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CHOICE OF THE CONTROL VARIABLES OF AN ISOLATED INTERSECTION BY GRAPH COLOURING

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Abstract: This paper deals with the problem of grouping traffic streams into signal groups on a signalized intersection. Determination of the complete sets of signal groups, i.e. the groups of traffic streams on one intersection, controlled by one control variable is defined in this paper as a graph-coloring problem. The complete sets of signal groups are obtained by coloring the complement of the graph of identical indications. It is shown that the minimal number of signal groups in the complete set of signal groups is equal to the chromatic number of the complement of the graph with identical indications. The problem of finding all complete sets of signal groups with minimal cardinality is formulated as a linear programming problem where the values of variables belong to a set $\{0,1\}$.

Keywords: Traffic control, Signalized intersection, Signal group, Graph coloring, Optimization.

MSC: 90C35.

1. INTRODUCTION

Vehicles approaching an intersection are ready to perform certain "maneuver", i.e. to drive straight through, turn left, or turn right at the intersection. The vehicles which perform the same maneuver and form the same queue on an approach, in one or several lanes, represent a flow component that can be considered separately from other flow components that perform other maneuvers [1], [2]. Such an arrival flow component is termed as a traffic stream. In fact, this is the smallest flow component that can be controlled by a separate traffic signal, i.e. by a sequence of signal indications different from the sequences on other signals.

Traffic streams on an intersection are elements of the set of traffic streams $\mathscr{S}$ i.e.

$$\mathscr{S} = \{\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots, \sigma_{\mathrm{I}}\}, \tag{1}$$

where $i \in \mathscr{J}$, and $\mathscr{J}$ is the set of traffic stream indices:

$$\mathscr{J} = \{1,2,\ldots,i,\ldots,\mathrm{I}\} = \{1,2,\ldots,i,\ldots,\mathrm{I}',\ldots,\mathrm{I}\}.$$

Indices $i = 1,2,\ldots,\mathrm{I}'$ are assigned to vehicle traffic streams, and indices $i = \mathrm{I}'+1,\ldots,\mathrm{I}$ to a pedestrian and other traffic streams.

Elements of set $\mathscr{S}$ are components of vector $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{\mathrm{I}})$, which describes the uncontrolled system input and represent passenger vehicle flows, pedestrian flows, flows of public transport vehicles, etc.

For an exact statement and solution of traffic control problems, it is necessary to study the relations in the set of traffic streams $\mathscr{S}$. The most important relations are: *conflictness, non-conflictness* and *compatibility.*

1.1. Conflictness of Traffic Streams

Some pairs of traffic streams use along the part of their trajectories, through the intersection, the same space, so-called *the conflict area*. Trajectories of these streams cross or merge. Between such streams, there exists *a conflict*.

The set of all pairs of traffic streams where elements of a pair are in conflict represents the *conflictness relation.* Thus, the conflictness relation C_1 can be defined in the following way:

$$C_1 \subset \mathscr{S} \times \mathscr{S} \tag{2}$$

$$C_1 = \{(\sigma_i, \sigma_j) \mid \text{trajectories of } \sigma_i \text{ and } \sigma_j \text{ cross or merge}, \; i,j \in \mathscr{J}\} \tag{3}$$

The graph of conflictness G_k is defined by the set $\mathscr{S}$ and the relation C_1:
$G_k = (\mathscr{S}, C_1)$.

Since there is a conflict between any two streams whose trajectories cross or merge, it is obvious that the conflictness relation is symmetrical:

$$(\sigma_i,\sigma_j) \in C_1 \Rightarrow (\sigma_j,\sigma_i) \in C_1, \quad i,j \in \mathscr{J}. \tag{4}$$

Relation C_1 is not reflexive (a stream cannot be in conflict by itself). Therefore, $(\sigma_i,\sigma_i) \notin C_1$, $(i \in \mathscr{J})$.

1.2. Non-conflictness of Traffic Streams

The non-conflictness relation of traffic streams represents a set of ordered pairs of traffic streams, where the trajectories of the elements of the pairs neither cross nor merge. Thus, this relation is the set of all pairs of traffic streams that are not mutually in conflict:

$$C_2' = \overline{C}_1 = (\mathscr{S} \times \mathscr{S}) \setminus C_1 \tag{5}$$

The graph of non-conflictness is defined by the set $\mathscr{S}$ and the relation C_2', as $G_k' = (\mathscr{S}, C_2')$. Trajectories traversed by different traffic streams through the intersection have to be known in order to determine whether a pair of traffic streams can simultaneously gain the right-of-way, i.e. whether the streams are *compatible.*

1.3. Compatibility of Traffic Streams

Since the main objective of the traffic control by traffic lights is to give the right-of-way to some traffic streams, and to stop others in the set of traffic streams of an intersection, it is necessary to find the traffic streams which can simultaneously get the right-of-way. Therefore, *the traffic stream compatibility relation* is introduced. It is defined by a set of traffic streams pairs, such that elements of a pair can simultaneously get the right-of-way.

