of graphs, or the lower and upper bounds. The strong metric dimension of some special classes of graphs is presented in Section 3. Section 4 describes a mathematical programming model of the strong metric dimension problem. Section 5 is devoted to the existing metaheuristic approaches for solving the problem of determining the strong metric dimension. Finally, Section 6 contains concluding remarks.

2. THEORETICAL RESULTS

The concept of the metric dimension was motivated by the problem of determining the location of an intruder in a network [9,12]. Namely, if S is a resolving set of a graph G then, vertex u where intruder is located, is uniquely determined by metric coordinates of u with respect to S. However, vectors of the metric coordinates with respect to S do not determine the graph G uniquely.

For example, consider graphs G_2 and G_3 with $V(G_2)=V(G_3)$ on Fig. 2. The set $S=\{A,B\}$ is a metric basis for both graphs and vertices have the same vectors of metric coordinates with respect to S. Nevertheless, graphs G_2 and G_3 are different.

It was observed in [11] that if S is a strong resolving set of G then, set $\{r(v,S)|v\in V\}$ uniquely determines graph G in the following sense. If G' is a graph with $V(G')=V(G)$ such that S is a strong resolving set in G', and if for all vertices $v\in V(G')=V(G)$ we have $r_G(v,S) = r_{G'}(v,S)$ then, $G=G'$. Namely, by the definition of strongly resolved vertices, and using strong resolving set S and the corresponding vectors of metric coordinates, it is possible to reconstruct the distances between all vertices in G, which uniquely determines G.

In [9], it has been proved that the strong metric dimension problem is closely related to well known NP-hard vertex covering problem (VCP). This connection is based on the concepts of the strong resolving graph G_{SR} of graph G.

Definition 1 [9]. Vertex u is **maximally distant** from vertex v if for all vertices w adjacent to u, it follows that $d(w,v) \le d(u,v)$.

Definition 2 [9]. The **strong resolving graph** G_{SR} of G is a graph with the following properties: $V(G_{SR}) = V(G)$, and $E(G_{SR})$ contains all mutually maximally distant pairs of vertices from G.

Definition 3 [10] The **boundary** ∂G of graph G is defined as a set of all u in V for which there exists $v \in V$ such that u,v are mutually maximally distant.

It is easy to see that $V(G_{SR}) = \partial G$. The following theorem connects SMDP and VCP:

Theorem 3 [9]. For any connected graph G, $sdim(\mathrm{G}) = \alpha(G_{SR})$.

Moreover, as can be seen from [9], there exist a polynomial time transformation of the VCP to the SMDP, which implies that the strong metric dimension problem is NP-hard.

Although the strong metric dimension problem is NP-hard in general, for some classes of graphs it can be solved in polynomial time. For example, this holds for distance hereditary graphs [7]. A graph G is **distance hereditary** if every connected induced subgraph H of G is isometric, i.e. for each $u,v \in V(H)$, it follows $d_H(u,v) = d_G(u,v)$. Note that trees are a special case of this class. In [7], an algorithm for finding the strong metric dimension of distance hereditary graphs with $O(|V|\cdot|E|)$ complexity is presented.

A concise survey of theoretical results on the strong metric dimension is given in Table 1. The first column contains the definition of graph G, the second column lists the assumptions, the third column gives the exact value of the strong metric dimension of G , and the last column points to the references. In Table 1, the following definiton is used:

Table 1: Strong metric dimension of some general classes of graphs

G	*Assumptions*	$sdim(G)$	*ref.*
$H\square K$	H,K connected	$\alpha(H_{SR}\times K_{SR})$	
$H\square K$	H,K connected	$\|\partial(H)\|\cdot\|\partial(K)\|$ - $in(H_{SR}\times K_{SR})$	
$H\square K$	H,K connected, H_{SR},K_{SR} vertex transitive	$min\{\|\partial(H)\|\cdot sdim(K), \|\partial(K)\|\cdot sdim(H)\}$	
$H\square K$	H,K connected 2-antipodal	$\|V(H)\|\cdot\|V(K)\|/2$	
$H\square K$	H connected 2-antipodal, K connected with $\|\partial(K)\|=\|\sigma(K)\|$	$\|V(H)\|\cdot\|\sigma(K)\|/2$	
$H\square K$	H,K connected, H_{SR},K_{SR} regular, at least one is bipartite	$\|\partial(H)\|\cdot\|\partial(K)\|/2$	
$H\square T$	H connected 2-antipodal, T is a tree, $l(T)$ -number of leaves in T	$\|V(H)\|\cdot l(T)/2$	[10]
$H\square K_r$	H connected 2-antipodal, K_r complete graph on r vertices	$\|V(H)\|\cdot r/2$	
$H\square K$	H distance regular, K connected, K_{SR} regular bipartite	$\|V(H)\|\cdot\|\partial(K)\|/2$	
$H\square K$	H,K connected, $\partial(H)=\sigma(H)$, $\partial(K)=\sigma(K)$	$min\{\|\partial(H)\|\cdot(\|\partial(K)\|-1), \|\partial(K)\|\cdot(\|\partial(H)\|-1)\}$	
H	H connected, $\|H\|\geq 2$, $diam(H)=2$	$\|H\|$ - $\varpi(H)$	
$H\odot K_1$		$\|H\|$ - 1	[6]
$H\odot K$	G_i subgraph corresponding to i-th copy of K, $\|H\|=n$	$sdim(K_1 + \bigcup_{i=1}^{n} G_i)$	

Table 2, which is organized in a similar way as Table 1, contains the lower or upper bounds for the strong metric dimension of some general classes of graphs.

Table 2: Bounds for the strong metric dimension

G	*Assumptions*	$sdim(G)$	*ref.*
$H\square K$	H,K connected	$\leq \min\{\lvert\partial(H)\rvert\cdot sdim(K), \lvert\partial(K)\rvert\cdot sdim(H)\}$	
$H\square K$	H,K connected	$\geq sdim(H) \cdot sdim(K)$	
G		$\lvert\sigma(G)\rvert -1 \leq sdim(G) \leq \lvert\partial(G)\rvert -1$	[10]
$H\square K$	H,K connected, K_2 complete graph	$\geq \mu(H_{SR}) \cdot sdim(H\square K_2) \geq 2\cdot\mu(H_{SR}) \cdot\mu(K_{SR})$	
H	H connected, $\lvert H\rvert\geq 2$	$\leq \lvert H\rvert - \varpi(H)$	[6]

In [10] is also proven that:

- $sdim(G) = 1$ if and only if $G \cong P_n$;
- $sdim(H\square K) = 2$ if and only if H and K are both paths.

3. STRONG METRIC DIMENSION OF SOME SPECIAL CLASSES OF GRAPHS

In Table 3, the strong metric dimension of some special classes of graphs is concisely presented. Here C_n, K_n, P_n denote the cycle, the complete graph, and the path on n vertices, respectively. Notation T represents a tree, with the number of leaves $l(T)$. The Hamming graph $H_{n,k}$ is the Cartesian product $K_k\square K_k\square\cdots\square K_k$, with n factors. Hypercube Q_n is defined as $H_{n,2}$. The convex polytope D_n ($n\geq 5$) is defined as follows: $V(D_n) = \{a_i, b_i, c_i, d_i \mid i = 0, 1, \ldots, n-1\}$, $E(D_n) = \{(a_i,a_{i+1}), (a_i,b_i), (b_i,c_i), (b_{i+1},c_i), (c_i,d_i), (d_i,d_{i+1}) \mid i = 0, 1, \ldots, n-1\}$, where indices are taken modulo n. Similarly, the convex polytope T_n ($n \geq 5$) is defined as: $V(T_n) = \{a_i, b_i, c_i, d_i \mid i = 0, 1, \ldots, n-1\}$, $E(T_n) = \{(a_i,a_{i+1}), (a_i,b_i), (a_{i+1},b_i), (b_i,b_{i+1}), (b_i,c_i), (c_i,c_{i+1}), (c_i,d_i), (c_{i+1},d_i), (d_i,d_{i+1}) \mid i = 0, 1, \ldots, n-1\}$, where indices again are taken modulo n.

