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ON RESERVE AND DOUBLE COVERING PROBLEMS FOR THE SETS WITH NON-EUCLIDEAN METRICS

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Abstract: The article is devoted to Circle covering problem for a bounded set in a two-dimensional metric space with a given amount of circles. Here we focus on a more complex problem of constructing reserve and multiply coverings. Besides that, we consider the case where covering set is a multiply-connected domain. The numerical algorithms based on fundamental physical principles, established by Fermat and Huygens, are suggested and implemented. This allows us to solve the problems for the cases of non-convex sets and non-Euclidean metrics. Preliminary results of numerical experiments are presented and discussed. Calculations show the applicability of the proposed approach.

Keywords: Covering Problem, Fermat Principle, Huygens Principle, Wave Front, Non-Euclidean Metric, Reserve Covering, Double Covering, Computational Experiment.

MSC: 65K10, 90B06.

1. INTRODUCTION

Circle covering is the dual problem of circle packing problem. The covering problem means to locate congruent geometric objects in a metric space so that its given area lies entirely within their union. Usually, scientists deal with the circles covering problem (CCP) in special cases, when the covered area is a square, a circle, a rectangle or a regular triangle. In fact, there are two popular directions of studying this problem. The first one is to improve the best-known covering without any proof of optimality; the second is to verify the optimality of a particular circle configuration. Some analytical results can be found in [12, 13, 19, 22].

The first computer results for circle coverings in the plane were obtained by Zahn [28]. He divided a big circle into small cells and applied local optimization methods to cover as many of these cells as possible while keeping the radius of the small circles constant. The computational results on coverings of a square seems to be first presented by Tarnai and Gaspar, who found coverings with up to 10 circles [25].

Algorithms for covering of simply connected sets by congruent circles employing quasi-differentiability of the objective function are presented in [14], heuristic and metaheuristic methods can be found in [1, 5, 20, 27], algorithms of integer and continuous optimization are proposed by [19, 21, 22], geometric methods are suggested in [26]. A modification of feasible directions' method appears in [23] where optimal coverings are given for different $n \leq 100$. Note that the approach proposed in [19] is based on a simulated annealing algorithm using the Voronoi tessellation of the square with respect to the centers of the circles. A lot of computational results for the circle covering problem are available on the web, for example, Erich's Packing Center web-site.

This theoretical problem is widely used in solving practical tasks in various fields of human activity. Examples of such tasks are placement of cell towers, rescue points, police stations, ATMs, hospitals, schools [6, 8, 10, 11], designing energy-efficient monitoring of distributed objects by wireless sensor networks [2, 3, 4, 9, 24] etc.

Note that most of the known results are obtained for the case when covered areas or containers are subsets of the Euclidean plane or a multi-dimensional Euclidean space. In the case of a non-Euclidean metric, covering and packing problems are relatively poorly studied. In this paper, the authors deal with the circles covering problems of a multiply connected set with a non-Euclidean metric. We present a numerical algorithm for solving this problem and expand it to problems of reserve and multiple covering constructing. A similar multiply covering problem is considered in [11]. These problems appear in infrastructure logistics where there is a main servicing system and it is necessary to create a duplicate system to provide service in the case of failure of one or more (even all) nodes.

2. FORMULATION

Assume we are given a metric space X, a bounded domain $D \subset X$, compact sets $B_k \subset D, k = 1, ..., m$, and n of covering circles $C_i(O_i)$ with the centers $O_i = (x_i, y_i)$, $i = 1, ..., n$. Let $0 < f(x, y) \leq \beta$ be a continuous function, which makes sense of the instantaneous speed of movement at every point of D and $B = \bigcup\limits_{k=1}^{m} B_k$. Then, we define a closed multiply-connected set M:

$$M = \mathrm{cl}\,(D \setminus B) \subset X \subseteq \mathbb{R}^2 \ . \tag{1}$$

Here cl is the closure operator.

The distance in space X is determined as follows:

$$\rho(a, b) = \min_{\Gamma \in G(a,b)} \int\limits_{\Gamma} \frac{d\Gamma}{f(x, y)} \ , \tag{2}$$

where $G(a, b)$ is the set of all continuous curves, which belong to X and connect the points a and b. In other words, the shortest route between two points is a curve, that requires the least time to be spent.

Definition 1. $P_n(r) = \bigcup\limits_{i=1}^{n} C_i(O_i)$ *is a covering of multiply-connected set* M *with the radius* r *if* $\forall i = 1, ..., n : O_i \in M$ *and* $P_n(r) \cap D = D$.

