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AN LP BASED APPROXIMATE DYNAMIC PROGRAMMING MODEL TO ADDRESS AIRLINE OVERBOOKING UNDER CANCELLATION, REFUND AND NO-SHOW

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Abstract: In this paper we simultaneously address four constraints relevant to airline revenue management problem: flight cancellation, customer no-shows, overbooking, and refunding. We develop a linear program closely related to the dynamic program formulation of the problem, which we later use to approximate the optimal decision rule for rejecting or accepting customers. First, we give a novel proof that the optimal objective function of this linear program is always an upper bound for the dynamic program. Secondly, we construct a decision rule based on this linear program and prove that it is asymptotically optimal under certain circumstances. Finally, using Monte Carlo simulation, we demonstrate that, numerically, the result of the linear programming policy presented in this paper has a short distance to the upper bound of the optimal answer, which makes it a fairly good approximate answer to the intractable dynamic program.

Keywords: Revenue Management, Approximate Dynamic Programming, Overbooking, Cancellation, No-show, Refund.

MSC: 90B85, 90C26.

1. INTRODUCTION

The nature of the airline networks is more compatible with quantity control methods and therefore, the frequency of papers published in this area is by far higher than that of pricing methods. Our Linear Program Policy , in compar-

Referred to as LPP from here on

ison to dynamic programming is practically more important and faster due to its linear computational complexity. Dynamic programming formulations for real life problems suffer from Curse of Dimensionality, whereas LPP has a far faster performance at the cost of losing a portion of the answer precision.

We offer a linear-programming-based approximate dynamic programming method to provide quality solutions to airline network revenue management under the constraints, overbooking, cancellation, refund, and no-show. Also we provide a mathematical proof that dynamic programming formulation of this problem is bounded by the aforementioned linear program. Additionally we constructed an approximate decision rule, based on our linear program and we mathematically prove it is asymptotically optimal under certain circumstances.

1.1. Literature Review

1.1.1. Pricing Papers

The main efforts made in these papers is to develop a pricing framework to maximize the generated revenue. These papers do not address the problem of rejecting or accepting flight requests and are only concerned with the pricing aspect of the product. Gallego (1997) [13] developed a general framework for multi-product network dynamic pricing and showed it's application in network revenue management. The problem was how to price the final products in a way that maximizes total revenue generated. He also provides an upper bound for the revenue generated, based on a deterministic mathematical program.

Kleywegt (2001) [19] provides an optimal control for the dynamic pricing problem. He also considers the two constraints, cancellation, and refund and assumes they are independent.

1.1.2. Quantity Control Papers

These papers in general address the problem of whether accepting or rejecting the incoming requests. They are classified as either network or single-source optimization problems.

Single-source Problems

Arslan et al. (2015) [1] in his paper addresses the single-source problem, when the time is continuous. He uses a discrete-time operator based on dynamic programming and analyzes the value function and its properties. Beckman (1958) [2] calculates the optimum sale levels when distribution function of cancellation and no-shows are given. He uses the gamma function to approximate the cancellation and no-show processes. Chatwin (1998) [6] addresses a single-source and static problem in which request for low fare flights arrive sooner than the high fare requests. He also takes into account cancellation and no-shows and tries to calculate the booking levels for this kind of flight.

Coughlan (1999) [8] introduces an overbooking model for the Irish airline company Aier Lingus. He examines his model using the real life data of this company and reveals that revenue can increase between %1 to %2. Karaesmen (2004) [18] addresses overbooking where different classes of airline are substitutable. He solves a two stage optimization problem. In the first stage flight requests are either rejected or accepted with regard to their cancellation probabilities. In the second stage customers cancel their flights and the remainder of which are allocated to different classes of airline. The allocation is optimized to maximize the revenue. Eventually he reveals that the substitutability of different classes of airline has a great impact in reduction of losses and increase of revenue.

Thompson (1961) [26] addresses the status of different airlines regarding the overbooking problem and explains the challenges they should face. He also discovers deficiencies of these systems and provides solutions for improvement.

Gosavi (2002) [17] provides a machine learning approach to address a single leg airline revenue management problem. He takes into consideration both cancellation and no-shows and solves the problem with regard to them. He states that solving this problem using dynamic programming requires a state variable with a large state space, while his method has no such constraint.

Network Problems

Network models perform capacity allocation throughout the whole network, contrary to previous models which do this for a small portion of the whole network. Single-source problems take into account only one leg of a network while network models address multiple related legs simultaneously. The collection of related legs in a network is technically referred to as a *hub and spoke* network. We will address this concept, in section 2. Majority of papers cited here reflect the fact that network revenue management increase revenue far greater than using single-source models multiple times. [23]

Mathematical Programming Models

Bertsimas (2003) [3] solves a mathematical program which is the simplified form of a dynamic program. He shows that his mathematical-programming-based algorithm has better performance than conventional bid-pricing methods which rely on Lagrange coefficients. Farias (2003) [12] states that the curse of dimensionality makes the dynamic programming of network revenue management intractable and provides a stochastic linear program to solve this problem.

Talluri (1998) [24] uses bid-pricing to solve network revenue management problem and shows that this approach is not generally optimal. He shows that if the leg capacities and flight demands grow at the same rate then the bid-pricing policy becomes asymptotically optimal. Weatherford (2012) [27] encourages researchers to use a mixed linear and dynamic program approach and shows that using these two methods simultaneously yield better results than using them separately. He implemented his method in networks with over 100 legs and provided very good results. He also shows that the running time of his optimization models shows great improvements over the previous models.