Table 3: Strong metric dimension of some special graph classes

G	*Assumptions*	*sdim*(*G*)	*ref.*
Q_n		2^{n-1}	[3]
$H_{n,k}$		$(n-1)\cdot n^{k-1}$	[3,10]
$K_{n_1} \square K_{n_1} \square .. \square K_{n_r}$		$n_1 \cdot n_2 \cdot \ldots \cdot n_r - \max\limits_{1 \le i \le r} \left\{ \frac{n_1 \cdot n_2 \cdot \ldots \cdot n_r}{n_i} \right\}$	
$K_n \square P_r$		n	[10]
$C_n \square P_r$		n	
$K_n \square K_r$		$min\{n\cdot(r-1), r\cdot(n-1)\}$	
$K_n \square C_{2r}$		$n \cdot r$	
$K_n \square C_{2r+1}$		$min\{n\cdot(r+1), (n-1)\cdot(2r+1)\}$	
$T \square P_r$		$l(T)$	
$T \square C_{2r}$		$l(T) \cdot r$	
$T \square C_{2r+1}$		$min\{l(T)\cdot(r+1), (2r+1)\cdot(l(T)-1)\}$	
$T \square K_n$		$min\{l(T)\cdot(n-1), n\cdot(l(T)-1)\}$	
$T_1 \square T_2$		$min\{l(T_1)\cdot(l(T_2)-1), l(T_2)\cdot(l(T_1)-1)\}$	
$C_n \square C_{2r}$		$n \cdot r$	
$C_{2n+1} \square C_{2r+1}$		$min\{(2n+1)\cdot(r+1), (n+1)\cdot(2r+1)\}$	[9]
$C_{2n+1} \times C_{2n+1}$		$(2n+1)\cdot(n+1)$	
$K_n \times K_r$	$n,r \ge 3$	$max\{n\cdot(r-1), r\cdot(n-1)\}$	
$K_n \times P_r$	$n \ge 3, r \ge 2$	$n \cdot \lceil r/2 \rceil$	
$K_n \times C_r$	$n \ge 3, r \ge 4$	$\begin{cases} n\cdot(r-1), & r \in \{4,5\} \\ \frac{n \cdot r}{2}, & r = 2k, r \ge 6 \\ n\cdot(r - \lfloor r/3 \rfloor), & otherwise \end{cases}$	[10]
Convex polytopes D_n	$n \ge 5$	$\begin{cases} 13, & n = 6 \\ 19, & n = 8 \\ 5n/2, & n = 2k, n \notin \{6,8\} \\ 2n, & n = 2k+1 \end{cases}$	[4]
Convex polytopes T_n	$n \ge 5$	$\begin{cases} 5n/2, & n = 2k+1, \\ 2n, & n = 2k \end{cases}$	

4. INTEGER LINEAR PROGRAMMING FORMULATION

The first integer linear programming (ILP) formulation of the strong metric dimension problem is given in [4]. It has been usefull for finding the strong metric dimension of some special classes of graphs in singular cases. For example, values $sdim(D_6)$=13 and $sdim(D_8)$=19 from Table 3 have been obtained using the ILP formulation.

Given a simple connected undirected graph $G = (V,E)$, where $V=\{1,2,..., n\}$, $|E|=m$, it is easy to determine the length $d(u,v)$ of a shortest u-v path for all $u,v \in V$, using any shortest path algorithm. The coefficient matrix A can be defined as follows:

$$A_{(u,v),i} = \begin{cases} 1, & d(u,i) = d(u,v) + d(v,i) \\ 1, & d(v,i) = d(v,u) + d(u,i) \\ 0, & otherwise \end{cases} \tag{1}$$

where $1 \le u < v \le n$, $1 \le i \le n$. Variable y_i described by (2) determines whether vertex i belongs to a strong resolving set S.

