Yugoslav Journal of Operations Research
26 (2016) Number 1, 103-120
DOI: 10.2298/YJOR140101021S

MANUFACTURER-SUPPLIER COOPERATIVE INVENTORY MODEL FOR DETERIORATING ITEM WITH TRAPEZOIDAL TYPE DEMAND

Narayan SINGH
Department of Mathematics D.N. (P.G) College, Meerut (U.P), India
narayansingh1980@gmail.com

Bindu VAISH
Department of Mathematics D.N. (P.G) College, Meerut (U.P), India
bindu58@rediffmail.com

S. R. SINGH
Department of Mathematics D.N. (P.G) College, Meerut (U.P), India
shivrajpundir@gmail.com

Received: January 2014 / Accepted: June 2014

Abstract: This paper investigates a Supply Chain System for deteriorating items in which a supplier supplies a manufacturer with raw material, and the manufacturer produces the finished goods. Demand rate is assumed to be time-sensitive in nature (Trapezoidal type), which allows three-phase variation in demand, and production rate is demand dependent. Our adoption of trapezoidal type demand reflects a real market demand for newly launched product. We show that the total cost function is convex. With the convexity, a simple solution algorithm is presented to determine the optimal order quantity and optimal cycle time of the total cost function. Numerical examples are given and the results are discussed.

Keywords: Supply chain, Trapezoidal type demand, Deterioration.

MSC: 90B05.

1. INTRODUCTION

There has been a growing interest in supply chain management in recent years. The supply chain, which is also referred to as the logistic network, consists of supplier distribution centres and retailer outlets, as well as raw material, work in process inventory and finished goods that flow between the facilities. Quite a lot of researchers and enterprises have shown a growing interest for efficient supply chain management. This is due to the rising cost of manufacturing, transportation, the globalization of market economies and the customer demand for diverse products of short life cycles, which are all factors that increase competition among companies.

For reducing the costs, a common strategy is made and through its coordination, the number of deliveries is derived to achieve a minimum overall integrated cost. Clark and Scarf (1960) were the first authors to consider the multi echelon supply chain in inventory research with the assumption of constant demand rate. In the growth and/or end stage life cycle, demand rate may well be approximated by a linear function. Resh et al. (1976) and Donaldson (1977) were the first who studied a model with linearly time varying demand. In the most of the papers, two types of time varying demand rate have been considered: (i) Linear positive/ negative trend in demand rate (ii) Exponentially increasing/decreasing demand rate (cannot increase continuously over time). Hill (1995) proposed an inventory model with increasing demand followed by a constant demand. However, it is observed that the demand rate of a new brand of consumer goods increases at the beginning of the season up to a certain moment (say, μ), and then remains to be constant for the rest of the time. For example, the demand rate is increasing during the growth stage and then the market grows into a stable stage that the demand becomes constant until the end of the inventory cycle. The term ‘‘ramp type’’ is used to represent such a demand pattern. Therefore, a ramp type demand rate function has two different time segments. In its first segment, the demand is an increasing function of time. But the demand remains constant in its second time segment. It is obvious that any ramp type demand function has at least one break point μ between two time segments at which it is not differentiable. This non-differentiable break point μ makes the analysis of the problem more complicated. This has urged researchers to study inventory models with ramp type demand patterns. Mandal and Pal (1998) extended the inventory model with ramp type demand for deterioration items and allowing shortage. Wu and Ouyang (2000) extended the inventory model to include two different replenishment policies: (a) models starting with no shortage and (b) models starting with shortage. Deng, Lin and Chu (2007) pointed out some questionable results of Mandal and Pal (1998) and Wu and Ouyang (2000) and then resolved the similar problem by offering a rigorous and efficient method to derive the optimal solution. Wu (2001) further investigated the inventory model with ramp type demand rate such that the deterioration is followed by Weibull distribution. Giri, Jalan and Chaudhari (2003) extended the ramp type demand inventory model with a more generalized Weibull deterioration distribution. Various types of order level inventory models for deteriorating items with time dependent demand were discussed by several authors as Manna and Chaudhari (2006), Panda, Senapati and Basu (2008). After that, several authors have discussed the time dependent demand in EOQ/EPQ inventory models as well as in multi-echelon supply chain models for inventory like Goyal and Gunasekaran (1995), they observed an integrated production–

inventory marketing model to determine economic production quantity and economic order quantity for raw materials in a multi-echelon production system.

Zhou et al. (2008) addressed a two echelon supply chain coordination model with one manufacturer and one retailer where the demand for the product at the retailer is dependent on the on-hand inventory. Skouri et al. (2009) developed an inventory model with general ramp type demand rate, Weibull deterioration rate and partial backlogging of unsatisfied demand. They discussed two cases in their model. First is starting with no shortage, and the second is starting with shortage. Singh and Singh (2010) discussed supply chain model with stochastic lead time under imprecise partially backlogging and fuzzy ramp-type demand for expiring items. He et al. (2010) developed a two echelon supply chain inventory model of deteriorating items for manufacturers selling goods to multiple markets with different selling seasons. Singh et al. (2010) discussed a time sensitive demand, Pareto distribution for deterioration and backlogging under trade credit policy. Skouri et al. (2011) analysed a supply chain models for deteriorating products with ramp type demand rate under permissible delay in payments. Recently, Taleizadeh et al. (2012) investigated an inventory model for a multi-product, multi-chance constraint, multi-buyer and single-vendor, considering uniform distribution demand and lot size dependent lead time partial backlogging. Singh et al (2012) discussed shortage in an economic production lot-size model with rework and flexibility. Sarkar (2012) extended an EOQ model for time-varying demand, and deteriorating items with discounts on purchasing costs under the environment of delay-in-payments. Goyal et al. (2013) discussed a production policy for amelioration/deteriorating items with ramp type demand.

In the following, we extend Hill's ramp type demand rate to Trapezoidal type demand rate. This type of demand pattern is generally seen in the case of any fad or seasonal goods coming to market. The demand rate for such items increases with time up to a certain moment then ultimately stabilizes and becomes constant, finally it approximately decreases to a constant or zero, and then the next replenishment cycle begins. This type of demand rate is quite realistic and a useful inventory replenishment policy. In this paper, we propose a two echelon inventory model with trapezoidal type production rate and demand rate for both supplier and manufacturer, and a constant rate for deterioration is also considered. Its study requires exploring the feasible ordering relations between the times parameters appeared, which leads to multiple models. For each model, the optimal replenishment policy is determined. The necessary and the sufficient conditions of the existence and uniqueness of the optimal solutions are also provided. An easy-to-use algorithm is proposed to find the optimal replenishment/production policy and the optimal order quantity. Numerical examples are presented to demonstrate the developed model and the solution procedure. The paper is organized as follows: The notation and assumptions used are given in Section 2. In Section 3 and 4, models are formulated for the supplier of raw material and the manufacturer, and Sections 5 is devoted to the formulation of the integrated supply chain models and their derivation of the optimal production policy. Numerical examples highlighting the results obtained are given in Section 6. The paper closes with concluding remarks in section 7.