Decomposition Models

Goensch (2013) [14] addresses network revenue management problem for a special category of products called opaque products. These products reveal the airline, and departure time of the flight only after customer purchases the product. Naturally these products have lower prices than the non-opaque ones. He shows that solving this problem for large networks becomes intractable and uses decomposition to overcome this issue. He also states that opacity should be used moderately and excess in use of this technique can result in revenue loss. Goensch (2014) [15] incorporates upgrades in the formulation of his network revenue management problem and shows that conventional decomposition methods are incapable of solving this problem as the state space of this problem grows exponentially. He offers a new decomposition method that can be applied to this problem. He also mentioned that this is certainly not the only decomposition method that can be applied to this problem, and other decomposition methods are likely to yield different results. He encourages researchers to explore other decomposition methods that can be applied to his model.

Farias (2007) [11] studies an airline network in which customer arrivals follow a Markovian Stochastic Process. He approximates his dynamic programming value function with a concave function that can be decomposed into legs of the network. He shows that his model can increase revenue by up to 8% in comparison to linear programming models.

Birbil (2013) [4] decomposes the network airline revenue management problem into origins and destinations. He offers a holistic decomposition framework for airline revenue management problems and states that it can also be used for single-source problems. He also solves multiple networks and multi-source problems, using his framework and shows its good performance in comparison to previous models. His model is capable of incorporating robust optimization and customer behavior.

Erdelyi (2010) [9] addresses capacity allocation and overbooking simultaneously in an airline network. He first decomposes the problem into flight legs and shows that even for single-source problems it is still intractable. He reduces the state space into a scalar variable to overcome this issue. In the results section he reveals that his decomposition method performs better than most previous models. Kunnumkal (2008) [20] shows that if the dumping penalty function is a separable function then the optimality equation for capacity allocation with overbooking can be decomposed into flight legs. He compares his approach with a linear-programming based and a recent dynamic programming model and shows that the improvement is noticeable. He also observed improvements in duration of the solving time of the problem.

Zhang (2011) approximates the value function of an airline revenue management problem, with a non-linear non-separable function, and incorporates customer behavior, in his model. The result of his work is a non-linear problem with non-linear constraints. He shows, in his model that heuristic control policies provided can perform better than previous models.

Simulation-based Models

Simulation-based models make attempts at approximating the value function using a variety of methods.

Gosavi (2006) [16] used a simulation-based optimization for airline network capacity allocation. He takes into account realistic constraints like cancellation. His method is capable of solving both single-source and multi-leg networks. The main feature of his work is that his approach does not require a mathematical model and only needs a discrete-event simulation and a numerical optimization method like gradient ascent or a meta-heuristic model. He proposes focusing on the optimization time of the problem as one of the windows of opportunity for further research.

2. PROBLEM STATEMENT

The problem we are going to address in this paper is relevant to an airline network. This network includes some origins and destinations and one or more hubs. This configuration is referred to as *hub and spoke* network in the literature, and has fundamental differences from the traditional *point to point* network as shown in figure 1.

A leg is defined as any path from two adjacent nodes in the graph (e.g. B to Hub, or Hub to H). An itinerary is defined as a path between two nodes (wether adjacent or non-adjacent). Our resources in this problem is the number of seats available at each flight leg at the begging of the simulation. During the time horizon of our optimization, flight request arrive and our decision is either to reject or accept these requests. For each request that we accept we give up some resources at some flight legs and receive revenues proportional to the fare price of the itinerary we sold. Customers may or may not cancel their flights during the time horizon of the problem. At the end of the time horizon, not all of the customers show up to board their planes. This gives us the incentive to overbook our resources at each leg. We refund both cancellation and no-show occurrences.

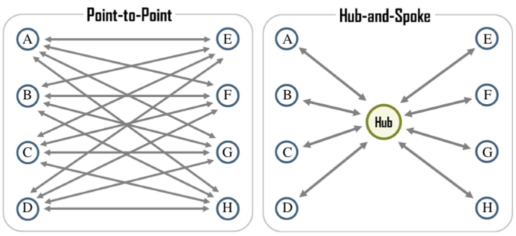

Figure 1: Point to Point versus Hub, and Spoke Airline Network

Our objective is to decide on rejecting or accepting customer requests in a way that at the end of the the time horizon we have maximized our revenue. This includes following considerations:

- We should decide wether it's better to wait for the customers with higher fare prices or avoid the risk of empty seats at the end of the time horizon and sell our resources right away.

- We should determine how many customers are not going to show up at the time of the flight, how many are going to cancel their flights and accordingly overbook our resources.
- After airline network is filled to capacity and high fare itineraries arrive we should consider dumping low fare customers and accepting recently received high fare flight requests.
- At the departure time when some seats are overbooked we should decide upon which customers to dump so we incur the least penalty while considering the capacity constraints of the airline network.

3. Problem Formulation

3.1. Parameters, and Variables

- $\{\tau, \dots, 1\}$ is the time horizon of the problem. τ represents the time we start to sell the flights and 1 represents the last time interval in which we sell our flights. Furthermore, 0 shows the time at which planes departure. Time intervals are small enough to allow at most one occurrence of cancellation or itinerary request happen (characteristic of Poisson distribution).
- L is the set that includes all the legs in the airline network.

$$L = \{1, \dots, m\}$$

- J is the set that includes all the itineraries ready to be sold in the airline network.

$$J = \{1, \dots, n\}$$

- c_i is the capacity of the ith leg.
- p_{jt} is the chance that jth itinerary receives a flight request at the time t
- q'_{jt} is the probability of cancellation occurrence at the time interval t for itinerary j. This probability is independent from cancellation at other time intervals, and other itineraries, and is also independent from flight requests for other itineraries.
- f_j is the revenue generated from selling the jth itinerary.
- r_j is the sum refunded to the person who has booked the jth itinerary, has not cancelled his flight during the time horizon of problem and has not shown up at departure time.