The traffic stream compatibility relation plays an important role in solving traffic control problems related to:

- Deciding whether a traffic control by traffic lights should be introduced at an intersection,
- Assigning control variables to traffic streams or to subsets of traffic streams,
- The traffic control process on an intersection.

The factors to be considered when defining the compatibility relation are:

- The intersection geometry,
- Factors related to the traffic safety process, for which traffic engineers' expert estimations are needed.

The analysis of the intersection geometry considers mutual relations of trajectories of traffic streams. Obviously, when trajectories of two traffic streams do not cross, these streams can simultaneously get the right-of-way, i.e. they are *compatible*. On the other hand, when trajectories of two traffic streams do cross, the streams are in a *conflict* and their simultaneous movement through the intersection should not be permitted. However if volumes are not high, a "filtering" of one stream through another stream can be permitted in some cases. When determining the compatibility relation, some special requirements should be taken into account, e.g., it is required sometimes that some streams have to pass through the intersection without any disturbance although, filtering could be permitted if only their volumes are considered. These requirements are usually achieved by so called *directional signals*.

When only geometrical factors are considered, the *relation of conflictness* and the *relation of non-conflictness* can be defined. It means that when determining the compatibility relation of traffic streams, besides data on geometrical features of traffic stream trajectories, it is necessary to consider some other factors, i.e. it is necessary to list:

- Pairs of conflicting traffic streams that can simultaneously get the right-of-way,
- The traffic streams required to pass through the intersection without any disturbance (the streams to which the right-of-way is given by directional signals).

Some pairs of conflicting traffic streams can be, at the same time, the pairs of compatible streams (although the streams are conflicting). Therefore, it is necessary to divide the conflicts into allowed and forbidden [3]. Forbidden conflicts can be regulated only by traffic lights, while allowed conflicts are solved by traffic participants themselves, respecting priority rules prescribed by traffic regulations. Without traffic lights, conflicts are solved by "filtering" one stream through another. Obviously, the possibility of filtering depends on vehicle spacing interval, which depends on volume of traffic streams. Since the volumes change during a day, and there are periods with very high volume differences, such as morning peak, afternoon peak, off-peak and night periods, situations may arise that two conflicting traffic streams may simultaneously have the right-of-way in one period but not in some other.

The set of traffic streams pairs, which comprise *conditionally compatible* streams, i.e. conflicting streams allowed to pass simultaneously through an intersection, can be thus defined as follows:

$$\begin{aligned} C_2'' = \; & \{(\sigma_i,\sigma_j) \mid (\sigma_i,\sigma_j) \in C_1, \\ & i,j \in \mathscr{J},\; \textit{streams } \sigma_i \textit{ and } \sigma_j \textit{ can simultaneously} \\ & \quad \textit{get the right-of-way}\} \end{aligned} \tag{6}$$

The problem of introducing traffic signals for the traffic control on an intersection is actually a problem of the same kind. It is necessary to determine when traffic lights have to be introduced in order to remove conflicts, i.e. to determine the values of traffic stream volumes when filtering is not possible any more. Before the traffic signals were introduced, traffic participants themselves, using filtering and respecting priority rules, were solving all the conflicts.

When volumes of conflicting traffic streams reach a level where filtering becomes difficult, the introduction of traffic lights becomes unavoidable because traffic participants themselves cannot solve the conflicts. The values of traffic stream volumes that justify an introduction of the signalization of an intersection are given in tables in traffic-engineering handbooks. Not introducing traffic lights when these levels are reached can lead to many negative effects, such as an enormous number of stops and delays, increase in the number of traffic accidents, etc. Therefore, conflicts at all conflict points on a not signalized intersection are prevented by traffic participants, respecting priority rules, while at a signalized intersection traffic lights are used in order to avoid conflicts at most of the conflict points, with a possibility of conflicts in some conflict points still left for "self-regulation" by traffic participants.