2. ASSUMPTIONS AND NOTATIONS

The following assumptions and notations are considered to develop the model.

2.1. Assumptions

1. A Single supplier, a Single manufacturer and a single item are considered.
2. Replenishment rate is infinite, thus, replenishment is instantaneous.
3. Deterioration rate is constant, and deteriorated items are not repaired or replaced during a given cycle.
4. The planning horizon is finite.
5. Shortages are not allowed.
6. Lead time is assumed to be negligible.
7. Production rate $P(t)$ is demand dependent and demand rate $d(t)$ is a Trapezoidal type function of time given by

   $d(t) = \begin{cases} f(t) & 0 \le t \le \mu \\ d_0 & \mu \le t \le \delta \\ g(t) & \delta \le t \le T \end{cases}$ where $f(t)$ is a positive continuous increasing function of $t \in [0, \mu]$ and $g(t)$ is a continuous decreasing function of time $t \in [\delta, T]$ and $P = k\, d(t)$ where $k > 1,\ \delta > \mu$.

2.2. Notations

T	the constant cycle time
θ_1	deterioration rate for raw material
θ_2	deterioration rate for manufacture's finished goods
C_{1w}	supplier's ordering cost per order cycle
C_{1m}	manufacturer's ordering and set-up cost per order cycle
C_{2w}	raw material per unit holding cost per unit time
C_{2m}	manufacturer's finished goods per unit holding cost per unit time
C_{3w}	the cost incurred from the deterioration of one unit for raw material
C_{3m}	the cost incurred from the deterioration of one unit for manufacturer
$I_w(t)$	raw materials inventory level at any time t, $0 \le t \le T_1$
$I_m(t)$	manufacturer's finished goods inventory level at anytime t $0 \le t \le T$
TC_w	total cost per unit time for raw material
TC_m	total cost per unit time for manufacturer
TC	total system cost per unit time

3. SUPPLIER'S RAW MATERIAL INVENTORY SYSTEM

The supplier's raw materials inventory level $I_w(t), 0 \leq t \leq T_1$ from Fig. 2 (a), Fig. 2 (b) and Fig. 2(c) at any time t can be represented by the following differential equation:

$$\frac{dI_w(t)}{dt} = -P(t) - \theta_1 I_w(t) \quad 0 \leq t \leq T_1 \tag{1}$$

with boundary condition $I_w(T_1) = 0$

There are three possible relations between parameters $\mu, \ \delta \ and \ T_1$:

(I) $0 \leq \mu \leq \delta \leq T_1$ (II) $\mu \leq T_1 \leq \delta$ and (III) $T_1 \leq \ \mu \leq \delta$

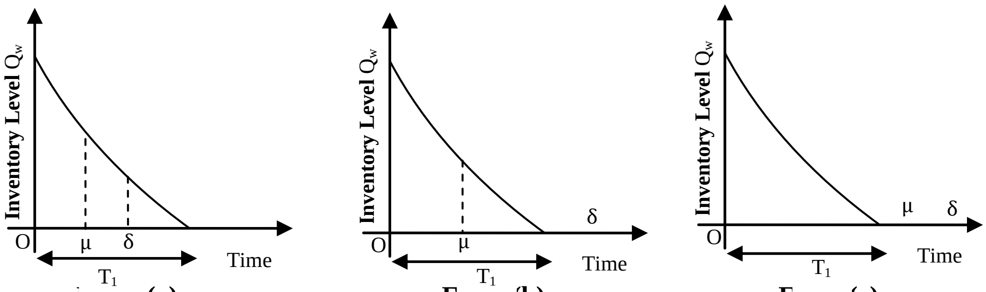


Fig. 2 (a) Fig. 2 (b) Fig. 2 (c)

Case I when $0 \leq \mu \leq \boldsymbol{\delta} \leq \boldsymbol{T_1}$

In this case, eq. (1) reduces to the following three:

$$\frac{dI_{w1}(t)}{dt} = -kf(t) - \theta_1 I_{w1}(t) \quad 0 \leq t \leq \mu \tag{2}$$

$$\frac{dI_{w2}(t)}{dt} = -kd_0 - \theta_1 I_{w2}(t) \quad \mu \leq t \leq \delta \tag{3}$$

$$\frac{dI_{w3}(t)}{dt} = -kg(t) - \theta_1 I_{w3}(t) \quad \delta \leq t \leq T_1 \tag{4}$$

with boundary conditions $I_{w1}(\mu_-) = I_{w2}(\mu_+)$, $I_{w2}(\delta_-) = I_{w3}(\delta_+)$ and $I_{w3}(T_1) = 0$

The solution of eqs. (2) - (4) are

$$I_{w1}(t) = ke^{-\theta_1 t}\left\{\int_t^{\mu} e^{\theta_1 x} f(x)dx + d_0\int_{\mu}^{\delta} e^{\theta_1 x}dx + \int_{\delta}^{T_1} e^{\theta_1 x} g(x)dx\right\} \quad 0 \leq t \leq \mu \tag{5}$$

$$I_{w2}(t) = ke^{-\theta_1 t}\left\{d_0\int_t^{\delta} e^{\theta_1 x}dx + \int_{\delta}^{T_1} e^{\theta_1 x} g(x)dx\right\} \quad \mu \leq t \leq \delta \tag{6}$$

$$I_{w3}(t) = ke^{-\theta_1 t}\int_t^{T_1} e^{\theta_1 x} g(x)dx \quad \delta \leq t \leq T_1 \tag{7}$$

The total amount of deteriorated items during $[0, T_1]$ is

$$D_{w1} = I_{w1}(0) - \int_0^{T_1} P(\mathrm{t})dt$$

$$D_{w1} = k\int_0^{\mu}\left(e^{\theta_1 t} - 1\right) f(t)dt + kd_0\int_{\mu}^{\delta}\left(e^{\theta_1 t} - 1\right)dt + k\int_{\delta}^{T_1}\left(e^{\theta_1 t} - 1\right) g(t)dt \tag{8}$$

The total inventory carried during the interval $[0, T_1]$ is

$$H_{w1} = \int_0^{T_1} I_w(t)dt = \int_0^{\mu} I_{w1}(t)dt + \int_{\mu}^{\delta} I_{w2}(t)dt + \int_{\delta}^{T_1} I_{w3}(t)dt \tag{9}$$

Then, the average total cost of raw material per unit time under the condition $0 \le \mu \le \delta \le T_1$ can be given by

$$TC_{w1}(T_1) = \frac{1}{T}\left(c_{w1} + c_{w2}H_{w1} + c_{w3}D_{w1}\right) \tag{10}$$