- r'_j is the sum refunded to the person who has booked the jth itinerary and cancels his flight before the departure time. Essentially $\forall j \;\; r_j \leq r'_j$ always holds since typically refund sums paid to cancellations are more than those of no-shows.
- q_j is the probability that the customer who has purchased the jth itinerary shows up at the departure time (provided he does not cancel his flight, during the time horizon of the problem)
- q'_j is the probability that a customer who has booked the jth itinerary cancels his flight before the end of time horizon. This probability is independent from his place in the queue of the individuals who have booked the jth itinerary and is also independent from the time interval at which he has purchased his ticket.
- a_{ij} if we accept the flight request for itinerary j, we will consume a_{ij} of our resources in leg i.
- γ_j is the penalty we incur if we dump a customer who has booked the itinerary j.
- x_{jt} is the total number of reservations for itinerary j at the start of the time interval t. x_t is the vector, that shows the status of reservations for the whole network and its elements are x_{jt}.

$$x_t = (x_{1t}, \ldots, x_{nt})$$

- x_{j0} is the total number of reservations for itinerary j right before departure time.
- z_j is the total number of people we are going to accept during the time horizon of problem (relevant to deterministic model).
- y_j is the total number of people we are going to dump at the end of the time horizon (relevant to deterministic model).
- Z_j^* is the random variable representing the total number of passengers we accept under the dynamic programming optimal policy.
- Y_j^* is the random variable representing the total number of passenger we dump at the end of time horizon under the dynamic programming optimal policy.
- S_j^* is the random variable representing the total number of passengers that board the planes under the dynamic programming optimal policy.

- $s_j(k)$ is a Bernoulli variable with parameter q_j which at the event that the kth customer of itinerary j, who *has not* cancelled his flight during the time horizon of the problem, shows up at the departure time takes value 1 and otherwise its value is 0. This is equivalent to the event that the customer who has booked the flight j and has not cancelled this flight during the time horizon of the problem, shows up at the departure time.

$$s_j(k) \sim B\big(1, q_j\big)$$

- $s'_j(k)$ is a Bernoulli variable with parameter $q_j(1-q'_j)$ which at the event that the kth customer of itinerary j shows up at the departure time takes value 1 and otherwise its value is 0. This is equivalent to the event that the customer who has booked the flight j does not cancel his flight and shows up at the flight time.

$$s_j(k) \sim B\big(1, q_j(1-q'_j)\big)$$

- $s_{j0}(x_{j0})$ is the total number of reservations that are present at the time of departure. Given that decisions of the customers to show up at the time of departure are independent from each other, $s_{j0}(x_{j0})$ has a binomial distribution with parameters, x_{j0}, and q_j.

$$\forall j \quad s_j(k) \sim B(1, q_j), \quad \forall i, j \quad \mathrm{Cov}\Big(s_j(k), s_i(k)\Big) = 0$$
$$\Rightarrow \sum_{k=0}^{x_{j0}} s_j(k) = s_{j0}(x_{j0}) \sim B(x_{j0}, q_j)$$

- $s_0(x_0)$ is the vector that shows the number of reservations, that are present at the time of departure.

$$s_0(x_0) = \Big(s_{10}(x_{10}), \ldots, s_{n0}(x_{n0})\Big)$$

- d_{jt} is the Bernoulli random variable with parameters p_{jt} which takes value 1 if request for itinerary j arrives at time interval t and otherwise its value is 0.

$$d_{jt} \sim B(1,\ p_{jt})$$

- d'_{jt} is the Bernoulli random variable with parameters q'_{jt} which takes value 1 if request for itinerary j is cancelled at time interval t and otherwise its value is 0.

$$d'_{jt} \sim B(1,\ q'_{jt})$$

- D_j is the random variable that shows the total number of requests arrived for itinerary j during the time horizon of the problem.

$$D_j = \sum_{t=1}^{\tau} d_{jt}$$

- D'_j is the random variable representative of the total number of cancelled reservations for itinerary j during the time horizon of the problem.

$$D'_j = \sum_{t=1}^{\tau} d'_{jt}$$

3.2. The Dynamic Programming Model

We define a dynamic program with the following specifics:

1. Step: Each time interval in the set $\{\tau, \dots, 1\}$ is a step for our dynamic program.
2. State: Our state variable at this mathematical model is $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$. Our state variable is a vector with n dimensions. This problem as it seems, suffers form Curse of Dimensionality.
3. Boundary Condition: We define the value function at the last interval as follows:

$$u_0(x_0) = -\mathbb{E}\{V(s_0(x_0))\}$$

in which the operator $\mathbb{E}$ represents the expected value of a random variable. We define $V(s_0(x_0))$ as follows:

$$
\begin{aligned}
V(s_0(x_0)) = \min \sum_{j=1}^{n} \gamma_j y_j + \Big(x_{j0} - s_j(x_{j0})\Big) r_j & & & (1) \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij}[s_{j0}(x_{j0}) - y_j] \le c_i & \qquad & i = 1, \dots, m & \quad (2) \\
y_j \le s_{j0}(x_{j0}) & & j = 1, \dots, n & \quad (3) \\
y_j \in \mathbb{Z}_+ & & j = 1, \dots, n & \quad (4)
\end{aligned}
$$

The objective function in problem 1 to 4 is equal to the dumping penalty incurred plus the refund sums.
Problem 1 through 4 is an integer program. The decision variable y_j is the number of people we are going to dump from itinerary j. Constraint 2 ensures the total number of people boarding the planes does not exceed airline network capacity. Constraint 3 ensures the total number of dumped people do not exceed the total number of people present at the departure time.

