Yugoslav Journal of Operations Research
25 (2015), Number 2, 291–297
DOI: 10.2298/YJOR130917012N

THE STUDY OF THE SET OF NASH EQUILIBRIUM POINTS FOR ONE TWO-PLAYERS GAME WITH QUADRATIC PAYOFF FUNCTIONS

Mikhail Sergeevich NIKOLSKII
Steklov Mathematical Institute, Russian Academy of Sciences, 119991, Moscow-RUSSIA, Gubkina str., 8.
mni@mi.ras.ru

Aboubacar MOUSSA
Department of Mathematics and Computer Science, Faculty of Sciences, Abdou Moumouni University, BP: 10662 Niamey-NIGER
moussa@mail.ru

Received: September 2013 / Accepted: April 2014

Abstract: In this paper, the general two-players game on the square with quadratic payoff functions is considered. We have studied the problem of determination of the set of Nash equilibrium points, and here we present a constructive graphical method for determination of the required set, which we have developed.

Keywords: Two-players game, Nash equilibrium, quadratic payoff functions.

MSC: 91A05.

1. INTRODUCTION

We consider a general two-players game (see, for example [1]-[7]) on the square with quadratic payoff functions. Here, the problem of determination of the set of Nash equilibrium points is studied, and our newly developed constructive graphical method for determination of the required set is presented. We construct a general example of our game in which the set of Nash equilibrium points contains a non-zero length segment .

Note that there are many articles (see, for example [1]-[7], etc.) devoted to properties of the set of Nash equilibrium. Our paper provides an useful illustrative material for the game theory, granting frequent use of quadratic criteria in this theory.

2. PROBLEM FORMULATION

We consider a two-players game on the square $K = [0,1] \times [0,1]$. The payoff function of the 1st player is

$$f(x,y) = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y, \tag{1}$$

where $(x,y) \in K$, a_i are arbitrary fixed numbers, and $a_1 \neq 0$. The payoff function of the 2nd player is

$$g(x,y) = b_1 y^2 + b_2 xy + b_3 x^2 + b_4 y + b_5 x, \tag{2}$$

where $(x,y) \in K$, b_i are arbitrary fixed numbers, and $b_1 \neq 0$. By selecting $x \in [0,1]$, the first player strives to maximize his payoff $f(x,y)$. By selecting $y \in [0,1]$, the second player strives to maximize his payoff $g(x,y)$. We will study the Nash equilibrium points for the formulated game.

3. SOLUTION OF THE PROBLEM

Definition. The point $(x_0, y_0) \in K$ is called the Nash equilibrium for the given game if these conditions:

$$f(x, y_0) \leqslant f(x_0, y_0) \quad \forall x \in [0,1]\,,$$

$$g(x_0, y) \leqslant g(x_0, y_0) \quad \forall y \in [0,1]$$

are realized.

We denote by $\mathfrak{A}$ the set of points of the Nash equilibrium in our game. In general, the set $\mathfrak{A}$ may be empty.

Using the well-known theorem 7.2.2 of [1], it is easy to show that in our game $\mathfrak{A} \neq \varnothing$ for $a_1 < 0$, $b_1 < 0$. If at least one of the numbers a_1, b_1 is positive, then the problem of non- emptiness of the set $\mathfrak{A}$ is more difficult to solve. Denoted by $\omega_1(y)$ for $y \in \mathbf{R}^1$ is the set of points $x \in [0,1]$, which maximizes the function $f(x,y)$ on $x \in [0,1]$ and by $\omega_2(x)$ for $x \in \mathbf{R}^1$ is denoted the set of points $y \in [0,1]$, which maximizes the function $g(x,y)$ on $y \in [0,1]$ (Here the symbol $\mathbf{R}^1$ denotes the number line). It is easy to observe that the set $\omega_1(y) \neq \varnothing$, and that it is a compact set for any $y \in \mathbf{R}^1$. Similarly, the set $\omega_2(x) \neq \varnothing$, and it is a compact set for any $x \in \mathbf{R}^1$. The symbolic definitions of the sets $\omega_1(y)$, $\omega_2(x)$ are written as:

$$\omega_1(y) = \operatorname*{Arg\,max}_{x \in [0,1]} f(x,y),$$

$$\omega_2(x) = \operatorname*{Arg\,max}_{y \in [0,1]} g(x,y).$$

The set-valued functions $\omega_1(y)$, $\omega_2(x)$ are examples of multi-valued maps. The theory of multi-valued maps is widely used in modern mathematics (see for example [6]). To obtain an analytical description of the required set $\mathfrak{A}$, it is useful to consider the set

