Yugoslav Journal of Operations Research
24 (2014) Number 2, 299 - 303
DOI: 10.2298/YJOR130202010L

NOTE ON FUZZY SETS

Robert LIN

Department of Information Management Science,
Oriental Institute Technology, Taiwan, R. O. C.
hjlin@mail.oit.edu.tw

Received: February 2013 / Accepted: April 2013

Abstract: We present some improvements for the landmark paper of fuzzy sets for distributive law, convex combination and convex fuzzy sets. Our enhancement will help researcher absorb the original paper of fuzzy sets.

Keywords: Fuzzy sets, convex combination.

MSC: 94D05.

1. INTRODUCTION

Zadeh [11] is the pioneer paper for fuzzy sets that opens a new academic subject on vague objects and uncertain linguistic description. We may demonstrate the tremendous influence of this paper by the following two statistic data. There are 22 scientific journals with "fuzzy" in their title and according to Google Scholar there are more than 26,000 papers that have cited Zadeh [11] in their references. Nevertheless, in this note, we provide three comments for Zadeh [11] to help researchers realize full importance of Zadeh [11].

Fuzzy set theory has been studied a lot and applied to many diverse areas. For examples, Brown [1] considered a note on fuzzy sets to obtain some analytical findings on convex fuzzy sets, star-shaped fuzzy sets, and arcwise connected fuzzy sets. Mizumoto and Tanaka [7] found some properties of fuzzy sets for intuitive fuzzy sets. They applied the extension principle to show that convex fuzzy grades form a commutative semi-ring and normal convex fuzzy grades form a distributive lattice under join and meet. Cheng [2] developed a new approach for ranking fuzzy numbers to calculate the distance to the centroid point with the special characteristics: higher mean and lower spread to incorporate with human intuition. Kumar et al. [3] investigated some operations on intuitive fuzzy sets and then verified some propositions and provided some examples. For medical diagnosis, Kumar et al. [4] presented an application of intuitive fuzzy sets for decision making problems. Li [5] constructed multiattribute decision making models under intuitive fuzzy sets environment to derive a pair of maximum and minimum linear programming problems by combining two linear programming under the same constraints.

Mandal et al. [6] formulated multi-objective economic order quantity model with shortages and demand dependent unit cost under storage space constraint. They represented the cost parameters as triangular shaped fuzzy numbers with different types of left and right branch membership functions. Yadav et al. [10] developed a model to optimize the possibility measure of the fuzzy goal of the objective functions, and the constraints satisfy some pre-defined necessity. Pandey and Kumar [9] improved the model of Bector and Chandra on duality in fuzzy linear programming by using non-linear membership functions. Othman et al. [8] proposed a multicriteria analysis in ranking the quality of teaching using fuzzy rule. The proposed method applied fuzzy sets and approximate reasoning in deciding the ranking of the quality of teaching in several courses.

2. REVIEW OF DISTRIBUTIVE LAW AND OUR IMPROVEMENT

First, we recall the distributive law of Zadeh [11]. For three fuzzy sets, A, B and C, after the union and intersection operations were defined, Zadeh [11] tried to prove the distributive law

$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B) \tag{1}$$

the corresponding relation in terms of membership functions is

$$Max\left[f_C, Min[f_A, f_B] \right] = Min\left[Max[f_C, f_A], Max[f_C, f_B] \right] \tag{2}$$

which can be verified to be an identity by considering the six cases:

$f_A(x) > f_B(x) > f_C(x)$, $f_A(x) > f_C(x) > f_B(x)$, $f_B(x) > f_A(x) > f_C(x)$, $f_B(x) > f_C(x) > f_A(x)$, $f_C(x) > f_A(x) > f_B(x)$ and $f_C(x) > f_B(x) > f_A(x)$.

From the above partition, it was revealed that Zadeh [7] overlooked some cases for example, $f_A(x) = f_B(x)$, $f_C(x) = f_B(x)$ or $f_A(x) = f_C(x)$.

In the following, we will provide a different approach to discuss the distributive law. To simplify the expressions, we assume that $a = f_A(x)$, $b = f_B(x)$ and $c = f_C(x)$.