The compatibility relation of traffic stream pairs whose elements can simultaneously get the right-of-way is:

$$C_2 = C_2' \cup C_2'' \tag{7}$$

In some cases, it may be necessary to control the traffic in such a way that certain streams can pass through an intersection without conditional conflicts. Then they cannot gain the right-of-way simultaneously with any other conflicting streams although, it would be justified if only volumes were considered. For controlling these streams, the directional signals are used.

If the set of streams that have to pass through the intersection without any conflict is denoted by $\mathscr{S}'$, where $\mathscr{S}' \subset \mathscr{S}$, then the set of pairs of traffic streams that can simultaneously get the right-of-way is defined by the following expression:

$$C_3 = C_2 \setminus \{(\sigma_i, \sigma_j) \mid (\sigma_i, \sigma_j) \in C_2'', \quad (\sigma_i \text{ or } \sigma_j \in \mathscr{S}')\} \tag{8}$$

Assuming that each traffic stream is compatible with itself then, in order to define the set of pairs that determine the compatibility relation, set of pairs C_3 should be extended by the diagonal Δ_S in set $\mathscr{S}$.

Therefore, the compatibility relation can be defined as:

$$C = C_3 \cup \Delta_S, \tag{9}$$

where

$$\Delta_S = \{(\sigma_i, \sigma_i) \mid, \; i \in \mathscr{I}\} \tag{10}$$

Relation C is symmetric and reflexive.

Compatibility graph of traffic streams is defined by the set of traffic streams $\mathscr{S}$ and the compatibility relation C:

$$G_c = (\mathscr{S}, C). \tag{11}$$

Since the set $\mathscr{S}$ is finite, and the relation C is symmetric and reflexive, graph G_c is a finite, non-oriented graph, with a loop at each node. The incidence matrix of this graph is $B = [b_{ij}]_{I \times I}$, where $I = card\,\mathscr{S}$. Elements of the adjacency matrix are defined as

$$b_{ij} = \begin{cases} 1, & (\sigma_i, \sigma_j) \in C \\ 0, & (\sigma_i, \sigma_j) \notin C \end{cases}, \quad (i, j \in \mathscr{I}) \tag{12}$$

A compatibility graph does not have to be a connected graph.

2. CONTROL VARIABLE

Introduction of a traffic control system on an intersection means to install signals that will control traffic streams by different light indications. The basic intention of traffic signals introduction is to prevent simultaneous movement of incompatible traffic streams.

The traffic control at an intersection comprises thus, giving and canceling the right-of-way to particular traffic streams. Giving and canceling the right-of-way is performed by different signal indications. The indications get the meanings by convention. Green indication for vehicles means *allowed* passage, while red means *forbidden* passage. Amber indication appearing after green indication, as well as after red/red-amber informs drivers that the right-of-way will be changed. The duration of amber and red-amber intervals in some countries are determined by traffic regulations and most frequently, it is specified as 3s for amber and 2s for red-amber indication. Signals that control pedestrian streams usually have only two indications: red ("stop") and green ("walk").

The most frequently used sequence of signal indications for vehicles and pedestrians is presented in Figure 1. However, in some countries there are other sequences, such as flashing amber before a steady amber indication, or direct switching from red to green, etc.

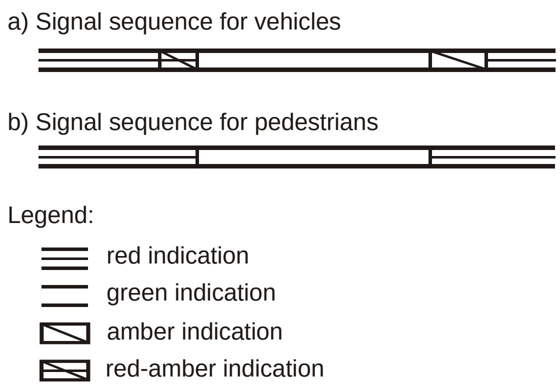

Figure 1: The sequences of signal indications for vehicles and pedestrians

The control of traffic lights, i.e. forming and implementing of specified signal sequences is performed by an electronic device – *a traffic controller*. The controller changes signal indications by using sequence of pulses.

Changes of signal indications are described by a mathematical variable, so-called *control variable*. Control variable can be assigned to *every* traffic stream. However, as compatible traffic streams can simultaneously gain and loose the right of way, it is possible that a subset of traffic streams, comprising several compatible streams, can be controlled by a single control variable [1].