Case II when $0 \le \boldsymbol{\mu} \le \boldsymbol{T_1} \le \boldsymbol{\delta}$

In this case, eq. (1) reduces to the following two:

$$\frac{dI_{w1}(t)}{dt} = -kf(t) - \theta_1 I_{w1}(t) \qquad 0 \le t \le \mu \tag{11}$$

$$\frac{dI_{w2}(t)}{dt} = -kd_0 - \theta_1 I_{w2}(t) \qquad \mu \le t \le T_1 \tag{12}$$

with boundary conditions $I_{w1}(\mu_-) = I_{w2}(\mu_+)$ and $I_{w2}(T_1) = 0$, their solutions are:

$$I_{w1}(t) = ke^{-\theta_1 t}\left\{\int_t^{\mu} e^{\theta_1 x} f(x)dx + d_0\int_{\mu}^{T_1} e^{\theta_1 x}dx\right\} \qquad 0 \le t \le \mu \tag{13}$$

$$I_{w2}(t) = kd_0 e^{-\theta_1 t}\int_t^{T_1} e^{\theta_1 x}dx \qquad \mu \le t \le T_1 \tag{14}$$

The total amount of deteriorated items during $[0, T_1]$ is

$$D_{w2} = I_{w1}(0) - \int_0^{T_1} P(\mathrm{t})dt = k\int_0^{\mu}\left(e^{\theta_1 t} - 1\right) f(t)dt + kd_0\int_{\mu}^{T_1}\left(e^{\theta_1 t} - 1\right)dt \tag{15}$$

The total inventory carried during the interval $[0, T_1]$ is

$$H_{w2} = \int_0^{T_1} I_w(t)dt = \int_0^{\mu} I_{w1}(t)dt + \int_{\mu}^{T_1} I_{w2}(t)dt \tag{16}$$

Then, the average total cost of raw material per unit time under the condition $0 \le \mu \le T_1 \le \delta$ can be given by

$$TC_{w2}\left(T_1\right)=\frac{1}{T}\left(c_{w1}+c_{w2}H_{w2}+c_{w3}D_{w2}\right) \tag{17}$$

Case III when $\mathbf{0 \le T_1 \le \mu \le \delta}$

In this case, eq. (1) becomes:

$$\frac{dI_{w1}(t)}{dt}=-kf(t)-\theta_1 I_{w1}(t) \qquad 0\le t\le T_1 \tag{18}$$

with boundary condition $I_{w1}\left(T_1\right)=0$, the solution is

$$I_{w1}(t)=ke^{-\theta_1 t}\int_t^{T_1} e^{\theta_1 x} f(x)dx \qquad 0\le t\le T_1 \tag{19}$$

The total amount of deteriorated items during $[0, T_1]$ is

$$D_{w3}=I_{w1}(0)-\int_0^{T_1} P(\mathrm{t})dt=k\int_0^{T_1}\left(e^{\theta_1 t}-1\right)f(t)dt \tag{20}$$

The total inventory carried during the interval $[0, T_1]$ is

$$H_{w3}=\int_0^{T_1} I_w(t)dt=\int_0^{T_1} I_{w1}(t)dt \tag{21}$$

Then, the average total cost of raw material per unit time under the condition $0 \le T_1 \le \mu \le \delta$ can be given by

$$TC_{w3}\left(T_1\right)=\frac{1}{T}\left(c_{w1}+c_{w2}H_{w3}+c_{w3}D_{w3}\right) \tag{22}$$

4. MANUFACTURER'S FINISHED GOODS INVENTORY SYSTEM

The manufacture's inventory system is shown in Fig. 3(a), Fig.3 (b) and Fig. 3 (c). During time period T_1 there is an inventory build up, and hence deterioration becomes effective. At the $t = T_1$ the production stops and the inventory level increases to its maximum MI_m. There is no production during time period$(T - T_1)$ and inventory level decreases due to demand and deterioration and becomes zero at $t = T$.

The manufacturer's finished goods inventory level $I_m(t), 0 \le t \le T$ from Fig. 3 (a), Fig. 3 (b) and Fig. 3 (c) at any time t can be represented by the following differential equation;

$$\frac{dI_m(t)}{dt}=P(t)-d(t)-\theta_2 I_m(t) \qquad 0\le t\le T_1 \tag{23}$$

with boundary condition $I_m(0) = 0$ and

$$\frac{dI_m(t)}{dt}=-d(t)-\theta_2 I_m(t) \qquad T_1\le t\le T \tag{24}$$

with boundary condition $I_m(T) = 0$

There are three possible relations between parameters $\mu,\ \delta,\ T_1 \,and\, T$:
(1) $\mu \le \delta \le T_1 \le T$ (2) $0 \le \mu \le T_1 \le \delta \le T$ and (3) $0 \le T_1 \le \mu \le \delta \le T$

$\boldsymbol{Case\ 1\ when}\ 0 \le \mu \le \boldsymbol{\delta} \le \boldsymbol{T_1} \le T$

In this case, eq. (23) and (24) reduce to the following four:

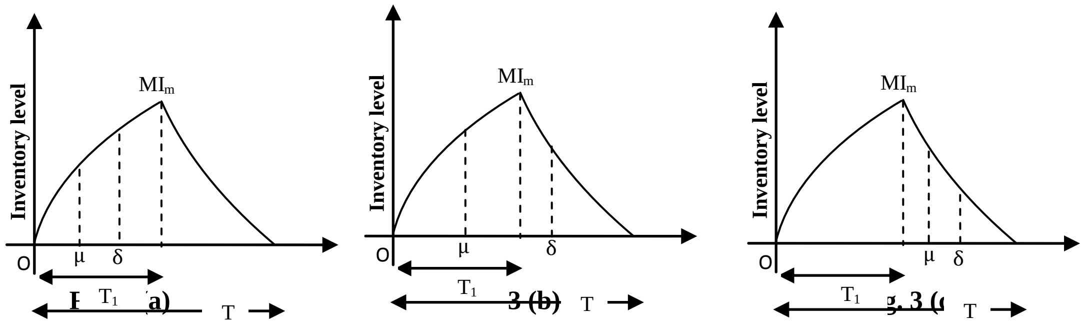
$$\frac{dI_{m1}(t)}{dt} = (k-1)\,f(t) - \theta_2 I_{m1}(t) \qquad 0 \le t \le \mu \tag{25}$$

$$\frac{dI_{m2}(t)}{dt} = (k-1)\,d_0 - \theta_2 I_{m2}(t) \qquad \mu \le t \le \delta \tag{26}$$

$$\frac{dI_{m3}(t)}{dt} = (k-1)\,g(t) - \theta_2 I_{m3}(t) \qquad \delta \le t \le T_1 \tag{27}$$

$$\frac{dI_{m4}(t)}{dt} = -g(t) - \theta_2 I_{m4}(t) \qquad T_1 \le t \le T \tag{28}$$

with boundary conditions $I_{m1}(0) = 0$, $I_{m1}(\mu_-) = I_{m2}(\mu_+)$, $I_{m2}(\delta_-) = I_{m3}(\delta_+)$ and $I_{m4}(T) = 0$