4. value function:

$$u_t(x_t) = \sum_{j=1}^{n} \Big[\underbrace{p_{jt}}_{P(A):\text{Arrival}} \max\big\{f_j + u_{t-1}(x_t + e_j),\ u_{t-1}(x_t)\big\}\Big] \tag{5}$$

$$+ \sum_{j=1}^{n} \Big[\underbrace{q'_{jt}}_{P(B):\text{Cancellation}} \Big[-r'_j + u_{t-1}\Big(g(x_t - e_j)\Big)\Big]\Big] \tag{6}$$

$$+ \underbrace{\Big[1 - \sum_{j=1}^{n} \big(p_{jt} + q'_{jt}\big)\Big]}_{1-P(A\cup B):\text{Neither}} u_{t-1}(x_t) \tag{7}$$

$g(x) : \mathbb{R}^{n\times 1} \to \mathbb{R}^{n\times 1}$ is the vector that its elements are Heaviside Functions:

$$g^T(x) = \Big[g_1(x_1) = H(x_1),\ g_2(x_2) = H(x_2), \ldots,\ g_n(x_n) = H(x_n)\Big]$$

5. Optimal Policy: Upon receiving a customer request we accept a request, if following holds, and reject the request otherwise.

$$f_j + u_{t-1}(x_{t-1} + e_j) \geq u_{t-1}(x_{t-1}) \tag{8}$$

We can consider the $u_{t-1}(x_{t-1}) - u_{t-1}(x_{t-1} + e_j)$ as the bidding price for itinerary j below which we do not accept any requests.

3.3. The Linear Program

The value functions in the dynamic program presented is intractable to compute and so is the bidding price. The original idea behind the linear program we are going to present is that if all stochastic variables acquire their *expected* values then we can develop a linear model to address this counterpart deterministic problem. Later we show the relationship between this linear program and the dynamic program.The linear program is as follows:

$$\max \quad \sum_{j=1}^{n} f_j z_j - \sum_{j=1}^{n} \gamma_j y_j - \sum_{j=1}^{n} \Big[(1-q_j)(1-q'_j) r_j z_j\Big] - \sum_{j=1}^{n} q'_j r'_j z_j \tag{9}$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij} \big[q_j (1-q'_j) z_j - y_j\big] \le c_i \quad i = 1, \dots, m \tag{10}$$

$$z_j \le \sum_{t=1}^{\tau} p_{jt} \quad j = 1, \dots, n \tag{11}$$

$$y_j - q_j (1-q'_j) z_j \le 0 \quad j = 1, \dots, n \tag{12}$$

$$z_j, y_j \ge 0 \tag{13}$$

The objective function in problems 9 through 13 represents the revenue gained through accepting arriving itinerary requests minus following items:

1. The total penalty we incurred because of the passengers we dumped
2. Refund to no-shows
3. Refund to cancellations

Constraint 10 ensures that the expected number of people present at the time of departure minus the number of people we are going to dump, does not exceed the capacities of our flight legs. Constraint 11 ensures that the total number of people we accept, does not exceed the expected number of people who send requests to our airline network. Constraint 12 ensures that the total number of people we reject does not exceed the expected number of people that are present at the departure time.

Linear Programming Policy

Here we provide a policy to make decisions about the dynamic program presented at 5 through 7. The dynamic program policy is essentially an optimal policy but suffers from Curse of Dimensionality and thus is intractable. The Linear Programming Policy (LPP) we provide here has linear computational complexity which makes it a good choice for practical purposes. At the same time we show that under certain circumstances the value function under this policy asymptotically converges toward the optimal value function.

We use the linear program presented at 9 to 13 to decide upon accepting or rejecting flight requests. $\mu^* = (\mu_1^*, \dots, \mu_m^*)$ is the optimal values of Lagrange Coefficients relevant to constraint 10 in problem 9 to 13. We use the μ_i^* to calculate the opportunity cost of consuming one unit of ith leg capacity.

If the revenue generated from accepting a request exceeds the sum of opportunity costs for the resources it uses across our airline network, we accept the request. Likewise if the revenue generated is greater than the expected penalty we will be paying at the event of dumping that passenger, again we accept the request. In other cases we simply reject the request.

We present the LPP in the form of following decision rule:

$$f_j \geq \min \left\{ q_j(1-q'_j)\sum_{i=1}^{m} a_{ij}\mu_i^*,\ q_j(1-q'_j)\gamma_j \right\} \tag{14}$$

The decision rule 14 comprises of two different parts. Firstly, if the opportunity cost of resources consumed is lower than the fair for itinerary requested we accept the request. Secondly, if the expected value of the penalty paid is lower than revenue generated by selling the itinerary we again accept the request. In fact, if $f_j \geq q_j(1-q'_j)\gamma_j$ holds, we can expect to generate revenue equal to $f_j - q_j(1-q'_j)\gamma_j$.