Therefore, among the first problems to be solved when introducing traffic lights control at an intersection is the problem of establishing a correspondence between traffic streams and traffic signal sequences, i.e. to determine the control variables which control the traffic streams. The simplest way to assign control variables to traffic streams is to assign one control variable to one traffic stream. However, there are practical reasons why this assignment is not always used.

Technical and economic considerations cause a tendency to minimize the number of control variables. Namely, the traffic controller should be simpler, with a smaller number of modules that form control variables and thus, it would give a cheaper solution.

Modern traffic controllers can implement more complex control algorithms than those used before their introduction. By increasing the number of control variables, the combinatorial nature of traffic control problems is emphasized, which gives way to improve the performances of the control system.

3. SIGNAL GROUP

Various intersection performance indices depend on the choice of the traffic control system for an intersection. Among these performance indices are: total delay or total number of vehicle stops in a defined interval, total flow through the intersection (for saturated intersections), capacity factor, linear combination of delays and number of stops, etc. Values of these performance indices depend on the assignment of control variables to traffic streams. The best results are, obviously, obtained if each traffic stream is controlled by one control variable.

If the number of control variables is smaller than the number of traffic streams, certain constraints have to be introduced, expressing the requirement that several traffic streams simultaneously get and loose the right-of-way. The consequence of introducing such constraints is the "corruption" of optimum values of performance indices, compared with the case when each traffic stream is controlled by its own control variable.

Reduction in the number of control variables results in simplification of traffic control problems, and also in a possibility to use cheaper and simpler traffic controllers.

In real-time traffic control systems, in which data on current values of traffic stream parameters are used for determine values of control variables, a particular attention has to be paid on choosing the appropriate set of control variables and assigning them to traffic streams.

Determination of the set of control variables is very complex due to all mentioned reasons. This problem, in fact, is the problem of partitioning the set of traffic streams S into subsets of traffic streams so that control of each subset can be performed by a single control variable. A subset of traffic streams that can simultaneously gain and loose the right-of-way, i.e. which can be controlled by a single control variable, is called a *signal group*.

A signal group can also be defined as: A signal group is a set of traffic streams controlled by identical traffic signal indications. Some authors define a signal group as the set of signals on various traffic lights that always show the same indication [4]. For traffic equipment manufacturers, a signal group is a controller module, which always produces one sequence of traffic signal indications.

It is obvious that the traffic streams belonging to the same signal group have to be mutually compatible. However, this condition is not sufficient. Namely, signals used for control of traffic streams of various types - vehicle, pedestrian, tram, etc., cannot always have the same indications, which is necessary if they are to belong to the same signal group. Vehicle streams are, for example, controlled by signal sequences with four indications, while for pedestrian streams only two indications are used. Therefore, signal groups are formed so to contain only the same types of traffic streams and the set of traffic streams $\mathscr{S}$ has to be partitioned in several subsets: the subset of *vehicle traffic streams*, the subset of *pedestrian traffic streams*, etc.

According to the signal group definition, for the intersection presented in Figure 2 together with its compatibility graph, the signal groups are the following subsets: $D_1 = \{\sigma_1, \sigma_2, \sigma_5\}$, $D_2 = \{\sigma_1, \sigma_3\}$, $D_3 = \{\sigma_6\}$, etc.

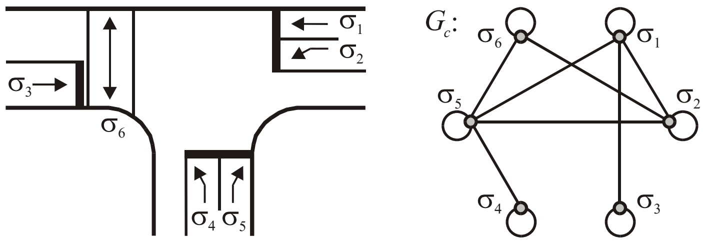

Figure 2: Intersection and its compatibility graph

A signal group D_p represents a subset of the set of traffic streams $\mathscr{S}$ and can be presented as follows:

$$D_p = \{\sigma_{p1}, \sigma_{p2}, \ldots, \sigma_{pe}, \ldots, \sigma_{p\mathrm{E}(p)}\} \tag{13}$$

where $\sigma_{pe} \in \mathscr{S}$, $e \in \mathscr{E}_p$, and $\mathscr{E}_p$ is the set of traffic stream indices in signal group D_p, i.e.

$$\mathscr{E}_p = \{1, 2, \ldots, e, \ldots, \mathrm{E}(p)\}.$$

3.1. The Relation of Identical Signal Indications (Identity Relation)

In order to form signal groups, it is necessary to determine for each pair of compatible traffic streams whether they can be controlled by traffic lights which always have identical indications. The set of such traffic streams pairs represents a relation in the set of traffic streams $\mathscr{S}$. Since this relation determines whether identical traffic light indications can be used for controlling traffic the streams pairs, it is called the relation of identical signal indications, or the identity relation.