The solution of eqs. (25) - (28) are

$$I_{m1}(t) = (k-1)e^{-\theta_2 t}\int_0^t e^{\theta_2 x} f(x)\,dx \qquad 0 \le t \le \mu \tag{29}$$

$$I_{m2}(t) = (k-1)e^{-\theta_2 t}\left\{\int_0^\mu e^{\theta_2 x} f(x)\,dx - d_0\int_\mu^t e^{\theta_2 x}dx\right\} \qquad \mu \le t \le \delta \tag{30}$$

$$I_{m3}(t) = (k-1)e^{-\theta_2 t}\left\{\int_0^\mu e^{\theta_2 x} f(x)\,dx - d_0\int_\mu^\delta e^{\theta_2 x}dx - \int_\delta^t e^{\theta_2 x} g(x)\,dx\right\} \qquad \delta \le t \le T_1 \tag{31}$$

$$I_{m4}(t)=e^{-\theta_2 t}\int_t^T e^{\theta_2 x} g(x)dx \quad T_1 \le t \le T \tag{32}$$

The cumulative inventory carried in the interval $[0.T]$ is

$$H_{m1}=\int_0^T I_m(t)dt=\int_0^{\mu} I_{m1}(t)dt+\int_{\mu}^{\delta} I_{m2}(t)dt+\int_{\delta}^{T_1} I_{m3}(t)dt+\int_{T_1}^{T} I_{m4}(t)dt \tag{33}$$

The total amount of deteriorated items during $[0.T]$ is

$$D_{m1}=\int_0^{T_1} P(\mathrm{t})dt-\int_0^T d(t)dt$$

$$D_{m1}=(k-1)\left\{\int_0^{\mu} f(t)dt+d_0(\delta-\mu)+\int_{\delta}^{T} g(t)dt\right\}-\int_{T_1}^{T} g(t)dt \tag{34}$$

Then, the average total cost of finished product per unit time under the condition $0 \le \mu \le \delta \le T_1 \le T$ can be given by

$$TC_{m1}(T_1)=\frac{1}{T}\left(c_{m1}+c_{m2}H_{m1}+c_{m3}D_{m1}\right) \tag{35}$$

Case* 2 *when $0 \le \boldsymbol{\mu} \le \boldsymbol{T_1} \le \boldsymbol{\delta} \le T$

In this case, eq. (23) and (24) reduce to the following four:

$$\frac{dI_{m1}(t)}{dt}=(k-1)f(t)-\theta_2 I_{m1}(t) \qquad 0 \le t \le \mu \tag{36}$$

$$\frac{dI_{m2}(t)}{dt}=(k-1)d_0-\theta_2 I_{m2}(t) \qquad \mu \le t \le T_1 \tag{37}$$

$$\frac{dI_{m3}(t)}{dt}=-d_0-\theta_2 I_{m3}(t) \qquad T_1 \le t \le \delta \tag{38}$$

$$\frac{dI_{m4}(t)}{dt}=-g(t)-\theta_2 I_{m4}(t) \qquad \delta \le t \le T \tag{39}$$

with boundary conditions $I_{m1}(0)=0$, $I_{m1}(\mu_-)=I_{m2}(\mu_+)$, $I_{m3}(\delta_-)=I_{m4}(\delta_+)$ and $I_{m4}(T)=0$

The solutions of eqs. (36) - (39) are

$$I_{m1}(t)=(k-1)e^{-\theta_2 t}\int_0^t e^{\theta_2 x} f(x)dx \qquad 0 \le t \le \mu \tag{40}$$

$$I_{m2}(t)=(k-1)e^{-\theta_2 t}\left\{\int_0^{\mu} e^{\theta_2 x} f(x)dx-d_0\int_{\mu}^{t} e^{\theta_2 x}dx\right\} \qquad \mu \le t \le T_1 \tag{41}$$

$$I_{m3}(t)=e^{-\theta_2 t}\left\{d_0\int_t^{\delta}e^{\theta_2 x}dx+\int_{\delta}^{T}e^{\theta_2 x}g(x)dx\right\} \qquad T_1\le t\le\delta \tag{42}$$

$$I_{m4}(t)=e^{-\theta_2 t}\int_t^{T}e^{\theta_2 x}g(x)dx \qquad \delta\le t\le T \tag{43}$$

The cumulative inventory carried in the interval $[0.T]$ is

$$H_{m2}=\int_0^T I_m(t)dt=\int_0^{\mu}I_{m1}(t)dt+\int_{\mu}^{T_1}I_{m2}(t)dt+\int_{T_1}^{\delta}I_{m3}(t)dt+\int_{\delta}^{T}I_{m4}(t)dt \tag{44}$$

The total amount of deteriorated items during $[0.T]$ is

$$D_{m2}=\int_0^{T_1}P(\mathrm{t})dt-\int_0^{T}d(t)dt$$

$$D_{m2}=(k-1)\left\{\int_0^{\mu}f(t)dt+d_0(T_1-\mu)\right\}-\left\{d_0(\delta-T_1)+\int_{\delta}^{T}g(t)dt\right\} \tag{45}$$

Then, the average total cost of finished product per unit time under the condition $0\le\mu\le\delta\le T_1\le T$ can be given by

$$TC_{m2}(T_1)=\frac{1}{T}\left(c_{m1}+c_{m2}H_{m2}+c_{m3}D_{m2}\right) \tag{46}$$

***Case* 3 *when* $\mathbf{0}\le\boldsymbol{T_1}\le\boldsymbol{\mu}\le\delta\le T$**

In this case, eq. (23) and (24) reduce to the following four:

$$\frac{dI_{m1}(t)}{dt}=(k-1)f(t)-\theta_2 I_{m1}(t) \qquad 0\le t\le T_1 \tag{47}$$

$$\frac{dI_{m2}(t)}{dt}=-f(\mathrm{t})-\theta_2 I_{m2}(t) \qquad T_1\le t\le\mu \tag{48}$$

$$\frac{dI_{m3}(t)}{dt}=-d_0-\theta_2 I_{m3}(t) \qquad \mu\le t\le\delta \tag{49}$$

$$\frac{dI_{m4}(t)}{dt}=-g(t)-\theta_2 I_{m4}(t) \qquad \delta\le t\le T \tag{50}$$

with boundary conditions $I_{m1}(0)=0$, $I_{m2}(\mu_-)=I_{m3}(\mu_+)$, $I_{m3}(\delta_-)=I_{m4}(\delta_+)$ and $I_{m4}(T)=0$