In order to use this policy we decide about the requests based on the rule 14. If the inequality 14 holds we accept the request and we update the right hand side of the problem 9 to 13 proportional to the resources consumed; otherwise we reject the request and without revising the right hand side values wait for the next request to arrive. In case of cancellation, again, we update the right hand side of the linear program. We expect the right hand side of a leg in network have higher shadow prices at the event that many itineraries use this leg or it is part of high-fare itineraries. It is logical that the customer should pay higher fares if he uses this leg in his itinerary.

3.4. Theoretic Findings

At this juncture, we provide theoretic findings that describe the relation between dynamic program in 5 to 7, and the linear program in 9 to 13.

We categorize results provided, in this section in two lemmata and one theorem. Lemma 1 explains that the optimal value of objective function in linear program 9 to 13 serves as an upper bound for the optimal value of value function 5 to 7. lemma 5 explains that value function of the optimal policy converges toward the optimal objective function of the linear program as time horizon of the problem grows.

Theorem 6 explains that the LPP policy provided here if the time horizon of the problem is large enough, is asymptotically optimal.

Lemma 1. *The optimal value of objective function of linear program is an upper bound for the value function of the dynamic program. In other words:*

$$Z_{LP} \geq u_\tau(\bar{0})$$

Proof. We define:

- Z_{LP} is the optimal objective function of the linear program.
- $u_\tau(\bar{0})$ is the optimal value function of the dynamic program

Following equations always hold:

- Total number of boarded passengers minus the total number of dumped passengers should not exceed our resources.

$$\sum_{j=1}^{n} a_{ij}(S_j^* - Y_j^*) \leq c_i \quad i = 1, 2, \ldots, m \tag{15}$$

- Total number of people accepted should be less than the total number of requests arrived.

$$Z_j^* \leq D_j \tag{16}$$

- Total number of dumped passengers should not exceed total number of boarded passengers.

$$Y_j^* \leq S_j^* \tag{17}$$

For $u_\tau(\bar{0})$ we have :

$$\begin{aligned} u_\tau(x_t = \bar{0}) = &\sum_{j=1}^{n} f_j \mathbb{E}(Z_j^*) - \sum_{j=1}^{n} \gamma_j \mathbb{E}(Y_j^*) - \\ &\sum_{j=1}^{n} (1 - q_j)(1 - q_j') r_j \mathbb{E}(Z_j^*) - \sum_{j=1}^{n} q_j' r_j' \mathbb{E}(Z_j^*) \end{aligned} \tag{18}$$

If Z_j^{**}, and Y_j^{**} are the optimal answers of the linear program we have

$$\begin{aligned} Z_{LP} = &\sum_{j=1}^{n} f_j (Z_j^{**}) - \sum_{j=1}^{n} \gamma_j (Y_j^{**}) - \\ &\sum_{j=1}^{n} (1 - q_j)(1 - q_j') r_j (Z_j^{**}) - \sum_{j=1}^{n} q_j' r_j' (Z_j^{**}) \end{aligned} \tag{19}$$

Additionally:

$$\begin{aligned} \mathbb{E}(S_j^*) &= \mathbb{E}\Big[\mathbb{E}\big(\sum_{k=1}^{Z_j^*} s_j'(k) \mid Z_j^*\big)\Big] = \mathbb{E}\Big[\mathbb{E}\big(s_j'(k)\big) Z_j^*\Big] = \mathbb{E}\Big[q_j(1 - q_j') Z_j^*\Big] \\ &= q_j(1 - q_j')\mathbb{E}(Z_j^*) \Longrightarrow \boxed{\mathbb{E}(S_j^*) = q_j(1 - q_j')\mathbb{E}(Z_j^*)} \end{aligned} \tag{20}$$

Also:

$$D_j = \sum_{t=1}^{\tau} d_{jt} \Rightarrow \mathbb{E}\{D_j\} = \sum_{t=1}^{\tau} \mathbb{E}\{d_{jt}\} = \sum_{t=1}^{\tau} p_{jt} \Longrightarrow \boxed{\mathbb{E}\{D_j\} = \sum_{t=1}^{\tau} p_{jt}} \quad (21)$$

Now calculating the expected values of 15, 16, and 17 and substituting values from 20, and 21, equations 22 through 24 result.

$$\sum_{j=1}^{n} a_{ij}\left[q_j(1-q_j')\mathbb{E}(Z_j^*) - \mathbb{E}(Y_j^*)\right] \leq c_i \qquad i = 1, \ldots, m \quad (22)$$

$$\mathbb{E}(Z_j^*) \leq \sum_{t=1}^{\tau} p_{jt} \qquad j = 1, \ldots, n \quad (23)$$

$$\mathbb{E}(Y_j^*) - q_j(1-q_j')\mathbb{E}(Z_j^*) \leq 0 \qquad j = 1, \ldots, n \quad (24)$$

Equations 22 through 24 show that vectors $\mathbb{E}(Z^*) = \Big[\mathbb{E}(Z_1^*), \mathbb{E}(Z_2^*), \ldots, \mathbb{E}(Z_n^*)\Big]$, and
$\mathbb{E}(Y^*) = \Big[\mathbb{E}(Y_1^*), \mathbb{E}(Y_2^*), \ldots, \mathbb{E}(Y_n^*)\Big]$ are feasible answers for the linear program 9 through 13. In addition we can infer from equality of 18, and 19 and the fact that the objective function of a feasible solution is less than that of the optimal solution:

$$Z_{LP} \geq Z_{\mathbb{E}(Z^*)} = u_\tau(\bar{0})$$

and proof is complete.
□

Before explaining lemma 5 we provide two definitions.