The identity relation C_α is defined as:

$$C_\alpha = \{(\sigma_i, \sigma_j) \mid \textit{traffic streams } \sigma_i, \sigma_j \textit{ can be controlled by a siongle control vriable } i, j \in \mathscr{J}\} \tag{14}$$

Relation C_α can be presented as:

$$C_\alpha = C \setminus C_4, \qquad i, j \in \mathscr{J}$$

where

$$C_4 = \{(\sigma_i, \sigma_j) \mid ((\sigma_i, \sigma_j) \in C) \wedge \wedge (\sigma_i \in \mathscr{S}^f, \sigma_j \in \mathscr{S}^l, f, l \in \mathscr{F}, f \neq l),\ i, j \in \mathscr{J}\} \tag{15}$$

The subsets $\mathscr{S}^1, \mathscr{S}^2, \ldots, \mathscr{S}^f, \ldots, \mathscr{S}^{\mathrm{F}}$ represent subsets of the same type (vehicles, pedestrians, trams, etc.) of traffic streams. Traffic streams of one type are controlled by the signals which have the same sequences of indications. For vehicle traffic streams, for example, this sequence is: green, amber, red, red-amber.

The set $\mathscr{F}$ is the index set of traffic stream types, i.e. signal types:

$$\mathscr{F} = \{1, 2, \ldots, f, \ldots \mathrm{F}\} \tag{16}$$

The collection

$$\overline{\mathscr{S}} = \{\mathscr{S}^1, \mathscr{S}^2, \ldots, \mathscr{S}^f, \ldots, \mathscr{S}^{\mathrm{F}}\} \tag{17}$$

represents a partition of set $\mathscr{S}$. Hence, we have:

$$\bigcup_{f=1}^{\mathrm{F}} \mathscr{S}^f = \mathscr{S} \tag{18}$$

$$\mathscr{S}^f \bigcap \mathscr{S}^l = \emptyset, \quad (f \in \mathscr{F}, \quad l \in \mathscr{F}, \quad f \neq l) \tag{19}$$

The relation of identical traffic signal indications C_α is:

a) Reflexive, i.e.

$$(\sigma_i, \sigma_i) \in C_\alpha, \quad (i \in \mathscr{J}) \tag{20}$$

b) Symmetric, i.e.

$$(\sigma_i, \sigma_j) \in C_\alpha \Rightarrow (\sigma_j, \sigma_i) \in C_\alpha, \quad (i, j \in \mathscr{J}) \tag{21}$$

The identity relation corresponds to an identity graph:

$$G_\alpha = (\mathscr{S}, C_\alpha) = (\mathscr{S}, \Gamma_\alpha), \tag{22}$$

where Γ_α is

$$\Gamma_\alpha : \mathscr{S} \rightarrow \mathcal{P}(\mathscr{S}).$$

The identity graph given in Figure 3 refers to the intersection with 6 traffic streams presented in Figure 2 together with its identity graph. There are two traffic stream types.

Vehicle traffic streams belong to subset $\mathscr{S}^1 = \{\sigma_1, \sigma_2, ..., \sigma_5\}$ and pedestrian traffic stream belong to type two, i.e. $\mathscr{S}^2 = \{\sigma_6\}$.

If traffic streams of various types pass through an intersection (F>1), the identity graph G_α is a non-connected graph. The number of connected components is equal to or greater than the number of stream types F. Graph G_α is a non-oriented graph with a loop in each node.

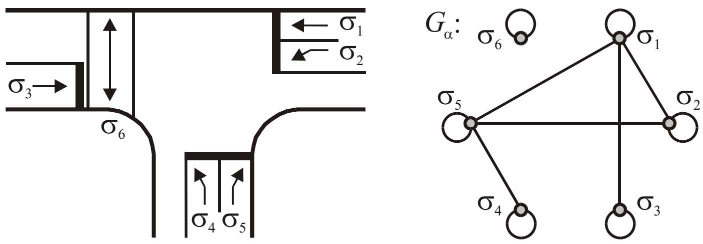

Figure 3: The intersection with 6 traffic streams and its identity graph

Since graphs $G_c = (\mathscr{S}, C)$ and $G_\alpha = (\mathscr{S}, C_\alpha)$ have the same set of nodes, and $C_\alpha \subseteq C$ then, the identity graph G_α is a spanning subgraph of the compatibility graph G_c.