Definition 2. $P_n^*(r)$ *is an optimal covering of* M *if* r *is minimal.*

Thus, it is necessary to find a partition of D on n segments D_i, $i = 1, ..., n$, and the location of the circles centers $\{O_i^*\} \in M$, which provide minimum for

$$R_* = \max_{i=\overline{1,n}} \rho\,(O_i, \partial D_i)\,, \tag{3}$$

where $\rho\,(O_i, \partial D_i)$ is the distance from the circle center O_i to the closed boundary of the corresponding segment D_i

$$\rho(O_i, \partial D_i) = \min_{x \in \partial D_i} \rho(O_i, x) \ . \tag{4}$$

Definition 3. *A covering* $P_a^*(r') = \bigcup\limits_{j=1}^{a} C_j(O_j)$ *is called a d-reserve covering of* M *for* $P_n^*(r)$ *if* $\forall \overline{1, n}, \forall j = \overline{1, a} : \rho(O_i, O_j) \geq d$.

Definition 4. *A covering* $P_{nb}(r, r') = P_n^*(r) \cup P_b^*(r'')$, *which consists of* n *circles* C_i *of the radius* r *and* b *circles* C_l *of the radius* r'' *is called a double covering of* M *if* $\forall Z \in D$: $Z \in C_i$ *and* $Z \in C_l$, $i, l = \overline{1, n + b}, i \neq l$ *and* $\forall i = \overline{1, n}, \forall l = \overline{1, b} :$ $\rho(O_i, O_l) \geq d$.

In other words, every point of D must be covered at least by two circles.

3. SOLUTION METHOD

In this section, the authors propose methods for constructing d-reserve and double coverings, based on the analogy between the propagation of the light wave, and finding the minimum of the functional integral (2). This analogy is a consequence of the physical laws of Fermat and Huygens. This approach is described more detailed in [15, 16, 17, 18].

Algorithm for circles covering constructing

1. Randomly generate initial coordinates of the circles centers $O_i \in M$, $i = \overline{1,n}$.
2. From $O_i, i = \overline{1,n}$, we initiate the light waves using the algorithm [15]. It allows us to divide set D on n segments D_i and to find their boundaries ∂D_i, $i = \overline{1,n}$.
3. Boundary ∂D_i of segment D_i is approximated by the closed polygonal line with nodes at the points $A_l, l = \overline{1,q}$.
4. From $A_l, l = \overline{1,q}$, we initiate the light waves using the algorithm [15] as well.
5. Every point $(x, y) \in D_i$, first reached by one of the light waves is marked (here and further on, we assume using an analytical grid for x and y). We memorize time $T(x, y)$, which is required to reach (x, y).
6. Find $\bar{O}_i = \underset{(x,y)\in D_i \setminus B}{\arg\max}\ T(x, y)$. Then, the minimum radius of a circle that covers D_i, is given by

$$R_{i\,\min} = \max_{l=\overline{1,q}} \rho(\bar{O}_i, A_l)\ .$$

   Steps 3–6 are carried out independently for each segment $D_i, i = \overline{1,n}$.
7. Find $R_{\min} = \max\limits_{i=1,..n} R_{i\,\min}$. Then go to step 2 with $O_i = \bar{O}_i$, $i = \overline{1,n}$.
   Steps 2–7 are being carried out while $R_{\min}$ is decreasing, then the current covering

$$P_n = \bigcup_{i=1}^{n} C_i(\bar{O}_i, R_{\min})$$

   is memorized as a solution.
8. The counter of initial coordinates generations $Iter$ is incremented. If $Iter$ becomes equal to some preassigned value, then the algorithm is terminated and $P_n^* = P_n$ be a solution with $r = R_{\min}$. Otherwise, go to step 1.

Using this algorithm as the base, we introduce two following algorithms.

Algorithm for d-reserve covering constructing

1. Using the algorithm for circles covering constructing, we obtain the optimal covering $P_n^*(r)$ for the set M, which consists of circles C_i with centers in O_i, $i = \overline{1,n}$.

2. From each $O_i, i = \overline{1,n}$, we initiate the light wave to find such a front that is d units distant from O_i. Then we get the set $H = \{s(x,y) : \rho(s, O_i) \leq d, \forall i = \overline{1,n}\}$.
3. Using the algorithm for circles covering constructing, we construct the optimal covering $P^*_a(r')$ for the set $M \setminus H$, which consists of circles C_j with centers in O_j, $j = \overline{1,a}$.