$$Gr\omega_1 = \{(x, y) \in K : x \in \omega_1(y), y \in [0, 1]\} \tag{3}$$

and the set

$$Gr\omega_2 = \{(x, y) \in K : x \in [0, 1], y \in \omega_2(x)\}. \tag{4}$$

The sets $Gr\omega_1$, $Gr\omega_2$ are called graphs of multi-valued maps ω_1, ω_2, respectively. From the theory of multi-valued maps (see, for example [6]), it follows that the sets $Gr\omega_1$, $Gr\omega_2$ are compact sets. The well-known fact of game theory is the equality (see, for example [1]-[4], [6])

$$\mathfrak{A} = Gr\omega_1 \bigcap Gr\omega_2. \tag{5}$$

The formula (5) is of great interest for our game. We show that the multi-valued maps ω_1, ω_2 have a fairly simple structure that allows constructive (graphically) use of the formula (5) for calculating $\mathfrak{A}$. To build the multi-valued mapping $\omega_1(y)$, it is useful to consider the functions

$$h_1(\xi, \eta) = -\xi^2 + c_2\xi\eta + c_3\eta^2 + c_4\xi + c_5\eta,$$

$$h_2(\xi, \eta) = \xi^2 + d_2\xi\eta + d_3\eta^2 + d_4\xi + d_5\eta,$$

where c_i, d_j are arbitrary fixed numbers. Let us maximize the function $h_1(\xi, \eta)$ on $\xi \in [0, 1]$ for $\eta \in \mathbf{R}^1$. Extracting out the perfect square, we obtain the following representation:

$$h_1(\xi, \eta) = -\left(\xi - \frac{c_2\eta + c_4}{2}\right)^2 + \frac{(c_2\eta + c_4)^2}{4} + c_3\eta^2 + c_5\eta.$$

Hence, for the function

$$\omega_3(\eta) = \operatorname*{Arg\,max}_{\xi \in [0,1]} h_1(\xi, \eta),$$

we obtain the following formula:

$$\omega_3(\eta) = k\left(\frac{c_2\eta + c_4}{2}\right),$$

where the function $k(\zeta) = 0$ for $\zeta < 0$, $k(\zeta) = \zeta$ for $\zeta \in [0, 1]$ and $k(\zeta) = 1$ for $\zeta > 1$. Note that the function $k(\zeta)$ is a one-valued scalar function. Similarly, arguing for the problem of maximizing of the function $h_2(\xi, \eta)$ on $\xi \in [0, 1]$, we obtain for the set-valued function

$$\omega_4(\eta) = \operatorname*{Arg\,max}_{\xi \in [0,1]} h_2(\xi, \eta),$$

the following formula:

$$\omega_4(\eta) = l\left(\frac{d_2\eta + d_4}{2}\right),$$

where the set-valued function $l(\zeta) = 0$ for $\zeta < -\frac{1}{2}$, $l(\zeta) = 1$ for $\zeta > -\frac{1}{2}$ and $l\left(-\frac{1}{2}\right)$ is the two-points set $\{0, 1\}$. Note that the function $l(\zeta)$ is one-valued everywhere except at $-\frac{1}{2}$ and at $-\frac{1}{2}$ it is two-valued. To construct $\omega_1(y)$, we represent the function $f(x, y)$ in the form

$$f(x, y) = |a_1|\tilde{f}(x, y),$$

where

$$\tilde{f}(x, y) = \operatorname{sign} a_1.(x^2 + \tilde{a}_2xy + \tilde{a}_3y^2 + \tilde{a}_4x + \tilde{a}_5y),$$

here $\tilde{a}_i = \frac{a_i}{a_1}$, $i = 2, ..., 5$. It is easy to see that

$$\omega_1(y) = \operatorname*{Arg\,max}_{x\in[0,1]} \tilde{f}(x, y).$$

If the function f is of the first type, i.e. for $\operatorname{sign} a_1 < 0$, then by the above

$$\omega_1(y) = k\left(\frac{-\tilde{a}_2y - \tilde{a}_4}{2}\right). \tag{6}$$

If the function f is of the second type, i.e. for $\operatorname{sign} a_1 > 0$, then

$$\omega_1(y) = l\left(\frac{\tilde{a}_2y + \tilde{a}_4}{2}\right). \tag{7}$$

Similarly arguing for the function g of the first type, i.e. for $\operatorname{sign} b_1 < 0$, we obtain the following formula

$$\omega_2(x) = k\left(\frac{-\tilde{b}_2y - \tilde{b}_4}{2}\right), \tag{8}$$

and for the function g of the second type, i.e. for $\operatorname{sign} b_1 > 0$, we obtain the following formula

$$\omega_2(x) = l\left(\frac{\tilde{b}_2y + \tilde{b}_4}{2}\right), \tag{9}$$

where $\tilde{b}_2 = \frac{b_2}{b_1}$, $\tilde{b}_4 = \frac{b_4}{b_1}$.