Definition 2 (Policy). *A policy is a rule for decision making and thus a policy can be dependent upon the history of a process from beginning to current time when we wish to make a decision. It can be also completely stochastic*

Example 3 (Dynamic Programming Optimal Policy). *The optimal policy in dynamic program 5 to 7 is defined as the function* $f : \mathbb{R}^{n\times 1} \longrightarrow \{0,1\}$ *where* $\mathbb{R}^{n\times 1}$ *is the state space of the vector* x_t*, and the set* $\{0,1\}$ *is representative of the decision space.* 1 *stands for accepting a flight request, and* 0 *stands for rejecting a flight request. We may show the optimal policy as follows:*

$$f(x_t) = \begin{cases} 1 & \textit{if} \quad f_j \geq u_{t-1}(x_{t-1}) - u_{t-1}(x_{t-1} + e_j) \\ 0 & \textit{Otherwise} \end{cases}$$

Definition 4 (Optimal Policy). *Policy π^* is an optimal policy only if:*

$$u_{\pi^*}(x_t) = \sup_{\pi} u_\pi(x_t)$$

where π represents all possible policies.

Lemma 5. *If the time horizon of the problem is large enough and quantity of requests arrived are large enough then the optimal value function of dynamic program converges toward optimal objective function of linear program, or:*

$$\text{if} \quad \tau \longrightarrow \infty, \; \forall \, \epsilon > 0 \quad |Z_{LP} - u_\tau(\bar{0})| < \epsilon$$

Proof. To prove this lemma we first define First Come First Served policy . FCFS policy is such that until the total number of requests accepted for itinerary j has not reached $\mathbb{E}(Z^{**})$ we accept all flight requests. Since $\tau \longrightarrow \infty$ and we have no limitation on the requests arrived we can reach the maximum capacity of each leg after a long time regardless of the arrival probabilities of the itineraries. At the end of time horizon we will not allow $\mathbb{E}(Y_j^{**})$ customers to board planes. We then have:

$$u_\tau^{FCFS}(\bar{0}) = Z_{LP} \tag{25}$$

In addition according to lemma 1 we have:

$$u_\tau^{FCFS}(\bar{0}) \leq u_\tau(\bar{0}) \leq Z_{LP} \tag{26}$$

and thus:

$$\text{if} \quad \tau \longrightarrow \infty, \quad u_\tau^{FCFS} \longrightarrow Z_{LP} \tag{27}$$

then according to 26 and squeeze theorem we will have:

$$u_\tau(\bar{0}) \longrightarrow Z_{LP} \tag{28}$$

and proof is complete.

□

Theorem 6. *LPP is asymptotically optimal if the time horizon of the problem is large enough.*

We refer to this as FCFS from here on

Proof. LPP is as follows:

$$f(x_t) = \begin{cases} 1 & \text{if} \quad f_j \geq \min \left\{ q_j(1-q'_j) \sum_{i=1}^{m} a_{ij}\mu_i^*,\ q_j(1-q'_j)\gamma_j \right\} \\ 0 & \text{Otherwise} \end{cases} \tag{29}$$

Value 1 represents accepting a request and 0 represents rejecting it. μ_i^* is the shadow prices relevant to the constraints of the following linear program 9 through 13.

We update the right hand side of this linear program upon consumption or release (flight cancellation) of resources. Since LPP uses the information received during the time horizon of the problem it will perform better than FCFS which does not use this information. Thus:

$$u_\tau^{FCFS}(\bar{0}) \leq u_\tau^{LPP}(\bar{0}) \leq u_\tau(\bar{0}) \leq Z_{LP} \tag{30}$$

Now using lemma 5, and:

$$\tau \longrightarrow \infty : \quad u_\tau^{FCFS} \longrightarrow Z_{LP}$$

Using squeeze theorem and 30:

$$\tau \longrightarrow \infty : \quad u_\tau^{LPP} \longrightarrow Z_{LP} = \sup_\pi u_\pi(x_t) \Longrightarrow \boxed{u_\tau^{LPP}(\bar{0}) = \sup_\pi u_\pi(x_t)} \tag{31}$$

Thus proof is complete □

4. NUMERIC FINDINGS

Figure 2 shows the sample airline network we use to test the performance of LPP. It comprises of one hub, one origin, and one destination. It consequently has two legs namely Origin-Hub and Hub-Destination and three itineraries Origin-Hub (itinerary 1), Hub-Origin (itinerary 2) and Origin-Destination (itinerary 3). All parameters relevant to no-shows, refunds, cancellations, arrival probabilities, itinerary fares, and dumping penalties are given in figure bellow.

We benchmark LPP performance against the First Come First Served (FCFS) policy. The results are shown in following tables. We used Monte Carlo simulation to evaluate the average performance of each algorithm. A total of 2500 samples of simulating network 4 is aggregated to create table 1 and 3. The columns show their relevant average under different cancellation rates starting from %1 to %30.

The results relevant to LPP is shown in table 3 and 4. LPP consistently offers close-to-upper-bound results except for the very high cancellation rates. The upper bound gap has a one figure value which is acceptable. The upper bound gap when the cancellation rate is exceptionally high is %11.5 which is still considerably better than the FCFS results.