We know that

$$Max\{a, b\} = \frac{a + b + |a - b|}{2} \tag{3}$$

and

$$Min\{a,b\} = \frac{a+b-|a-b|}{2} \tag{4}$$

such that we can rewrite the left hand side of (2) as

$$\frac{1}{4}\left(2c+a+b-|a-b|+\left|2c-a-b+|a-b|\right|\right) \tag{5}$$

and the right hand side of (2) as

$$\frac{1}{4}\left(a+c+|a-c|+b+c+|b-c|-\left|\left(a+c+|a-c|\right)-\left(b+c+|b-c|\right)\right|\right). \tag{6}$$

Hence, to verify the equality of (2) is equivalent to prove the following

$$\left|2c-a-b+|a-b|\right|+\left|a-b+|a-c|-|b-c|\right| = |a-c|+|b-c|+|a-b|. \tag{7}$$

We can simplify the right hand side of (7) as

$$2Max\{a,b,c\}-2Min\{a,b,c\}. \tag{8}$$

We can derive the following lemmas.

Lemma 1. *If* $c \ge Max\{a,b\}$, *then* $\left|2c-a-b+|a-b|\right| = 2c-2Min\{a,b\}$ *and* $\left|a-b+|a-c|-|b-c|\right| = 0$.

Lemma 2. *If* $Max\{a,b\} > c$ *, then* $\left|2c-a-b+|a-b|\right| = 2Mid\{a,b,c\}-2Min\{a,b,c\}$ *and* $\left|a-b+|a-c|-|b-c|\right| = 2Max\{a,b,c\}-2Mid\{a,b,c\}$.

If we combine our derivations for (8), Lemmas 1 and 2, then we prove the distributive law.

3. REVIEW OF CONVEX COMBINATION AND OUR REVISIONS

Let us recall the convex combination in Zadeh [11]. He assumed that *A*, *B*, and Λ were three arbitrary fuzzy sets. The convex combination of *A*, *B*, and Λ is denoted by $(A,B;\Lambda)$ and is defined by the relation

$$(A,B;\Lambda) = \Lambda A + \Lambda' B \tag{9}$$

where Λ' is the complement of Λ.

Written out in terns of membership functions, (9) reads

$$f_{(A,B;\Lambda)}(x) = f_\Lambda(x)\, f_A(x) + \left[1-f_\Lambda(x)\right] f_B(x), \quad x \in X. \tag{10}$$

Zadeh [11] mentioned that it is of interest to observe that, given any fuzzy set C satisfying $A \cap B \subset C \subset A \cup B$, one can always find a fuzzy set Λ such that $C = (A, B; \Lambda)$. The membership function of this set is given by Zadeh [11] as

$$f_\Lambda(x) = \frac{f_C(x) - f_B(x)}{f_A(x) - f_B(x)}, \quad x \in X. \tag{11}$$

However, we must point out that the above relation only holds when $f_A(x) \neq f_B(x)$

If $f_A(x) = f_B(x)$, then the expression in (11) is not well defined.

In the following, we will provide an improvement for the convex combination in Zadeh [7]. For a given $x \in X$, when $f_A(x) = f_B(x)$, owing to $A \cap B \subset C \subset A \cup B$, we know that $f_A(x) = f_C(x)$ and $f_\Lambda(x)$ can be an arbitrary value with $0 \leq f_\Lambda(x) \leq 1$. Hence, we evaluate that

$$f_{(A,B;\Lambda)}(x) = f_\Lambda(x) f_A(x) + [1 - f_\Lambda(x)] f_B(x) = f_C(x) \tag{12}$$

to provide a revision for his results.

4. REVIEW OF CONVEX FUZZY SETS AND OUR MODIFICATIONS

There are two definitions in Zadeh [11] for convex fuzzy sets. The first definition: if $A = \{(x, f_A(x)) : x \in A\}$ is a fuzzy set, Γ_α is a convex set in the real number for any $\alpha \in [0,1]$, where $\Gamma_\alpha = \{x \mid f_A(x) \geq \alpha\}$. The second definition is

$$f_A(\lambda x_1 + (1-\lambda) x_2) \geq Min\{f_A(x_1), f_A(x_2)\} \tag{13}$$

for all x_1 and x_2 in X and all λ in $[0,1]$.