3.2 The Complete Set of Signal Groups

The identity relation C_α given by (14) defines the set of traffic streams pairs that can be controlled by identical signal indications, while the identity graph G_α enables determination of all subsets of set $\mathscr{S}$ that represent signal groups.

A set of nodes of any subgraph of identity graph G_α, such that the subgraph is a complete graph, represents, in fact, a signal group. Since a complete subgraph of a graph represents a clique, a signal group can be also defined in the following way:

A signal group is a clique (in Berge's sense [5]) of the graph of identical signal indications G_α.

Therefore, for traffic control at an intersection, it is necessary to determine a set of signal groups such that each element of set $\mathscr{S}$ belongs to one and only one signal group, i.e. to a clique of graph G_α. Such a set of signal groups is called *the complete set of signal groups*, and it represents a partition of set $\mathscr{S}$.

For one graph of identical signal indications, there exist several complete sets of signal groups. This means that one intersection can be controlled in several ways, based on the choice of the complete set of signal groups. Introducing an appropriate measure for comparison of complete sets of signal groups, the choice of the complete set can be formulated as an optimization problem: Find a complete set of signal groups such that the value of the chosen performance index is optimal. The set of feasible solutions for this problem is the collection of all complete sets of signal groups.

The performance index that is often optimized is the cost of the traffic controller. Having in mind that one control variable is assigned to each signal group and that each control variable is realized by a separate module of control equipment, it is obvious that the equipment cost depends on the number of signal groups in a chosen complete set of signal groups.

4. PROBLEM STATEMENT

In this paper, we propose the method for solving the following problem: Find all **complete sets of signal groups** when the graph of identical indications is given.

The solution of this problem includes the solution of the following problem important for the practice: Find **a complete set of signal groups which contains the minimal number of signal groups.**

The problem of finding a complete set of signal groups is the problem of partitioning a set $\mathscr{S}$.

Since a signal group is a clique of graph G_α , it is necessary to determine all cliques of graph G_α, and to choose a set of cliques such that each element of the set $\mathscr{S}$ belongs to one and only one clique in the chosen set of cliques of graph G_α .

The introduction of a graph G_α', which is the complement of a graph G_α, enables a transformation of the problem: Find all **complete sets of signal groups** if the graph of identical indications is given, as a problem of coloring graph G_α'. It is enabled by the fact that for each clique (signal group) in graph G_α , there is a correspondent stable set in graph G_α'. The stable or the independent set is a set of vertices in a graph where no two vertices are adjacent. It is obvious that vertices of each stable set, i.e. the signal group, can be colored by one color. It means that complete set of signal groups can be obtained by coloring vertices of each stable set of the graph G_α' by the same color. Then, the stable sets are called the color classes. An assignment of the colors to color classes is a vertex coloring of a graph G_α'.

More formally, a vertex coloring with k colors means assigning k colors to the vertices of the graph $G_\alpha' = (\mathscr{S}, C_\alpha')$. It is defined by the next function:

$$c: \mathscr{S} \to \{1,2,\ldots,k\}, \text{ such that } c(u) \neq c(v) \text{ for each edge } (u,v) \in G_\alpha'.$$

The Chromatic number $\chi(G_\alpha')$ of the graph G_α' is the minimal value of k such that there exists a vertex coloring of the graph G_α' with k colors [6].

The maximal number of colors for the vertices coloring of graph G_α' is equal to the number of traffic streams I, i.e.

$$k_{max} = |\mathscr{S}| = I,$$

and the minimal value of k is equal to the chromatic number $\chi(G_\alpha')$, i.e.

$$k_{min} = \chi(G_\alpha').$$

It means that all complete sets of signal groups can be determined by the vertex coloring of the graph $G_\alpha^{'}$, with k colors for all integer values of k between k_{min} and k_{max} i. e. for

$$\chi\left(G_a^{'}\right) \le k \le I .$$

5. METHOD FOR PROBLEM SOLUTION

Method for finding all complete sets of signal groups consists of the following steps:

1. Determination of all color classes of graph $G_\alpha^{'}$;
2. Finding the chromatic number of graph $G_\alpha^{'}$;
3. Coloring of graph $G_\alpha^{'}$ for all integers values of k belonging to the interval $[\chi(G_\alpha^{'}, I]$.