The solutions of eqs. (47) - (50) are

$$I_{m1}(t)=(k-1)e^{-\theta_2 t}\int_0^{t}e^{\theta_2 x}f(x)dx \qquad 0\le t\le T_1 \tag{51}$$

$$I_{m2}(t) = e^{-\theta_2 t}\left\{\int_t^{\mu} e^{\theta_2 x} f(x)dx + d_0\int_{\mu}^{\delta} e^{\theta_2 x}dx + \int_{\delta}^{T} e^{\theta_2 x} f(x)dx\right\} \qquad T_1 \le t \le \mu \tag{52}$$

$$I_{m3}(t) = e^{-\theta_2 t}\left\{d_0\int_t^{\delta} e^{\theta_2 x}dx + \int_{\delta}^{T} e^{\theta_2 x} g(x)dx\right\} \qquad \mu \le t \le \delta \tag{53}$$

$$I_{m4}(t) = e^{-\theta_2 t}\int_t^{T} e^{\theta_2 x} g(x)dx \qquad \delta \le t \le T \tag{54}$$

The cumulative inventory carried in the interval $[0,T]$ is

$$H_{m3} = \int_0^T I_m(t)dt = \int_0^{T_1} I_{m1}(t)dt + \int_{T_1}^{\mu} I_{m2}(t)dt + \int_{\mu}^{\delta} I_{m3}(t)dt + \int_{\delta}^{T} I_{m4}(t)dt \tag{55}$$

The total amount of deteriorated items during $[0,T]$ is

$$D_{m3} = \int_0^{T_1} P(\mathrm{t})dt - \int_0^{T} d(t)dt$$

$$D_{m3} = (k-1)\int_0^{T_1} f(\mathrm{t})\,\mathrm{dt} - \int_{T_1}^{\mu} f(\mathrm{t})\,\mathrm{dt} - \int_{\mu}^{\delta} d_0 dt - \int_{\delta}^{T} g(t)dt \tag{56}$$

Then, the average total cost of finished product per unit time under the condition $0 \le T_1 \le \mu \le \delta \le T$ can be given by

$$TC_{m3}(T_1) = \frac{1}{T}\left(c_{m1} + c_{m2}H_{m3} + c_{m3}D_{m3}\right) \tag{57}$$

5. INTEGRATED SUPPLY CHAIN MODEL

It is obvious that the results of section 3 & 4 are non-integrated. The determination of the total cost function requires further examination of the ordering relations between δ, T_1, T . This work aims to minimize the annual integrated cost TC defined as

$$TC = TC_w + TC_m$$

We treat TC as defined in $k, T_1 > 0$. Fixing k, the following results are obtained

$$TC_1(T_1) = TC_{w1}(T_1) + TC_{m1}(T_1) \quad \text{where} \quad 0 \le \mu \le \delta \le T_1 \le T \tag{58}$$

$$TC_2(T_1) = TC_{w2}(T_1) + TC_{m2}(T_1) \quad \text{where} \quad 0 \le \mu \le T_1 \le \delta \le T \tag{59}$$

$$TC_3(T_1) = TC_{w3}(T_1) + TC_{m3}(T_1) \quad \text{where} \quad 0 \le T_1 \le \mu \le \delta \le T \tag{60}$$

By setting $f(t) = a_1 + b_1 t \ \mathit{and}\ g(t) = a_2 - b_2 t$

5.1.The optimal production policy for the Case ($\mathbf{0 \le \mu \le \delta \le T_1 \le T}$)

The results in the previous sub-section lead to the following total system cost function:

$TC_1(T_1) = TC_{w1}(T_1) + TC_{m1}(T_1)$

The first order condition for a minimum of $TC_1(T_1)$ is:

$\frac{dTC_1(T_1)}{dT_1} = h_1(T_1) = 0$ where

$$h_1(T_1) = \frac{1}{T}\left[\begin{array}{l} k\left(c_{3w} + \frac{c_{2w}}{\theta_1}\right)\left(e^{\theta_1 T_1} - 1\right)\left(a_2 - T_1 b_2\right) + b_2 T_1 (k-2)\left(\frac{c_{2m}}{\theta_2} - c_{3m}\right) \\ -c_{2m} d_0 (k-1)(\delta - \mu) \end{array}\right.$$
$$-\frac{c_{2m}}{\theta_2^2}(k-2)(a_2\theta_2 + b_2) - \frac{c_{2m}}{\theta_2^2}\left(\theta_2(a_2 - Tb_2) + b_2\right)e^{\theta_2(T-T_1)} + ka_2c_{3m} \quad (61)$$
$$\left. +c_{2m}(k-1)\left(\left(a_1\mu + b_1\mu^2\right) + \frac{e^{\theta_2\delta}}{\theta_2^2}\left(\theta_2(a_2 - \delta b_2) + b_2\right)\right)\right]$$

It is easily verified that $h_1(0) < 0$

(since $\left(c_{2m}(k-1)\left(\left(a_1\mu + b_1\mu^2\right) + \frac{e^{\theta_2\delta}}{\theta_2^2}\left(\theta_2(a_2 - \delta b_2) + b_2\right)\right) + ka_2c_{3m}\right)$

$< \left(c_{2m}d_0(k-1)(\delta-\mu) + \frac{c_{2m}}{\theta_2^2}(k-2)(a_2\theta_2 + b_2) + \frac{c_{2m}}{\theta_2^2}\left(\theta_2(a_2 - Tb_2) + b_2\right)e^{\theta_2 T}\right)$),

$h_1(T) > 0$

(Because $\left(k\left(c_{3w} + \frac{c_{2w}}{\theta_1}\right)\left(e^{\theta_1 T} - 1\right)\left(a_2 - Tb_2\right) + b_2 T (k-2)\left(\frac{c_{2m}}{\theta_2} - c_{3m}\right) + ka_2c_{3m}\right)$

$+c_{2m}(k-1)\left(\left(a_1\mu + b_1\mu^2\right) + \frac{e^{\theta_2\delta}}{\theta_2^2}\left(\theta_2(a_2 - \delta b_2) + b_2\right)\right) > \left(\begin{array}{l} c_{2m}d_0(k-1)(\delta-\mu) \\ + \frac{c_{2m}}{\theta_2^2}(k-2)(a_2\theta_2 + b_2) \end{array}\right.$

$\left. + \frac{c_{2m}}{\theta_2^2}\left(\theta_2(a_2 - Tb_2) + b_2\right)e^{\theta_2 T}\right)$
)

and $h_1^{|}(T_1) > 0$. So the derivative $dTC_1(T_1)/dT_1$ vanishes at $T_{1.1}$ with $\delta \le T_{1.1} \le T$ which is root of $h_1(T_1) = 0$. For this $T_{1.1}$, we have

$$\frac{d^2TC_1(T_1)}{dT_1^2} = k\left(c_{3w} + \frac{c_{2w}}{\theta_1}\right)\left(a_2 - \left(\theta_1 T_{1.1} + 1\right)b_2\right)e^{\theta_1 T_{1.1}} + b_2(k-2)\left(\frac{c_{2m}}{\theta_2} - c_{3m}\right)$$
$$+\frac{c_{2m}}{\theta_2}\left(\theta_2(a_2 - Tb_2) + b_2\right)e^{\theta_2(T-T_{1.1})} > 0$$