Note that r and r' are different.

Algorithm for double covering constructing

Here the first and the second step are similar to the corresponding steps of the previous algorithm.

1. Using the algorithm for circles covering constructing, we obtain the optimal covering $P^*_n(r)$ for the set M, which consists of circles C_i with centers in O_i, $i = \overline{1,n}$.
2. From each $O_i, i = \overline{1,n}$, we initiate the light wave to find such a front that is d units distant from O_i. Then we get the set $H = \{s(x,y) : \rho(s, O_i) \leq d, \forall i = \overline{1,n}\}$.
3. Find set L, all points of which belong to only one circle of $P^*_n(r)$:

$$V = \left\{ v(x,y) : v(x,y) \in M \setminus \bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} C_i(O_i, r) \cap C_j(O_j, r) \right\}.$$

4. Using the algorithm for circles covering constructing, we construct the optimal covering $P^*_b(r'')$ for the set $M \setminus (V \cup H)$, which consists of circles C_j with centers in O_j, $j = \overline{1,b}$.

4. COMPUTATIONAL EXPERIMENTS

Testing of the algorithm proposed in the previous section was carried out using the PC of the following configuration: Intel (R) Core i7-5500U (2.4 GHz, 8 GB RAM) and Windows 10 operating system. The algorithm is implemented in C# using the Visual Studio 2013.

Example 1. This example considers the case when the metric is given by formula (2), where $f(x,y) = v_0(1 + ky)$, v_0, k are the given constants (here $v_0 = 1, k = 0.1$). It means that the speed of wave propagation increases linearly along the coordinate y. The computational results are presented in Table 1 and Fig. 1. Here R is the radius of the best covering, R_0 is the radius of the best covering without barriers, ΔR is a relative error, t is computation (processor) time. The origin is located in the upper left corner.

Recall that the radius R here means the time of moving from center to the boundary of the circle.

Table 1: Covering of a convex polygon with barriers

n	R	R_0	$\Delta R(\%)$	$t(sec)$
1	8,348	6,443	29,567	2,34
2	6,810	5,136	32,593	5,50
3	5,683	4,182	35,892	11,86
4	4,836	3,54	36,610	14,15
5	4,364	3,09	41,230	20,22
6	4,011	2,832	41,631	22,41
7	3,632	2,678	35,624	35,73
8	3,348	2,442	37,101	50,95
9	3,252	2,259	43,958	98,48

It is easy to observe that the results correspond to the theoretical ones obtained by A. Borovskikh [7]; he proved that covering objects are circles with displaced centers for the given metric.

Note, the presence of barriers (grey polygons in Fig. 1) substantially (30-45%) impairs the radius of covering.

Example 2. This example shows the effect of small perturbation on the solution. Let $f(x,y) = \dfrac{x^2+y^2}{x^2+y^2+1} + 0.1$. This metric is close to the Euclidean with the distance from the origin. Parameter $d = 5$ and number of iterations $Iter = 100$. In Table 2, R_1 and R_2 are radii of double covering for the given metric, ΔR_1 and ΔR_2 are deviations from the case with Euclidean metric. In Fig. 2, thin circles have radius R_1, bold ones have radius R_2, grey ones show forbidden zones for double covering.