Zadeh [11] provided a proof to verify that these two definitions are equivalent. Under the second definition, we know that when $x_1 \in \Gamma_\alpha$, then $f_A(x_1) \geq \alpha$. However, in Zadeh [11], he only assumed that $\alpha = f_A(x_1)$ and Γ_α is viewed as the set of all points x_2 which satisfies that $f_A(x_2) \geq f_A(x_1)$.

We must point out that the above argument does not cover all situations, for example, the case of $f_A(x_1) > \alpha$ is overlooked by Zadeh [11]. Hence, in the following, we provide an improvement. We assume that x_3 and $x_4 \in \Gamma_\alpha$ which is defined as $f_A(x_3) \geq \alpha$ and $f_A(x_4) \geq \alpha$. If we assumed that A is a convex fuzzy set under the second definition, then we need to prove that Γ_α is a convex set.

Our goal is to verify that $\lambda x_3 + (1-\lambda) x_4$ is still in Γ_α for $0 \le \lambda \le 1$. We derive that

$$f_A\left(\lambda x_3 + (1-\lambda) x_4\right) \ge Min\left\{ f_A(x_3), f_A(x_4)\right\} \ge \alpha . \tag{14}$$

Therefore, $\lambda x_3 + (1-\lambda) x_4 \in \Gamma_\alpha$ to imply Γ_α is a convex set in the real number.

5. CONCLUSION

We provide three minor revisions for the greatest paper proposed by Zadeh [11] that contains many important ideas about fuzzy sets. However, as there are some small drawbacks in his mathematical derivations, we present revisions for distributive law, convex combination and convex fuzzy sets. Our results will facilitate practitioners to have a more sound mathematical background for fuzzy sets.

REFERENCES

[1] Brown, J.G., "A note on fuzzy sets", *Information and Control*, 18 (1971) 32-39.

[2] Cheng, C.H., "A new approach for ranking fuzzy numbers by distance method", *Fuzzy Sets and Systems*, 95 (1998) 307- 317.

[3] Kumar, S., Biswas, R., and Roy, A.R., "Some operations on intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, 114 (2000) 477-484.

[4] Kumar, S., Biswas, R., and Roy, A.R., "An application of intuitionistic fuzzy sets in medical diagnosis", *Fuzzy Sets and Systems*, 117 (2001) 209-213.

[5] Li, D.F., "Multiattribute decision making models and methods using intuitionistic fuzzy sets", *Journal of Computer and System Sciences*, 70 (2005) 73–85.

[6] Mandal, N.K., "Fuzzy Economic Order Quantity Model with Ranking Fuzzy Number Cost Parameters", *Yugoslav Journal of Operations Research*, 22 (2) (2012) 247-264.

[7] Mizumoto, M., and Tanaka, K., "Some Properties of Fuzzy Sets of Type 2", *Information and Control*, 31 (1976) 312-340.

[8] Othman, M., Ku-Mahamud, K.R., and Abu Bakar. A., "Fuzzy Evaluation Method Using Fuzzy Rule Approach in Multicriteria Analysis", *Yugoslav Journal of Operations Research*, 18 (1) (2008), 95-107.

[9] Pandey, D., and Kumar, S., "Fuzzy Optimization of Primal-Dual Pair Using Piecewise Linear Membership Functions", *Yugoslav Journal of Operations Research*, 22 (2) (2012) 97-106.

[10] Yadav, D., and Singh, S.R., and Kumari, R., "Inventory Model of Deteriorating Items with Two-Warehouse and Stock Dependent Demand Using Genetic Algorithm in Fuzzy Environment", *Yugoslav Journal of Operations Research*, 22 (1) (2012) 51-78.

[11] Zadeh, L.A., "Fuzzy Sets", *Information and Control*, 8 (1965) 338-353.