Therefore, we have

Property1. The deteriorating supply chain inventory model under the condition $\delta \leq T_1 \leq T$, $TC_1(T_1)$ obtains its minimum value at $T_1 = T_{1.1}$, where $h_1(T_{1.1}) = 0 \; if \; \delta < T_{1.1}$. On the other hand, $TC_1(T_1)$ obtains its minimum at $T_{1.1} = \delta \; if \; T_{1.1} < \delta$. Therefore, the optimal order quantity of raw material for supplier, $Q_w^* = I_{w1}(0)$

$Q_w^* = k\int_0^{T_{1,1}} e^{\theta_1 t} f(t)\,dt$, and the optimal production quantity of finished good

$$Q_m^* = \int_0^{T_{1,1}} P(\mathrm{t})dt$$

5.2.The optimal production policy for the Case $(\mathbf{0} \leq \boldsymbol{\mu} \leq \boldsymbol{T_1} \leq \boldsymbol{\delta} \leq \boldsymbol{T})$

The results in the previous sub-section lead to the following total system cost function:

$TC_2(T_1) = TC_{w2}(T_1) + TC_{m2}(T_1)$

The first order condition for a minimum of $TC_2(T_1)$ is

$\dfrac{dTC_2(T_1)}{dT_1} = h_2(T_1) = 0$ where

$$h_2(T_1) = \frac{1}{T}\left[kd_0\left(c_{3w} + \frac{c_{2w}}{\theta_1}\right)\left(e^{\theta_1 T_1} - 1\right) + (k-1)c_{2m}\left(a_1\mu + b_1\mu^2 - d_0(T_1 - \mu)\right) - 2d_0c_{2m}(\delta - T_1)\right.$$

$$\left. - c_{2m}\left(\left(\frac{a_2 - Tb_2}{\theta_2} + \frac{b_2}{\theta_2^2}\right)e^{\theta_2 T} - \left(\frac{a_2 - \delta b_2}{\theta_2} + \frac{b_2}{\theta_2^2}\right)e^{\theta_2 \delta}\right) + kd_0c_{2m}\right] \quad (62)$$

It is easily verified that $h_2(0) < 0$ (since $\left(c_{2m}\left(a_1\mu + b_1\mu^2 + d_0\mu\right) + kd_0c_{3m}\right)$

$$< c_{2m}\left(2d_0\delta + \left(\frac{a_2 - Tb_2}{\theta_2} + \frac{b_2}{\theta_2^2}\right)e^{\theta_2 T} - \left(\frac{a_2 - \delta b_2}{\theta_2} + \frac{b_2}{\theta_2^2}\right)e^{\theta_2 \delta}\right)$$

$h_2(T) > 0$ (because $\left(d_0\left(c_{3w} + \dfrac{c_{2w}}{\theta_1}\right)\left(e^{\theta_1 T} - 1\right) + c_{2m}\left(a_1\mu + b_1\mu^2 + d_0(T + \mu)\right) + d_0c_{3m}\right)$

$> c_{2m}\left(2d_0\delta + \left(\dfrac{a_2 - Tb_2}{\theta_2} + \dfrac{b_2}{\theta_2^2}\right)e^{\theta_2 T} - \left(\dfrac{a_2 - \delta b_2}{\theta_2} + \dfrac{b_2}{\theta_2^2}\right)e^{\theta_2 \delta}\right)$) and $h_2^{1}(T_1) > 0$.

So the derivative $dTC_2(T_1)/dT_1$ vanishes at $T_{1.2}$ with $\mu \leq T_{1.2} \leq \delta$ which is root of $h_2(T_1) = 0$. For this $T_{1.2}$, we have

$$\frac{d^2TC_2(T_1)}{dT_1^2} = kd_0\left(c_{2w} + \theta_1 c_{3w}\right)e^{\theta_1 T_1} - c_{2m}d_0(k-3) > 0$$

Therefore, we have

Property2. The deteriorating supply chain inventory model under the condition $\mu \leq T_1 \leq \delta$, $TC_2(T_1)$ obtains its minimum value at $T_1 = T_{1.2}$, where $h_2(T_{1.2}) = 0 \; if \; \mu < T_{1.2} < \delta$. On the other hand, $TC_2(T_1)$ obtains its minimum at $T_{1.2} = \mu \; if \; T_{1.2} < \mu$ and $TC_2(T_1)$ obtains its minimum at $T_{1.2} = \delta \; if \; \delta < T_{1.2}$. Therefore, the optimal order quantity of raw material for supplier, $Q_w^* = I_{w1}(0)$

$Q_w^* = k\int_0^{T_{1.2}} e^{\theta_1 t} f(t)\,dt$, and the optimal production quantity of finished goods

$$Q_m^* = \int_0^{T_{1.2}} P(\mathrm{t})dt$$

5.3.The optimal production policy for the Case $(\mathbf{0 \leq T_1 \leq \mu \leq \delta \leq T})$

The results in the previous sub-section lead to the following total system cost function:

$$TC_3(T_1) = TC_{w3}(T_1) + TC_{m3}(T_1)$$

The first order condition for a minimum of $TC_3(T_1)$ is:

$\dfrac{dTC_3(T_1)}{dT_1} = h_3(T_1) = 0$ where

$$h_3(T_1) = \frac{1}{T}\left[\begin{array}{l} k\left(c_{3m} + \dfrac{c_{2w}}{\theta_1}\left(e^{\theta_1 T_1} - 1\right) + \dfrac{c_{3w}}{\theta_1}\left(\theta_1 e^{\theta_1 T_1} - 1\right)\right)(a_1 + b_1 T_1) + c_{2m}\left(k - (k-1)e^{-\theta_2 T_1}\right) \\ \left(\dfrac{a_1}{\theta_2} - \dfrac{b_1}{\theta_2^2}\right) + \dfrac{c_{2m}(k-2)b_1 T_1}{\theta_2} \\ + \dfrac{kc_{3w}b_1}{\theta_1^2} - c_{2m}\left(\begin{array}{l}\left(\dfrac{a_1 + b_1\mu}{\theta_2} - \dfrac{b_1}{\theta_2^2} - \dfrac{d_0}{\theta_2}\right)e^{\theta_2\mu} \\ + \left(\dfrac{a_2 - Tb_2}{\theta_2} + \dfrac{b_2}{\theta_2^2}\right)e^{\theta_2 T} - \left(\dfrac{a_2 - b_2\delta}{\theta_2} + \dfrac{b_2}{\theta_2^2} - \dfrac{d_0}{\theta_2}\right)e^{\theta_2\delta}\end{array}\right)\end{array}\right] \quad (63)$$