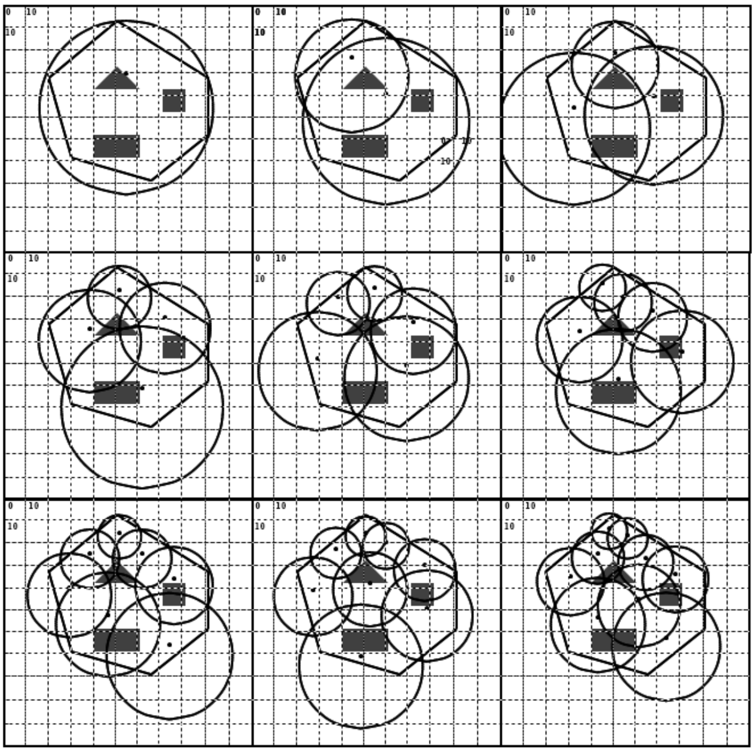

Figure 1: Covering of the convex polygon with linear metric

Table 2 shows that the deviations are rather small and increase insignificantly with increasing the number of circles. Computation time is acceptable.

Example 3. The metrics, like those described below, arise in infrastructure logistics when we want to locate some objects in the highlands. Here the speed of movement depends on the angle of ascent or descent. Therefore, the wavefronts are strongly distorted.

$$a_1(x,y) = \frac{(x-2.5)^2+(y-2.5)^2}{1+(x-2.5)^2+(y-2.5)^2}, f_1(x,y) = \begin{cases} 0, a_1(x,y) \geq 0.8, \\ a_1(x,y), \end{cases}$$

$$a_2(x,y) = \frac{(x-2.5)^2+(y-7.5)^2}{1+(x-2.5)^2+(y-7.5)^2}, f_2(x,y) = \begin{cases} 0, a_2(x,y) \geq 0.8, \\ a_2(x,y), \end{cases}$$

$$a_2(x,y) = \frac{(x-7.5)^2+(y-2.5)^2}{1+(x-7.5)^2+(y-2.5)^2}, f_3(x,y) = \begin{cases} 0, a_3(x,y) \geq 0.8, \\ a_3(x,y), \end{cases}$$

$$a_3(x,y) = \frac{(x-7.5)^2+(y-7.5)^2}{1+(x-7.5)^2+(y-7.5)^2}, f_4(x,y) = \begin{cases} 0, a_4(x,y) \geq 0.8, \\ a_4(x,y), \end{cases}$$

$$F(x,y) = f_1(x,y) + f_2(x,y) + f_3(x,y) + f_4(x,y),$$

Table 2: The double covering of a convex polygon

n	b	R_1	R_2	$\Delta R_1(\%)$	$\Delta R_2(\%)$	$t(sec)$
1	1	27,288	28,064	0,010	0,000	4,02
1	2	27,288	17,297	0,010	0,015	5,15
1	3	27,288	14,799	0,010	0,014	7,34
1	4	27,288	11,818	0,010	0,026	10,01
2	1	17,297	29,598	0,015	0,170	7,02
2	2	17,297	17,930	0,015	0,223	9,24
2	3	17,297	15,693	0,015	0,016	11,35
2	4	17,297	11,957	0,015	0,009	17,78
3	1	14,799	27,288	0,014	0,010	11,9
3	2	14,799	17,491	0,014	0,572	15,09
3	3	14,799	15,903	0,014	0,223	19,58
3	4	14,799	12,129	0,014	0,862	21,2
4	1	11,818	27,288	0,026	0,010	17,89
4	2	11,818	16,589	0,026	0,024	19,05
4	3	11,818	14,799	0,026	0,010	24,67
4	4	11,818	13,016	0,026	0,077	27,31

$$f(x,y) = \begin{cases} 0.4, 0 < F(x,y) \leq 0.4, \\ F(x,y), \\ 0.8, F(x,y) = 0. \end{cases}$$

Here we construct d-reserve and double coverings for the same set to compare their radii. Summary results are presented in Table 3 and Fig. 3. In Table 3, R_1 is the radius of the best covering, R_2 is the radius of the reserve covering, R_3 is the radius of double covering, $d = 15$, $Iter = 100$.

It is easy to see that $R_3 \leq R_2$ because to construct the double covering, we solve a regular circle covering problem twice. First, we solve it for the set M, and then for set $Q = \{(x,y) \in C_i, (x,y) \notin C_j, i,j = 1,...,n, i \neq j\}$. The area of Q is obviously smaller than that of M. Note, if $d = 0$, then $R_3 \leq R_1$.