All color classes can be found by using the program CLIQUE [2]. It means that the result of the application of the program CLIQUE is a collection of all color classes:

$Q^{'} = \{Q_1, Q_2,\ldots,Q_l,\ldots,Q_L\}$. It means that l is the element of the index set

$\mathscr{L} = \{1,2,\ldots l,\ldots,L\}$.

One partition of set $\mathscr{S}$ is obtained by k – coloring of graph $G_\alpha^{'}$. But, coloring of graph $G_\alpha^{'}$ by using k colors is not a unique process. It means that there can be more possibilities for coloring the graph $G_\alpha^{'}$ by k colors. The color classes used for coloring one partition of the set $\mathscr{S}$ are elements of the collection $Q^{'}$. The choice of the color classes can be exactly described by the introduction of the selection vector x:

$x=[x_1, x_{2,\ldots,} x_l,\ldots,x_L], \quad x_l \in \{0,1\}$.

The assignment of the separate values to the variable x_l has the next meaning:

$$x_l = \begin{cases} 1, & \text{if the color class } Q_l \text{ is included in the chosen partition of the set } \mathscr{S}, \\ 0, & \text{if it is not} \end{cases}$$

Since k – coloring is a partition of the set $\mathscr{S}$, every element σ_i of the set $\mathscr{S}$ has to be present in only one color class included in that partition.

The problem of determining the chromatic number and the corresponding partition sets of set $\mathscr{S}$, containing the minimal number of elements, has now the following exact formulation:

Find all selection vectors that enable the achievement of the minimal value of the function:

$$P = a^T \cdot x = \sum_{l=1}^{L} x_l , \quad \text{where } a^T = [a_1, a_{2,\ldots,} a_{l,\ldots,} a_L] = [1,1,\ldots,1,\ldots,1],$$

subject to constraints

$Hx = b,$

$x_l \in \{0,1\}, \quad l \in \{1,2,...,L\}$

$H = [h_{il}]_{I \times L}$, where

$$h_{il} = \begin{cases} 1, & \text{if traffic stream } \sigma_i \text{ is an element of the color class } Q_l \\ 0, & \text{if it is not} \end{cases}$$

$$b = [b_1, b_2, ..., b_I]^T = [1,1,...,1]^T .$$

The program COMP [2] can be used, if necessary, besides the partition sets with minimal cardinality, to find all complete sets of signal groups.

The use of the proposed method is presented by the next example.

6. EXAMPLE

For the intersection and its identity graph and the complement of that graph, presented in Figure 4, find the chromatic number $\chi(G_\alpha^{'})$ of graph $G_\alpha^{'}$, i.e. of the complement of the identity graph G_α , and find also a vertex coloring of graph $G_\alpha^{'}$ with the cardinality $\chi(G_\alpha)$.

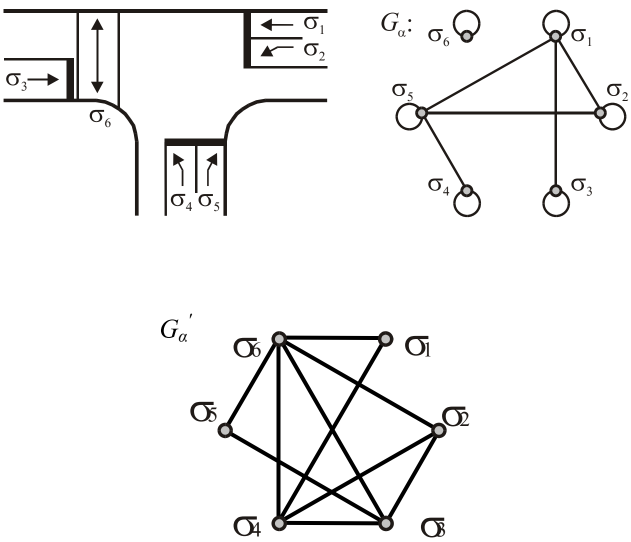

Figure 4: The intersection with 6 traffic streams its identity graph and complement graph of identity graph

All color classes of graph $G_\alpha^{'}$ are obtained by using the program CLIQUE.