It can be easily verified that $h_3(0) < 0$

(since $\left(ka_1\left(c_{3m} + \dfrac{c_{3w}}{\theta_1}(\theta_1 - 1)\right) + c_{2m}\left(\dfrac{a_1}{\theta_2} - \dfrac{b_1}{\theta_2^2}\right) + \dfrac{kc_{3w}b_1}{\theta_1^2}\right)$

$< c_{2m}\left(\left(\dfrac{a_1 + b_1\mu}{\theta_2} - \dfrac{b_1}{\theta_2^2} - \dfrac{d_0}{\theta_2}\right)e^{\theta_2\mu} + \left(\dfrac{a_2 - Tb_2}{\theta_2} + \dfrac{b_2}{\theta_2^2}\right)e^{\theta_2 T} - \left(\dfrac{a_2 - b_2\delta}{\theta_2} + \dfrac{b_2}{\theta_2^2} - \dfrac{d_0}{\theta_2}\right)e^{\theta_2\delta}\right)$),

$h_3(T) > 0$,

Because

$$k\left(c_{3m}+\frac{c_{2w}}{\theta_1}\left(e^{\theta_1 T}-1\right)+\frac{c_{3w}}{\theta_1}\left(\theta_1 e^{\theta_1 T}-1\right)\right)\left(a_1+b_1T\right)+c_{2m}\left(k-\left(k-1\right)e^{-\theta_2 T}\right)\left(\frac{a_1}{\theta_2}-\frac{b_1}{\theta_2^2}\right)$$

$$+\frac{c_{2m}\left(k-2\right)b_1T}{\theta_2}+\frac{kc_{3w}b_1}{\theta_1^2}\Bigg)>c_{2m} \qquad)$$

$$\left(\left(\frac{a_1+b_1\mu}{\theta_2}-\frac{b_1}{\theta_2^2}-\frac{d_0}{\theta_2}\right)e^{\theta_2\mu}+\left(\frac{a_2-Tb_2}{\theta_2}+\frac{b_2}{\theta_2^2}\right)e^{\theta_2 T}-\left(\frac{a_2-b_2\delta}{\theta_2}+\frac{b_2}{\theta_2^2}-\frac{d_0}{\theta_2}\right)e^{\theta_2\delta}\right)$$

and $h_3^!(T_1)>0$. So, the derivative $dTC_3(T_1)/dT_1$ vanishes at $T_{1.3}$ with $0\le T_{1.3}\le\mu$ which is root of $h_3(T_1)=0$. For this $T_{1.3}$, we have

$$\frac{d^2TC_3\left(T_1\right)}{dT_1^2}=kc_{2w}\left(\left(a_1+b_1T_1\right)e^{\theta_1T_1}+\frac{b_1}{\theta_1}\left(e^{\theta_1T_1}-1\right)\right)$$

$$+kc_{3w}\left(\left(a_1+b_1T_1\right)\theta_1e^{\theta_1T_1}+\frac{b_1}{\theta_1}\left(\theta_1e^{\theta_1T_1}-1\right)\right)$$

$$+c_{2m}\left(\frac{\left(k-2\right)b_1}{\theta_2}+\left(k-1\right)\left(a_1-\frac{b_1}{\theta_2}\right)e^{-\theta_2T_1}\right)+c_{3m}ka_1>0$$

Therefore, we have

Property3. The deteriorating supply chain inventory model under the condition $0\le T_1\le\mu$, $TC_3(T_1)$ obtains its minimum value at $T_1=T_{1.3}$, where $h_3(T_{1.3})=0\ if\ T_{1.3}<\mu$. On the other hand, $TC_3(T_1)$ obtains its minimum at $T_{1.3}=\mu\ if\ T_{1.3}\ge\mu$. Therefore, the optimal order quantity of raw material for supplier, $Q_w^*=I_{w1}(0)$

$Q_w^*=k\int_0^{T_{1.3}}e^{\theta_1 t}f\left(t\right)dt$, and the optimal production quantity of finished goods

$$Q_m^*=\int_0^{T_{1.3}}P(\mathrm{t})dt$$

Then a procedure is proposed to determine the optimal replenishment/production policy:

Step1. Find the global minimum of $TC(T_1)$, say $T_{1.1},T_{1.2},T_{1.3}$ as follow

Step2. Compute T_1 from (61), (62) and (63) if $\delta<T_1<T,\mu<T_1<\delta\ and\ 0<T_1<\mu$ respectively, then set $T_1=T_{1.1},T_1=T_{1.2}\ and\ T_1=T_{1.3}$ and compute $TC_1(T_{1.1}),TC_2(T_{1.2})andTC_3(T_{1.3})$ else go to step 3.

Step3. Find the min $\{TC_1(T_{1.1}),TC_2(T_{1.2}),TC_3(T_{1.3})\ \}$ and accordingly select the optimal value for T_1

6. NUMERICAL EXAMPLES

In this section, we provide some numerical examples to illustrate the theoretical results obtained in the previous sections.

Example 1 ($\delta < T_1 < T$). The values of input parameters are given as follow; $c_{1w} = \$200, c_{1m} = \$500, c_{2w} = \$2\ per\ unit, c_{2m} = \$5\ per\ unit, c_{3w} = \$6\ per\ unit, c_{3m} = \$8\ per\ unit,\ \theta_1 = 0.2, \theta_2 = 0.4, \mu = 2\ weeks, \delta = 4\ weeks, T = 12\ weeks, k = 2, f(t) = a_1 + b_1 t,\ g(t) = a_2 - b_2 t\ where\ a_1 = 100\ unit, b_1 = 5\ unit, a_2 = 130\ unit, b_2 = 5\ unit,$. Based on the solution procedure as the above, and with the help of software (Mathematica-7.0) from (61), we have $h_1(\delta) = -943.41 < 0\ and\ \mathrm{h}_1^{!}(T) = 2590.42 > 0$, then it yields that the optimal replenishment or production time $T_1 = T_{1.1} = 5.18\ weeks$ and the total minimum system cost $TC_1(T_{1.1}) = \$4046.42$.

Example 2 ($\mu < T_1 < \delta$). The values of input parameters are given as follow; $c_{1w} = \$200, c_{1m} = \$500, c_{2w} = \$4\ per\ unit, c_{2m} = \$1\ per\ unit, c_{3w} = 5\ per\ unit, c_{3m} = \$3\ per\ unit,\ \theta_1 = 0.2, \theta_2 = 0.4, \mu = 2\ weeks, \delta = 6\ weeks, T = 10\ weeks, k = 2, f(t) = a_1 + b_1 t,\ g(t) = a_2 - b_2 t\ where\ a_1 = 100\ unit, b_1 = 5\ unit, a_2 = 140\ unit, b_2 = 5\ unit,$. Based on the solution procedure as the above, and with the help of software (Mathematica-7.0) from (62), we have $h_2(\mu) = -768.39 < 0\ and\ \mathrm{h}_2^{!}(T)(\delta) = 376.57 > 0$, then it yields that the optimal replenishment or production time $T_1 = T_{1.2} = 4.98\ weeks$ and the total minimum system cost $TC_2(T_{1.2}) = \$3580.43$.