Table 3: Optimal reserve and double covering

n	R_1	R_2	R_3
1	82,52004	112,21367	112,21367
2	77,16915	91,95601	89,83781
3	70,68287	73,55724	71,53404
4	59,00788	62,48673	61,49187

5. CONCLUSIONS

The circles covering problem, which is one of the classical mathematical problems, was considered in a large number of papers. As a rule, authors studied

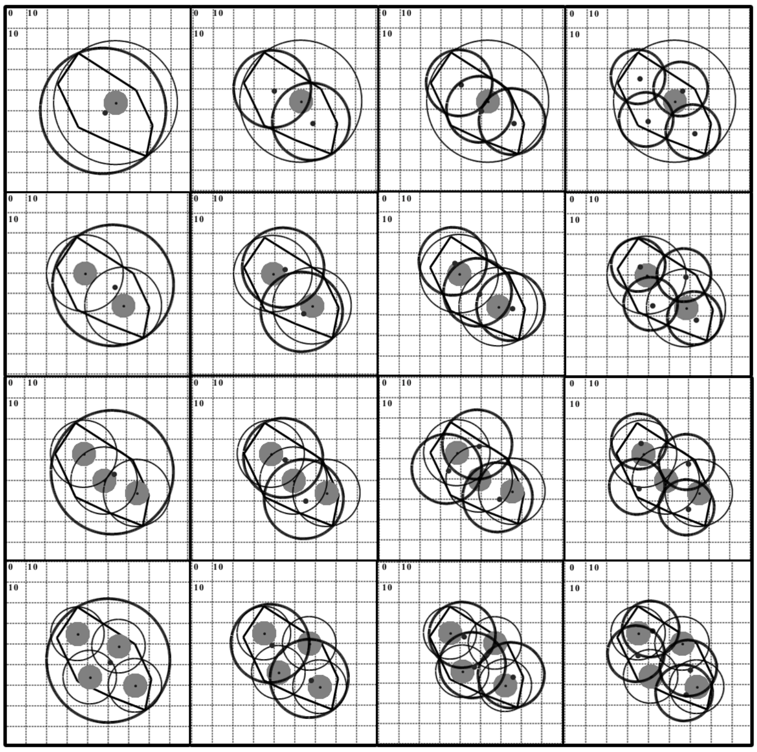

Figure 2: The double covering of a convex polygon

convex sets and Euclidean distances between points. The case of a non-Euclidean metric has rearly been studied, especially if a multiply connected set is covered. Such a problem is the subject of consideration in this article. Moreover, we address a more complex problem of constructing reserve and multiply coverings. We present two methods, based on optical-geometrical approach, which expand our previous results.

The results of the computational experiment make it possible to conclude that the proposed approach is applicable. In particular, our computational results coincide with the theoretical ones (if they exist).

Finally, note that although the proposed algorithms are used for illustration purposes, they can be applied to continuous location problems where the objective is to locate n facilities in order to service a given domain M in some "optimal" way.

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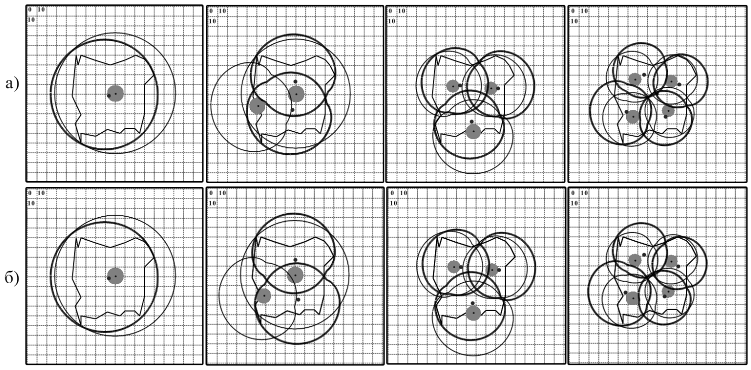

Figure 3: a) Reserve covering b) Double covering

REFERENCES

[1] Al-Sultan, K.S., Hussain, M.F., and Nizami, J.S., "A genetic algorithm for the set covering problem", *Journal of the Operational Research Society*, 47 (5) (1996) 702–709.

[2] Aldynool, T.A., Erzin, A.I., and Zalyubovskiy, V.V., "The coverage of a planar region by randomly deployed sensors", *Journal of Mathematical Sciences*, 10 (4) (2010) 7–25.

[3] Astrakov, S.N., and Erzin, A.I., "Efficient band monitoring with sensors outer positioning", *Optimizat. J. Math. Program and Operat. Res.*, 62 (10) (2013) 1367–1378.

[4] Astrakov, S.N., Erzin, A.I., and Zalyubovskiy, V.V., "Sensor networks and covering of plane by discs", *Journal of Applied and Industrial Mathematics*, 16 (3) (2009) 3–19.