$$y_i = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases} \tag{2}$$

The ILP model of the strong metric dimension problem can now be formulated as:

$$\min \sum_{i=1}^{n} y_i \tag{3}$$

subject to:

$$\sum_{i=1}^{n} A_{(u,v),i} \cdot y_i \ge 1 \qquad 1 \le u < v \le n \tag{4}$$

$$y_i \in \{0,1\} \quad 1 \le i \le n. \tag{5}$$

The objective function (3) represents the cardinality of a strong resolving set S, and constraints (4) ensure that each pair of vertices $u,v \in V$ is strongly resolved by at least one vertex $i \in S$. Constraints (5) represent binary nature of decision variables y_i. Note that ILP model (3)-(5) has only n variables and $\binom{n}{2}$ linear constraints.

5. METAHEURISTIC APPROACHES

In this section, we give an overview of the three existing heuristics based on meta-heuristic approaches proposed for solving SMDP: Electromagnetism-like approach - EM [5], Genetic algorithm - GA [2], and Variable neighborhood search - VNS [8].

The electromagnetism-like (EM) metaheuristic is a population-based algorithm for global optimization, which is also used for combinatorial optimization as a stand-alone approach or as an accompanying algorithm for other methods. The population contains N_{pop} real vectors p_k, $k = 1, 2, ... , N_{pop}$, of length n, transformed into the SMDP feasible solutions. In the first iteration, real vectors p_k are randomly generated from a set $[0,1]^n$. For each p_k, the corresponding strong resolving set S is initially established by rounding in the following way: if i-th coordinate of p_k is greater or equal than 0.5 then, S contains the vertex i. If set S is not a strong resolving set then, it is repaired by adding randomly new vertices from $V \setminus S$, until S becomes a strong resolving set. The objective function value is the cardinality of S. After the objective function is computed, its possible improvement is tried by a local search procedure, which removes some elements of S in such a way that S remains a strong resolving set. Next, a scaling procedure is applied, which additionally moves points towards solutions obtained by the local search. Afterwards, the calculation of charges and forces using EM attraction-repulsion mechanism is applied, resulting in moving towards a local minimum. A detailed description of EM for SMDP can be seen in [5].

In [2], a genetic algorithm (GA) for SMDP is presented. The GA uses a binary encoding of the individuals, where each solution S (i.e. a candidate for a strong resolving set) is represented by a binary string of length n. Digit 1 at the i-th position of the string denotes that the vertex i belongs to S, while 0 shows the opposite. In case when S is not a strong resolving set, the same reparation technique as in EM is applied. The objective function value is again the cardinality of S. The population of the first generation is randomly generated, providing the maximal diversity of the genetic material. In order to prevent undeserved domination of some individuals in the current population, an elitist strategy is used. Duplicated individuals are removed from the current population by setting their fitness to zero. The fine-grained tournament selection, the one-point crossover and the simple mutation are implemented. The run-time performance of GA is improved by a caching technique.

The variable neighborhood search (VNS) approach for the SMDP [8] is based on the idea of decomposition. The initial set S is obtained by a simple procedure, which starts from the empty set and adds randomly chosen vertices from V until S becomes a strong resolving set. For a given strong resolving set S, the last element is deleted to obtain set S'. Here the objective function value is computed as the number of pairs of vertices from V which are not strongly resolved with respect to S'. In other words, unlike the EM and GA, VNS objective function measures the infeasibility of set S'. In case when the objective value is equal to zero, $S := S'$ is a new strong resolving set with smaller cardinality, and new S' is generated. The neighborhood $N^k(S')$ contains all sets obtained from S' by deleting k of its elements and replacing them by k elements from $V \setminus S'$. A local search procedure tries to improve randomly generated $S'' \in N^k(S')$ by interchanging one element from set S'' with one element of its complement, and updates S whenever a new strong resolving set with smaller cardinality is generated.