After the construction of the set-valued maps $\omega_1(y)$, $\omega_2(x)$,we can use the formulas (3)- (5) to calculate the required set $\mathfrak{A}$. Note that in the case of one-point set $\omega_1(y)$ for $y \in [0, 1]$ and the set $\omega_2(x)$ for $x \in [0, 1]$, both of these functions are continuous, respectively, for $y \in [0, 1]$ and for $x \in [0, 1]$. In this case, on K we can consider the map

$$m(x, y) = (\omega_1(y), \omega_2(x)).$$

It is obvious that $m(x,y)$ transform the square K in itself. Applying the well-known Brouwer fixed point theorem (see for example [6]), we can affirm that the continuous mapping $m(x,y)$ has at least one fixed point of $(x_0, y_0) \in K$ which satisfies the relations

$$x_0 = \omega_1(y_0), \quad y_0 = \omega_2(x_0).$$

From here, it easily follows that $\mathfrak{A}$ is not empty, and the point $(x_0, y_0) \in \mathfrak{A}$. From the above, it follows that only if at least one of the inequalities $a_1 > 0$, $b_1 > 0$ is realized, the emptiness of the set $\mathfrak{A}$ is possible, and in the case $a_1 > 0$, the relation:

$$-\frac{1}{2} \in \mathrm{co}\left\{\frac{\tilde{a}_4}{2}, \frac{\tilde{a}_2 + \tilde{a}_4}{2}\right\},$$

should be realized, i.e.

$$-1 \in \mathrm{co}\left\{\tilde{a}_4, \tilde{a}_2 + \tilde{a}_4\right\},$$

and in the case $b_1 > 0$, the relation

$$-1 \in \mathrm{co}\left\{\tilde{b}_4, \tilde{b}_2 + \tilde{b}_4\right\},$$

should be realized. Here co denotes convexification operation of the set. Now we take up the construction of a special case of our game, for which $\mathfrak{A}$ contains some segment of non-zero length. Fix in (1), (2) the numbers: $a_1 = -1$, $b_1 = -1$. In connection with formula (6), we consider for $y \in [0,1]$ the function

$$h(y) = \frac{1}{2}\left(a_2 y + a_4\right), \tag{10}$$

where a_2 is a non-zero number, and the conditions

$$a_4 \in [0,2], \quad a_2 + a_4 \in [0,2] \tag{11}$$

are realized. When the relations (11) are realized for $y \in [0,1]$, we have

$$h(y) \in [0,1].$$

According to the formula (6), and the definition of the function $k(\zeta)$ from (10), (11), it follows that for $y \in [0,1]$

$$\omega_1(y) = \frac{1}{2}\left(a_2 y + a_4\right).$$

Now, we will move to the construction of the constants b_2, b_4 (for $b_1 = -1$). Consider the equation

$$x = \frac{1}{2}\left(a_2 y + a_4\right).$$

From it, we get the equation

$$y = \frac{2x - a_4}{a_2}. \tag{12}$$

Assuming (see (8), (12))

$$\frac{b_2 x + b_4}{2} \equiv \frac{2x - a_4}{a_2},$$

we obtain the relations

$$b_2 = \frac{4}{a_2}, \quad b_4 = -\frac{2a_4}{a_2}.$$

The constants a_3, a_5, b_3, b_5 can be chosen arbitrary. It is easy to see that for selected $a_1, a_2, a_4, b_1, b_2, b_4$, the set $\mathfrak{A}$ belongs to a non-zero length of the segment on the plane of the variables (x, y) connecting the points with the coordinates $\left(\frac{a_4}{2}, 0\right)$, $\left(\frac{a_2 + a_4}{2}, 1\right)$.