[5] Azimi, Z.N., Toth, P., and Galli, L., "An electromagnetism metaheuristic for the unicost set covering problem", *European Journal of Operational Research*, 205 (2) (2010) 290–300.

[6] Bánhelyi, B., Palatinus, E., and Lévai B.L., "Optimal circle covering problems and their applications", *Central European Journal of Operations Research*, 23 (2015) 815–832.

[7] Borovskikh, A.V., "The two-dimensional eikonal equation", *Siberian Mathematical Journal*, 47 (5) (2006) 813–834.

[8] Brusov, V.S., and Piyavskii, S.A., "A computational algorithm for optimally covering a plane region", *Computational Mathematics and Mathematical Physics*, 11 (2) (1971) 17–27.

[9] Cardei, M., Wu, J., and Lu, M., "Improving netwotk lifetime using sensors with ad-justible sensing ranges", *Int. J. Sensor Networks*, 1 (2006) 41–49.

[10] Drezner, Z., *Facility location: A survey of applications and methods.* Springer-Verl, New York, 1995.

[11] Galiev, S.I., and Karpova M.A., "Optimization of multiple covering of a bounded set with circles", *Computational Mathematics and Mathematical Physics*, 50 (4) (2010) 721–732.

[12] Heppes, A., "Covering a planar domain with sets of small diameter", *Periodica Mathematica Hungarica*, 53 (2006) 157–168.

[13] Heppes, A., and Melissen, J.B.M., "Covering a rectangle with equal circles", *Periodica Mathematica Hungarica*, 34 (1997) 65–81.

[14] Jandl, H., and Wieder, K., "A continuous set covering problem as a quasidifferentiale optimization problem", *Optimizat. J. Math. Program and Operat. Res.*, 19 (6) (1988) 781–802.

[15] Kazakov, A.L., and Lempert, A.A., "An approach to optimization in transport logistics", *Automation and Remote Control*, 72 (7) (2011) 1398–1404.

[16] Kazakov, A.L., Lempert, A.A., and Bukharov, D.S., "On segmenting logistical zones for servicing continuously developed consumers", *Automation and Remote Control*, 74 (6) (2013) 968–977.

[17] Lempert, A.A., Kazakov, A.L., and Bukharov, D.S., "Mathematical model and program system for solving a problem of logistic object placement", *Automation and Remote Control*, 76 (8) (2015) 1463–1470.

[18] Lempert, A.A., and Kazakov, A.L., "On mathematical models for optimization problem of logistics infrastructure", *International Journal of Artificial Intelligence*, 13 (1) (2015) 200–210.

[19] Melissen, J.B.M., and Schuur, P.C., "Covering a rectangle with six and seven circles", *Discrete Appl. Math.*, 99 (2000) 149–156.

[20] Nasab, H.H., Tavana, M., and Yousef M., "A new heuristic algorithm for the planar minimum covering circle problem", *Production & Manufacturing Research*, 2 (1) (2014) 142–155.

[21] Nurmela, K.J., "Conjecturally optimal coverings of an equilateral triangle with up to 36 equal circles", *Experiment. Math.*, 9 (2) (2000) 241–250.

[22] Nurmela, K.J., and Ostergard, P.R.J., "Covering a square with up to 30 equal circles", Technical report, Lab. Technol. Helsinki Univ., 2000.

[23] Stoyan, Y.G., and Patsuk, V.M., "Covering a compact polygonal set by identical circles", *Comput. Optim. Appl.*, 46 (2010) 75–92.

[24] Tabirca, T., Yang, L.T., and Tabirca, S., "Smallest number of sensors for k-covering", *Int. J. Comput. Commun. Control*, 8 (2) (2013) 312–319.

[25] Tarnai, T., and Gaspar, Zs., "Covering a square by equal circles", *Elem. Math.*, 50 (1995) 167–170.

[26] Ushakov, V.N., and Lebedev, P.D., "Algorithms for the construction of an optimal cover for sets in three-dimensional euclidean space", *Proc. of the Steklov Institute of Mathematics*, 293 (S1) (2016) 225–237.

[27] Watson-Gandy, C.D.T., "Heuristic procedures for the m-partial cover problem on a plane", *European Journal of Operational Research*, 11 (2) (1982) 149–157.

[28] Zahn Jr., C.T., "Black box maximization of circular coverage", *Journal of Research of the National Bureau of Standards*, 66 (1962) 181–216.