$Q^{'} = \{Q_1, Q_2, \ldots, Q_l, \ldots, Q_{12}\} = \{\{\sigma_1\},\{\sigma_2\},\{\sigma_3\},\{\sigma_4\},\{\sigma_5\},\{\sigma_6\},\{\sigma_1,\sigma_2\},\{\sigma_1,\sigma_3\},$

$\{\sigma_1, \sigma_5\},\{\{\sigma_2, \sigma_5\},\{\sigma_4,\sigma_5\},\{\{\sigma_1,\sigma_2, \sigma_5\}\}.$

$$a = [a_1, a_2, ..., a_{12}]^T = [1,1,...,1]^T .$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = [b_1, b_2, ..., b_6]^T = [1,1,...,1]^T .$$

The formulated problem is solved by coloring graph $G_\alpha^{'}$, using the program MINA [1].

Chromatic number of graph $G_\alpha^{'}$ is

$$P_{min} = \chi(G_a^{'}) = 4 .$$

The *optimal* collections of the color classes (complete sets of signal groups containing minimal number of elements) are:

$$D_1^* = \{\{\sigma_3\}, \{\sigma_6\}, \{\sigma_1, \sigma_2\}, \{\sigma_4, \sigma_5\}\}$$

$$D_2^* = \{\{\sigma_4\}, \{\sigma_6\}, \{\sigma_1, \sigma_3\}, \{\sigma_2, \sigma_5\}\}$$

$$D_3^* = \{\{\sigma_2\}, \{\sigma_6\}, \{\sigma_1, \sigma_3\}, \{\sigma_4, \sigma_5\}\}$$

$$D_4^* = \{\{\sigma_3\}, \{\sigma_4\}, \{\sigma_6\}, \{\sigma_1, \sigma_2, \sigma_5\}\}$$

The complete set of signal groups $D_1^* = \{\{\sigma_3\}, \{\sigma_6\}, \{\sigma_1, \sigma_2\}, \{\sigma_4, \sigma_5\}\}$ is presented in Figure 5.

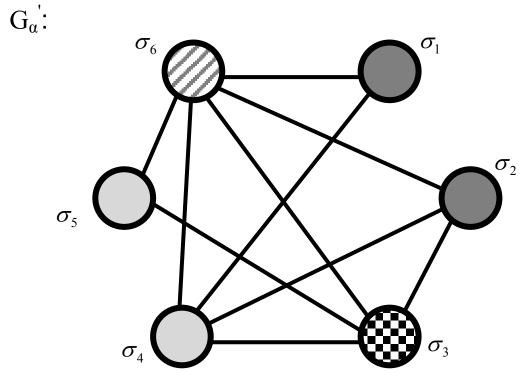

Figure 5: The complement graph of the identity graph colored by the minimal number of colors

7. CONCLUSION

The exact definition of the signal group, based on the relations and the corresponding graphs, defined over the set of traffic streams is introduced.

The graph of identical indications and its complement graph $G_\alpha' = (\mathscr{S}, C_\alpha')$ are used to show that k-coloring of the complement of the graph of identical indications can be used for finding the complete sets of signal groups on an signalized intersection. The minimal number of signal groups in one complete set of signal groups is equal to the chromatic number of that graph. The color classes are equivalent to signal groups. The program CLIQUE [2] is used to find all color classes, i.e. all subsets of the set $\mathscr{S}$ colored by the same color. The program COMP [2] is developed to find all complete sets of signal groups.

The problem of finding all complete sets of signal groups with the minimal cardinality, which is equal to the chromatic number $\chi(G_\alpha')$, is formulated as a linear programming problem where the values of variables belong to set {0,1}. The program MINA [1] is developed to solve this problem.

REFERENCES

[1] Guberinić, S., Šenborn, G., Lazić, B., *Optimal Traffic Control: Urban Intersection,* CRC Press, Boca Raton, 2007.

[2] Guberinić S., Mitrović Minić S., "Signal Group: Definitions and Algorithms*", Yugoslav Journal of Operational Research*, 3 (1993) 219-240.

[3] Forchhammer N., Poulsen L., *Traffic Streams in Relation to Traffic Lights – the Operative Conflict Matrix*, L.M. Ericsson Technical Note, Unpublished, 1968.

[4] Pavel, G., *Planen von Signalanlagen für den Strassenverkehr*, Kirschbaum Verlag, Bonn-Bad Godesberg, 1974.

[5] Berge, C.*, Théorie des graphes et ses application*, Dunod, Paris, 1958.

[6] Cvetković D., Kovačević-Vujčić V., *Kombinatorna optimizacija,* DOPIS, Beograd, 1996.