Table 1: Average of FCFS policy performance (Monte Carlo simulation results)

Cancellation Probability (%)	1	5	10	20	30
Upper Bound Gap(%)	33.92	32.14	27.43	18.31	14.59
Upper Bound ($)	31514.33	31231.22	30901.25	30340.00	29957.32
Revenue Generated ($)	20824.43	21194.72	22425.96	24784.94	25585.96
Penalty Paid ($)	1474.08	1735.84	1451.76	490.80	56.00
Dumping Penalty Ratio(%)	7.51	8.70	6.81	2.04	0.24
Cancellation Loss ($)	504.41	2463.48	5327.35	11815.87	18645.24
Cancellation Loss Ratio (%)	2.43	11.65	23.86	48.04	73.62
No-Shows Loss ($)	1233.26	1106.56	1154.69	1083.05	886.06
No-Shows Loss Ratio(%)	5.85	5.17	5.14	4.37	3.48
Load Factor (%)	98.82	98.42	96.95	90.90	78.31
Itinerary 1 Dumping Rate(%)	3.40	3.83	3.77	1.57	0.25
Itinerary 2 Dumping Rate (%)	2.98	3.56	2.52	0.61	0.00
Itinerary 3 Dumping Rate (%)	0.00	0.00	0.00	0.00	0.00
Average Dumping Rate (%)	2.13	2.46	2.10	0.73	0.08
No-Shows Rate (%)	7.95	6.92	6.15	4.60	3.27
Cancellation Rate (%)	2.30	10.71	19.61	35.30	47.33
Average Optimization Time (Seconds)	1.20	1.22	1.20	1.18	1.15

Table 2: Standard deviation of FCFS policy performance (Monte Carlo simulation results)

Cancellation Probability (%)	1	5	10	20	30
Upper Bound Gap(%)	7.06	7.31	7.35	7.79	9.91
Upper Bound ($)	0.00	0.00	0.00	0	0.00
Revenue Generated ($)	2224.79	2282.05	2326.86	2364.11	2969.14
Penalty Paid ($)	1293.21	1567.44	1573.04	1006.04	367.98
Dumping Penalty Ratio(%)	7.12	8.41	7.81	4.31	1.65
Cancellation Loss ($)	446.53	949.91	1375.83	2108.13	2396.68
Cancellation Loss Ratio (%)	2.14	4.45	6.12	9.44	11.26
No-Shows Loss ($)	594.63	579.88	533.81	543.69	523.66
No-Shows Loss Ratio(%)	2.60	2.57	2.30	2.14	2.03
Load Factor (%)	1.93	2.06	3.27	5.57	7.78
Itinerary 1 Dumping Rate(%)	3.62	4.46	5.02	3.66	1.51
Itinerary 2 Dumping Rate (%)	3.56	4.34	3.71	2.08	0.00
Itinerary 3 Dumping Rate (%)	0.00	0.00	0.00	0.00	0.00
Average Dumping Rate (%)	1.82	2.19	2.25	1.47	0.50
No-Shows Rate (%)	2.49	2.23	1.80	1.53	1.21
Cancellation Rate (%)	1.26	2.51	2.99	3.27	3.15
Average Optimization Time (Seconds)	0.02	0.03	0.03	0.03	0.03

Table 3: Average performance of LPP (Monte Carlo simulation results)

Cancellation Probability (%)	1	5	10	20	30
Upper Bound Gap (%)	7.68	6.50	6.51	7.87	11.15
Upper Bound ($)	31514.33	31231.22	30901.25	30340.00	29957.32
Revenue Generated ($)	29092.84	29201.26	28888.32	27951.12	26617.83
Penalty Paid ($)	2702.00	1944.16	1311.76	450.88	28.96
Dumping Penalty Ratio(%)	9.48	6.83	4.60	1.57	0.10
Cancellation Loss	658.70	3076.53	6379.55	12890.40	19538.67
Cancellation Loss Ratio (%)	2.27	10.62	22.27	46.62	74.22
No-Shows Loss ($)	1708.08	1621.63	1498.39	1177.64	923.90
No-Shows Loss Ratio (%)	5.81	5.54	5.18	4.22	3.50
Load Factor (%)	99.26	98.57	97.00	90.40	77.61
Itinerary 1 Dumping Rate(%)	8.15	5.79	4.02	1.36	1.23
Itinerary 2 Dumping Rate (%)	8.02	5.41	3.18	0.91	0.03
Itinerary 3 Dumping Rate(%)	0.00	0.00	0.00	0.00	0.00
Network Dumping Rate(%)	5.39	3.73	2.40	0.75	0.05
No-Shows Rate (%)	10.05	8.95	7.70	5.62	4.05
Cancellation Rate (%)	2.79	12.72	24.19	43.79	59.13
Average Optimization Time (Seconds)	1.17	1.19	1.19	1.16	1.14

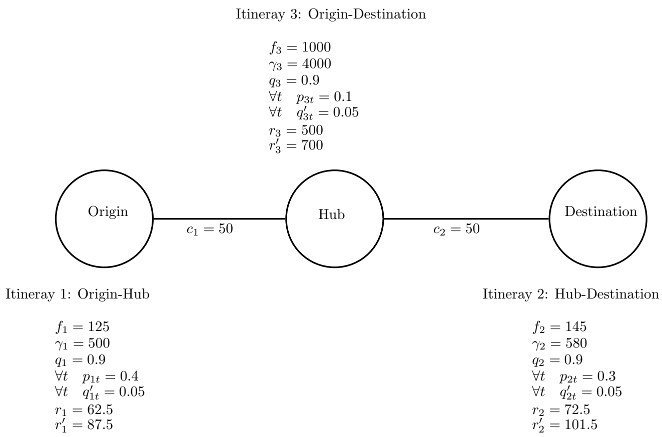

Figure 2: Sample Airline Network for benchmarking LPP performance

Table 4: Standard deviation of LPP performance (Monte Carlo simulation results)