When $\mathfrak{A}$ is not empty, the estimate from above of the number of its points is of great interest. Here we can use the definition of the functions $k(\zeta)$, $l(\zeta)$ for $\zeta \in \mathbf{R}^1$, and the formula (6)-(8). Note that for $a_1 < 0$, the graph of the function $x = \omega_1(y)$ (see (6)) where $y \in \mathbf{R}^1$ on the plane of variables (x, y) in the general case consists of 3 pieces of lines, situated respectively on three lines

$$x = 0; \quad x = 1; \quad x = -\frac{1}{2}(\tilde{a}_2 + \tilde{a}_4). \tag{13}$$

For $a_1 > 0$, the graph of $\omega_1(y)$ (see (7)) is generally composed of two linear pieces respectively situated on two lines:

$$x = 0, \quad x = 1. \tag{14}$$

For the graph of $y = \omega_2(x)$ (see (8), (9)), in a similar way it appears for $b_1 < 0$

$$y = 0; \quad y = 1; \quad y = -\frac{1}{2}\left(\tilde{b}_2 + \tilde{b}_4\right), \tag{15}$$

and for $b_1 > 0$, the lines are:

$$y = 0; \quad y = 1. \tag{16}$$

Assume that $a_1 < 0$, $b_1 < 0$. In this case, three lines from (13) and three lines from (15) have some connections. If $\tilde{a}_2 \neq 0$, $\tilde{b}_2 \neq 0$ and $\frac{\tilde{a}_2}{2} \neq \frac{2}{\tilde{b}_2}$, then each of the 3 lines of (13) has on each of the line of (15) only one point of intersection, and so here it follows from the formula (5) a rough estimate of the number of points $\mathfrak{A}$ above by the number 9. This estimation is rough, as the graphs $x = \omega_1(y)$ and $y = \omega_2(x)$ belong to the square K, and this was not taken into account when we counted up the number of intersections of corresponding lines.

Also consider the cases:

1) $a_1 < 0, \quad b_1 > 0$; 2) $a_1 > 0, \quad b_1 > 0$; 3) $a_1 > 0, \quad b_1 < 0$.

By virtue of what has been said above, for $\mathfrak{A} \neq \emptyset$: in the first case the number of points in $\mathfrak{A}$ is not greater than two, and (see (6), (9))

$$\mathfrak{A} \subset \{(\omega_1(0), 0), (\omega_1(1), 1)\};$$

in the second case, the number of points in $\mathfrak{A}$ is not greater than four, and (see (7), (9))

$$\mathfrak{A} \subset \{(0,0),(0,1),(1,0),(1,1)\};$$

in the third case, the number of points in $\mathfrak{A}$ is not greater than two, and (see (7),(8))

$$\mathfrak{A} \subset \{0,\omega_2(0)),(1,\omega_2(1))\}.$$

In conclusion, let us clarify the conditions under which we can guarantee that the set $\mathfrak{A}$ is a one-point set. To do this, we apply theorems 2, 6 from [7], fixing the constants $r_1 = 1$, $r_2 = 1$ from this Theorems. In order to guarantee that $\mathfrak{A}$ is a singleton, in accordance with the prescription of [7], it is necessary to calculate for $a_1 < 0$, $b_1 < 0$ some symmetric (2x2)-matrix $G(z,r) + G'(z,r)$, where $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the prime denotes the transpose of the matrix, and to guarantee its negative definiteness. Simple calculation is carried out in our case, and we receive the requirement of a negative definiteness of symmetric matrix $\begin{pmatrix} 4a_1 & a_2+b_2 \\ a_2+b_2 & 4b_1 \end{pmatrix}$. With the help of Sylvester criterion (in addition to the inequalities $a_1 < 0$, $b_1 < 0$), we obtain inequality

$$16a_1b_1 > (a_2+b_2)^2.$$

Acknowledgements. This work is supported by the Russian Foundation of Basic Research (projects 11-01-12112-ofi-m-2011, 12-01-00175-a, 12-01-00506, 13-01-00685, 13-01-12446 fi-m2).

Authors are very grateful to Prof. V.I. Zhukovsky and Prof. V.V. Morozov for their remarks and discussions about this paper.

REFERENCES

[1] Parhasarathi, T., and Raghavan, T., Some questions of the theory of two-person games, M.: Mir, 1974. (in Russian)

[2] Petrosyan, L.A., Zenkevich, N.A., and Semina, E.A., Game theory, M.: High School, 1998. (in Russian)

[3] Vasin, A.A., and Morozov, V.V., Introduction to game theory with applications to economics, M.: 2003. (in Russian)

[4] Maschenko, S.O., "The concept of Nash equilibrium and its development (review)", *Journal of Comput. and Appl. Mathematics*, 1(107) (2012) 40-65. (in Russian)

[5] Zhukovsky, V.I., Introduction to differential games with uncertainty. Nash equilibrium, M., KRASAND, 2010. (in Russian)

[6] Aubin, J.P., Nonlinear analysis and its economic application, M.: Mir, 1988. (in Russian)

[7] Rosen, J. B., "Existence and uniqueness of equilibrium points for concave N-person games", *Econometrica*, 33(3) (1965) 520-534.