Example 3 ($0 < T_1 < \mu$). The values of input parameters are given as follow; $c_{1w} = \$200, c_{1m} = \$500, c_{2w} = \$3\ per\ unit, c_{2m} = \$2\ per\ unit, c_{3w} = \$5\ per\ unit, c_{3m} = \$6\ per\ unit,\ \theta_1 = 0.2, \theta_2 = 0.4, \mu = 4\ weeks, \delta = 6\ weeks, T = 10\ weeks, k = 2, f(t) = a_1 + b_1 t,\ g(t) = a_2 - b_2 t\ where\ a_1 = 100\ unit, b_1 = 5\ unit, a_2 = 150\ unit, b_2 = 5\ unit,$. Based on the solution procedure as the above, and with the help of software (Mathematica-7.0) from (63), we have$h_3(0) = -318.28 < 0\ and\ \mathrm{h}_3^{!}(T)(\mu) = 1534.19 > 0$, and then it yields that the optimal replenishment or production time $T_1 = T_{1.3} = 2.56\ weeks$ and the total minimum system cost $TC_3(T_{1.3}) = \$3945.38$.

7. CONCLUDING REMARK

In this paper, a Supplier-Manufacturer Cooperative Inventory Model for deteriorating items with Trapezoidal Type Demand and Production rate is studied. , where ordering relations between the times parameters appeared leads to three different situations. For each situation, the optimal production policy has been derived, and convexity has also been provided. An easy to use algorithm is proposed to find the optimal production policy and optimal production time, and numerical examples are studied to illustrate the model. Our model can be used for the determination of the total system cost and optimal production time when all the parties work together. This paper may be extended by using a two-parameter Weibull distribution deterioration rate and under the condition of permissible delay.

Acknowledgement: The authors would like to thank the anonymous referees and editors for their detailed and constructive comments and the first author is very much thankful to the council of scientific and industrial research (CSIR) New Delhi, India for the financial assistance in the form of Junior Research Fellowship (JRF).

REFERENCES

[1] Cleark, A.J., and Scarf, H., "Optimal policy for a multi-echelon inventory problem", *Management Science*, 6 (1960) 475-490.

[2] Donaldson, W.A., "Inventory replenishment policy for a linear trend in demand: an analytic solution", *Operational Research quarterly*, 28 (1977) 663-670.

[3] Deng, P.S., Lin, R. H. J., and Chu, P., "A note on the inventory models for deteriorating items with ramp type demand rate", *European Journal of Operational Research*, 178 (1) (2007)112–120.

[4] Goyal, S.K., and Gunasekaran, A., "An integrated production–inventory–marketing model for deteriorating items", *Computers and Industrial Engineering*, 28 (1995) 755–762.

[5] Giri, B.C., Jalan, A.K. and Chandhuri, K.S., "Economic order quantity model with weibull deterioration distribution, shortage and ramp type demand", *International Journal of System Science*, 34 (4) (2003) 237-243.

[6] Goyal, S.K., Singh, S.R. and Dem, H., "Production policy for ameliorating/deteriorating items with ramp type demand", *International Journal of Procurement Management*, 6 (4) (2013) 444-465.

[7] Hill, R.M., "Inventory model for increasing demand followed level demand", *Journal of the Operational Research Society*, 46 (1995) 1250-1259.

[8] He, Y., Wang, S.Y., and Lai, K.K., "An optimal production-inventory model for deteriorating items with multiple-market demand", *European Journal of Operational Research*, 203 (2010), 593–600.

[9] Mandal, B., and Pal, A.K., "Order level inventory system with ramp type demand rate for deteriorating items", *Journal of Interdisciplinary Mathematics*, 1(1998) 49-66.

[10] Manna, S.K., and Chandhuri, K.S., "An EOQ model with ramp type demand, time dependent deterioration rate, unit production cost and shortages", *European Journal of Operational Research*, 171 (2006) 557-566.

[11] Panda, S., Senapati, S., and Basu, M., "Optimal replenishment policy for perishable seasonal products in a season with ramp type time dependent demand", *Computer Industrial Engineering*, 54 (2008)301-314.

[12] Resh, M., Friedman, M., and Barbosa, L.C., "On a general solution of the deterministic lot size problem with time proportional demand", *Operations Research*, 24 (1976) 718-725.

[13] Skouri, K., Konstantaras, I., Papachristos, S., and Ganas, I., "Inventory models with ramp type demand rate partial backlogging and weibull deterioration rate", *European Journal of Operational Research*, 192 (2009) 79-92.

[14] Singh, S.R., and Singh, C., "Supply chain model with stochastic lead time under imprecise partially backlogging and fuzzy ramp-type demand for expiring items", *International Journal of Operational Research*, 8 (4) (2010), 511-522.

[15] Singh, N., Vaish, B., and Singh, S.R., "An EOQ model with Pareto distribution for deterioration, Trapezoidal type demand and backlogging under trade credit policy", *The IUP Journal of Computational Mathematics*, 3 (4) (2010)30-53.

[16] Skouri, K., Konstantaras, I., Papachristos, S., and Teng, J. T., "Supply chain models for deteriorating products with ramp type demand rate under permissible delay in payments", *Expert Systems with Applications*, 38 (2011) 14861-14869.

[17] Singh, N., Vaish, B., and Singh, S.R., "An economic production lot-size (EPLS) model with rework and flexibility under allowable shortages", *International Journal of Procurement Management*, 5 (1) (2012) 104-122.

[18] Sarkar, B., "An EOQ model with delay in payments and time varying deterioration rate", *Mathematical and Computer Modeling*, 55(2012) 367–377.

[19] Taleizadeh, A.A., Niaki, S.T.A., and Makui, A., "Multi-product multiple-buyer single vendor supply chain problem with stochastic demand, variable lead-time, and multi-chance constraint", *Expert Systems with Applications*, 39(2012) 5338–5348.

[20] Wu, K. S., and Ouyang, L.Y., "A replenishment policy for deteriorating items with ramp type demand rate", Proceedings of the National Science Council, ROC and Part A: Physical Science and Engineering, 24 (4) (2000) 279-286.

[21] Wu, K. S., "An EOQ inventory model with ramp type demand rate for items with weibull deterioration ramp type demand and partial backlogging", *Production Planning and Control*, 12(8)(2001)787-793.

[22] Zhau, Y.W., Min, J., and Goyal, S.K., "Supply chain coordination under an inventory level dependent demand rate", *International Journal of Production Economics*, 113 (2) (2008) 518-527.