Cancellation Probability (%)	1	5	10	20	30
Upper Bound Gap (%)	2.66	3.67	3.70	3.80	4.01
Upper Bound ($)	0.00	0.00	0.00	0.00	0.00
Revenue Generated ($)	839.47	1146.12	1144.60	1154.04	1201.05
Penalty Paid ($)	2350.44	1798.67	1544.32	974.96	257.95
Dumping Penalty Ratio(%)	8.43	6.52	5.41	3.45	0.84
Cancellation Loss	615.82	1267.98	1749.28	2111.02	2335.58
Cancellation Loss Ratio (%)	2.15	4.46	6.46	8.95	11.60
No-Shows Loss ($)	809.66	774.10	699.83	600.20	516.08
No-Shows Loss Ratio (%)	2.59	2.59	2.34	2.07	1.94
Load Factor (%)	1.55	2.19	3.58	6.31	7.89
Itinerary 1 Dumping Rate(%)	2.19	1.74	0.91	0.63	0.36
Itinerary 2 Dumping Rate (%)	2.02	1.33	0.90	0.49	0.02
Itinerary 3 Dumping Rate(%)	0.00	0.00	0.00	0.00	0.00
Network Dumping Rate(%)	1.40	1.02	0.60	0.37	0.13
No-Shows Rate (%)	10.05	8.95	7.70	5.62	4.05
Cancellation Rate (%)	1.77	3.15	3.58	3.90	3.47
Average Optimization Time (Seconds)	0.03	0.03	0.04	0.02	0.03

Optimization time is one of the most important metrics in evaluation of an algorithm's performance. This point becomes more relevant after we consider the fact that commercial revenue management models need to present results in real-time. The calculation time under Bellman Equation has exponential computational complexity as more seats are added to flight legs since state space too, grows exponentially. This makes the dynamic programming intractable in large Airline Networks.

Figure 3 shows the average optimization time as a function of number of seats at each flight leg and time horizon of the problem. The average optimization time under LPP is clearly independent from number of seats at each flight leg. Furthermore with time horizon of the problem increasing optimization time grows linearly. In fact the optimization time has no non-linear relationship with these two factors.

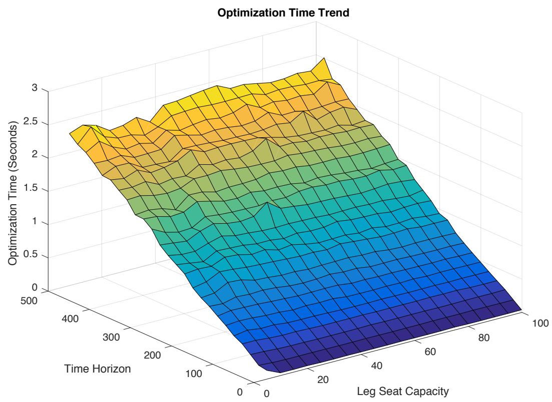

Figure 3: Average optimization time as a function of time horizon, and flight leg seats

5. CONCLUSIONS and SUGGESTIONS

In this paper we simultaneously addressed the overbooking, no-shows, cancellation, and refunding constraints in airline revenue management problem. We developed a dynamic program to address this problem and we showed it follows a bid-pricing policy for which the value function is intractable. We created a closely related linear program to tackle this challenge and proved that our linear programming formulation serves as an upper bound to the dynamic program. In addition, we constructed a decision rule based on our linear program to approximate the optimal decision rule of the dynamic program. We showed that under certain circumstances our decision rule is asymptotically optimal. Finally, we provided numerical results to represent the quality of our approximate solution. For further research we propose inclusion of upgrades, group arrivals and capacity substitutability in this model as it will add further to how realistic it is in addressing airline revenue management networks.

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	Objective function			deviation (%)		Running time	
p	Best known	VNS	VNS+	VNS	VNS+	VNS	VNS+
2	815313.31	815313.31	815313.31	0.00	0.00	0.08	0.00
3	551062.88	551062.88	551062.88	0.00	0.00	0.41	0.25
4	288191.00	288191.00	288191.00	0.00	0.00	0.00	0.01
5	209068.80	209068.80	209068.80	0.00	0.00	0.00	0.00
6	180488.20	180488.22	180488.22	0.00	0.00	0.00	0.00
7	163704.17	163704.17	163704.17	0.00	0.00	0.00	0.27
8	147050.80	147050.80	147050.80	0.00	0.00	0.86	0.48
9	130936.12	130936.13	130936.13	0.00	0.00	0.83	0.56
10	115339.03	115339.03	115339.03	0.00	0.00	5.95	0.00
11	100133.20	100133.20	100133.20	0.00	0.00	0.00	0.00
12	94152.05	94152.05	94152.05	0.00	0.00	0.00	2.94
13	89376.81	89454.76	89454.76	0.09	0.09	2.00	3.22
14	84807.67	84807.67	84807.67	0.00	0.00	6.58	9.33
15	80177.04	80177.04	80198.74	0.00	0.03	4.59	5.59

Table 5: ARLS results for $n = 654$ and small values of p with $\alpha = 1/2$ and $g = 3$