Experimental results and comparison of the previous three approaches are presented on two different ORLIB classes of graph instances: crew scheduling, and graph coloring. The GA tests were performed on an AMD Sempron 1.6 GHz with 256 MB memory [2], under Linux (Knoppix 5.0) operating system, while the EM and VNS are tested on an Intel Pentium IV 2.5 GHz with 4 GB memory [5,8]; all presented running times are in seconds. All three methods have been run 20 times for all instances, and the results are summarized in Table 4 and Table 5. Sign “-” denotes that the running time exceeds 5 hours. The tables are organized as follows:

Table 4: Results on Crew scheduling OR-LIB instances

Inst.	*n*	*m*	*EM*		*GA*		*VNS*	
			best	*t*	*best*	*t*	*best*	*t*
csp50	50	173	29	123	29	27	29	0.35
csp100	100	715	62	889	61	528	61	4.90
csp150	150	1355	106	3152	98	3166	98	25
csp200	200	2543	154	6768	144	8048	142	83
csp250	250	4152	181	13214	178	17060	172	198
csp300	300	6108	-	-	-	-	224	464
csp350	350	7882	-	-	-	-	237	817
csp400	400	10760	-	-	-	-	288	1363
csp450	450	13510	-	-	-	-	316	2097
csp500	500	16695	-	-	-	-	367	3474

- The first three columns contain the test instance name, the number of nodes and edges, respectively;
- The fourth and fifth columns contain the best EM solution, and the average running time (named *best* and *t*) obtained in 20 runs;
- Results related to GA and VNS are presented in the same way, in the last four columns.

As can be seen from Table 4 and Table 5, VNS approach outperforms EM and GA meta-heuristics both in the solution quality and running times. Moreover, VNS finds solutions of large-scale instances in the cases when EM and GA fail (csp300-csp500, gcol21-gcol30).

6. CONCLUSIONS

The existing results from the literature devoted to the strong metric dimension can be divided into two groups. The first group contains theoretical results related to both general properties and the explicit values of strong metric dimension for some special classes of graphs. The second group of papers introduces metaheuristic approaches for solving the strong metric dimension problem. In this survey, we unify these two aspects in order to give a complete overview of the related literature.

Table 5: Results on Graph coloring OR-LIB instances

Inst.	*n*	*m*	*EM*		*GA*		*VNS*	
			best	*t*	*best*	*t*	*best*	*t*
gcol1	100	2487	93	1106	91	173	91	10.8
gcol2	100	2487	93	1104	91	183	91	10.8
gcol3	100	2482	93	1107	91	199	91	10.8
gcol4	100	2503	93	1101	91	173	91	10.8
gcol5	100	2450	92	1109	91	166	91	11.0
gcol6	100	2537	93	1080	91	195	91	10.5
gcol7	100	2505	92	1091	91	171	91	10.8
gcol8	100	2479	92	1123	90	177	90	10.8
gcol9	100	2486	92	1099	91	178	91	10.9
gcol10	100	2506	93	1099	91	168	91	10.8
gcol11	100	2467	92	1111	91	169	91	10.9
gcol12	100	2531	93	1076	91	166	91	10.7
gcol13	100	2467	93	1107	91	171	91	10.9
gcol14	100	2524	93	1088	91	170	91	10.7
gcol15	100	2528	92	1084	91	183	91	10.6
gcol16	100	2493	93	1100	91	174	91	10.8
gcol17	100	2503	93	1102	91	173	91	10.8
gcol18	100	2472	93	1115	91	173	91	10.8
gcol19	100	2527	93	1089	91	163	91	10.6
gcol20	100	2420	93	1127	91	176	91	11.1
gcol21	300	22482	-	-	-	-	288	906
gcol22	300	22569	-	-	-	-	288	902
gcol23	300	22393	-	-	-	-	289	908
gcol24	300	22446	-	-	-	-	288	915
gcol25	300	22360	-	-	-	-	288	907
gcol26	300	22601	-	-	-	-	288	912
gcol27	300	22327	-	-	-	-	288	913
gcol28	300	22472	-	-	-	-	288	902
gcol29	300	22520	-	-	-	-	288	974
gcol30	300	22543	-	-	-	-	288	964

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